About families of orthogonal polynomials satisfying Heun's differential equation.

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October 18, 2020 version 0.3 (incomplete and unfinished)

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Abstract. We consider special families of orthogonal polynomials satisfying differential equations. Besides known hypergeometric cases, we look especially for ***Heun***'s differential equations.We show that such equations are satisfied by orthogonal polynomials related to some classical weight functions modified by Dirac weights or by division of powers of binomials. An appropriate set of biorthogonal rational functions, or 2-point Padé approximations, is also described.

Keywords: orthogonal polynomials, Padé approximation, Heun's differential equation.

2010 Mathematics Subject Classification:
34M35 Differential equations in the complex domain, singularities, monodromy, local behavior of solutions, normal forms, 34M55 Painlevé and other special equations; classification, hierarchies; 41A21 Padé approximation 42C05 Orthogonal functions and polynomials, general theory, 81Q05 Closed and approximate solutions to the Schrödinger eq. etc.

1. Introduction.

Classical orthogonal polynomials satisfy hypergeometric differential equations. Their description fills many textbooks and handbooks [1, 19] [82, §18.3] etc.

The hypergeometric differential equations are second order differential equations of Fuchsian type with three singular points. The usual canonical setting sends one singular point to ∞ and the equation is

$$(u-\mathfrak{a})(u-\mathfrak{b})\left\{\frac{d^2Y}{du^2}+\left[\frac{\gamma}{u-\mathfrak{a}}+\frac{\delta}{u-\mathfrak{b}}\right]\frac{dY}{du}\right\}-\lambda Y=0. \tag{1}$$

For instance Jacobi polynomials, orthogonal with respect to the weight function $(1-x)^p(1+x)^q$ on $(-1,1)$ satisfy (1) with $\mathfrak{a}=-1, \mathfrak{b}=1, \gamma=q+1, \delta=p+1, \lambda=n(n+p+q+1)$ [1, 22.6.1] [82, §18.3] etc.

The simplest second order differential equation of Fuchsian type just above the hypergeometric one is the Heun's equation, after Karl Heun 1859-1929 [89]. It has now four singular points and, when one of them is sent to ∞, has the form

$$(u-\mathfrak{a})(u-\mathfrak{b})(u-\mathfrak{c})\left\{\frac{d^2Y}{du^2}+\left[\frac{\gamma}{u-\mathfrak{a}}+\frac{\delta}{u-\mathfrak{b}}+\frac{\epsilon}{u-\mathfrak{c}}\right]\frac{dY}{du}\right\}+(Pu-Q)Y=0. \qquad (2)$$

Near $\mathfrak{a}, \mathfrak{b}$, or $\mathfrak{c}$, a solution behaves like a power $(u-\mathfrak{a})^\rho$ etc. with $\rho(\rho-1)+\gamma\rho=0$, whence a regular solution ($\rho=0$) and a solution showing the exponent $1-\gamma$, and similar results at the other singular points.

When $u\to\infty$, a u^ρ behaviour implies now $\rho(\rho-1)+(\gamma+\delta+\epsilon)\rho+P=0\Rightarrow\rho=-\alpha$ and $-\beta$, where $\alpha+\beta=\gamma+\delta+\epsilon-1$ and $\alpha\beta=P$ [61, Example 3, p. 394], [11,27,37,58,60,87,89,92], that's where the symbols α and β are used, and they will not be found with another meaning here.

However, orthogonal polynomials related to what seems the simplest generalization of a Jacobi weight function $w(x)=x^p(1-x)^q(x-\mathfrak{t})^r$ are shown to satisfy a differential equation more complicated than (2). This will be considered in § 2.2.

Non classical orthogonal polynomials satisfying differential relations and equations have been studied in the 19th century by Laguerre [73] these polynomials are sometimes called semi-classical, the corresponding measure may contain point masses

$$d\mu(x)=w(x)dx+\sum_k\rho_k\delta(x-\xi_k)dx,$$

where w satisfies $W(x)w'(x)=V(x)w(x)$ with polynomials W,V [39,56,85,86]. The orthogonal polynomials $P_0,P_1,\ldots$ are found to be solutions of linear second order differential equations, a different equation for each new degree. See also Atkinson & Everitt [6], Shohat [91].

Semi-classical orthogonal polynomials which are not classical are not eigenfunctions of a second order differential operator. A very small number of higher order operators with a set of polynomial eigenfunctions of all degrees have been found [72, 76, 107]

Recently, more second order operators with polynomial eigenfunctions have been investigated, where some degrees are missing, although the set of eigenfunctions is complete in a reasonable L^2-space. These new families are often called "exceptional orthogonal polynomials" [12, 13, 43–45, 59, 83].

After some transformations, the invariant, or Schrödinger, form is considered,

$$LY=d^2Y/dX^2+\mathcal{V}Y,$$

leading to research on interesting new special potentials $\mathcal{V}$ with simple formulas of eigenfunctions [62, 63, 83, 84],

Heun's differential equation is of course the first choice of this theory [20, 21, 27–33, 50, 63, 95] , [61, Example 3, p. 394], we give a short survey and some new proofs in § 4.

2. Semi-classical orthogonal polynomials satisfying Heun's differential equations.

After an introduction to basic tools (recurrence relation, etc.) in § 2.1, and excerpts of the Laguerre theory of differential equations in § 2.2, we conclude that the measure must be a Jacobi measure plus a point mass, a complete proof being given. Addition of a point mass is discussed in § 2.4, using a technique of W. Hahn, also following Belmehdi, Maroni, and Kiesel & Wimp. The last sections give more details on the recurrence relation and the differential equation, special and limit cases (Laguerre-type and Hermite-type).

2.1. **Recurrence relations and Stieltjes-Markov function.**

Let $P_0, P_1, \dots,$ be orthogonal polynomials with respect to the measure $d\mu(x) = w(x)dx + \sum_k \rho_k \delta(x-\xi_k)dx$, $\displaystyle\int_a^b P_m(x)P_n(x)d\mu(x) = 0$ if $m \neq n$.

Here, w is piecewise analytic and need not be a positive weight function, not even a true integrable function on (a,b), the integral may have to be considered on a particular contour, including a double loop called Pochhammer contour [82, pp. 326, 389], such contours are needed with Jacobi weights with negative exponents [102, §12.43, p.257].

A finite number of point masses may also be present. Of course, we could use a more abstract setting with a linear form [16, 17, 78, 80], but the function w will be too useful later on.

Such orthogonal polynomials must satisfy a recurrence relation of the following form [16, 17, 19, 34, 41, 65] etc.

$$P_{n+1}(x) = (r_n x + r'_n)P_n(x) - s_n P_{n-1}(x), \quad P_{-1}(x) \equiv 0, P_0(x) \equiv 1. \tag{3}$$

P_n has degree n and its main coefficient is $X_n = r_0 \cdots r_{n-1}$. Sometimes we will need the coefficient $Y_n = X_n t_n$ of x^{n-1} as well: $Y_{n+1} = r_n Y_n + r'_n X_n$, or $t_{n+1} = Y_{n+1}/X_{n+1} = t_n + r'_n/r_n$, so

$$P_n(x) = r_0 \cdots r_{n-1}(x^n + t_n x^{n-1} + \cdots), \qquad t_n = \frac{r'_0}{r_0} + \cdots + \frac{r'_{n-1}}{r_{n-1}}. \tag{4}$$

and a relation for the square norms $\|P_n\|^2 = \displaystyle\int_a^b P_n^2(x)d\mu(x) = r_0 \cdots r_{n-1} \int_a^b x^n P_n(x)d\mu(x)$:

$0 = \displaystyle\int_a^b x^{n-1}P_{n+1}(x)d\mu(x) = r_n \int_a^b x^n P_n(x)d\mu(x) - s_n \int_a^b x^{n-1}P_{n-1}(x)d\mu(x)$, or

$$r_n\|P_n\|^2 = r_{n-1}s_n\|P_{n-1}\|^2 \tag{5}$$

We will also need the **numerator** polynomials satisfying the same recurrence relation (3):

$$N_{n+1}(x) = (r_n x + r'_n)N_n(x) - s_n N_{n-1}(x), \quad N_0(x) \equiv 0. \tag{6}$$

$$\frac{N_n(x)}{P_n(x)} = \cfrac{N_1}{r_0 x + r'_0 - \cfrac{s_1}{\ddots \, \overline{r_{n-2}x + r'_{n-2} - \cfrac{s_{n-1}}{r_{n-1}x + r'_{n-1}}}}}$$

$P_0 = 1, P_1(x) = r_0 x + r'_0, P_2(x) = (r_0 x + r'_0)(r_1 x + r'_1) - s_1$.
$N_0 = 0, N_1$ is a constant, $N_2(x) = (r_1 x + r'_1)N_1$, etc.

Note that the degree of N_n is $n-1$. One sometimes prefers to write $N_n(x) = P_{n-1}(x;1)$, called the first *associated* polynomial to $\{P_n\}$ [74, 75, 98, 104].

The ***Stieltjes-Markov function*** related to the measure $d\mu$ is

$$S(x) = \int_a^b \frac{d\mu(t)dt}{x-t} = \frac{\mu_0}{x} + \frac{\mu_1}{x^2} + \cdots, \quad x \notin [a,b] \tag{7}$$

where $\mu_k = \int_a^b t^k d\mu(t)$, $k = 0, 1, \dots$, are the power moments of $d\mu$. The shown expansion in negative powers of x is an asymptotic expansion at $x \to \infty$. If (a,b) is bounded, the expansion is convergent for all real or complex x with $|x| > \max(|a|, |b|)$ [98], [101, chap. XIII, XIV].

A very important solution of the recurrence relations (3-6) is given by the ***functions of the second kind***

$$Q_n(x) = \int_a^b \frac{P_n(t)d\mu(t)}{x-t} = P_n(x)S(x) - N_n(x), n = 0, 1, \ldots \tag{8}$$

[97, §5.4] [98]. Under mild conditions (determinated case of the moment problem), if x is not in the support of $d\mu$, $Q_n(x)/P_n(x) \to 0$ which also means that $N_n(x)/P_n(x) \to S(x)$ when $n \to \infty$, [40], [41, Thms 1.41, 1.43].

Of course, $S = Q_0$. Next, $Q_1(x) = r_0(x - \mu_1/\mu_0)(\mu_0/x + \mu_1/x^2 + \mu_2/x^3 + \cdots) - r_0\mu_0 = r_0(\mu_2 - \mu_1^2/\mu_0)/x^2 + o(x^{-2}) = s_1\mu_0/(r_1x^2) + o(x^{-2})$. We also have $P_n(x)Q_n(x) = \displaystyle\int_a^b \frac{P_n^2(t)d\mu(t)}{x-t}$ [98, (2.8)] [97, §5.4.3], so that the expansion of $Q_n(x)$ in negative powers of x starts with $\dfrac{\|P_n\|^2 = s_1\cdots s_n r_0\mu_0/r_n}{r_0\cdots r_{n-1}\, x^{n+1}}$.

There is another interesting property of $P_n(x)Q_n(x)$ related to the inverse of the tridiagonal matrix of the recurrence relation (3) see Wall [101, §60]

$$\begin{bmatrix} r_0x+r_0' & -1 & & & \\ -s_1 & r_1x+r_1' & -1 & & \\ & -s_2 & r_2x+r_2' & -1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} AP_0(x) \\ \vdots \\ AP_{n-1}(x) \\ AP_n(x) \\ BQ_{n+1}(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

giving the n^{th} column of the inverse matrix if $AP_n(x) = BQ_n(x)$ and $\underbrace{-s_nAP_{n-1}(x) + (r_nx + r_n')AP_n(x)}_{AP_{n+1}(x)} - BQ_{n+1}(x) = 1$, so the n^{th} (starting with 0) diagonal element of the inverse matrix is $AP_n(x) =$

$\dfrac{P_n(x)}{P_{n+1}(x) - P_n(x)Q_{n+1}(x)/Q_n(x)} = \dfrac{P_n(x)Q_n(x)}{s_1\cdots s_nr_0\mu_0}$ (Casorati)

$\dfrac{P_n(x)Q_n(x)}{s_1\cdots s_nr_0\mu_0} = \dfrac{1}{r_nx} - \dfrac{r_n'}{r_n^2x^2} + o(x^{-2})$, or

$P_n(x)Q_n(x) = s_1\cdots s_nr_0\mu_0\left[\dfrac{1}{r_nx} - \dfrac{r_n'}{r_n^2x^2}\right] + o(x^{-2})$

Divide by $P_n(x) = r_0\cdots r_{n-1}x^n\left[1 + t_n/x + o(x^{-1})\right]$:

$Q_n(x) = \dfrac{s_1\cdots s_nr_0\mu_0}{r_0\cdots r_nx^{n+1}}\left[1 - t_{n+1}/x\right] + o(x^{-n-2})$ from $t_{n+1} = t_n + r_n'/r_n$ in (4).

We shall need another division by P_n

$$\frac{Q_n(x)}{P_n(x)} = \frac{s_1\cdots s_nr_0\mu_0}{r_0^2\cdots r_{n-1}^2r_nx^{2n+1}}\left[1 - (t_n + t_{n+1})/x\right] + o(x^{-2n-2}). \tag{9}$$

2.2. **Differential relations and equations.**

Differential properties are not to be expected in general, unless the weight function itself satisfies a differential equation. In the simplest case, if w'/w is a rational function piecewise in (a, b), i.e., up to a finite number of singular points, the same rational function V/W on each subinterval, the Stieltjes-Markov function satisfies

$$WS' = VS + U \tag{10}$$

with polynomial coefficients, the Laguerre's starting point [73] (almost the same notation).

Indeed, we perform (7) on subintervals (a_i, b_i) where w is regular, and add the effect of possible Dirac masses as a rational function R:

$S(x) = \sum_i \int_{a_i}^{b_i} \frac{w(t)dt}{x-t} + R(x),$

$W(x)S(x) = \sum_i \left[\int_{a_i}^{b_i} \frac{[W(x)-W(t)]w(t)dt}{x-t} + \int_{a_i}^{b_i} \frac{W(t)w(t)dt}{x-t} \right] + W(x)R(x)$, where the integrals involving $(W(x)-W(t))/(x-t)$ are polynomials, as well as the product WR, if we care to have W to vanish at every singular point. We derivate, and perform integration by parts, using the rational function $w'/w = V/W$,

$(W(x)S(x))' = \sum_i \int_{a_i}^{b_i} \frac{(V(t)+W'(t))w(t)dt}{x-t} +$ polynomial ,

and we get (10), where U is still another polynomial, if we added a sufficient number of common factors to V and W to have a polynomial product $(V+W')R$. For instance we will find in the next section an example with $R(x) = \kappa/(x-a)$ and $W(x) = (x-a)^2(x-b)$, $V(x)$ being $(x-a)$ times a first degree polynomial.

We may also perform all the operations of calculus on distributions containing Dirac distributions and their derivatives $x\delta(x) = 0, x\delta'(x) = -\delta(x)$ [24, Problem 9.8].

Let the expansions of the polynomials W and V be $W(x) = \sum_0^{s+2} w_k x^k$ and $V(x) = \sum_0^{s+1} v_k x^k$, then putting these expansions in (10), using the power moments expansion (7), we get the recurrence relation for these moments

$$-\sum_{k=0}^{s+2} w_k(n+k)\mu_{n-1+k} = \sum_{k=0}^{s+1} v_k \mu_{n+k}, n = 0, 1, \dots \tag{11}$$

which is equivalent to (10) [80, eq. (7.2)] [81, Prop. 1.21].

From (10), Laguerre achieved a number of remarkable results culminating in the differential equation

$$P_n'' + \left\{ \frac{V+W'}{W} - \frac{\Theta_n'}{\Theta_n} \right\} P_n' + \frac{K_n}{W\Theta_n} P_n = 0, \tag{12}$$

[73], where Θ_n and K_n are polynomials. In particular,

$$\begin{aligned} \Theta_n &= W(N_n P_n' - N_n' P_n) + V N_n P_n + U P_n^2 \\ &= P_n^2 \left[W \left(S - \frac{N_n}{P_n} \right)' - V \left(S - \frac{N_n}{P_n} \right) \right] \\ &= P_n^2 \left[W \left(\frac{Q_n}{P_n} \right)' - V \left(\frac{Q_n}{P_n} \right) \right] \end{aligned} \tag{13}$$

from (8)

As $S(x) - N_n(x)/P_n(x) = Q_n(x)/P_n(x) = O(x^{-2n-1})$ (Padé property), the degree of Θ_n is $s =$ max (degree$(W)-2$, degree$(V)-1$), the ***class*** of the semi-classical orthogonal polynomials [9,10].

We look at the coefficients of x^s and x^{s-1} of $\Theta_n(x)$, let

$W(x) = W_{s+2}x^{s+2} + W_{s+1}x^{s+1} + \cdots$, and $V(x) = V_{s+1}x^{s+1} + V_s x^s + \cdots$,

then (13) becomes

$\Theta_n(x) = s_1 \cdots s_n r_0 \mu_0 / r_n \left[x^{2n} + 2t_n x^{2n-1} + \cdots \right] \left\{ (W_{s+2}x^{s+2} + W_{s+1}x^{s+1} + \cdots) \right.$
$\left[-(2n+1)x^{-2n-2} + (2n+2)(t_n + t_{n+1})x^{-2n-3} + \cdots \right] - (V_{s+1}x^{s+1} + V_s x^s + \cdots)$
$\left. \left[x^{-2n-1} - (t_n + t_{n+1})x^{-2n-2} + \cdots \right] \right\}$, from (4) and (9) put in (13), or

$$\Theta_n(x) = -\frac{s_1\cdots s_n r_0\mu_0}{r_n}\{((2n+1)W_{s+2}+V_{s+1})x^s$$
$$+[((2n+1)W_{s+2}+V_{s+1})(t_n-t_{n+1})+(2n+1)W_{s+1}+V_s-W_{s+2}(t_n+t_{n+1})]\,x^{s-1}+\cdots\} \quad (14)$$

When $s=1$, there is no $\cdots$ anymore, and $\Theta_n(x)=\Theta'_n(x-\theta_n)$, with

$$\theta_n = t_{n+1}-t_n+\frac{W_3(t_n+t_{n+1})-(2n+1)W_2-V_1}{(2n+1)W_3+V_2}. \quad (15)$$

With a Jacobi weight, $W(x)=(x-a)(x-b)$, Θ_n is a constant, and we have only two singular points in the finite plane. No wonder, as one just recovers then the hypergeometric differential equation of Jacobi orthogonal polynomials!

However, with the simplest generalized Jacobi weight $(x-e)^p(x-d)^q(x-c)^r$, the degree of Θ_n now raises to 1, and we have *four* singular points in the finite plane, one too much!

(12) becomes (when $c=0, d=1, e=\mathfrak{t}$)

$P''_n+\left\{\dfrac{r+1}{x}+\dfrac{q+1}{x-1}+\dfrac{p+1}{x-\mathfrak{t}}-\dfrac{1}{x-\theta_n}\right\}P'_n+\dfrac{\lambda_n x^2+\omega_n x+\omega'_n}{x(x-1)(x-\mathfrak{t})(x-\theta_n)}P_n=0$, which is NOT of Heun's type, and where θ_n,λ_n etc. depend on n and, if also considered as functions of $\mathfrak{t}$, may be shown to be Painlevé functions [22, 77].

Invariant, or Schrödinger form: we see that $Z_n(z)=x^{(r+1)/2}(x-1)^{(q+1)/2}(x-\mathfrak{t})^{(p+1)/2}(x-\theta_n)^{-1/2}P_n(x)$ satisfies the equation without first order derivative $Z''_n+I_nZ_n=0$, with the invariant $I_n(x)=-\left(\dfrac{Z'_n}{Z_n}\right)'-\left(\dfrac{Z'_n}{Z_n}\right)^2$. Here,

$$I_n(x)=\frac{(r+1)/2}{x^2}+\frac{(q+1)/2}{(x-1)^2}+\frac{(p+1)/2}{(x-\mathfrak{t})^2}-\frac{1/2}{(x-\theta_n)^2}+\frac{\lambda_n x^2+\omega_n x+\omega'_n}{x(x-1)(x-\mathfrak{t})(x-\theta_n)}$$
$$-\left(\frac{(r+1)/2}{x}+\frac{(q+1)/2}{x-1}+\frac{(p+1)/2}{x-\mathfrak{t}}-\frac{1/2}{x-\theta_n}\right)^2 \quad (16a)$$

considered by D. Chudnovsky [22, eq. (3.5) p.401] to compare with Heun's eq. with invariant

$$I_h(x)=\frac{Px-Q}{x(x-1)(x-\mathfrak{c})}-\frac{1}{4}\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\epsilon}{x-\mathfrak{c}}\right)^2+\frac{1}{2}\left(\frac{\gamma}{x^2}+\frac{\delta}{(x-1)^2}+\frac{\epsilon}{(x-\mathfrak{c})^2}\right) \quad (16b)$$

[33, 63]

Of course, the only way to have $I_n(x)\equiv I_h(x)$ in (16a) and (16b) is by the confluence of two singular points, here, $\mathfrak{t}\to 0$. Then, $\gamma=p+r+2, \delta=q+1, \epsilon=-1, \mathfrak{c}=\theta_n$, and we will also have to check that $\omega'_n=0$.

We return to the example of the simplest generalized Jacobi weight
$w(x)=(x-c)^r(x-d)^q(x-e)^p$, satisfying $Ww'=Vw$ with the polynomials
$W(x)=(x-c)(x-d)(x-e)$ and $V(x)=r(x-d)(x-e)+q(x-c)(x-e)+p(x-c)(x-d)$ and look at what happens when $d\to c$. However, the actual complete description of the weight function is some constant C times $(x-c)^r(d-x)^q(e-x)^p$ on the subinterval (or arc) (c,d), and another constant D times $(x-c)^r(x-d)^q(e-x)^p$ on (d,e), see fig. 1.

If we make $d\to c$, do we return to a simple Jacobi weight $D(x-c)^{r+q}(e-x)^p$ on the whole interval (c,e)? No. If C depends on $d-c$ so that the total weight $\int_c^d w(x)dx$ on (c,d) has a nonzero limit, say, κ, the true limit of the measure on its whole support $[c,e]$ is $d\mu(x)=D(x-c)^{r+q}(e-x)^pdx+\kappa\delta(x-c)dx$ [9, eq. 4.27, p. 265](C must have a $(d-c)^{-r-q-1}$ behavior).

Remark also that $x-c$ is a common factor of the limits $W(x)\to(x-c)^2(x-e)$ and

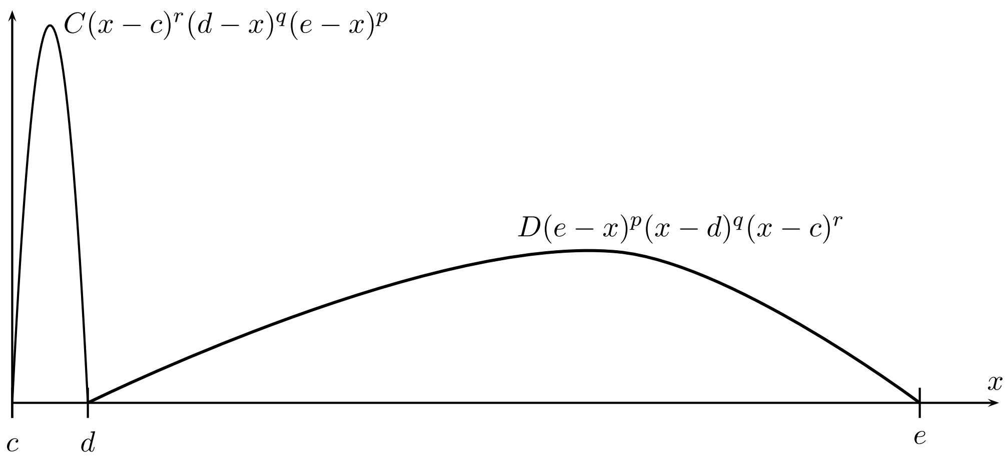

FIGURE 1. Generalized Jacobi weight on two subintervals (c,d) and (d,e).

$V(x) \to (x-c)[(r+q)(x-e)+p(x-c)]$, so that $\delta(x-c)$ is indeed a particular solution of $Wy' = Vy$, see the simplest example of Encyc [100], also Davies [24, Problem 9.8] $x^2\delta'(x)$ and $x\delta(x) \equiv 0$.

Here is a complete proof that orthogonal polynomials satisfying Heun's equations must be related to a Jacobi weight augmented by a point mass:

2.3. **Theorem.** *Let $\{P_n\}_0^\infty$ be a sequence of orthogonal polynomials, degree$(P_n) = n$. Then, if each P_n satisfies a Heun's differential equation*

$$(x-\mathfrak{a}_n)(x-\mathfrak{b}_n)(x-\mathfrak{c}_n)\left\{\frac{d^2P_n(x)}{dx^2} + \left[\frac{\gamma_n}{x-\mathfrak{a}_n} + \frac{\delta_n}{x-\mathfrak{b}_n} + \frac{\epsilon_n}{x-\mathfrak{c}_n}\right]\frac{dP_n(x)}{dx}\right\} + K_n(x)P_n(x) = 0, \tag{17}$$

with distinct $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n$ in the finite complex plane, and first degree polynomials K_n, we have

(1) *two of $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n$ are independent of n, say, $\mathfrak{a}_n = a, \mathfrak{b}_n = b$,*
(2) *the orthogonality measure has support $[a,b]$, and is $d\mu(x) = w(x)dx + \kappa\delta(x-c)dx$, where w is a Jacobi weight function $\lambda(b-x)^p(x-a)^q$, and $c = a$ or $c = b$,*
(3) *let $c = a$, then $\gamma_n = q+2, \delta_n = p+1, \epsilon_n = -1$.*

The full formulas for $\mathfrak{c}_n$ and the coefficients of the first degree polynomials K_n will be given in sections 2.4-2.8.

Proof. The first part of the proof comes from a paper [54] where W. Hahn supposes the existence of second order differential equations (not necessarily of the Heun kind) for each P_n, and supposes nothing on the orthogonality measure, which could not exist, but even in the most formal setting [16, 17, 34, 42, 78, 80], we have the recurrence relation (3), where one only supposes $s_n \neq 0$, $n = 1, 2, \ldots$ so that P_n and P_{n-1} have never common zeros. Then, the (difficult) proof of Hahn produces various differential relations resulting in the disclosure of a linear recurrence relation of the form (11), so that we are back to the Laguerre-Shohat-Atkinson-Everitt theory of (10). From this equivalence of Laguerre and Hahn theories, the name Laguerre-Hahn is sometimes used [14, 39, 85].

We compare now the coefficient of $dP_n(z)/dz$ in (12) and (17):

$$\frac{V(x)+W'(x)}{W(x)} - \frac{\Theta_n'(x)}{\Theta_n(x)} = \frac{\gamma_n}{x-\mathfrak{a}_n} + \frac{\delta_n}{x-\mathfrak{b}_n} + \frac{\epsilon_n}{x-\mathfrak{c}_n}$$

If degree$(W) \leqslant 2$, we are back to the hypergeometric case instead of a true Heun situation,

if degree$(W) \geqslant 3$, we saw that degree$(\Theta_n) \geqslant 1$, and that makes at least four bounded poles, **unless**

W has a double zero, say $W(x) = (x-a)^2(x-b)$, so the three poles are a, b, and the zero θ_n of Θ_n.

As V/W has simple poles, V must vanish at the double zero of W:

$V(x) = (x-a)[p(x-a)+q(x-b)]$, so $\dfrac{w'(x)}{w(x)} = \dfrac{\gamma-2}{x-a} + \dfrac{\delta-1}{x-b}$. Let $\gamma = q+2, \delta = p+1, w(x) = \lambda(x-a)^q(b-x)^p$ on (a,b).

Finally, $WS' - VS = U$, what is U? Let $S(x) = \displaystyle\int_a^b \frac{w(t)dt}{x-t} + R(x)$, where R takes the singular part of the measure into account. Let $\{\tilde{\mu}_n\}_0^\infty$ be the power moments of w alone.

We start with $(x-a)(x-b)S'(x) - [p(x-a)+q(x-b)]S(x)$, which is not $W(x)S'(x) - V(x)S(x) = U(x)$, but the rational function $U(x)/(x-a)$.

$$(x-a)(x-b)S(x) = N(x) - \int_a^b \lambda \frac{(b-t)^{p+1}(t-a)^{q+1}}{x-t} dt + (x-a)(x-b)R(x),$$

where N is the polynomial $N(x) = \displaystyle\int_a^b \frac{(x-a)(x-b)-(t-a)(t-b)}{x-t} w(t)dt = \tilde{\mu}_0 x + \cdots$, and $\tilde{\mu}_0 = \displaystyle\int_a^b w(t)dt = \lambda \int_a^b (b-t)^p(t-a)^q dt = \lambda(b-a)^{p+q+1}B(q+1,p+1)$
$= \lambda(b-a)^{p+q+1}\dfrac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$.

$d[(x-a)(x-b)S(x)]/dx = \tilde{\mu}_0 - \lambda \displaystyle\int_a^b \frac{d[(b-t)^{p+1}(t-a)^{q+1}]/dt}{x-t} dt + d[(x-a)(b-x)R(x)]/dx =$
$\tilde{\mu}_0 - (p+q+2)\tilde{\mu}_0 + [(p+1)(x-a)+(q+1)(x-b)][S(x)-R(x)] + [(x-a)(x-b)R(x)]'$,

$(x-a)(x-b)S'(x) - (p(x-a)+q(x-b))S(x)$
$= -(p+q+1)\tilde{\mu}_0 + (x-a)(x-b)R'(x) - [p(x-a)+q(x-b)]R(x)$, see Grosjean [49, eq. (80) p. 283], which is the rational function $U(x)/(x-a)$, so we have a differential equation for R. The general solution of the homogeneous equation is constant times $(x-a)^q(x-b)^p$ discarded if $p+q+1 \neq 0$, as $R(x) \sim (\mu_0 - \tilde{\mu}_0)/x$ for large x. A particular solution is $\kappa/(x-a)$. Indeed, $U(x) = -(p+q+1)\tilde{\mu}_0(x-a) - \kappa[p(x-a)+(q+1)(x-b)]$, so that $\kappa = U(a)/[(q+1)(b-a)]$. Of course, $\kappa/(x-a)$ is the Stieljes-Markov function of $\kappa\delta(x-a)$, and we complete the proof:

$d\mu(x) = \lambda(x-a)^q(b-x)^p dx + \kappa\delta(x-a)dx$. □

See also Belmehdi [9, eq. (4.27), p. 265].

More details on the recurrence coefficients, on $\theta_n = \mathfrak{c}_n$, and $K_n(0)$ in $K_n(x) = K_n'x + K_n(0)$, with $K_n' = -n(n+p+q+1)$, will now be given.

2.4. **Addition of a mass point.**

We survey known results about a single mass point addition.

Let us consider the measure

$$d\mu(x) = w(x)dx + \kappa\delta(x-c)dx, \tag{18}$$

where w is a known weight function. The orthogonal polynomials P_n with respect to $d\mu$ are often given by their expansion in the basis of the orthogonal polynomials $\tilde{P}_m$ with respect to w alone, and this involves the corresponding kernel polynomial of degree n (simplest Uvarov formula, [65, 2.9] [88]).

Indeed, if $m < n$, $0 = \displaystyle\int_a^b P_n(x)\tilde{P}_m(x)d\mu(x) = \int_a^b P_n(x)\tilde{P}_m(x)w(x)dx + \kappa P_n(c)\tilde{P}_m(c)$, so that the m^{th} coefficient in the expansion of P_n in the $\{\tilde{P}_m\}$ basis is $-\kappa P_n(c)\tilde{P}_m(c)/\|\tilde{P}_m\|^2$, and the closure $P_n(c) = P_n(c)$ involves the kernel polynomial $\tilde{k}_{n-1}(x;c) = \sum_0^{n-1} \tilde{P}_m(x)\tilde{P}_m(c)/\|\tilde{P}_m\|^2$. Building and using kernel polynomials may be difficult in general, but $\tilde{k}_n$ is known to be the orthogonal polynomial of degree n with respect to $(x-c)w(x)$ which may be easier to handle. For instance, we shall use the Jacobi weight $w(x) = (b-x)^p(x-a)^q$ on (a,b) and $c = a$ so that $(x-c)w(x)$ is still a Jacobi weight (with parameters p and $q+1$).

So, we start with the set $\{\hat{P}_n\}$ of orthogonal polynomials with respect to
$\hat{w}(x)dx = (x-c)w(x)dx$, which is the same measure as $(x-c)d\mu(x)$, as $(x-c)\delta(x-c) \equiv 0$ [24, Problem 9.8] so $\hat{P}_n$ is some constant, say C_n times the kernel polynomial k_n (and also another constant $\tilde{C}_n$ times $\tilde{k}_n$, which will not be needed here).

$$\hat{P}_n(x) = C_n k_n(x;c) = C_n \sum_0^n \frac{P_m(c)P_m(x)}{\|P_m\|^2} = C_n \frac{P_n(c)P_{n+1}(x) - P_{n+1}(c)P_n(x)}{r_n\|P_n\|^2(x-c)} \tag{19}$$

the last term being Christoffel-Darboux. We shall need the relations between the moments μ_n of $d\mu$ and $\hat{\mu}_n$ of $\hat{w}(t)dt = (t-c)w(t)dt = (t-c)d\mu(t)$:

$$\hat{\mu}_n = \int_a^b t^n(t-c)d\mu(t) = \mu_{n+1} - c\mu_n, \quad n = 0, 1, \ldots \tag{20}$$

so that $\mu_1 = \hat{\mu}_0 + c\mu_0, \mu_2 = \hat{\mu}_1 + c\hat{\mu}_0 + c^2\mu_0$ etc. if the $\hat{\mu}_n$s are known. The value of μ_0 remains a degree of liberty.

First polynomials are $P_0 \equiv 1$ with $\|P_0\|^2 = \mu_0$, and P_1 with $P_1(x) = r_0(x - \mu_1/\mu_0) = r_0(x - c - \hat{\mu}_0/\mu_0)$, $\|P_1\|^2 = r_0^2(\mu_2 - \mu_1^2/\mu_0) = r_0^2(\hat{\mu}_1 - c\hat{\mu}_0 - \hat{\mu}_0^2/\mu_0) = r_0^2\hat{\mu}_0(-\hat{r}_0'/\hat{r}_0 - c - \hat{\mu}_0/\mu_0)$

The first C_ns are $C_0 = \|P_0\|^2 = \mu_0; C_1 =$ the ratio of the main coefficients of $\hat{P}_1$ and k_1: $C_1 = \dfrac{\hat{r}_0}{r_0 P_1(c)/\|P_1\|^2} = \dfrac{\hat{r}_0\hat{\mu}_0(-\hat{r}_0'/\hat{r}_0 - c - \hat{\mu}_0/\mu_0)}{-\hat{\mu}_0/\mu_0}$,

$$C_0 = \mu_0; \quad C_1 = \mu_0(\hat{r}_0 c + \hat{r}'_0) + \hat{\mu}_0\hat{r}_0. \tag{21}$$

We have

$$P_n(x) = \frac{\|P_n\|^2}{P_n(c)}(k_n(x;c) - k_{n-1}(x;c)) = \frac{\|P_n\|^2}{P_n(c)}\left(\frac{\hat{P}_n(x)}{C_n} - \frac{\hat{P}_{n-1}(x)}{C_{n-1}}\right) \tag{22}$$

giving P_n as a combination of $\hat{P}_n$ and $\hat{P}_{n-1}$ (the coefficients $C_n, P_n(c)$, etc. being still unknown).

W. Hahn discussed in [51,52], how to form a new sequence of orthogonal polynomials $\{P_n\}$ from a known set $\{\hat{P}_n\}$ satisfying a differential equation (12). He found the new differential equation solved by a combination of $\hat{P}_n$ and $\hat{P}_{n-1}$ provided that the combination is such that it makes an actual set of orthogonal polynomials satisfying a recurrence relation of the required form (3). There is no weight function discussion there, but Hahn's results on P_n and $\hat{P}_n$ are equivalent to (19).

When there are several mass points, P_n is a combination of a higher number of $\hat{P}_n, \ldots, \hat{P}_{n-s}$ which may be condensed as a polynomial combination $\hat{P}_n$ and $\hat{P}_{n-1}$, as found in Ronveaux & Marcellán [88, eq. (13)], also Kiesel & Wimp [69, 105].

We now look for a recurrence relation for the C_ns. Hahn [51, p.95-96], [52], and Kiesel & Wimp [105] relate the recurrence coefficients of the P_ns and the $\hat{P}_n$s and find a four-term recurrence relation for C_n, curiously simplified as a three-term one. We follow the much faster derivation of Maroni [79], the result is that C_n satisfies the recurrence relation of $\hat{P}_n$ at $x = c$:

$$C_{n+1} = (\hat{r}_n c + \hat{r}'_n)C_n - \hat{s}_n C_{n-1}, \quad n = 1, 2, \ldots \tag{23}$$

Zhedanov [107, § 6] gives the same result from papers of Geronimus of 1940.

Indeed, it is obvious from (22) that $P_n(x)$ is orthogonal to all polynomials $(x-c)p(x)$ with degree $p < n-1$, as $\hat{P}_n$ and $\hat{P}_{n-1}$ are orthogonal to polynomials up to the degree $n-2$ with respect to $\hat{w}(x) = (x-c)w(x) = (x-c)d\mu(x)$. It only remains to settle C_n so that P_n in (22) is orthogonal to the constants with respect to $d\mu$:

$0 = \int_a^b P_n(x)d\mu(x) = \dfrac{\|P_n\|^2}{P_n(c)}\left(\dfrac{\int_a^b \hat{P}_n(x)d\mu(x)}{C_n} - \dfrac{\int_a^b \hat{P}_{n-1}(x)d\mu(x)}{C_{n-1}}\right)$, and it follows that $\int_a^b \hat{P}_n(x)d\mu(x)/C_n$ is a constant independent of n, and is $\mu_0/\mu_0 = 1$ from (21) at $n=0$:

$$C_n = \int_a^b \hat{P}_n(x)d\mu(x), \quad \text{for } n = 0, 1, \dots, \tag{24}$$

and the 3-term recurrence relation

$C_{n+1} - (\hat{r}_n c + \hat{r}'_n)C_n + \hat{s}_n C_{n-1} = \displaystyle\int_a^b [\hat{P}_{n+1}(x) - (\hat{r}_n c + \hat{r}'_n)\hat{P}_n(x) + \hat{s}_n \hat{P}_{n-1}(x)]d\mu(x)$

$= \hat{r}_n \displaystyle\int_a^b \hat{P}_n(x)(x-c)w(x)dx = 0, n = 1, 2, \dots$ follows.

C_n is therefore a combination of $\hat{P}_n(c)$ and $\hat{N}_n(c)$, say, $\sigma\hat{N}_n(c) + \tau\hat{P}_n(c)$, $n = 0, 1, \dots$ At $n = 0, \tau = \mu_0$ follows; at $n = 1, C_1 = \mu_0(\hat{r}_0 c + \hat{r}'_0) + \hat{\mu}_0\hat{r}_0 = \sigma\hat{\mu}_0\hat{r}_0 + \mu_0(\hat{r}_0 c + \hat{r}'_0)$, so, $\sigma = 1$:

$$C_n = \hat{N}_n(c) + \mu_0\hat{P}_n(c), n = 0, 1, \dots \tag{25}$$

The Stieltjes-Markov functions of $d\mu$ and $d\hat{\mu}$ are related by $\hat{S}(x) = \displaystyle\int_a^b \frac{(t-c)w(t)dt}{x-t} = \int_a^b \frac{(t-c)d\mu(t)}{x-t} = -\mu_0 + (x-c)S(x)$. From $\tilde{S}(x) = \displaystyle\int_a^c \frac{\tilde{w}(t) = w(t)}{x-t}dt$, $\tilde{S}(x) = \displaystyle\int_a^c \frac{\hat{w}(t)dt}{(t-c)(x-t)} + \int_c^b \frac{\hat{w}(t)dt}{(t-c)(x-t)}$, and its first moment $\tilde{\mu}_0 = \displaystyle\int_a^b \tilde{w}(t)dt = \int_a^c \frac{\hat{w}(t)dt}{t-c} + \int_c^b \frac{\hat{w}(t)dt}{t-c}$. We still have $\hat{S}(x) = -\tilde{\mu}_0 + (x-c)\tilde{S}(x)$, so

$S(x) = \tilde{S}(x) + \dfrac{\mu_0 - \tilde{\mu}_0}{x-c}$, confirming the point mass $(\mu_0 - \tilde{\mu}_0)\delta(t-c)$ [79].

2.5. **Jacobi weight** $+\kappa\delta(x-a)$.

2.5.1. *Formula for* P_n.

We consider $d\mu(x) = \lambda(x-a)^q(b-x)^p + \kappa\delta(x-a)dx$.

We start with $\hat{w}(x) = \lambda(b-x)^p(x-a)^{q+1}$ corresponding to Jacobi polynomials [82, §18.3.1]

$\hat{P}_n(x) = P_n^{(p,q+1)}\left(\dfrac{2x-a-b}{b-a}\right) = \dfrac{(n+p+q+2)(n+p+q+3)\cdots(2n+p+q+1)}{2^n\, n!}$

$\times \left[\left(\dfrac{2x-a-b}{b-a}\right)^n + \dfrac{n(p-q-1)}{2n+p+q+1}\left(\dfrac{2x-a-b}{b-a}\right)^{n-1} + \cdots\right]$, so that

$$\hat{r}_0 \cdots \hat{r}_{n-1} = \frac{\Gamma(2n+p+q+2)}{(b-a)^n n! \Gamma(n+p+q+2)}, \hat{t}_n = -n(a+b)/2 + \frac{n(p-q-1)}{2n+p+q+1}\,\frac{b-a}{2}. \tag{26}$$

Recurrence relation from [82, 18.9.1,2]

$$\hat{P}_{n+1}(x) = \frac{2n+p+q+3}{(n+1)(n+p+q+2)}\left[\frac{2n+p+q+2}{b-a}(x + \hat{t}_{n+1} - \hat{t}_n)\hat{P}_n(x) - \frac{(n+p)(n+q+1)}{2n+p+q+1}\hat{P}_{n-1}(x)\right]. \tag{27}$$

Also, $\hat{P}_n(a) = P_n^{(p,q+1)}(-1) = (-1)^n \binom{n+q+1}{n}$ [82, 18.5.8]

Moments $\hat{\mu}_0 = \lambda(b-a)^{p+q+2}B(p+1,q+2) = (b-a)^{p+q+2}\dfrac{\Gamma(p+1)\Gamma(q+2)}{\Gamma(p+q+3)}$,

$\hat{\mu}_1/\lambda = -(b-a)^{p+q+3}B(p+2,q+2) + b(b-a)^{p+q+2}B(p+1,q+2) = \dfrac{(q+2)b+(p+1)a}{p+q+3}\mu_0^{(p,q+1)}$,

$\hat{S}(x) = \hat{\mu}_0\left[\dfrac{1}{x} + \dfrac{(q+2)b+(p+1)a}{(p+q+3)x^2} + \dfrac{(q+2)(q+3)b^2+2(p+1)(q+2)ab+(p+1)(p+2)a^2}{(p+q+3)(p+q+4)x^3} + \cdots\right]$

Finally we need from (25) at $c = a$, $C_n = \hat{N}_n(a) + \mu_0\hat{P}_n(a)$ asking for special values of $\hat{N}_n$ satisfying the Jacobi polynomials recurrence relation (26-27)

$2(n+1)(n+p+q+2)(2n+p+q+1)\left\{\hat{N}_{n+1}(a) - (2n+p+q+1)(2n+p+q+2)(2n+p+q+3)\right.$
$\left.\left(\dfrac{(n+1)(p-q-1)}{2n+p+q+3} - \dfrac{n(p-q-1)}{2n+p+q+1} - 1\right)\hat{N}_n(a)\right\}$
$+\,2(n+p)(n+q+1)(2n+p+q+3)\hat{N}_{n-1}(a) = 0$

First items are $\hat{N}_0(a) = 0, \hat{N}_1(a) = \dfrac{\hat{\mu}_0(p+q+3)}{b-a}, \hat{N}_2(a) = -\dfrac{\hat{\mu}_0(p+q+4)(pq+4p+5q+8+q^2)}{2(p+q+3)(b-a)}$

Special solutions of the form $(-1)^n\dfrac{\Gamma(n+u)}{\Gamma(n+v)}$ must satisfy $\dfrac{2(n+1)(n+p+q+2)(2n+p+q+1)(n+u)}{n+v} =$
$(2n+p+q+2)[(2n+p+q+1)(2n+p+q+3)-p^2+(q+1)^2] - \dfrac{2(n+p)(n+q+1)(2n+p+q+3)(n+v-1)}{n+u-1}$

Solved by $(u,v) = (q+2,1)$, corresponding to $P_n^{(p,q+1)}(-1) = (-1)^n\dfrac{(q+2)\cdots(q+n+1)}{n!}$, [82, table 18.6.1] and also by $(u,v) = (p+1,p+q+2)$.

So, $\hat{N}_n(a)$ is a linear combination of $(-1)^n\dfrac{\Gamma(n+q+2)}{\Gamma(n+1)}$ and $(-1)^n\dfrac{\Gamma(n+p+1)}{\Gamma(n+p+q+2)}$ and we match $n=0$ and $n=1$:

$\hat{N}_n(a) = (-1)^{n-1}\dfrac{\hat{\mu}_0(p+q+2)}{(q+1)(b-a)}\left[\dfrac{\Gamma(n+q+2)}{\Gamma(q+2)\Gamma(n+1)} - \dfrac{\Gamma(p+q+2)\Gamma(n+p+1)}{\Gamma(p+1)\Gamma(n+p+q+2)}\right]$

Interesting confirmation by Lewanowicz [74], also Wimp [104], who considers associated polynomials of level ℓ, $R_n^{(\alpha,\beta)}(x;\ell) = P_n^{(\alpha,\beta)}(2x-1;\ell)$ [104, eq.(13)], $P_n^{(\alpha,\beta)}(-1;\ell)$
$= \dfrac{(-1)^n}{\beta(\alpha+\beta+2\ell)}\left[\dfrac{\Gamma(\ell+1)\Gamma(n+\beta+\ell+1)(\alpha+\beta+\ell)}{\Gamma(\beta+1)\Gamma(n+\ell+1)} - \dfrac{\ell\Gamma(\alpha+\beta+\ell+1)\Gamma(n+\alpha+\ell+1)}{\Gamma(\alpha+\ell)\Gamma(n+\alpha+\beta+\ell+1)}\right]$ [104, (33) p.990], which we apply with $p, q+1, n-1$ and $\ell = 1$:

$P_n^{(p,q+1)}(-1;1) = \dfrac{(-1)^{n-1}}{(q+1)(p+q+3)}\left[\dfrac{\Gamma(n+q+2)(p+q+2)}{\Gamma(q+2)n!} - \dfrac{\Gamma(p+q+3)\Gamma(n+p+1)}{\Gamma(p+1)\Gamma(n+p+q+2)}\right]$

see also [49, 75]

$$\hat{N}_n(a) = \frac{(-1)^{n-1}\mu_0^{(p,q+1)}}{(b-a)(q+1)}\left[(p+q+2)\frac{(q+2)(q+3)\cdots(q+n+1)}{n!} - \frac{(p+1)\cdots(p+n)}{(p+q+3)\cdots(p+q+n+1)}\right]$$
$$= \hat{\mu}_0\frac{1}{(b-a)(q+1)}\left[-(p+q+2)P_n^{(p,q+1)}(-1) + (-1)^n\frac{(p+1)\cdots(p+n)}{(p+q+3)\cdots(p+q+n+1)}\right]$$

We shall write $C_n = \hat{N}_n(a) + \dfrac{(p+q+2)\hat{\mu}_0}{(b-a)(q+1)}\hat{P}_n(a) + \kappa\hat{P}_n(a)$ instead of $\hat{N}_n(a) + \mu_0\hat{P}_n(a)$, and we show that κ is the Dirac mass, indeed,

$\mu_0^{(p,q)} = \tilde{\mu}_0 = \lambda\int_a^b(b-t)^p(t-a)^q dt = \lambda(b-a)^{p+q+1}B(p+1,q+1) = \lambda(b-a)^{p+q+1}\dfrac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} =$
$\dfrac{\hat{\mu}_0(p+q+2)}{(b-a)(q+1)}$, so, $\mu_0 = \mu_0^{(p,q)} + \kappa$, and $d\mu(t) = \lambda(b-t)^p(t-a)^q + \kappa\delta(t-a)$ is confirmed.

Final formula for C_n is $(-1)^n\left[\dfrac{\mu_0^{(p,q)}(p+1)\cdots(p+n)}{(p+q+2)\cdots(p+q+n+1)}+\kappa\dfrac{(q+2)\cdots(q+n+1)}{n!}\right]$, so

$$C_n=(-1)^n\left[\frac{\lambda(b-a)^{p+q+1}\Gamma(p+n+1)\Gamma(q+1)}{\Gamma(p+q+n+2)}+\kappa\frac{(q+2)\cdots(q+n+1)}{n!}\right] \tag{28}$$

See also Zhedanov [107, eq. (7.7)].

Final check from (24) $C_n=\int_a^b \hat{P}_n(x)d\mu(x)=\int_a^b \hat{P}_n(x)[\lambda w(x)+\kappa\delta(x-a)]dx$ $=\int_a^b \hat{P}_n(x)\left[\lambda\dfrac{\hat{w}(x)}{x-a}+\kappa\delta(x-a)\right]dx=\lambda\int_a^b P_n^{(p,q+1)}((2x-a-b)/(b-a))(x-a)^q(b-x)^p dx+$ $\kappa P_n^{(p,q+1)}(-1)=\lambda(b-a)^{p+q+1}\dfrac{\Gamma(q+1)\Gamma(p+n+1)(-1)(-2)\cdots(-n)}{n!\Gamma(p+q+n+2)}+\kappa P_n^{(p,q+1)}(-1)$ from [38, §16.4, (1), (2) p.284] with $\alpha=p,\beta=q+1,\sigma=q$ (a $n!$ seems to be missing in the denominator of (1)).

Remark that $\dfrac{C_n}{C_{n-1}}=-\dfrac{p+n}{p+q+n+1}$ if $\kappa=0$; $-\dfrac{q+n+1}{n}$ if $\lambda=0$.

The polynomial $P_n(x)$ is a constant times $\hat{P}_n(x)-\dfrac{C_n\hat{P}_{n-1}(x)}{C_{n-1}}=P_n^{(p,q+1)}(x)-\dfrac{C_nP_{n-1}^{(p,q+1)}(x)}{C_{n-1}}$ which is here $\dfrac{(n+p+q+2)(n+p+q+3)\cdots(2n+p+q+1)}{(b-a)^n\ n!}$

$$\times\left[x^n-n(a+b)x^{n-1}/2+(b-a)\frac{n(p-q-1)}{2(2n+p+q+1)}x^{n-1}+\cdots\right]$$
$$-\frac{C_n}{C_{n-1}}\frac{(n+p+q+1)(n+p+q+2)\cdots(2n+p+q-1)}{(b-a)^{n-1}\ (n-1)!}[x^{n-1}+\cdots].$$

This means that $t_n=-\dfrac{n(a+b)}{2}+\dfrac{(b-a)n(p-q-1)}{2(2n+p+q+1)}-\dfrac{C_n}{C_{n-1}}\dfrac{(b-a)n(n+p+q+1)}{(2n+p+q)(2n+p+q+1)}$.

We will need later
$t_{n+1}=-\dfrac{(n+1)(a+b)}{2}+\dfrac{(b-a)(n+1)(p-q-1)}{2(2n+p+q+3)}-\dfrac{C_{n+1}}{C_n}\dfrac{(b-a)(n+1)(n+p+q+2)}{(2n+p+q+2)(2n+p+q+3)}$,
with the recurrence relation (26-27) at $x=a$ giving C_{n+1}:
$C_{n+1}=\dfrac{2n+p+q+3}{(n+1)(n+p+q+2)}\left[\dfrac{2n+p+q+2}{b-a}(a+\hat{t}_{n+1}-\hat{t}_n)C_n-\dfrac{(n+p)(n+q+1)}{2n+p+q+1}C_{n-1}\right]$,
$\dfrac{(n+1)(n+p+q+2)}{2n+p+q+3}\dfrac{C_{n+1}}{C_n}=(2n{+}p{+}q{+}2)\left(-\dfrac{1}{2}+\dfrac{(n+1)(p-q-1)}{2(2n+p+q+3)}-\dfrac{n(p-q-1)}{2(2n+p+q+1)}\right)-$ $\dfrac{(n+p)(n+q+1)}{2n+p+q+1}\dfrac{C_{n-1}}{C_n}$,
$t_{n+1}=-\dfrac{(n+1)(a+b)}{2}+(b-a)\left(\dfrac{1}{2}+\dfrac{n(p-q-1)}{2(2n+p+q+1)}\right)+\dfrac{(n+p)(n+q+1)(b-a)}{(2n+p+q+1)(2n+p+q+2)}\dfrac{C_{n-1}}{C_n}$,

2.5.2. *Heun's differential equation for P_n.*

We build (12) with $V(x)/W(x)=w'(x)/w(x)=q/(x-a)+p/(x-b)$ and $W(x)=(x-b)(x-a)^2$, so

$$P_n''(x)+\left(\frac{p+1}{x-b}+\frac{q+2}{x-a}-\frac{1}{x-\theta_n}\right)P_n'(x)-\frac{n(p+q+n+1)x-K_n(0)}{(x-a)(x-b)(x-\theta_n)}P_n(x)=0 \tag{29}$$

where $\theta_n=t_{n+1}-t_n+\dfrac{W_3(t_n+t_{n+1})-(2n+1)W_2-V_1}{(2n+1)W_3+V_2}$ is given by (15) with $W_3x^3+W_2x^2+W_1x+W_0=(x-a)^2(x-b)$ and $V_2x^2+V_1x+V_0=p(x-a)^2+q(x-a)(x-b)$:

$$\theta_n = \frac{(2n+p+q+2)t_{n+1} - (2n+p+q)t_n + (2n+1)(2a+b) + 2pa + q(a+b)}{2n+p+q+1}$$
$$= \frac{(4n+3p+q+2)a + (q-p)b}{2(2n+p+q+1)} + (b-a)\frac{2n+p+q+2}{2n+p+q+1}\left(\frac{1}{2} + \frac{n(p-q-1)}{2(2n+p+q+1)}\right)$$
$$+\frac{(n+p)(n+q+1)(b-a)}{(2n+p+q+1)^2}\frac{C_{n-1}}{C_n} - \frac{2n+p+q}{2n+p+q+1}\frac{(b-a)n(p-q-1)}{2(2n+p+q+1)} + \frac{(b-a)n(n+p+q+1)}{(2n+p+q+1)^2}\frac{C_n}{C_{n-1}}$$
$$= a + \frac{(n+q+1)(b-a)}{2n+p+q+1}\left(1 + \frac{n+p}{2n+p+q+1}\frac{C_{n-1}}{C_n}\right)$$
$$+ \frac{n(b-a)}{(2n+p+q+1)^2}\left(p-q-1 + (n+p+q+1)\frac{C_n}{C_{n-1}}\right)$$

One readily checks that $\theta_n = a$ when $\kappa = 0$ (then, $C_n/C_{n-1} = -(p+n)/(p+q+n+1)$ as seen above), and also when $\lambda = 0$ (with $C_n/C_{n-1} = (q+n+1)/n$). As $\theta_n - a$ is a quadratic form in C_n and C_{n-1}, so of κ and λ, divided by $C_{n-1}C_n$, only the $\lambda\kappa$ term remains. So we expand $C_nC_{n-1}(\theta_n - a) = \dfrac{b-a}{(2n+p+q+1)^2}\{(n+q+1)((2n+p+q+1)C_{n-1}C_n + (n+p)C_{n-1}^2) + n((p-q-1)C_{n-1}C_n + (n+p+q+1)C_n^2)\}$ and keep only the $\lambda\kappa$ terms:

$$C_nC_{n-1}(\theta_n - a) = \lambda\kappa\frac{(b-a)^{p+q+2}}{(q+1)(2n+p+q+1)^2}\frac{\Gamma(q+1)}{(\Gamma(q+2))^2}\times$$
$$\left[-[(n+q+1)(2n+p+q+1) + n(p-q-1)]\left(\frac{\Gamma(p+n+1)\Gamma(q+n+1)}{\Gamma(p+q+n+2)(n-1)!} + \frac{\Gamma(p+n)\Gamma(q+n+2)}{\Gamma(p+q+n+1)n!}\right)\right.$$
$$\left.+2(n+q+1)(n+p)\frac{\Gamma(p+n)\Gamma(q+n+1)}{\Gamma(p+q+n+1)(n-1)!} + 2n(n+p+q+1)\frac{\Gamma(p+n+1)\Gamma(q+n+2)}{\Gamma(p+q+n+2)n!}\right]$$

neatly simplifies as

$$\begin{aligned}\theta_n &= a - (b-a)\frac{(p+1)\cdots(p+n-1)(q+1)^2(q+2)\cdots(q+n)\mu_0^{(p,q)}\kappa}{n!(p+q+2)\cdots(p+q+n+1)C_{n-1}C_n} \\ &= a - (b-a)^{p+q+2}\frac{(q+1)\Gamma(p+n)\Gamma(q+n+1)\ \lambda\kappa}{n!\Gamma(p+q+n+2)\ C_{n-1}C_n}\end{aligned} \tag{30}$$

with C_{n-1} and C_n from (28).

We proceed now to the $K_n(x)$ term in the differential equation (29)

$$K_n(x) = -\frac{(x-a)(x-b)(x-\theta_n)P_n''(x) + \{[(p+q+3)x - a(p+1) - b(q+2)](x-\theta_n) - (x-a)(x-b)\}P_n'(x)}{P_n(x)}$$
$$\frac{P_n''(x)}{P_n(x)} = \frac{n(n-1)x^{n-2} + (n-1)(n-2)t_nx^{n-3} + \cdots}{x^n + t_nx^{n-1} + \cdots} = n(n-1)x^{-2} - 2(n-1)t_nx^{-3} + \cdots,$$
$$\frac{P_n'(x)}{P_n(x)} = \frac{nx^{n-1} + (n-1)t_nx^{n-2} + \cdots}{x^n + t_nx^{n-1} + \cdots} = nx^{-1} - t_nx^{-2} + \cdots,$$

we expand in decreasing powers of x, knowing that the final result is an exact polynomial of degree 1:

$$K_n(x) = -n(n-1)x + 2(n-1)t_n + (a+b+\theta_n)n(n-1) - (p+q+2)nx + (p+q+2)t_n$$
$$- n[-a(p+1) - b(q+2) - (p+q+3)\theta_n) + a + b]$$
$$= -n(n+p+q+1)x + (p+q+2n)t_n + n(n+p+q+2)\theta_n + n(n+p-1)a + n(n+q)b$$
$$= -n(n+p+q+1)(x-a) + (p+q+2n)(t_n + n(a+b)/2) + n(n+p+q+2)(\theta_n - a) + n(q-p)(b-a)/2$$
$$= -n(n+p+q+1)(x-a) + (p+q+2n)\frac{(b-a)n(p-q-1)}{2(2n+p+q+1)} - \frac{C_n}{C_{n-1}}\frac{(b-a)n(n+p+q+1)}{2n+p+q+1} +$$
$$n(n+p+q+2)(\theta_n - a) + n(q-p)(b-a)/2$$

As in the discussion of $\theta_n - a$ above, $C_nC_{n-1}K_n(a)$ is a quadratic form in λ and κ.

When $\kappa = 0$, the Heun's differential equation is $(x-a)$ times the hypergeometric equation, so $K_n(x) = \lambda_n(x-a) = -n(n+p+q+1)(x-a)$, $\theta_n = a$, $t_n = \tilde{t}_n = -n(a+b)/2 + \dfrac{n(p-q)}{2n+p+q}\,\dfrac{b-a}{2}$ from (26) (with (p,q) instead of $(p,q+1)$), and there is no λ^2 term in $C_nC_{n-1}K_n(a)$.

When $\lambda = 0$, $P_n(x) = P_n^{(p,q+1)}(x) - \dfrac{C_nP_{n-1}^{(p,q+1)}(x)}{C_{n-1}}$ with $\dfrac{C_n}{C_{n-1}} = -\dfrac{q+n+1}{n}$, so $t_n = -\dfrac{n(a+b)}{2} + \dfrac{(b-a)n(p-q-1)}{2(2n+p+q+1)} + \dfrac{q+n+1}{n}\dfrac{(b-a)n(n+p+q+1)}{(2n+p+q)(2n+p+q+1)}, \theta_n = a$ and $K_n(a) = (q+1)(b-a)$ remains, meaning that the κ^2 term in $C_nC_{n-1}K_n(a)$ is $\kappa^2(q+1)(b-a)\dfrac{\Gamma(n+q+2)\Gamma(n+q+1)}{n!(n-1)!(\Gamma(q+2))^2}$

Finally, we must look at the $\lambda\kappa$ term in $C_nC_{n-1}[(p+q+2n)(t_n+n(a+b)/2)+n(q-p)(b-a)/2] = (p+q+2n)C_nC_{n-1}\dfrac{(b-a)n(p-q-1)}{2(2n+p+q+1)} - C_n^2\dfrac{(b-a)n(n+p+q+1)}{2n+p+q+1} + n(q-p)C_nC_{n-1}(b-a)/2$ which is $\lambda\kappa\dfrac{n(b-a)^{p+q+2}}{2(2n+p+q+1)}\,\dfrac{\Gamma(p+n)\Gamma(q+1)\Gamma(q+n+1)}{\Gamma(p+q+n+2)n!\Gamma(q+2)} \times$
$\{-[(p+q+2n)(p-q-1)+(2n+p+q+1)(q-p)][(p+n)n+(p+q+n+1)(q+n+1)]$
$-\,4(n+p+q+1)(p+n)(q+n+1) = -2(q+1)(p+n)(2n+p+q+1)\}$,

and $-\lambda\kappa(b-a)^{p+q+2}\dfrac{\Gamma(p+n+1)\Gamma(q+n+1)}{\Gamma(p+q+n+2)(n-1)!}$ comes out, and we add $n(n+p+q+2)(\theta_n-a)$:

$$K_n(x) = -n(p+q+n+1)(x-a) - \lambda\kappa(b-a)^{p+q+2}\,\frac{(q+2)\Gamma(p+n)\Gamma(q+n+1)}{\Gamma(p+q+n+1)(n-1)!C_{n-1}C_n}$$
$$- (b-a)\kappa^2\frac{(q+1)(q+2)^2\cdots(q+n)^2(q+n+1)}{(n-1)!n!C_{n-1}C_n} \qquad (31)$$

For derivatives of Dirac distributions, see Arvesú et al. [5].

2.6. Point masses at the two endpoints.

Orthogonal polynomials with respect to a Jacobi weight plus point masses at the two endpoints have been considered by H.L. Krall and followers, see [72], also [107], for survey and new results, and were sometimes called Koornwinder polynomials for a short while after [71] was published, but the name Koornwinder is associated now to the much deeper subject of orthogonal polynomials in several variables.

Krall and followers looked for eigenproblems of high order for special choices of the Jacobi exponents; Koornwinder, and also Kiesel and Wimp [69], showed that a second order differential equation, normally not of spectral type, can always be exhibited. We show that this equation is normally not a Heun's equation, unless in special cases with a transformed variable.

The most elegant way to discuss the new polynomials is to extend the Hahn's trick of the preceding section by $P_n(x) =$ a combination of $\hat{P}_n(x), \hat{P}_{n-1}(x), \hat{P}_{n-2}(x)$, or $\hat{P}_n(x)$ and some product $(u_nx+v_n)\hat{P}_{n-1}(x)$, where the $\hat{P}_m$ are known Jacobi polynomials [69, 88].

We don't try to remake the full derivation here, we just consider a symmetric configuration where the results of the preceding section can be used.

So, let $d\mu(x) = \lambda(1-x^2)^r dx + (\kappa/2)(\delta(x-1)+\delta(x+1))dx$ on $[-1,1]$.

1. The polynomial of *even* degree $P_{2n}(x)$ is the orthogonal polynomial of degree n in the variable $y = 1-x^2$ with respect to the measure $\lambda y^r(1-y)^{-1/2}dy + \kappa\delta(y)$ on $y \in [0,1]$, and we apply the Heun equation (29) with $a = 0, b = 1, p = -1/2, q = r$:

$d^2P_{2n}(\sqrt{1-y})/dy^2 + \left(\dfrac{1/2}{y-1} + \dfrac{r+2}{y} - \dfrac{1}{y-\theta_n}\right) dP_{2n}(\sqrt{1-y})/dy$
$-\dfrac{n(r+n+1/2)y - K_n(0)}{y(y-1)(y-\theta_n)} P_{2n}(\sqrt{1-y}) = 0$, or[1]
$P''_{2n}(x) + 2\left(-\dfrac{r+2}{1-x^2} + \dfrac{1}{1-x^2-\theta_n}\right) xP'_{2n}(x) - \dfrac{4K_n(1-x^2)}{(1-x^2)(1-x^2-\theta_n)} P_{2n}(x) = 0$, with θ_n, K_n given by (28), (30), (31) with $[a,b,p,q] = [0,1,-1/2,r]$.

2. The polynomial of *odd* degree $P_{2n+1}(x)$ is x times the orthogonal polynomial of degree n in the variable $y = 1-x^2$ with respect to the measure $\lambda y^r(1-y)^{1/2}dy + \kappa\delta(y)$ on $y \in [0,1]$, and we apply the Heun equation (29) with $a=0, b=1, p=1/2, q=r$:
$\dfrac{d^2}{dy^2}\dfrac{P_{2n+1}(x)}{x} + \left(\dfrac{3/2}{y-1} + \dfrac{r+2}{y} - \dfrac{1}{y-\theta_n}\right)\dfrac{d}{dy}\dfrac{P_{2n+1}(x)}{x} + \dfrac{K_n(y)}{y(y-1)(y-\theta_n)}\dfrac{P_{2n+1}(x)}{x} = 0$, or
$P''_{2n+1}(x) + 2\left(-\dfrac{r+2}{1-x^2} + \dfrac{1}{1-x^2-\theta_n}\right) xP'_{2n+1}(x)$
$+\left(\dfrac{2(r+2)}{1-x^2} - \dfrac{2}{1-x^2-\theta_n} - \dfrac{4K_n(1-x^2)}{(1-x^2)(1-x^2-\theta_n)}\right) P_{2n+1}(x) = 0$, with appropriate $\theta_n, K_n(0)$ by (28), (30), (31) with $[a,b,p,q] = [0,1,1/2,r]$.

2.7. **Laguerre-type polynomials.** We consider the limit $\lambda x^q e^{-x} + \kappa\delta(x)$ of $\lambda x^q(1-x/L)^L + \kappa\delta(x)$ when $L \to \infty$.

We build (12) with $V(x)/W(x) = w'(x)/w(x) = q/x - 1 =$ limit of $q/x + p/(x-L)$ and $W(x) = (x-L)x^2$, so p and $b = L \to \infty$ together, and we have

$$P''_n(x) + \left(-1 + \frac{q+2}{x} - \frac{1}{x-\theta_n}\right) P'_n(x) + \frac{K_n(x)}{x(x-\theta_n)} P_n(x) = 0. \tag{32}$$

where $K_n(x)$ is the limit of $K_n(x)$ of (31) divided by $-L$. We have therefore a *confluent* Heun equation [89, Part B, eq. (1.2.27)].

In (32), θ_n and $K_n(x)$ depend on the limit of C_n of (28) where λ must be replaced by λ/L^L. We also take the Jacobi polynomials times $(-1)^n$ for convenience (the limits are Laguerre polynomials with main coefficients alternating sign with n). Result is

$C_n = \lambda\Gamma(q+1) + \kappa\dfrac{(q+2)\cdots(q+n+1)}{n!}$. It figures: from the general theory of (23) in § 2.4, the auxiliary polynomials $\hat{P}_n$ are the Laguerre polynomials $L_n^{(q+1)}$, so that C_n must be solution of the recurrence relation of $L_n^{(q+1)}(0) : (n+1)C_{n+1} = (2n+q+2)C_n - (n+q+1)C_{n-1}$. By (21), $C_0 = \mu_0 = \int_0^\infty \lambda x^q e^{-x}dx + \kappa = \lambda\Gamma(q+1) + \kappa$, $C_1 = \mu_0(\hat{r}_0 c + \hat{r}'_0) + \hat{\mu}_0\hat{r}_0 = (q+2)(\lambda\Gamma(q+1)+\kappa) - \lambda\Gamma(q+2)$, $C_n = \lambda\Gamma(q+1) + \kappa L_n^{(q+1)}(0)$ and we recover the formula above. Value $\lambda + \kappa(n+1)$ when $q=0$ is found by Ronveaux & Marcellán [88, §4].

From (30), θ_n is the limit at $p = b = L \to \infty$ of $-L^{L+q+2}\dfrac{(q+1)\Gamma(q+n+1)\ \lambda\kappa/L^L}{n!L^{q+2}\ (-C_{n-1}C_n)}$, so
$\theta_n = \dfrac{\kappa\lambda(q+1)\Gamma(q+n+1)}{n!C_nC_{n-1}}$. When $q=0, \theta_n = \dfrac{\kappa\lambda}{(\lambda+\kappa n)(\lambda+\kappa(n+1))}$ [76], [88, eq. (38)].

$K_n(x)$ is the $K_n(x)$ of (31) with $p = b = L \to \infty$ divided by $-L$:
$K_n(x) = nx + \lambda\kappa\dfrac{(q+2)\Gamma(q+n+1)}{(n-1)!C_{n-1}C_n} + \kappa^2\dfrac{(q+1)(q+2)^2\cdots(q+n)^2(q+n+1)}{(n-1)!n!C_{n-1}C_n}$
$= nx + \kappa\dfrac{n!\lambda(q+2)\Gamma(q+n+1) + \kappa(q+1)(q+2)^2\cdots(q+n)^2(q+n+1)}{(n-1)!n!C_{n-1}C_n}$.

[1] One of the authors is so unsure of his differential calculus knowledge that $dP_{2n}(\sqrt{1-y})/dy$ was obtained by considering that $P_{2n}(\sqrt{1-y})$ is a combination of even powers $(\sqrt{1-y})^{2s} = (1-y)^s$ derivated as $-s(1-y)^{s-1}$ whence $dP_{2n}(\sqrt{1-y})/dy = -P'_{2n}(x)/(2x)$, etc.

For derivatives of Dirac distributions, see Álvarez-Nodarse & Marcellán [3]

2.8. Hermite weight $+\kappa\delta(x)$.

Let the measure $d\mu(x) = |x|^r \exp(-x^2)dx + \kappa\delta(x)dx$, a (slightly) generalized) Hermite weight +a point mass at the origin.

The orthogonal polynomials of *odd* degree ignore the point mass, and are $xL_n^{((r+1)/2)}(x^2)$.

The orthogonal polynomials of *even* degree $P_{2n}(x)$ is the polynomial of degree n of the preceding section at x^2 , with $q = (r-1)/2$ and 2κ, so

$\dfrac{d}{dx^2}\left[\dfrac{dP_{2n}(x)}{dx^2}\right] + \left(-1 + \dfrac{(r+3)/2}{x^2} - \dfrac{1}{x^2-\theta_n}\right)\dfrac{dP_{2n}(x)}{dx^2} + \dfrac{nx^2+K_n(0)}{x^2(x^2-\theta_n)}P_{2n}(x) = 0$, which is Heun in x^2, but the final equation is not:

$$P''_{2n}(x) + \left(-1/x - 2x + \frac{r+3}{x} - \frac{2x}{x^2-\theta_n}\right)P'_{2n}(x) + \frac{4nx^2+4K_n(0)}{x^2-\theta_n}P_{2n}(x) = 0.$$

3. An example of biorthogonal rational functions set, or of 2−point Padé approximation.

3.1. Biorthogonal rational functions and rational interpolation.

Let A_m and B_n be polynomials of degres m and n, with $m, n = 0, 1, \ldots$ such that the rational functions $A_m(x)/(x-a)^{m+1}$ and $B_n(x)/(x-b)^{n+1}$ are orthogonal with respect to a (formal) measure $d\mu$ on an interval, or a contour, $[c, d]$:

$$\int_c^d \frac{A_m(x)}{(x-a)^{m+1}}\frac{B_n(x)}{(x-b)^{n+1}}d\mu(x) = 0, m \neq n$$

Existence and unicity depends on the non vanishing of determinants of moments $\int_c^d (x-a)^{-r}(x-b)^{-s}d\mu(x)$, see general expositions of biorthogonality [15, 25, 93, 108].

It follows then that A_m is orthogonal to all polynomials of degree $< m$ w.r.t. to $d\mu(x)/[(x-a)^{m+1}(x-b)^m]$, to $d\mu(x)/[(x-a)^n(x-b)^{n+1}]$ for B_n.

This cries for polynomials C_n orthogonal to all polynomials of degree $< n$ w.r.t. $d\mu(x)/[(x-a)(x-b)]^{n+1}$.

One often uses the variable $z = k(x-a)/(x-b) \Leftrightarrow x = (bz-ak)/(z-k)$, so that $1/(x-a) = (1-kz^{-1})/(b-a)$ and $k/(x-b) = (z-k)/(b-a)$, and the relevant rational functions are polynomials in z and polynomials in z^{-1} (*Laurent polynomials*).

For a very special case, consider the Jacobi weight $(b-x)^p(x-a)^q$ on (a, b): C_n is then the (rather unconventional) Jacobi polynomial of parameters $p-n-1$ and $q-n-1$ involving the hypergeometric expansion ${}_2F_1(-n, 1-p; q-n; z)$ with $z = (x-a)/(x-b)$ [1, 22.5.45] [57, p.31] [106, eq. (5.5)]. That's where the Pochhammer contour discussed above is needed [82, pp. 326, 389], [102, §12.43, p.257].

There are recurrence relations between the A, B and the Cs, for instance

$A_n(x) = \dfrac{C_n(x)A_{n+1}(b) - A_{n+1}(x)C_n(b)}{x-b}$, $C_n(x) = \dfrac{A_{n+1}(x)C_{n+1}(a) - C_{n+1}(x)A_{n+1}(a)}{x-a}$, up to multiplicative constants (Christoffel-Darboux-type [19, 41, 65] etc.), leading to a recurrence relation of the form

$C_{n+1}(x) = (\xi_n x + \eta_n)C_n(x) + \zeta_n(x-a)(x-b)C_{n-1}(x)$ (special case of type-II rec of Ismail & Masson [66, sec. 3])

We now call $P_n = C_n$, and show that P_n is the denominator of rational function achieving Taylor match of order n of the Stieltjes-Markov function (7) at a and b (two-point Padé approximation).

Indeed, the required interpolant to P_nS is

$$N_n(x) = \int_c^d \left\{ \frac{P_n(x)-P_n(t)}{x-t} + P_n(t)\frac{(t-a)^{n+1}(t-b)^{n+1}-(x-a)^{n+1}(x-b)^{n+1}}{(t-a)^{n+1}(t-b)^{n+1}(x-t)} \right\} d\mu(t)$$

from Hermite-Walsh formulas [25, Thm 3.6.1] seems to have degree $2n+1$ in x, but the terms $(t^k - x^k)/(x-t)$ turn in sums of products $t^s x^{k-1-s}$ killed by orthogonality with $P_n = C_n$ when $s < n$, so the powers $k-1-s \leqslant k-1-n \leqslant n+1$ remain for x.

3.2. **Chebyshev example.**

3.2.1. *Chebyshev polynomials formulas [66, Example 3.1].*

Consider the measure $d\mu(t)$ = a constant times $\sqrt{(d-t)(t-c)}\ dt$ on (c,d) leading to the Stieltjes-Markov function (7)

$S(x) = R(x) - x + (c+d)/2)$, where $R(x) = \sqrt{(x-c)(x-d)}$. We don't need to tell exactly how the square root is chosen, we only have to make two choices, at $x = a$ and at $x = b$ (the *germs* of [70]).

Let $I(x) := ((x-a)R(b) - (x-b)R(a))/(b-a)$ be the linear interpolant of R, and let $L(x) := ((x-a)S(b) - (x-b)S(a))/(b-a) = I(x) - x + (c+d)/2$ be the linear interpolant of S.

We start a continued fraction expansion as

$S(x) = L(x) + S(x) - L(x)$

$$= L(x) + \cfrac{(x-a)(x-b)}{\cfrac{(x-a)(x-b)}{S(x)-I(x)+x-(c+d)/2} = \cfrac{S(x)+x-(c+d)/2+I(x)}{[R^2(x)-I^2(x)]/((x-a)(x-b))}}$$

simplifies in $\sqrt{(x-c)(x-d)} + I(x) = 2I(x) + \dfrac{\gamma(x-a)(x-b)}{\sqrt{(x-c)(x-d)}+I(x)}$, where

$\gamma = 1 - [R(b)-R(a)]^2/(b-a)^2$ is the coefficient of x^2 in $R^2(x) - I^2(x) = \gamma(x-a)(x-b)$.

With the Laurent variable $z = k(x-a)/(x-b)$, $(k \neq 0)$, sending c and d to $c' = k(c-a)/(c-b)$ and $d' = k(d-a)/(d-b)$, one has $I(x) = [zR(b) - kR(a)]/(z-k)$, $(x-a)(x-b) = (b-a)^2 kz/(z-k)^2$,

$(x-c)(x-d) = k^2(b-a)^2(z-c')(z-d')/((c'-k)(d'-k)(z-k)^2)$, so,

$(a-c)(a-d) = (b-a)^2 c'd'/((c'-k)(d'-k)), (b-c)(b-d) = k^2(b-a)^2/((c'-k)(d'-k))$, and $R(a)$ and $R(b)$ are two independent choices of the square root in $(b-a)\sqrt{c'd'/((c'-k)(d'-k))}$ and $(b-a)k\sqrt{1/((c'-k)(d'-k))}$,

$\gamma = 1 - [k - \sqrt{c'd'}]^2/((c'-k)(d'-k)) = -k[c'+d'-2\sqrt{c'd'}]/((c'-k)(d'-k))$, finally, $I(x) = \dfrac{k(b-a)(z-\sqrt{c'd'})}{(z-k)\sqrt{(c'-k)(d'-k)}} = \dfrac{(b-a)(z-\sqrt{c'd'})\sqrt{-k\gamma}}{(\sqrt{c'}-\sqrt{d'})(z-k)}$.

Numerators and denominators are solutions of the recurrence relation

$P_{n+1}(x) = 2I(x)P_n(x) + \gamma(x-a)(x-b)P_{n-1}(x)$ with

$P_0(x) \equiv 1, P_1(x) = 2I(x)$; $N_0(x) = I(x) - x + (c+d)/2 = L(x)$,

$N_1(x) = 4I^2(x) + \gamma(x-a)(x-b) - (I(x) + x - (c+d)/2)P_1(x) = 2I(x)L(x) + \gamma(x-a)(x-b) \Rightarrow P_{-1}(x) \equiv 0, N_{-1}(x) \equiv 1$.

With the variable z, the recurrence relation becomes

$P_{n+1}(x) = 2\dfrac{(b-a)(z-\sqrt{c'd'})\sqrt{-k\gamma}}{(\sqrt{c'}-\sqrt{d'})(z-k)}P_n(x) + \dfrac{\gamma(b-a)^2kz}{(z-k)^2}P_{n-1}(x)$, or

$X_{n+1} = 2\dfrac{z-\sqrt{c'd'}}{\sqrt{z}(\sqrt{c'}-\sqrt{d'})}X_n - X_{n-1}$, where $X_n = \left[\dfrac{z-k}{(b-a)\sqrt{-k\gamma z}}\right]^n P_n(x)$,

and this recurrence relation is solved by Chebyshev polynomials

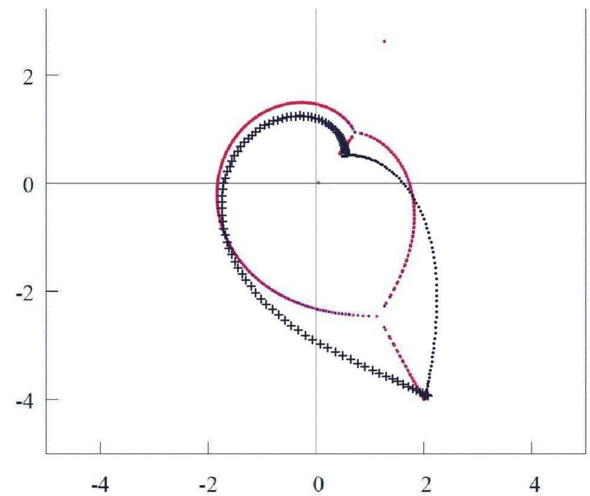

FIGURE 2. Red and blue dots: zeros and poles for the exponents $(1/4, -1/4)$ (Suetin [94]); black dots and + signs: divergence locus for the exponents $(1/2, 1/2)$ for two different choices of the sign of $\sqrt{cd}$. Here, $c = 0.45 + 0.55i$, $d = 2 - 4i$, $a = 0$ and $b = \infty$, the Laurent polynomials choice.

$$
\begin{aligned}
P_n(x) &= (b-a)^n(-\gamma kz)^{n/2}U_n\left(\frac{z-\sqrt{c'd'}}{\sqrt{z}(\sqrt{c'}-\sqrt{d'})}\right)/(z-k)^n,\\
N_n(x) &= L(x)P_n(x) - (b-a)^{n+1}(-\gamma kz)^{(n+1)/2}U_{n-1}\left(\frac{z-\sqrt{c'd'}}{\sqrt{z}(\sqrt{c'}-\sqrt{d'})}\right)/(z-k)^{n+1}, n = 0, 1, \ldots
\end{aligned}
\tag{33}
$$

Incidentally, convergence occurs in the non oscillatory region, the limit is $L(x) - (b-a)(-\gamma kz)^{1/2}Z(x)/(z-k)$, where $|Z(x)| < 1$ and $Z(x) + 1/Z(x) = 2\dfrac{z-\sqrt{c'd'}}{\sqrt{z}(\sqrt{c'}-\sqrt{d'})}$, so that $Z(x) = \dfrac{z-\sqrt{c'd'} \pm \sqrt{(z-c')(z-d')}}{\sqrt{z}(\sqrt{c'}-\sqrt{d'}} = (z-k)\dfrac{I(x) \pm \sqrt{(x-c)(x-d)}}{(b-a)\sqrt{-k\gamma z}}$. The limit is checked to be $L(x) - I(x) \mp \sqrt{(x-c)(x-d)}$, which is a determination of $S(x)$, as it should!

The divergence locus is the oscillatory region $\dfrac{z-\sqrt{c'd'}}{\sqrt{z}(\sqrt{c'}-\sqrt{d'})} = t \in [-1, 1]$, or $\mathrm{Im}\dfrac{z-2\sqrt{c'd'}+c'd'/z}{c'-2\sqrt{c'd'}+d'} = 0$, a part of a particular cubic curve (Deaux [26]),see fig. 2.

No similar simple results must be expected when the exponents p and q in $d\mu(t) = (d-t)^p(t-c)^q\, dt$ on (c, d) are not half integers! Komlov & Suetin [70] built an elaborate theory of asymptotic behavior; Zhedanov [106] showed that recurrence coefficients have normally no simple formulas.

3.2.2. *Differential equation.*

We start with the differential equation of U_n (NIST [82, 18.9.19-20] with $\lambda = 2$),

$\dfrac{d}{dX}\left[(1-X^2)^{3/2}\dfrac{dU_n(X)}{dX}\right]+n(n+2)(1-X^2)^{1/2}U_n(X) = 0$, where $X = \dfrac{z-\sqrt{c'd'}}{\sqrt{z}(\sqrt{c'}-\sqrt{d'})}$, from (33), and we see that $1-X^2 = -\dfrac{(z-c')(z-d')/z}{(\sqrt{c'}-\sqrt{d'})^2}$. The two solutions of the differential equation for U_n are $(X\pm\sqrt{X^2-1})^{n+1}/\sqrt{X^2-1}$ = a constant times $[\sqrt{z}-\sqrt{c'd'/z}\pm\sqrt{(z-c')(z-d')/z}]^{n+1}/\sqrt{(z-c')(z-d')/z}$.

We use $dX/dz = (z^{-1/2}+\sqrt{c'd'}\,z^{-3/2})/(2(\sqrt{c'}-\sqrt{d'}))$, $P_n(x)$ = constant times $(\sqrt{z}-k/\sqrt{z})^{-n}U_n(X)$ and get with the help of computer algebra,
$$\frac{d^2P_n(x)}{dz^2}+\left(\frac{3/2}{z-c'}+\frac{3/2}{z-d'}+\frac{2n}{z-k}-\frac{n}{z}-\frac{1}{z+\sqrt{c'd'}}\right)\frac{dP_n(x)}{dz}+\frac{R_n(z)P_n(x)}{z(z-c')(z-d')(z-k)^2(z+\sqrt{c'd'})}=0,$$
where R_n is a third degree polynomial.

This equation is certainly not Heun, as we already have singular points at $c', d', 0, \infty, k$, and $-\sqrt{c'd'}$. At $z=0$ and ∞, particular solutions behave like $z^{\pm(n+1)}$; at $z=k$ (corresponding to $x=\infty$ in $z=k(x-a)/(x-b)$), one has $R_n(k)/(k(k-c')(k-d')(k+\sqrt{c'd'})) = n(n-1)$, so that powers near $z=k$ are $(z-k)^{-n}$ and $(z-k)^{-(n-1)}$; finally, $-\sqrt{c'd'}$ is an apparent singularity.

We return to x in $z=k(x-a)/(x-b)$, to have
$$P_n''(x)-\left(\frac{n}{x-a}+\frac{n}{x-b}-\frac{3/2}{x-c}-\frac{3/2}{x-d}+\frac{1}{x-(ak+b\sqrt{c'd'})/(k+\sqrt{c'd'})}\right)P_n'(x)$$
$$+\frac{S_n(x)P_n(x)}{(x-a)(x-b)(x-c)(x-d)(k(x-a)+\sqrt{c'd'}(x-b))}=0,$$
where $S_n(x) = (x-b)^3R_n(k(x-a)/(x-b))/(k(k-c')(k-d'))$. Consider now the symmetric situation $a=-b, c=-d$, so that $c'=k(c-a)/(c+a), d'=k(c+a)/(c-a)$, and $\sqrt{c'd'}$ is either k or $-k$.

In the first case, $P_n''(x)-\left(\dfrac{2nx}{x^2-a^2}-\dfrac{3x}{x^2-c^2}+\dfrac{1}{x}\right)P_n'(x)+\dfrac{n(n-1)x^2P_n(x)}{(x^2-a^2)(x^2-c^2)}=0$, which does not always yield a polynomial of degree n, as P_1 is a mere constant.

In the second case $P_n''(x)-\left(\dfrac{2nx}{x^2-a^2}-\dfrac{3x}{x^2-c^2}\right)P_n'(x)+\dfrac{n[(n-2)x^2-((c^2-2a^2)/(a+c)+n(c-a)]P_n(x)}{(x^2-a^2)(x^2-c^2)}=0$, corresponds to the odd Stieltjes-Markov function $\sqrt{x^2-c^2}\ -x$.

The obvious change of variable is $x^2=t$, then $d/dx = 2\sqrt{t}d/dt$ and
$$\frac{d^2P_n(x)}{dt^2}-\left(-\frac{1}{2t}+\frac{n}{t-a^2}-\frac{3/2}{t-c^2}\right)\frac{dP_n(x)}{dt}+\frac{n[(n-2)t-((c^2-2a^2)/(a+c)+n(c-a)]P_n(x)}{4t(t-a^2)(t-c^2)}=0,$$
a Heun differential equation.

4. **Polynomial eigenfunctions of a Heun operator.**

4.1. **The Heun operator.**

Heun's differential equation appears in investigations of special cases of the Schrödinger equation, through the invariant form (16b) $Y''(x)-I_h(x)Y(x)=0$ [28–33, 62, 83] with relations to exceptional orthogonal polynomials, which are sets of eigenfunctions of some differential operators extending the classical setup [43, 48] which includes Heun examples [11, 95].

This field of particular Sturm-Liouville operators allowing a full set of polynomial eigenfunctions has expanded at an incredible rate in a few years. The name "exceptional orthogonal polynomials" is given to known non classical cases where the eigenfunctions do not have all possible degrees, for instance, the equation (34) has no constant solution if $q_0 \neq cp_0$. It is known however that the polynomial eigenfunctions make a complete set in the relevant L^2 weighted space, new corrected proofs have been published recently [36, 47].

The name "exceptional" will probably disappear, we will soon have to consider these new families within the realm of orthogonal polynomials.

The still called "exceptional" orthogonal polynomials related to Heun's differential equation are polynomial eigenfunctions of the operator L in

$$Ly := (x-a)(x-b)\left[y''(x) + \left(\frac{\gamma}{x-a} + \frac{\delta}{x-b} + \frac{\epsilon}{x-c}\right)y'(x)\right] + \frac{p_0x-q_0}{x-c}y = \lambda_n y \tag{34}$$

where y is a polynomial of degree n, but not for any non negative integer, for $n \in$ some set A.

The eigenfunctions of the operator L of (34) are orthogonal with respect to a weight function w of support $[a,b]$ if L can be shown to be (formally) selfadjoint in the relevant scalar product space, i.e., if $\int_a^b (Lf)g(x)w(x)dx$ is symmetric in f and g:

$$\int_a^b (Lf)g(x)w(x)dx = \int_a^b \left\{(x-a)(x-b)\left[f''(x) + \left(\frac{\gamma}{x-a} + \frac{\delta}{x-b} + \frac{\epsilon}{x-c}\right)f'(x)\right]\right.$$
$$\left. + \frac{p_0x-q_0}{x-c}f(x)\right\} g(x)w(x)dx,$$

we perform integration by parts on the term containing $f''(x)$, and kill the contributions of the unsymmetric $f'g$:

$$-[(x-a)(x-b)w(x)]' + (x-a)(x-b)\left(\frac{\gamma}{x-a} + \frac{\delta}{x-b} + \frac{\epsilon}{x-c}\right)w(x) = 0$$

leading to

$$w(x) = \text{ constant } \times (x-a)^{\gamma-1}(x-b)^{\delta-1}(x-c)^{\epsilon}, \tag{35}$$

[43, eq. (32), (36a)], [89, Part A, §5.2].

4.2. **Theorem.** *Non hypergeometric polynomial eigenfunctions of a Heun operator* (34) *are related to a Jacobi weight function divided by an even power* $(x-c)^{2m}$.

This means that ϵ is a negative integer $-2m$ in (34) and (35).

When $\epsilon = -2$, the polynomials are the X_1 exceptional Jacobi or Laguerre polynomials [43, 83, 95].

Exceptional X_ℓ polynomials are related to Jacobi or Laguerre weight functions divided by the square of a polynomial of degree ℓ [35, 45, 59, 84], they are not related to Heun's equation when $\ell > 1$.

Note already that the operator of (34) has no constant eigenfunction if $p_0c - q_0 \neq 0$.

It is known that exceptional polynomials are NOT semi-classical orthogonal polynomials [43, p.353].

Proof: we enter the polynomial eigenfunction $\sum_0^n c_k(x-c)^k$ in (34)

$(x-a)(x-b)\left[\sum_2^n k(k-1)c_k(x-c)^{k-2} + \epsilon\sum_1^n kc_k(x-c)^{k-2}\right]$
$+ [\gamma(x-b) + \delta(x-a)]\sum_1^n kc_k(x-c)^{k-1} + (p_0x-q_0)\sum_0^n c_k(x-c)^{k-1} = \lambda_n\sum_0^n c_k(x-c)^k$

and look at the coefficient of $(x-c)^k$:

$k(k-1+\epsilon)c_k + (2c-a-b)(k+1)(k+\epsilon)c_{k+1} + (c-a)(c-b)(k+2)(k+1+\epsilon)c_{k+2} + (\gamma+\delta)kc_k$
$+ [\gamma(c-b) + \delta(c-a)](k+1)c_{k+1} + p_0c_k + (p_0c-q_0)c_{k+1} - \lambda_nc_k = 0,\ k = 0,\dots,n.$ Of course, $c_{n+1} = c_{n+2} = 0$, so $n(n-1+\epsilon) + n(\gamma+\delta) + p_0 - \lambda_n = 0$ at $k = n$, giving λ_n. The coefficient of c_k is then $k(k-1+\epsilon) + k(\gamma+\delta) + p_0 - \lambda_n = (k-n)(k+n-1+\epsilon+\gamma+\delta)$ vanishing at $k = n$, and the recurrence relation is

$$(k-n)(k+n-1+\epsilon+\gamma+\delta)c_k + \chi_kc_{k+1} + (c-a)(c-b)(k+2)(k+1+\epsilon)c_{k+2} = 0, \tag{36}$$

for $k = n-1, n-2, \dots,$ where $\chi_k = [(2c-a-b)(k+\epsilon) + (\gamma(c-b) + \delta(c-a))](k+1) + p_0c - q_0$.

This recurrence relation (see Andrews & al. [4, App. F], [103]), appears everywhere in relation with Heun's equation, function, or polynomials Choun [20, 21] Erdelyi [37, §15.3] Hautot [55] Ronveaux [89, § 3.3, 3.6] NIST [82, §31.3], see also Alhaidari, [2], Grünbaum et al. [50], Ishkhanyan [64], for other polynomial expansions.

The recurrence relation at $k = n-1$ gives c_{n-1} as a multiple of c_n, then, c_{n-2}, etc. That must end! We must have $c_{-1} = 0$. Then, $c_{-2} = 0$ follows from (36) at $k = -2$, and all c_ks with negative index do vanish.

One has $(n+1)(n-2+\epsilon+\gamma+\delta)c_{-1} = (p_0c-q_0)c_0 + \epsilon(c-a)(c-b)c_1$ from (36) at $k=-1$, $c_{-1}=0$ certainly holds for all n when $\epsilon = 0$ and $p_0c - q_0 = 0$, we recover then the hypergeometric case.

In general, c_{-1} is a complicated function of n built by solving the steps of (36) from $k = n$ downto $k = -1$ (a continued fraction in [37, §15.3], also a determinant in [55]) , BUT, if ϵ is a negative integer, the c_{k+2} term vanishes in (36) at $k = -\epsilon-1$. We now have only to perform the recurrence steps from $k = -\epsilon-1$ downto $k = -1$, so that $c_{-1}/c_{-\epsilon}$ is a rational function of fixed degree in n.

When $\epsilon = -1$, $c_0 = \dfrac{\chi_0}{n(n-2+\gamma+\delta)}c_1$ and $c_{-1} = \dfrac{(p_0c-q_0)c_0 - (c-a)(c-b)c_1}{(n+1)(n-3+\gamma+\delta)}$ $= \left[\dfrac{(p_0c-q_0)\chi_0}{(n+1)(n-3+\gamma+\delta)n(n-2+\gamma+\delta)} - \dfrac{(c-a)(c-b)}{(n+1)(n-3+\gamma+\delta)}\right]c_1$, vanishing for an infinite set of values of n if $p_0c-q_0 = 0$ and $c = a$ or b, again the hypergeometric case.

When $\epsilon = -2$, $c_1 = \dfrac{\chi_1}{(n-1)(n-2+\gamma+\delta)}c_2$, $c_0 = \dfrac{\chi_0c_1 - 2(c-a)(c-b)c_2}{n(n-3+\gamma+\delta)}$

$$= \left[\frac{\chi_0\chi_1}{n(n-3+\gamma+\delta)(n-1)(n-2+\gamma+\delta)} - \frac{2(c-a)(c-b)}{n(n-3+\gamma+\delta)}\right]c_2,$$

$$(n+1)(n-4+\gamma+\delta)c_{-1} = (p_0c-q_0)c_0 - 2(c-a)(c-b)c_1$$

$$= \left[\frac{(p_0c-q_0)\chi_0\chi_1}{n(n-3+\gamma+\delta)(n-1)(n-2+\gamma+\delta)} - \frac{2(p_0c-q_0)(c-a)(c-b)}{n(n-3+\gamma+\delta)} - \frac{2\chi_1(c-a)(c-b)}{(n-1)(n-2+\gamma+\delta)}\right]c_2,$$

vanishes for all n if $\chi_1 = q_0 - p_0c = a+b-2c+\gamma(c-b)+\delta(c-a) = (1-\delta)a+(1-\gamma)b+(\gamma+\delta-2)c$, and if the residue

$$\chi_1 - \gamma(c-b) - \delta(c-a) - 2(c-a)(c-b)(2-\gamma-\delta)/\chi_1 = a+b-2c-2\frac{(c-a)(c-b)}{\dfrac{(1-\delta)a+(1-\gamma)b}{2-\gamma-\delta} - c} =$$

$$2((a+b)/2-c) - 2\frac{((a+b)/2-c)^2 - (b-a)^2/4}{\dfrac{(\delta-\gamma)(b-a)}{2(2-\gamma-\delta)} + (a+b)/2 - c}$$ vanishes too, an equation for c, solved by

$$c = \frac{a+b}{2} + \frac{(2-\gamma-\delta)(b-a)}{2(\delta-\gamma)}.$$

We recover the X_1-theory [43, eq. 36a] [44] [95, eq. 6.3].

When ϵ is a negative integer < -2, we only look at large n, $c_{-\epsilon-1}/c_{-\epsilon} \sim \chi_{-\epsilon-1}/n^2$, $c_{-\epsilon-2}/c_{-\epsilon} \sim -\epsilon(c-a)(c-b)/n^2$, and different behaviors occur at even and odd steps:

$c_{-\epsilon-2r} \sim u_r c_{-\epsilon}/n^{2r}$, $c_{-\epsilon-2r+1} \sim v_r c_{-\epsilon}/n^{2r}$, with $u_{r+1} = -(2r+1)(-\epsilon-2r)(c-a)(c-b)u_r$, $v_{r+1} = \chi_{-\epsilon-2r-1}u_r - 2r(-\epsilon-2r+1)(c-a)(c-b)v_r$.

If ϵ is the odd negative integer $-2m-1$, the main behavior of c_{-1}/c_{2m+1} is u_{m+1}/n^{2m+2} leading to $c = a$ or $c = b$, the hypergeometric case.

So, only even negative integers $\epsilon = -2m$ remain. □

It seems very unlikely that there is a solution when $m > 1$, so we conjecture that $\epsilon = -2$ is the only possible solution.

5. Conclusion.

Sequences of orthogonal polynomials satisfying Heun's differential equation are only found in two different classes.

When we try to extend the class of classical orthogonal polynomials $\{P_n\}_0^\infty$ by allowing them to satisfy Heun's differential equations (variable with the degree n), we only find classical weights modified by a point mass.

When we look at polynomial sequences $\{P_n\}_\ell^\infty$ made of eigenfunctions of a fixed Heun's operator, we only find classical weights divided by a power $(x-c)^2$ [43, 95].

The subfield relating exceptional polynomials to Heun's differential equation seems completely explored. For instance, A.M. Ishkhanyan [63] finds and discusses all the 35 possible forms for the potential function related in some way to Heun's differential equation (11 when some equivalence relations are taken into account).

This simply means that important classes of non hypergeometric orthogonal polynomials are related to Fuchsian differential equations of higher complexity than the Heun class [22].

However, many applications in mathematical physics are currently worked, as seen in recent papers [28–33, 62].

Progress on new soluble potentials must be expected in new directions [12, 13, 35, 46, 48, 96] no more related to Heun's equation.

There may be however interesting mathematical phenomena deserving more attention: for instance, there is a striking similarity between the apparent singularities phenomenon in the Laguerre theory of orthogonal polynomials [53, 54], and the theory of "exceptional" orthogonal polynomials, as it appears in the differential equations [67, 68, 90].

Recently, new asymptotic expansions involving the Heun's equation have been found [23, § 4], [18].

And now, discrete Heun operators enter the arena [7, 8, 11, 99].

6. Acknowledgements.

Many thanks to the referees who supplied numerous recent references, and to all the people who cared in the Journal for the manuscript.

A part of section 3 of the present study has been shown as a poster "Heun differential equation satisfied by some classical biorthogonal rational functions, preliminary notes", in the 14th International Symposium on Orthogonal Polynomials, Special Functions and Applications (OPSFA-14), University of Kent, Canterbury, UK, July 3-7, 2017. Many thanks to Peter Clarkson and Ana Loureiro.

A part of section 2 has been presented as the talk "About families of orthogonal polynomials satisfying Heun's differential equation" by one of us (F.N.) at the AIMS-Volkswagen Stiftung Workshop on Introduction to Orthogonal Polynomials and Applications, Douala, Cameroon, October 5 - 12, 2018. Many thanks to Mama Foupouagnigni, Wolfram Koepf, Francisco (Paco) Marcellán Español, Walter Van Assche.

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