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Chasing special biorthogonal rational functions

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Jes' Fine Fremount (the boy bug) of Walt Kelly's *Pogo.*

Classical orthogonal polynomials are simple hypergeometric expansions, they have funny names, as numerous as the craters of the Moon, but what about biorthogonal rational functions?

1. Biorthogonal rational functions and rational interpolation.

Biorthogonality refers to two sequences of objects, possibly of different kind, but able to interact through a bilinear form $\langle\ ,\ \rangle$: $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_m\}$ are biorthogonal if $\langle\mathcal{A}_n,\mathcal{B}_m\rangle = 0, m\neq n$. Here, $\mathcal{A}_n$ and $\mathcal{B}_m$ are rational functions

$$\frac{A_n(x)}{(x-a_0)\cdots(x-a_n)} \text{ and } \frac{B_m(x)}{(x-b_0)\cdots(x-b_m)},$$

and the bilinear form is $\int_c^d f(t)g(t)d\mu(t)$, (possibly formal).

'Normally', A_n (resp. B_m) is orthogonal to all polynomials of smaller degrees with respect to $d\mu(t)/((t-a_0)\cdots(t-a_n)(t-b_0)\cdots(t-b_{n-1}))$ (resp. $d\mu(t)/((t-a_0)\cdots(t-a_{m-1})(t-b_0)\cdots(t-b_m))$). This cries for C_n of degree n orthogonal to all polynomials of smaller degrees with respect to $d\mu(t)/((t-a_0)\cdots(t-a_n)(t-b_0)\cdots(t-b_n))$.

Let $P_{2n}=C_n$ and $P_{2n+1}=A_n$, $n=0,1,...,$

$$P_{n+1}(x) = P_n(x) + r_n\left(\begin{cases}(x-a_{n/2}) \text{ if } n \text{ is even}\\ (x-b_{(n-1)/2}) \text{ if } n \text{ is odd}\end{cases}\right)P_{n-1}(x), n = 0,1,\dots, P_{-1}=0$$ Ismail & Masson, Spiridonov & Zhedanov

A_n etc. are ***denominators of rational interpolants*** of $f(x) = \int_c^d(x-t)^{-1}d\mu(t)$: $A_n(x)f(x) =$ pol. interpolant at $a_0,\cdots,a_n,b_0,\dots,b_{n-1}+$ remainder, and the degree of the interpolant $=\int_c^d A_n(t)[1-(x-a_0)\dots/(t-a_0)\dots)](x-t)^{-1}d\mu(t)$ is only n instead of $2n$ thanks to orthogonality.

$$f(x)/f(a_0) = 1+\cfrac{r_0(x-a_0)}{1+\cfrac{r_1(x-b_0)}{1+\cfrac{r_2(x-a_1)}{1+\cfrac{r_3(x-b_1)}{1+\ddots}}}}$$ Multipoint Padé: Goncar, Lopez, Rahmanov, Stahl. Let $\rho_{2n}=r_{2n}(x-a_n)$, $\rho_{2n+1}=r_{2n+1}(x-b_n)$. If $a_n=b_n=a, n=0,1,\dots$:Padé,and the rational functions are polynomials of $1/(x-a)$, the orthogonal polynomials! If $a_n=a, b_n=b, n=0,1,\dots$:2-point Padé.

CLASSICAL functions must satisfy simple explicit recurrence relations, difference equations and relations, hypergeometric expansions, perhaps Rodrigues formulas.

We start with functions already encountered in classical orthogonal polynomials theory.

2. The exponential function.

Padé: $e^x = 1+\cfrac{x}{1+\cfrac{-x/2}{1+\cfrac{x/6}{1+\ddots}}}$: $r_n=(-1)^n/(2n+1+(-1)^n), n=1,2,\dots$

Cuyt, Perron, Wall, Euler 24 years old! (Knuth **2** §4.5.3, answers of §4.5.3, 16, Cretney 2014

$$\frac{1}{1},\frac{1+x}{1},\frac{1+x/2}{1-x/2},\frac{1+2x/3+x^2/6}{1-x/3},\frac{1+x/2+x^2/12}{1-x/2+x^2/12},\dots,\frac{{}_1F_1(-m,-m-n,x)}{{}_1F_1(-n,-m-n,-x)}$$

Khovanskii quotes Lagrange 1776 for more examples.

Rational interpolation on two lattices with the same step, say, $a+nh'$ and $b+nsh'$, where $s=1$ or -1. When $b=a+h'/2=a+h$, one interpolates on integer multiples of $h'/2=h$, it should have been done 100 years ago! Solved by Iserles in 1981.

$P_n(x)={}_2F_1(-n,(a-x)/h;-m-n;1-e^{-h})$, $N_m(x)={}_2F_1(-m,(a-x)/h;-m-n;1-e^{-h})$, $r_n=\dfrac{(-1)^n\exp((-1)^nh/2)}{h\cosh(h/2)[(n+(1+(-1)^n)/2)\coth(h/2)-1]}$, $n=1,2,\dots$

When $s=-1$ and $b=a-h$, we again have interpolation at equidistant points $nh, n\in\mathbb{Z}$, and a simple formula: $r_n=\dfrac{(-1)^n}{(2n+1+(-1)^n)\dfrac{h\exp(h)}{\exp(h)-1}}$

Do we have similar tricks when $b\neq a+h'/2$? We still have similar asymptotic behaviors when n is large: $s=1, r_n\sim\dfrac{(-1)^nX\exp((-1)^nh/4)}{n+s_n}$, with $X=2\dfrac{sinh(h/4)}{h\cosh^2(h/4)}$,
$s_n\approx\dfrac{1+(-1)^n}{2}-\dfrac{2(b-a)}{h}\tanh\dfrac{h}{4}-\dfrac{(b-a-h/2)(-1)^nX\exp(-(-1)^nh/4)}{n+\dots}$

3. Jacobi and Hahn's weights.

Jacobi

Gauss's ratio of hypergeometric functions Perron 1913, 1929 §59, 1957 §24; Wall chap. XVIII §89. ${}_2F_1(\alpha+1,1;\alpha+\beta+2;x)$ Denominators of approximants of order $2n$ and $2n-1$ are $x^nP_n^{(\alpha+1,\beta)}(1-2/x)$ and $x^nP_n^{(\alpha,\beta)}(1-2/x)=\dots x^n{}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;1/x)$, NIST 18.5.7

Erdelyi *IT* **2** §16.4(4) p. 284 $\int_{-1}^1(z-x)^{-1}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)dx = \dfrac{2^{\alpha+\beta+n+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)(z-1)^{n+1}}{}_2F_1(n+1,\alpha+n+1;\alpha+\beta+2n+2;2/(z-1))$

For the Jacobi measure $d\mu(t)=(d-t)^\alpha(t-c)^\beta dt$ classical 2-point Padé only if exponents are half integer, Heun JAT 2021

Hahn

Let $w_k=\dfrac{(\beta+1)\cdots(\beta+k)(\alpha+1)\cdots(\alpha+N-k-1)}{k!(N-1-k)!}$, $k=0,\dots,N-1$.

One has $\mu_0=\sum_0^{N-1}w_k=(\alpha+\beta+2)\cdots(\alpha+\beta+N)/(N-1)!$,

$f(x)=\sum_0^{N-1}\dfrac{w_k}{x-c-kh}$ leads to Hahn polynomials by Padé.

See that $(k+1)(\alpha+N-k-1)w_{k+1}=(\beta+k+1)(N-1-k)w_k$. Niki §2.4.6 Pearson

One finds the first order ***difference equation*** $(x-c+h)(x-d-\alpha h+h)f(x+h) = (x-c+\beta h+h)(x-d+h)f(x)-(\alpha+\beta+1)h\mu_0$, with $d=c+Nh$.

Interpolation at $x=c-\beta h, c-\beta h+h, c-\beta h+2h,\dots$:

Let $f_0(x)=f(x)/f(c-\beta h)$, $f_1(x)=\dfrac{r_0(x-c+\beta h)}{f_0(x)-1}$,

$r_0=\dfrac{\alpha+\beta}{h(\beta-1)(\alpha+\beta+N-1)}$, $r_{2n}=-n\dfrac{(\alpha+n)(N-n)}{h\,\Xi_n\Upsilon_{n+1}}$,
$r_{2n-1}=-\dfrac{(\beta-n)(\alpha+\beta-n)(\alpha+\beta+N-n)}{h\,\Xi_n\Upsilon_n}$, $n=1,2,\dots$

with $\Xi_n=\beta(\alpha+\beta+N-3n)-n(\alpha+2N)+3n^2$,
$\Upsilon_n=\beta(\alpha+\beta+N-3n+1)-n\alpha-(2n-1)N+3n^2-3n+1$
$=(\beta-n)(\alpha+\beta+N-2n+1)-(n-1)N+(n-1)^2$

$P_0=P_1=1$, $P_{2n}(x)=P_{2n-1}(x)+r_{2n-1}(x-c+\beta h-(2n-1)h)P_{2n-2}(x)=r_1r_3\cdots r_{2n-1}(x-c+\beta h)^n+\cdots,$

$$P_{2n+1}(x)=P_{2n}(x)+r_{2n}(x-c+\beta h-2nh)P_{2n-1}(x)$$
$$=r_2r_4\cdots r_{2n}(x-c+\beta h)^n\underbrace{\sum_0^n\frac{r_1r_3\cdots r_{2k-1}}{r_2r_4\cdots r_{2k}}}_{-\frac{r_1r_3\cdots r_{2n+1}\Xi_{n+1}}{hr_2r_4\cdots r_{2n}(\alpha+\beta-1)}}+\cdots$$ no proof

$P_{2n}(x)=$ const. ${}_3F_2(-n,\alpha+\beta-n,(x-c)/h+\beta-2n;\beta-2n,\alpha+\beta+N-2n;1)$,

$P_{2n+1}(x)=$ const. ${}_3F_2(-n,\alpha+\beta-n-1,(x-c)/h+\beta-2n-1;\beta-2n-1,\alpha+\beta+N-2n-1;1)$,

4.Classical biorthogonal rational functions.

Summary of 5000 years of difference calculus

Δ	$f(x+h)-f(x)$	
∇	$f(x)-f(x-h)$	
δ	$f(x+h/2)-f(x-h/2)$	
G	$f(qx)-f(x)$	
H	$f(qx+\omega)-f(x)$	
W	$f((t+i/2)^2)-f((t-i/2)^2)$, $x=t^2$	Some
AW	$f(\cos(\theta+\lambda))-f(\cos(\theta-\lambda))$, $x=\cos\theta$	
NSU	$f(x(s+1))-f(x(s))$, $x(s)=c_1q^s+c_2q^{-s}+c_3$ $x(s)=c_1s^2+c_2s+c_3$	
E	$f(\mathcal{E}(s+1))-f(\mathcal{E}(s))$, where $\mathcal{E}$ is an elliptic function.	

difference operators of ANSUW+E type. First column gives the name, or the context: G is for geometric or Heine, Jackson, H for Hahn,W for Wilson, AW for Askey-Wilson with $q=\exp(2i\lambda)$, NSU for Nikiforov-Suslov-Uvarov, E for elliptic (Baxter, Spiridonov & Zhedanov.

Conjecture. Let $f(x)=\sum_0^{N-1}\dfrac{(\mathfrak{y}(c+k)-\mathfrak{y}(c+k-1))w_k}{x-\mathfrak{x}(c+k)}$, where $\mathfrak{x}(c+k), c\in\mathbb{C}, k\in\mathbb{Z}$ is a lattice, or grid, of ANSUW+E kind, and let w_k satisfy
$U(\mathfrak{x}(c+k))\dfrac{w_{k+1}-w_k}{\mathfrak{x}(c+k+1)-\mathfrak{x}(c+k)}=V(\mathfrak{x}(c+k))\dfrac{w_{k+1}+w_k}{2}$, with U and V of degree $\leqslant 2$ (?) $\left(\mathfrak{y}(s)=\dfrac{\alpha\mathfrak{x}(s+1/2)+\beta}{\gamma\mathfrak{x}(s+1/2)+\delta}\right)$ JCAM 2009

If $\{a_0,\dots,a_m\}\bigcup\{b_0,\dots,b_n\}$ is a lattice with parameters in arithmetic progression $\{\mathfrak{x}(s_n-m),\mathfrak{x}(s_n-m+1),\dots,\mathfrak{x}(s_n+n+1)\}$, i.e., where a_0 and b_0 are NOT independent separate starting points ($m=n$ or $m=n\pm1$), and if one of the endpoints is a singular point, i.e. $\dfrac{U(\mathfrak{x}(s))}{\mathfrak{x}(s+1)-\mathfrak{x}(s)}\pm\dfrac{V(\mathfrak{x}(s))}{2}=0$, then, numerators and denominators of rational interpolants have simple hypergeometric expansions.

Use $(\mathcal{D}f)(\mathfrak{y}((s))=\dfrac{f(\mathfrak{x}(s+1))-f(\mathfrak{x}(s))}{\mathfrak{x}(s+1)-\mathfrak{x}(s)}$
$=-\sum\dfrac{(\mathfrak{y}(c+k)-\mathfrak{y}(c+k-1))w_k}{F(\mathfrak{x}(c+k),\mathfrak{y}(s))=(\mathfrak{y}(s)-\mathfrak{y}(c+k))(\mathfrak{y}(s)-\mathfrak{y}(c+k-1))}$
$=\sum\dfrac{w_{k+1}-w_k}{\mathfrak{y}(s)-\mathfrak{y}(c+k)}$, etc.

Wait! it's not finished. The conjecture is not an if-and-only-if

5. Known hypergeometric instances with separate $\{a_k\}$ and $\{b_k\}$ lattices.

An interpolatory example, M. Rahman 1981.

We interpolate $f=f_0$ at $a,b,a+h,b+h,a+2h,b+2h,\dots$, where f depends on two parameters ρ and σ, by

$f(x)=C\sum_{-\infty}^{\infty}\dfrac{\Gamma(k+1-a_0/h)\Gamma(k+1-b_0/h)}{\Gamma(k-\rho/h)\Gamma(k-\sigma/h)}\dfrac{1}{x/h-k}$. (Dougall). With C such that $f(a)=1$, One also shows
$(x-\rho)(x-\sigma)f(x+h)=(x-a+h)(x-b+h)f(x)+(a-h-\rho)(a-h-\sigma)$

Rahman finds $f(x)=1+\cfrac{r_0(x-a)}{1+\cfrac{r_1(x-b)}{1+\cfrac{r_2(x-a-h)}{\ddots}}}$, with $r_0=-\dfrac{h\alpha}{(b-h-\rho)(b-h-\sigma)}$,

$r_{2n}=\dfrac{nh}{(b-h-\rho)(b-h-\sigma)}$, $r_{2n-1}=\dfrac{h(\alpha+n)}{(b-h-\rho)(b-h-\sigma)}$, $n=1,2,\dots$ where $\alpha=(a+b-\rho-\sigma)/h-2>0$.

(Curiously, if one interpolates the same function at the single sequence $\{a,a+h,a+2h,\dots\}$, we still have closed forms, in agreement with the conjecture, but with much more complicated formulas for the r_ns). Return to Rahman's $\dfrac{P_{2n}(x)}{(x-a-h)\cdots(x-a-nh)}$
$=\dfrac{(\alpha+1)\cdots(\alpha+n)h^n}{(b-h-\rho)^n(b-h-\sigma)^n}\,{}_3F_2(-n,(a-\rho)/h,(a-\sigma)/h;\alpha+1,(a-x)/h+1;1)$

$\dfrac{P_{2n+1}(x)}{(x-b-h)\cdots(x-b-nh)}$
$=\dfrac{(\alpha+2)\cdots(\alpha+n+1)h^n}{(b-h-\rho)^n(b-h-\sigma)^n}\,{}_3F_2(-n,(b-\rho)/h,(b-\sigma)/h;\alpha+2,(b-x)/h+1;1)$

Rahman has an example with opposite directions too: $a+kh$ and $b-kh$

$R_n(x)={}_4F_3\left[\begin{matrix}-n,n+\rho+\sigma+1,-M-x/h,a/h-N-\sigma-1\\ \rho+1,-M-N,(a-x)/h\end{matrix}\right]$ and
$S_n(x)={}_4F_3\left[\begin{matrix}-n,n+\rho+\sigma+1,x/h-N,-b/h-M-1-\rho\\ \sigma+1,-M-N,(x-b)/h\end{matrix}\right]$ of poles $a+kh$ and $b-kh$, and a measure made of masses

$$w_k=\frac{\Gamma(M+\rho+k+1)\Gamma(N+1+\sigma-k)\Gamma(k+1-a/h)}{\Gamma(M+1+k)\Gamma(N+1-k)\Gamma(k-b/h)},$$

on $k=-M,-M+1,\dots,N$.

$w_{k+1}=\dfrac{(M+\rho+k+1)(N-k)(k-a/h+1)}{(M+1+k)(N+\sigma-k)(k-b/h)}w_k,$
$k=-M,-M+1,\dots N-1$.

Remark that w_k automatically vanishes at integers $<-M$ or $>N$.

The Stieltjes function $f(x)=\sum_{k=-M}^N\dfrac{w_k}{x-kh}$ satisfies $(x+(M+1)h)(x-Nh-\sigma h)(x-b)f(x+h)-(x+(M+\rho+1)h)(x-Nh)(x-a+h)f(x)=$ a rational function turning to be a polynomial of first degree

For $q-$analogues, Rahman & Suslov. See also

D. R. Masson, The Last of the Hypergeometric Continued Fractions , *in Mathematical Analysis, Wavelets, and Signal Processing An International Conference on Mathematical Analysis and Signal Processing January 3-9, 1994 Cairo University, Cairo, Egypt*, M. E. H. Ismail & *al.*, editors, *Contemporary Mathematics* Volume **190**, 1995, p. 287–294 (what a title!):

Let X be a monic polynomial of degree 8 (amazing!) with zeros $\mathfrak{a}q^2/s,\mathfrak{a}^2q^2/s,\mathfrak{a}^2q^3/s,\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f}$, with $s=\mathfrak{a}^3q^3/(\mathfrak{bcdef})$ (an appropriate balance condition in relevant basic hypergeometric functions). Then, $X(0)=$ product of the 8 zeros $=\mathfrak{a}^5q^7\mathfrak{bcdef}/s^3=\mathfrak{a}^8q^{10}/s^4$. Let also $Y(x)=(\mathfrak{a}q^2x-s)(x-\mathfrak{b})\cdots(x-\mathfrak{f})$, $Z(x)=(x-\mathfrak{b})\cdots(x-\mathfrak{f})$. Let
$\rho_{2n}=\dfrac{s(sq^{n-2}/\mathfrak{a})^6X(\mathfrak{a}q^{2-n}/s)}{(1-sq^{2n-1})(1-sq^{2n-2})Y(1)}$, $\rho_{2n+1}=\dfrac{X(\mathfrak{a}q^{n+1})s^3}{\mathfrak{a}^6q^{2n+7}Y(1)(1-sq^{2n})(1-sq^{2n-1})}$.

Check that $1+\rho_{2n}+\rho_{2n+1}$ has no residue at $1-sq^{2n-1}=0$,

Denominators of $f=\cfrac{1}{1+\cfrac{\rho_1}{1+\cfrac{\rho_2}{1+\ddots}}}$ $D_0=D_1=1, D_2=1+\rho_1=1+\dfrac{(s-\mathfrak{a}q)(s-\mathfrak{a}q^2)Z(\mathfrak{a}q)}{\mathfrak{a}^3q^3Y(1)(s-1)}$, etc. Start of basic hypergeometric expansions.

Interpolation setting: replace two zeros of X, say, $\mathfrak{b}$ and $\mathfrak{c}$ by $\sqrt{\mathfrak{bc}}\ e^\xi$ and $\sqrt{\mathfrak{bc}}\ e^{-\xi}$ keeping the same product. Then, with $\begin{cases}z_{2n}&=\mathfrak{a}q^{2-n}/s\\ z_{2n+1}&=\mathfrak{a}q^{n+1}\end{cases}$, the two corresponding factors of $X(z_n)/X(1)$ make $x+\dfrac{z_n}{(z_n-1)(z_n-\mathfrak{bc})}$, with $x=\dfrac{1}{1-2\sqrt{\mathfrak{bc}}\cosh\xi+\mathfrak{bc}}$ ($x=\sinh\xi$ in Masson §7 p.293)
So, $\rho_{2n}=r_{2n}(x-a_n)$, $a_n=-\dfrac{\mathfrak{a}q^{2-n}/s}{(\mathfrak{a}q^{2-n}/s-1)(\mathfrak{a}q^{2-n}/s-\mathfrak{bc})}$,
$\rho_{2n+1}=r_{2n+1}(x-b_n)$, $b_n=-\dfrac{\mathfrak{a}q^{n+1}}{(\mathfrak{a}q^{n+1}-1)(\mathfrak{a}q^{n+1}-\mathfrak{bc})}$.

See also Spiridonov & Zhedanov, *Comm. Math. Phys* 2000, §5.

Elliptic.

From Spiridonov & Zhedanov, *Comm. Math. Phys.* 2000, §7; Kluwer Academic Publishers. NATO Sci. Ser., 2001; *Ramanujan J.* 2007, the fully elliptic setting $R_n(z)={}_{10}E_9(\dots)$, of poles $\alpha_1,\dots,\alpha_n$ (2001, Thm 3, eq. (4.12)) recurrence rel.: $(z-\alpha_{n+1})R_{n+1}(z)=c_nR_n(z)-(d_n/(z-\alpha_n))R_{n-1}(z)$ with $c_n=z-\alpha_{n+1}+\dfrac{\epsilon_{n-1}\mathfrak{b}_n(z-\beta_{n-1})}{\epsilon_n\mathfrak{a}_n}+\dfrac{\mathfrak{c}_n(z-\lambda_1)}{\epsilon_n\mathfrak{a}_n}=(1+\rho_{2n}+\rho_{2n+1})\xi_{n+1}/\xi_n$,
$d_n=\dfrac{\epsilon_{n-1}\mathfrak{b}_n(z-\beta_{n-1})(z-\alpha_n)}{\epsilon_n\mathfrak{a}_n}=\rho_{2n-1}\rho_{2n}\xi_{n+1}/\xi_{n-1}$

Here, $\xi_{n+1}/\xi_n=\mathfrak{c}_n(z-\lambda_1)/(\epsilon_n\mathfrak{a}_n)$, ρ_n is a polynomial (depending on the evenness of n) divided by $[n-x_2][n+1-x_2]$ (θ functions).