Strong asymptotics of best rational approximation to the exponential function on a bounded interval.

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En outre, M. Liouville s'était offert à m'exposer un abrégé de la nouvelle théorie des fonctions elliptiques, professée par lui au Collège de France.
(Moreover, M.Liouville agreed to lecture me on an abridged version of the new theory of elliptic functions taught by him in the Collège de France.)
From a report of P.L. Chebyshev on a journey abroad, in his Œuvres, vol. 2, p. XIV.

Key words: rational approximation, exponential function, complex potential

2010 Mathematics Subject Classification:
30E10 Approximation in the complex domain 30E15 Asymptotic representations in the complex domain 31A15 Potentials and capacity, harmonic measure, extremal length 33E05 Elliptic functions and integrals 33F05 Numerical approximation and evaluation 41A20 Approximation by rational functions 41A25 Rate of convergence, degree of approximation 41A80 Remainders in approximation formulas

Abstract. We apply recent findings of complex approximation theory to best rational approximation of degree n to the exponential of $-(n+\nu)x$ on $[0, c]$. The error norm behaves like the nth power of the main approximation rate times the ν th power of a secondary approximation rate. The computation of the first rate is a consequence of works of A.A.Gonchar, E.A. Rakhmanov, and Herbert Stahl done in the 1980s; the complete asymptotic description has been achieved by A. Aptekarev in the first years of the 21st century. The solution is given in terms of elliptic integrals of the three kinds.

1. Introduction.

1.1. The subject matter.

Our subject is the asymptotic behaviour of best rational approximation to the exponential function on a real interval $[0, c]$.

This subject became heavily investigated when it appeared that the error norm decreases geometrically fast with the degree, even when the approximation interval is unbounded $c = \infty$ [22, 89]. A full asymptotic description was given by Gonchar and Rakhmanov as an application of their theory [36].

We consider here a normally bounded interval $[0, c]$ with $c < \infty$.

One will establish statements like $\exp(-x) - r_n(x) \sim s_n(x)$, with a formula for $s_n(x)$. The symbol $\sim$ means here that $A_n \sim B_n$ when $n \to \infty$ if $A_n/B_n \to 1$ then. When A_n and B_n depend on x, it usually means that the limit is reached uniformly in compacts in some set, sometimes uniformly in the whole set. We will sometimes encounter $A_n \sim B_n + C_n$, where B_n and C_n oscillate with n. The meaning is then $A_n = U_n + V_n$ with $U_n \sim B_n$ and $V_n \sim C_n$, see some cases in section 5.4.2.

Cleverly designed polynomial and rational approximations to holomorphic functions in some domain are known to involve a potential function related to the approximation region and a valuable holomorphy domain as will be recalled in § 2.

A proper scaling of the variable for the exponential function is necessary, we look at $\exp(-A(n)x)$ on a fixed interval $[0, c]$. It appears then that $A(n)$ must behave like a polynomial of first degree, see section 1.2.

We consider the best rational approximation $q_n(z)/p_n(z)$ of degree n to the exponential function $\exp(-(n+\nu)z)$ on the bounded interval $E = [0, c]$.

According to the Gonchar-Rakhmanov-Stahl theory [36], the root asymptotics

$$[p_n(z)]^{1/n} \to \exp[\mathcal{V}_p(z)],$$
$$[q_n(z) - p_n(z)\exp(-(n+\nu)z)]^{1/n} \to C\exp[2\mathcal{V}_z(z) - \mathcal{V}_p(z)],$$

hold, where the complex potentials

$$\mathcal{V}_p(z) = \int_F \log(z-t)d\mu_p(t),\ \mathcal{V}_z(z) = \int_E \log(z-t)d\mu_z(t)$$

involve the limit distributions of poles on an arc F joining a to b, and interpolation points on E. The final complex potential $\mathcal{V}(z) = \mathcal{V}_z(z) - \mathcal{V}_p(z)$ must have a constant real part on E, and the real part of $\mathcal{V}(z) + z/2$ must be (another) constant on F (the "external field" problem). Finally, $C = \exp(-2\mathcal{V}(a) - a)$, so that the n^{th} root of the error norm converges towards $|\exp(2\mathcal{V}(0) - 2\mathcal{V}(a) - a)|$.

Formulas for $\mathcal{V}, a, b, F$ will be given in § 3. The modulus $k = \sin(\theta/2)$, where $\theta = \arg(a/(a-c))$ (fig. 8) is of particular interest (§ 4). It will be found that k increases from 0, when $c = 0$, to $k_\infty = 0.9089...$ when $c = \infty$.

The geometry, or topological properties, of the limit locus of poles F is the same for $c = \infty$ as for $c < \infty$: it is a single arc joining a to b, where a and b depend on c (sections 3.2.1 and 3.2.4).

Things change when we come to $\exp(-x^2)$, where it seems that F is now made of TWO arcs [52, § 6] [56, § 5] [80, § 6.3], a case deserving more investigation.

Strong asymptotics are worked according to the theory of A. Aptekarev [6] in (5.1)

$$p_n(z) \sim [(z-a)(z-b)]^{-1/4} e^{(n+1/2)\mathcal{V}_z(z)} \exp[-(n+\nu)\mathcal{V}(z) + (\nu - 1/2)\mathcal{V}_*(z)],$$

$$p_n(z)\exp(-(n+\nu)z) - q_n(z) \sim C_n[(z-a)(z-b)]^{-1/4}$$
$$\exp[(n+1/2)\mathcal{V}_z(z) + (n+\nu)\mathcal{V}(z) - (\nu - 1/2)\mathcal{V}_*(z)],$$

where $\mathcal{V}_*(z) = C' \int_\infty^z \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}$ is an auxiliary complex potential whose real part is a constant on E and another constant on F, or any arc joining a to b without crossing E (usual plane condenser problem), and $C_n = \exp[-(n+\nu)(2\mathcal{V}(a) + a) + (2\nu - 1)\mathcal{V}_*(a)]$. Near the cuts E and F, one must add the contributions from the two sides in the formulas above.

It is useful to establish that the various features $\mathcal{V}, a$, etc. when $c < \infty$, do converge to the better known corresponding items in the $c = \infty$ case [21, 36, 52–54, 89]. This is seen in the last rows of table 1, in an incidental remark in § 4.3, at the end of § 4.3.3, at 4. and exercise of § 5.5.2, and in the second paragraph of § 5.5.3.

We remind in § 6 how the Adamyan- Arov- Krein (AAK) theory, initially a theory of approximation on the unit circle of meromorphic functions [2], has been brilliantly applied to approximation algorithms by Gutknecht and Trefethen [85, 86] in the 1980s.

Some curios follow, some of them may be considered as open problems: electrical images, integral Hankel operator, quadratic relations, B-spline shapes, "beyond ∞", or exploring the modulus $k > k_\infty$.

1.2. **Scaling.**

Best L_∞ rational approximation of degree n to $\exp(-x)$ on $[0, 1]$ yields, for $n = 1, 2, \ldots, 5$ the error norms $1.580...\ 10^{-3}, 1.645...\ 10^{-6}, 7.345...\ 10^{-10}, 1.822...\ 10^{-13}$, $2.875...\ 10^{-17}$.

The approximations of degree $1, 3, 5$ have a real negative pole at $-1.572...$, $-4.176...$, $-6.813...$.

The asymptotic history of this approximation is quite short, as zeros and poles recede to ∞, on such a large scale that the interval $[0,1]$ looks like a mere point, so that the approximation comes close to the ***Padé approximation***. This approximant has a real pole which is almost $-C''n$ with $C'' = 1.3255\ldots$ (Driver & Temme 1999 [24, p.10]).

Also, the asymptotic formula for the error norm (Meinardus ed. of 1967 [59, §9.3], Braess [16]) adapted[1] here to approximation to $\exp(-x)$ on $[0, L]$ is

$$\|e^{-x} - q_n(x)/p_n(x)\|_{\infty,[0,L]} \sim \frac{L^{2n+1}(n!)^2 \exp(-L/2)}{2^{4n+1}(2n)!(2n+1)!} = \frac{L^{2n+1}\exp(-L/2)\pi(n+1/2)}{2^{8n+2}(\Gamma(n+3/2))^2}$$

$\sim\ 2\exp(-L/2)\left(\dfrac{Le}{16n+8}\right)^{2n+1}$ (Stirling). The accuracy of the Meinardus-Braess formula is striking, as we have $L = 1$, $n = 1 : e^{-1/2}/384 = 1.579\ldots\ 10^{-3}$; $n = 2 : e^{-1/2}/368640 = 1.6453\ldots\ 10^{-6}$.

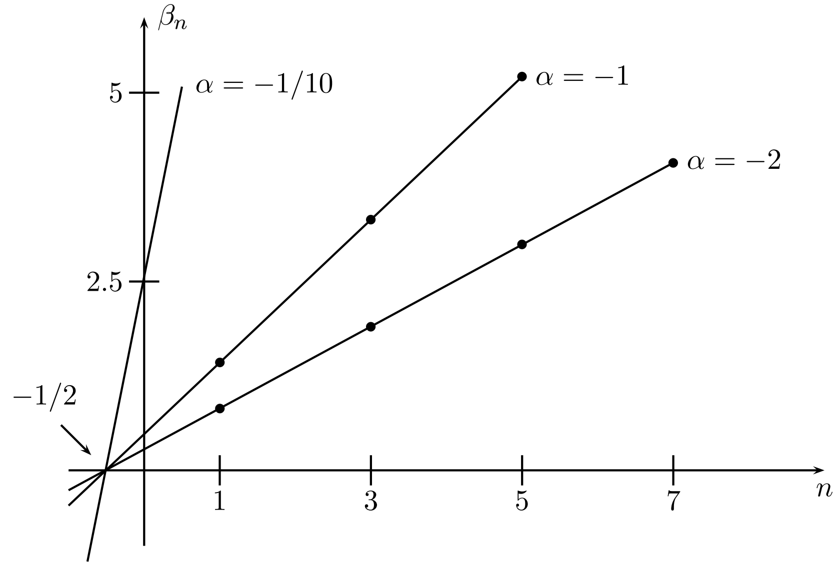

FIGURE 1. The importance of being $n+1/2$: in the best rational approximation of degree n to $\exp(-\beta_n x)$ on $[0,1]$, the parameter β_n is tuned so to have a real pole at a given point $\alpha < 0$. For each α, β_n tends to behave like a constant times $n+1/2$. If $c \neq 1$, use $c\beta_n$ and α/c. When $\alpha = -0.1$, the values are $\beta_1 = 7.49\ldots, \beta_3 = 17.60\ldots, \beta_5 = 27.68\ldots, \beta_7 = 37.76\ldots, \beta_9 = 47.88\ldots$.

We avoid the drift of the locus of poles by choosing a moving scale for the $x-$variable. A simple experiment is to freeze the real pole α of the odd degree approximants by the appropriate scaling $\beta_n x$. It is found in Fig. 1 that β_n is strikingly close to a constant times $n+1/2$. This feature will be discussed at the end of § 5.4.1 and in § 5.5.1.

So we may stick to $\exp(-(n+1/2)x)$ or, more generally, to $\exp(-(\gamma n+\delta)x)$. There is no loss of generality by taking $\gamma = 1$, we will work with $\exp(-(n+\nu)x)$ from section 3 onward.

We now consider the approximation of $\exp(-(n+1/2)x)$ on a fixed interval $[0, c]$. With x changed to $(n+1/2)x$, and $c = L/(n+1/2)$, the error norm behaves like $2\left(\dfrac{ce}{16}\right)^{2n+1}\exp(-(n+1/2)c/2) = 2[c^2e^{2-c/2}/256]^{n+1/2}$ for moderately small $c = L/n < 4 \times 1.325 = 5.3$.

A more detailed view of the best rational approximation of degree 5 on $[0,1]$ to $\exp(-5.5x)$ is given in Fig. 2.

[1]Using an idea of Németh and Newman [47], Braess considers Padé approximations to $\exp(z/2)$ and $\exp(\overline{z}/2)$, where $x = \cos\theta = (z+\overline{z})/2$. Here, $\exp(-x)$ on $[0, L] = \exp(-L/2)$ times $\exp(Lx/2)$ on $[-1, 1]$, the Meinardus error norm estimate has to be multiplied by $(L/2)^{2n+1}\exp(-L/2)$, provided L is not too large, as Padé poles must remain outside the unit disk in $z : L/4 < 1.325n$.

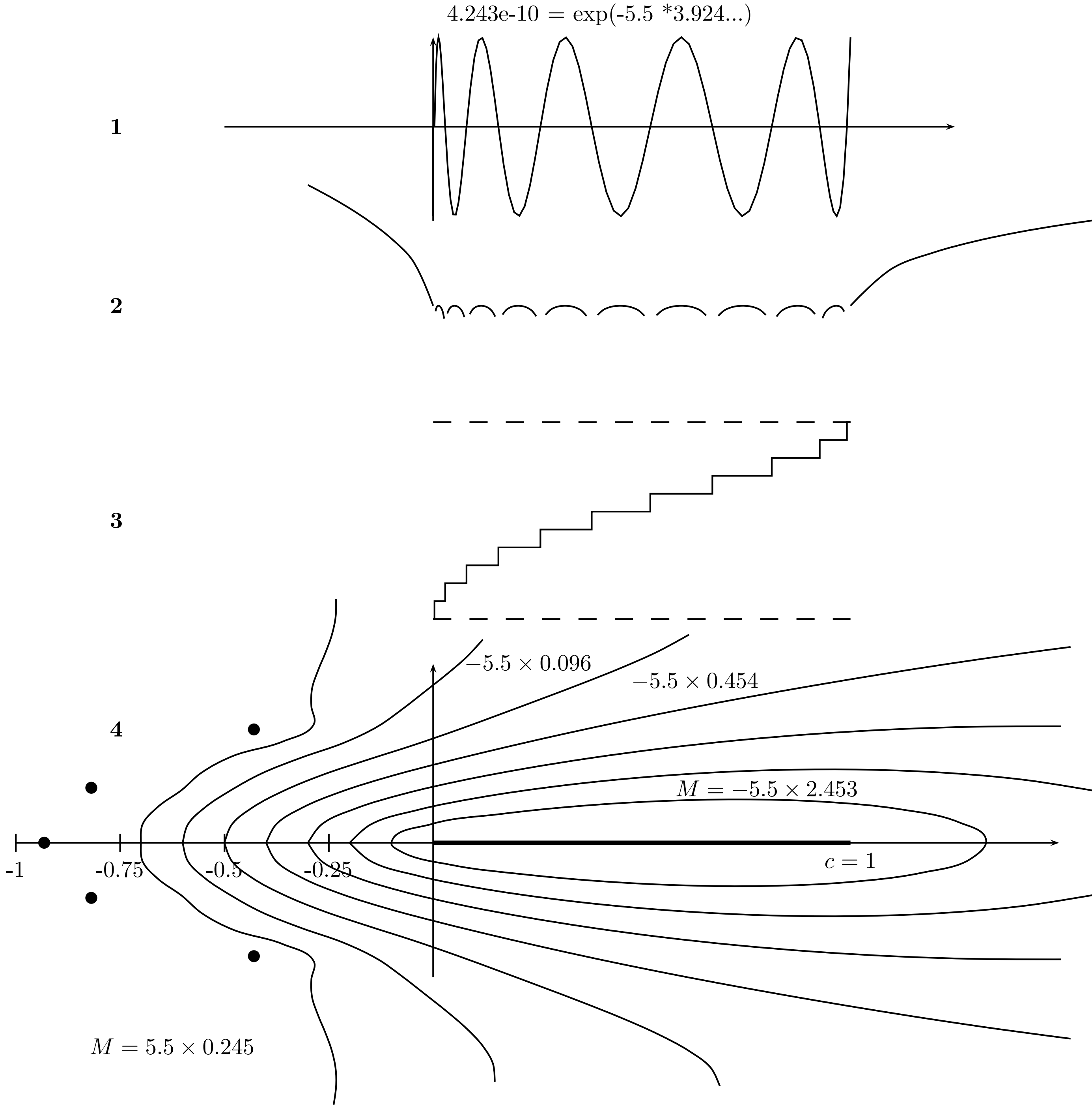

FIGURE 2. Best rational approximation of degree 5 to $\exp(-5.5x)$ on $[0,1]$: 1. graph of the error function; 2. $M(x)$ = logarithm of the absolute value of the error function at x; 3. distribution function $\mu_{z,5}$ of the 11 interpolation points; 4. a contour plot of M in the complex plane, and the 5 poles. Outside a contour containing the poles, $M(z = x + iy) \sim 5.5 \max(2V(z), -x)$, where V is the appropriate potential function. The y scales are not the same in the various parts of the figure.

2. Potential function and approximation in the complex plane

2.1. **Potential of a set of charged points.** Let us consider a system of charged particles in the complex plane and a function of the form $-\sum_k w_k \log|z-z_k|$, z and $z_k \in \mathbb{C}$, which is the *potential* function of the system. Remark that a positive charge w_k gives $+\infty$ at the particle position, and a negative charge creates a potential well at its position.

This writing is compatible with the notation seen before $\int_C \log|z-t|\, d\nu(t)$, by considering $d\nu(t)$ to be a *discrete* measure.

As an example of how such potential functions enter approximation, consider polynomial interpolation at $z_1, \ldots, z_n$ of a function defined in the complex plane by an integral $f(z) = \int_C \rho(t)dt/(z-t)$, where C may be a contour or a system of arcs, and $\rho(t)dt$ a real or complex measure (Markov-type, or Stieltjes-type functions), then the Hermite-Walsh formula [90, §3.1] of the interpolation error is

$$f(z) - p_n(z) = \int_C \prod_{k=1}^{n} \frac{z-z_k}{t-z_k} \frac{\rho(t)dt}{z-t},$$

so, $|f(z)-p_n(z)| \leqslant \exp(n[W_n(z) - \inf_{t\in C} W_n(t)]) \displaystyle\int_C \frac{|\rho(t)dt|}{|z-t|}$, where $W_n(z) = n^{-1}\sum_k \log|z-z_k|$ is here the potential of a total negative unit charge on the z_ks.

Up to now, the language of potential seems only to be a shorthand for interpolation error function description. But consider now the problem of polynomial *best approximation* of such a function f on a set E. The interpolation points z_k on E must now be such that the error function has equal local extremal values of E, as in the first part of fig. 2 (this is exact if f is real on the real set E). Let $V_{n,E}$ be the relevant potential of a total negative unit charge on the z_ks. Assuming the behaviour of the error function dominated by the $\exp(nV_{n,E}(z))$ term, we expect $V_{n,E}$ to be close to the harmonic function V_E taking a constant value on E (as in the second part of fig. 2). We are now at the center of potential theory, the determination of a harmonic function in a domain through boundary values! Actually, V_E is the *Green function* of E with a logarithmic singularity at ∞ : $V_E(z) - \log|z|$ bounded when $z \to \infty$ [28, chap. II,§3], [90, chap. 4].

For instance, if $E = [a,b]$, $V_E(z) = \log|\Phi(z)| +$ const., where $\Phi(z) = [z-(a+b)/2 + \sqrt{(z-a)(z-b)}]/(|b-a|/2)$ maps conformally the exterior of E to the exterior of the unit disk (and the reverse connection is $\Phi + \Phi^{-1} = 2(2z-a-b)/|b-a|$, the famous *Joukowski map*).

Let p_n be the denominator of a rational approximation of degree n to a function f, with zeros $p_1^{(n)}, \ldots, p_n^{(n)}$ in a set F, and q_n be the interpolation of $p_n f$ at $n+1$ points $z_1^{(n)}, \ldots, z_{n+1}^{(n)}$ in a set E. It is then possible to relate best rational approximations to functions analytic outside F and potential functions taking (different) constant values on the boundaries of E and F [8,28,29,90], also seen as condenser potential [7,23,31]. For instance, when E and F are intervals $[a,b]$ and $[c,d]$, the potential is the real part of $C\displaystyle\int_\infty^z \frac{dt}{\sqrt{(t-a)(t-b)(t-c)(t-d)}}$, which will be encountered in § 5.4.2.

Remark that Bogatyrev prefers Riemann surfaces to Green functions (criticism of Akhiezer's methods, [12, p. XX]).

2.2. **Rational interpolation at $2n+1$ points.** Rational approximation of degree n defined by n poles and interpolation at $n+1$ points does not always explain best approximation performance. Instead, we consider now rational approximation of numerator and denominator of degree n, the n poles being unknowns. These n new degrees of freedom allow normally interpolation at $2n+1$ points instead of $n+1$.

Let $\{f_n\}$ be a family of functions analytic in a region containing a contour C. So, for any z inside C,

$$f_n(z) = \int_C \frac{\rho_n(t)\,dt}{z-t}, \tag{2.1}$$

with $\rho_n(t) = -f_n(t)/(2\pi i)$. It may also happen that C shrinks towards an arc or a system of arcs, by deformation (through the point at infinity, if needed), $\rho_n(t)$ is then the difference $(f_n^-(t) - f_n^+(t))/(2\pi i)$ on the two sides of the cut, or even a real interval where the ρ_ns are positive functions (then, (2.1) is a (true) Markov function), see [32, 35], also the history in [71]. An example will be found at (5.7).

The best rational approximation of degree n on E to f_n is expected to interpolate $f_n(z) = \int_C \rho_n(t)(z-t)^{-1}dt$ at $2n+1$ points instead of $n+1$ points, say, $z_1^{(n)}, \dots, z_{2n+1}^{(n)}$, so

$$p_n(z)f_n(z) - q_n(z) = (z - z_1^{(n)})\cdots(z - z_{2n+1}^{(n)}) \int_C \frac{p_n(t)\rho_n(t)dt}{(z-t)(t - z_1^{(n)})\cdots(t - z_{2n+1}^{(n)})}, \tag{2.2}$$

by the Hermite-Walsh formula again, should the numerator q_n have degree $2n$, with
$q_n(z) = \int_C \frac{[(t - z_1^{(n)})\cdots(t - z_{2n+1}^{(n)})p_n(z) - (z - z_1^{(n)})\cdots(z - z_{2n+1}^{(n)})p_n(t)]\rho_n(t)dt}{(z-t)(t - z_1^{(n)})\cdots(t - z_{2n+1}^{(n)})}$. We find indeed
$q_n(z) = \int_C \frac{\{[-z^{2n} + [z_1^{(n)} + \cdots + z_{2n+1}^{(n)} - t]z^{2n-1} + \cdots]p_n(t) + \cdots\}\rho_n(t)dt}{(t - z_1^{(n)})\cdots(t - z_{2n+1}^{(n)})}$ of degree[2] $2n$, unless the integrals of $p_n(t), tp_n(t), \dots, t^{n-1}p_n(t)$ do vanish, i.e., when p_n is ***orthogonal*** to the polynomials of degree $< n$ on C with respect to the (probably complex) weight function $\frac{\rho_n(t)}{(t - z_1^{(n)})\cdots(t - z_{2n+1}^{(n)})}$, from $(z-t)^{-1} = z^{-1} + tz^{-2} + \cdots + t^{n-1}z^{-n} + t^n z^{-n}(z-t)^{-1}$.

By replacing $(z-t)^{-1}$ by its interpolation at the zeros of p_n, $t = p_1^{(n)}, \dots, p_n^{(n)}$, with interpolation error $p_n(t)/[(z-t)p_n(z)]$, so

$$f_n(z) - \frac{q_n(z)}{p_n(z)} = (z - z_1^{(n)})\cdots(z - z_{2n+1}^{(n)}) \int_C \frac{p_n^2(t)\rho_n(t)dt}{(z-t)p_n^2(z)(t - z_1^{(n)})\cdots(t - z_{2n+1}^{(n)})}. \tag{2.3}$$

We define now
$V_n(z) = V_z^{(n)}(z) - V_p^{(n)}(z) = \frac{1}{2n+1}\sum_1^{2n+1} \log|z - z_k^{(n)}| - \frac{1}{n}\sum_1^n \log|z - p_k^{(n)}|$,then

$$\limsup_{n\to\infty} |f_n(z) - q_n(z)/p_n(z)|^{1/n} \leqslant \exp[2V(z) - \min_{t\in C}(2V(t) - \phi(t))], \tag{2.4}$$

where $\phi(t) = \lim_{n\to\infty} n^{-1}\log|\rho_n(t)|$, should V_n have the limit V when $n \to \infty$.

2.3. **Conjectures and proofs.** The conditions of existence and the properties of the limit of V_n when $n \to \infty$ depend on our knowledge of asymptotic behaviour of orthogonal polynomials with respect to complex weights, which has been heavily investigated [31, 36, 37, 49–51, 55, 58, 66, 70–72, 76, 77, 79]. This research was mostly started by A.A. Gonchar, as early[3] as the 1960s-1970s [30, 31], and culminated in famous conjectures [33, 34], mostly proved from the no less famous works of H.Stahl [76–78].

The final proof by Gonchar & Rakhmanov 1987 [36, 37] establishes that the orthogonal polynomials p_n of above do satisfy $|p_n(z)|^{1/n} \to \exp(V_p(z))$ when $n \to \infty$, if the functions ρ_n of (2.1) are analytic in a domain containing C, and $n^{-1}\log\rho_n$ has a limit Φ when $n \to \infty$,

[2] The coefficients of $z^{2n}, z^{2n-1}, \dots, z^{n+1}$ are found by applying the Ruffini-Horner scheme to the division of $-(z - z_1^{(n)})\cdots(z - z_{2n+1}^{(n)})$ by $z - t$.

[3] The first author remembers vividly his surprise -and delight!- to have seen in [31] a short introduction to 2-dimensional condensers and their relation to rational approximation!

let ϕ be the real part of Φ, and where the final potential function $V(z) = V_z(z) - V_p(z) = \int_E \log|z-t|d\mu_z(t) - \int_{F\subseteq C} \log|z-t|d\mu_p(t)$, is a constant, say, $\gamma_E/2$, on E, $2V(z) - \phi(z) =$ is another constant, say, γ_F, on $F \subseteq C$, with $\gamma_F > \gamma_E$, $2V - \phi$ has equal exterior normal derivatives on the two sides of F (Stahl's symmetry property, or $S-$curve property [58, 70, 71]), in a form adapted to the Cauchy-like formula (2.1) by Aptekarev [6, §1.1].

Then, for the best rational approximations r_n on E, $V_n \to V$ when $n \to \infty$, and $\|f_n - r_n\|_E^{1/n} \to \rho := \exp(\gamma_E - \gamma_F) < 1$.

The convergence rate is also written $\rho = \exp(-2/\text{cap}(E, F, \phi))$, where $\text{cap}(E, F, \phi)) = 2/(\gamma_F - \gamma_E) = 1/[(V - \phi/2)_F - (V)_E]$ is the ***weighted condenser capacity*** of the system (E, F, ϕ), i.e., the ratio of a charge (negative unit charge on E, positive on F) and an augmented potential difference from E to F [74]. Aptekarev considers also more complete, and more symmetric, systems (E, F, ϕ, ψ) [6].

There is a wonderful association of the theory of best approximation and the theory of minimal capacity, at least in the weightless case ($\phi(t) \equiv 0$), but things are less obvious when $\phi(t) \not\equiv 0$ [19].

The complete theory and history are surveyed by Martínez-Finkelshtein and Rakhmanov [55, 58, 70, 71].

For strong asymptotics $\lim \|f_n - r_n\|_E/\rho^n$, see section 5.

2.4. **Complex potential.** Let $-\sum_k w_k \log(z - z_k)$ be a complex potential, where one must make an appropriate choice of the complex logarithm, such as the usual convention of a real logarithm for a positive argument, with a discontinuity on the negative half line. Then, with real values of (z_k, w_k), the imaginary part of the potential is a staircase function on the real values of z, such as the staircase function seen in part 3 of the figure 2.

A test particle of positive unit charge is submitted to a force which is the complex conjugate of the derivative of the complex potential. Gauss started studies relating complex function theory to the 2-dimensional mathematical physics [62, 91].

Note however that the actual poles and interpolation points of a particular best rational approximation problem are normally NOT exactly the solutions of a problem of electrostatics. Only the LIMIT distributions μ_p and μ_z have a physical meaning. As an example, fig. 5 in § 6.2 shows a set of poles not far, but not exactly the same, from a set of equilibrium points found by a discretization of the continuous problem.

The conditions on $\mathcal{V}$ are now [6, §1.2]

(I) the charge distributions μ_z and μ_p on the two regular compact sets E and F, making $\mathcal{V}(z) = \mathcal{V}_z(z) - \mathcal{V}_p(z) = \int_E \log(z-t)d\mu_z(t) - \int_{F\subseteq C} \log(z-t)d\mu_p(t)$, satisfy $\int_E d\mu_z(t) = \int_F d\mu_p(t) = 1$,

(II) the functions ρ_n of (2.1) are analytic in a domain containing C, and $n^{-1}\log\rho_n \to \Phi$ when $n \to \infty$,

(III) $\text{Re}(2\mathcal{V}(z) - \Phi(z)) =$ the constant γ_F on $F \subseteq C$,

(IV) $\text{Re}(2\mathcal{V}(z) - \Phi(z)) > \gamma_F$ on the remaining part of C,

(V) $\text{Re}\mathcal{V}(z) =$ the constant $\gamma_E/2$ on E,

(VI) $2\mathcal{V}'(z) - \Phi'(z)$ has opposite limit values on the two sides of F.

From Sokhotskyi-Plemelj [39, 3 §14.1], the limit values of $\mathcal{V}'(z) - \Phi'(z)/2 = \int_E \frac{d\mu_z(t)}{z-t} - \int_F \frac{d\mu_p(t)}{z-t} - \Phi'(z)/2$ on the two sides of F are $\int_E \frac{d\mu_z(t)}{z-t} - ⨍_F \frac{d\mu_p(t)}{z-t} \pm \pi i\mu_p'(z) - \Phi'(z)/2$, must be opposite, so,

$$\mathcal{V}'(z) = \frac{\Phi'(z)}{2} \pm \pi i \mu_p'(z), \quad z \in F, \tag{2.5}$$

where $\displaystyle⨍$ means principal value. The condition (VI) above means that the first integral of the right-hand side and the principal value add to $\Phi'(z)/2$ on F.

For $\mathcal{V}$ itself, $\mathcal{V}(z) = \mathcal{V}(a) + \Phi(z)/2 - \Phi(a)/2 \mp i\pi\mu_p(z)$ on the right side and the left side of F.

The central part of the theory is how to establish the asymptotic behaviour of the denominator p_n as the polynomial orthogonal to all polynomial of degree smaller than n with respect to the complex weight $\exp(n(\Phi(t) - 2\mathcal{V}_z(t)))$ on F, as seen in § 2.2. As $\mathcal{V}_p(z) = \int_F \log(z-t)d\mu_p(t)$ is the potential of the expected limit distribution of poles, $p_n(z) \sim \exp(n\mathcal{V}_p(z))$ is the obvious starting formula. However, $\mathcal{V}_p(z)$ is singular on F (discontinuous derivative when crossing F), and where are the zeros of p_n? An exponential function has no zero. A better tentative formula for z near F is $p_n(z) \sim A(z)\exp(n\mathcal{V}_{p,+}(z)) + B(z)\exp(n\mathcal{V}_{p,-}(z))$, referring to values on the right side and the left side of F.

Remind that the two parts making the complex potential function are written here $\mathcal{V}(z) = \mathcal{V}_z(z) - \mathcal{V}_p(z)$, where $\mathcal{V}_{z,p}(z) = \int_{E,F} \log(z-t)\, d\mu_{z,p}(t)$, μ_z and μ_p being positive measures of unit total weight on their supports E and F.

From above, $-\mathcal{V}_\pm(z) =$ constant $- \Phi(z)/2 \pm i\theta_q(z)$, where $\theta_q(z) = \pi(\mu_p(z) - \mu_p(a))$ on the right side of F increases from 0 to π when z runs from a to b on the right side of F (positive unit charge on F), and from π to 2π (or from $-\pi$ to 0) when z return to a on the left side of F.

Remark also that $d\theta_q/dz = i(\mathcal{V}'(z) - \Phi'(z)/2) = \pi\mu'(z)$ on the right side of F; $-\pi\mu'(z)$ on the left side.

So, as $\mathcal{V}_{p,\pm}(z) = \mathcal{V}_z(z) - \mathcal{V}_\pm(z) = \mathcal{V}_z(z) - \Phi(z)/2 \pm i\theta_q(z)$, our estimate of p_n is $p_n(z) \sim e^{n\mathcal{V}_z(z) - n\Phi(z)/2}[A(z)\exp(ni\theta_q(z)) + B(z)\exp(-ni\theta_q(z))]$, showing already a satisfactory oscillating behaviour on F! The still unknown functions A, B will follow from a check of orthogonality of p_n and all polynomials of degree $< n$. We use the test functions $p_n(z)/(z-p)$ for the n zeros p of p_n. Then,

$$\int_F p_n(t)\frac{p_n(t)}{t-p}\exp(n\Phi(t) - (2n+1)\mathcal{V}_z(t))dt$$
$$\sim \int_F \left[A(t)e^{ni\theta_q(t)} + B(t)e^{-ni\theta_q(t)}\right]^2 \frac{\exp(-\mathcal{V}_z(t))dt}{t-p}.$$

The integrals of the $e^{\pm 2ni\theta_q(t)}$ terms are high index Fourier coefficients, therefore have vanishing limit when $n \to \infty$. What remains is the integral of $2A(t)B(t)\dfrac{\exp(-\mathcal{V}_z(t))}{t-p}$ and does not depend on n. A more complete discussion of the principal value integral is needed and will be done in § 5.4.2.

3. The complex potential of the present problem.

3.1. **Theorem.** *Let q_n/p_n be the best rational approximant of degree n to the exponential function $\exp(-(n+\nu)x)$ on $E = [0, c]$. The limits*

$$(p_n(z))^{1/n} \to \exp[\mathcal{V}_p(z)],$$

$$(q_n(z) - p_n(z)\exp(-(n+\nu)z))^{1/n} \to C\exp[2\mathcal{V}_z(z) - \mathcal{V}_p(z)],$$

exist when $n \to \infty$, $z \notin E \bigcup F$, where F is an arc in the complex plane.
The complex potential $\mathcal{V} = \mathcal{V}_z - \mathcal{V}_p$, with

$$\mathcal{V}_p(z) = \int_F \log(z-t)d\mu_p(t), \ \mathcal{V}_z(z)) = \int_E \log(z-t)d\mu_z(t)$$

satisfies $\mathcal{V}(\infty) = 0$ *and*

$$\sqrt{\frac{z(z-c)}{(z-a)(z-b)}}\,\mathcal{V}'(z) = \frac{-1}{2\pi i}\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\,\frac{dt}{z-t}, \tag{3.1}$$

where the integral is taken on the right side of $F = [a, b]$*, the square roots being determined as being positive for large positive* z *and* t*, and continuous outside the cuts* $E = [0, c]$ *and* F*.*

One also has

$$\mathcal{V}''(z) = \frac{\nu_0 z + \nu_1}{\sqrt{z^3(z-c)^3(z-a)(z-b)}}, \tag{3.2}$$

where

$$\nu_0 x + \nu_1 = \frac{1}{4\pi i}\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\left[\frac{ab(x-c)}{t} + \frac{(a-c)(b-c)x}{t-c}\right] dt. \tag{3.3}$$

The unit charge conditions on E *and* F *lead to*

$$\int_a^b \frac{(\nu_0 x + \nu_1)dx}{\sqrt{x(x-c)^3(x-a)(x-b)}} = \pi i \tag{3.4}$$

Finally, the rate of error decrease is ρ *with*

$$\begin{aligned}\|\exp(-(n+\nu)x) - q_n(x)/p_n(x)\|_E^{1/n} \to \rho &= \exp[2V(0) - 2V(a) - \operatorname{Re} a] \\ &= \exp\left[2\operatorname{Re}\int_0^a \frac{(\nu_0 t + \nu_1)dt}{\sqrt{t(t-c)^3(t-a)(t-b)}}\right]\end{aligned} \tag{3.5}$$

3.2. **Proof.**

In (2.1) we have $\exp(-(n+\nu)z) = \int_C \rho_n(t)dt/(z-t)$ on a contour C containing z as an interior point, with $\rho_n(t) = -\exp(-(n+\nu)t)/(2\pi i)$, so that $\Phi(t) = \lim n^{-1}\log\rho_n(t) = -t$ everywhere in the complex plane.

3.2.1. *First formula.* The existence of the limits follows from the Gonchar-Rakhmanov-Stahl theory recalled in § 2.3, that we apply to the present problem.

As E is a real set, we enjoy an important simplification by trying a symmetric set F with respect to the real axis, so that the potential V satisfies the Schwarz's symmetry, or reflection, property $V(\overline{z}) = V(z)$. It also means that the complex potential $\mathcal{V}$ and its derivative $\mathcal{V}'$ at $\overline{z}$ are the complex conjugates of $\mathcal{V}(z)$ and $\mathcal{V}'(z)$ [40, vol. 1 Th. 7.7.2] [65, §5.5]. On the two sides of E, as the real potential V is constant, its gradient is vertical, so $\mathcal{V}'$ takes opposite pure imaginary values!

The two conditions on $\mathcal{V}'$ on the two sides of E and F will almost immediately give a formula for the complex potential, up to a small number of unknown constants.

We assume F to be an arc joining a to $b = \overline{a}$, the two main unknown constants to be determined later on.

As $\mathcal{V}'(z)$ must take opposite values on the two sides of $[0, c]$, $\sqrt{z(z-c)}\mathcal{V}'(z)$ is a meromorphic function outside the second cut F (elementary instance of homogeneous Privalov's problem [39, §14.8, example 1 with $d = 1/2$]) the same is true after division by $\sqrt{(z-a)(z-b)}$, so that $\sqrt{\frac{z(z-c)}{(z-a)(z-b)}}\,\mathcal{V}'(z) = \frac{1}{2\pi i}\oint \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\,\frac{\mathcal{V}'(t)\,dt}{z-t}$ on a contour shrinking to the two sides of the cut F. Let $\mathcal{V}'(t)$ be the value on the right side of F. When we proceed with the integral from b to a on the left side, the square root of $(t-a)(t-b)$ changes its sign, so that we have to consider the SUM of the values of $\mathcal{V}'(t) = -1/2 \pm \pi i \mu_p'(t)$ on the two sides of F, from (2.5). Then, (3.1) follows.

The values of $\mathcal{V}$ near E and F give μ_z and μ_p from the Sokhotskyi-Plemelj relations seen above, and $\mathcal{V}_z$ and $\mathcal{V}_p$ can then be reconstructed. However, curious quadratic relations (6.8) in § 6.3 do the job for our particular problem!

Remark that we do not need to know where F is: as Φ is analytic in a region containing F (here, $\Phi(z) = -z$), the right-hand side of (3.1) is the same for any arc joining a and b without crossing $E = [0, c]$. Later on, the true set F will be found as a part of the locus where $2V(z)+x = 2V(a) + \text{Re}\, a = 2V(b) + 2\text{Re}\, b$, see fig. 3.

It will also be necessary to find a contour C such that $2V(z) - \phi(z) = 2V(z) + x$ is larger on $C \setminus F$ than its constant value on F.

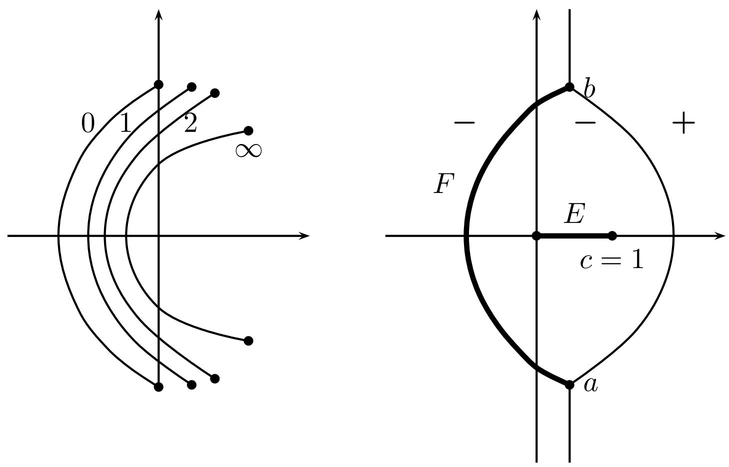

FIGURE 3. $F-$cut for various values of c; the full locus $V(z) + x/2 = V(a)+$ Re $a/2$ for $c = 1$. The sign of $V(z) + x/2 - V(a)-$ Re $a/2$ in the three regions is shown too. The contour C of the theory contains the arc F and is closed by an arc within the "+" region. Should the complex potential function be reduced to its first expansion term $\nu_0/(2z)$, with $\nu_0 < 0$, the locus should be the imaginary axis $x = 0$ and the circle $x^2 + y^2 = -\nu_0$ as first approximation with $a = -\sqrt{-\nu_0}\, i$.

3.2.2. *First equation for the parameters.* From $\mathcal{V}(z) = \int_E \log(z-t)d\mu_z(t) - \int_F \log(z-t)d\mu_p(t) = -\int_{E\bigcup F} \log(z-t)d\mu(t) = \dfrac{\mu_1}{z} + \dfrac{\mu_2}{2z^2} + \cdots$, $\mathcal{V}'(z)$ is only $O(z^{-2})$ at ∞, we have

$$\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\, dt = 0 \tag{3.6}$$

as a bonus!! (3.6) gives our first equation for a and b, knowing c, and another equation will be worked later on, from the unit charge condition $\int_E d\mu_z(t) = \int_F d\mu_p(t) = 1$.

When $c = 0$, (3.6) is solved when $b = -a$, as we integrate the odd function $t/\sqrt{t^2 - a^2}$ on $(a, -a)$. Moreover, as we know that F must be symmetric with respect to the real axis, we deduce that a and b are two opposite pure imaginary numbers. There is no more information in (3.6), but (3.1) yields the explicit formula $\dfrac{z}{\sqrt{z^2 - a^2}}\, \mathcal{V}'(z) = \dfrac{-1}{2\pi i} \displaystyle\int_a^{-a} \frac{t}{\sqrt{t^2 - a^2}} \frac{dt}{z - t} = \frac{1}{2} - \frac{z}{2\sqrt{z^2 - a^2}}$ from the Chebyshev-Markov example (5.7), so $\mathcal{V}'(z) = -\dfrac{1}{2} + \dfrac{\sqrt{z^2 - a^2}}{2z}$ shows a potential well of charge $-ia/2$ at the origin, whence $a = -2i$:

$$c = 0: \qquad \mathcal{V}(z) = -z/2 + \sqrt{z^2+4}/2 - \log[(\sqrt{z^2+4}+2)/z] \tag{3.7}$$

has indeed a potential well const. $+\log z$ coming from a negative unit charge at $z = 0$. Moreover, the derivative shows that $\mathcal{V}'(z)+1/2$ takes indeed opposite values on the two sides of any cut joining $-2i$ to $2i$ avoiding the origin. The actual line of poles is the locus where the two determinations of the square root of z^2+4 yield the same real part in the formula for $\mathcal{V}(z)$ [24], so, where $\mathcal{V}'(z)dz$ is pure imaginary, the quadratic differential $(z^2+4)dz^2/z^2 < 0$. For real negative $z = -2\sinh\theta$, $\cosh\theta - \log(\cosh\theta+1) = -\cosh\theta - \log(\cosh\theta - 1)$, or $2\cosh\theta = 2\log\coth(\theta/2)$, $\theta = 0.6219...$, $z = \alpha = -1.3255...$

In general, for a given $c \neq 0$, we shall have to work a non homegeneous equation expressing that E carries a negative unit charge, and F a positive unit charge. From Re$\mathcal{V}$ = constant on E, and Sokhotskyi-Plemelj [39, §14.1], $\mathcal{V}'(x \pm i0) = \mp\pi i\mu_i'(x)$ on $x \in E$, and we will need formulas for $\mathcal{V}$, so to use $\mathcal{V}(c+i0) - \mathcal{V}(i0) = -\pi i$.

Near F, a positive test particle is repelled, so, the gradient of $V(z) + x/2$ is directed towards F, and so is the complex derivative $\mathcal{V}' + 1/2$. On the right side of F, $\mathcal{V}' + 1/2$ has a negative real part, and the integral times dz from a to b has a negative imaginary part.

3.2.3. *Second derivative of the complex potential.* We need more formulas for $\mathcal{V}$ and its derivatives.

We build the differential equation (3.2) for $\mathcal{V}$ from (3.1), first multiply by $(z-a)(z-b)$, let $R(z) = \sqrt{z(z-c)(z-a)(z-b)}$:

$R(z)\mathcal{V}'(z) = \dfrac{1}{2\pi i}\displaystyle\int_a^b \dfrac{R(t)}{(t-a)(t-b)}\,\dfrac{t(z-t)+(t-a)(t-b)dt}{z-t}$, from $(z-a)(z-b) = (t-a)(t-b) + (z-t)(z+t-a-b)$, and using (3.6) as $\displaystyle\int_a^b \dfrac{R(t)dt}{(t-a)(t-b)} = 0$. Divide by $R(z)$, differentiate and integrate by parts:

$$\mathcal{V}''(z) = \frac{1}{2\pi i}\int_a^b \left[\frac{-tR'(z)R(t)}{(t-a)(t-b)R^2(z)} + \frac{[R'(t)/R(z) - R'(z)R(t)/R^2(z)]}{z-t}\right] dt,$$

$$R(z)\mathcal{V}''(z) = \frac{1}{2\pi i}\int_a^b R(t)\left[\frac{-tR'(z)}{(t-a)(t-b)R(z)} + \frac{[R'(t)/R(t) - R'(z)/R(z)]}{z-t}\right] dt,$$

which is therefore a rational function of poles $0, c, a$, and b, seeing that $2R'(z)/R(z) = 1/z + 1/(z-c) + 1/(z-a) + 1/(z-b)$. More precisely,

$$R(z)\mathcal{V}''(z) = \frac{1}{2\pi i}\int_a^b R(t)\left[\frac{-t}{(t-a)(t-b)}\left(\frac{1}{2z} + \frac{1}{2(z-c)} + \frac{1}{2(z-a)} + \frac{1}{2(z-b)}\right)\right.$$
$$\left. + \frac{1}{2tz} + \frac{1}{2(t-c)(z-c)} + \frac{1}{2(t-a)(z-a)} + \frac{1}{2(t-b)(z-b)}\right] dt$$
$$= \frac{1}{2\pi i}\int_a^b \frac{R(t)}{(t-a)(t-b)}\left[-\frac{(a+b)t-ab}{2tz} - \frac{(a+b-c)t-ab}{2(t-c)(z-c)} - \frac{b}{2(z-a)} - \frac{a}{2(z-b)}\right] dt$$

$= \dfrac{1}{4\pi i}\displaystyle\int_a^b \dfrac{R(t)}{(t-a)(t-b)}\left[\dfrac{ab}{tz} + \dfrac{(a-c)(b-c)}{(t-c)(z-c)}\right] dt$, using again (3.6). There is therefore no pole at $z = a$ and b, the residues at 0 and c are the integrals of ab/t and $(a-c)(b-c)/(t-c)$ divided by $4\pi i$, and this leads to (3.2-3.3).

The final result (3.2) for $\mathcal{V}''(z)$ is as in Gonchar & Rakhmanov [36, p. 323].

Here, ν_0 is the sum of the two residues, and ν_1 is $-c$ times the residue at 0. ν_0 is also $2\mu_1$ from the behaviour of $\mathcal{V}(z) = \mu_1/z + \cdots$ for large z in §3.2.2.

From the usual definition of the square root as positive for large positive z, and continuous outside the cuts, the denominator of (3.2) is negative imaginary on the upper side of the cut $[0, c]$, $\mathcal{V}''(z) = -\pi i\mu''(z)$ behaves like $\dfrac{\nu_0 c + \nu_1}{\sqrt{abc^3}}(z-c)^{-3/2}$ near $z = c$, $\mathcal{V}'(z) = -\pi i\mu'(z) \sim -2\dfrac{\nu_0 c + \nu_1}{\sqrt{abc^3}}(z-c)^{-1/2}$, and $\mathcal{V}(z) \sim$ constant $-4\dfrac{\nu_0 c + \nu_1}{\sqrt{abc^3}}(z-c)^{1/2}$. As μ' must be positive, $\nu_0 c + \nu_1 < 0$.

c	name	ρ	$a=\overline{b}$	k	ν_0	ν_1
0	Padé	0	$-2i$	0	-2	0
$c\to 0$		$c^2e^{2-c/2}/256$	$c/2-2i$	$c/4$	-2	c
0.5		1/177.934	$0.234-1.992i$	0.1245	-2.0155	0.565894
1		1/57.0700	$0.437-1.970i$	0.2458	-2.0608	1.27279
2		1/23.2287	$0.746-1.890i$	0.4626	-2.2248	3.11261
5		1/12.4330	$1.088-1.646i$	0.7696	-2.9314	11.1496
20		1/9.86048	$1.183-1.447i$	0.8865	-5.5344	83.6351
∞	'1/9'	1/9.28903	$1.195-1.389i$	0.9089	$-\infty$	∞

TABLE 1. c, ρ, a, k,ν_0, and ν_1.

Near $z=0$, $\mathcal{V}''(z)=-\pi i\mu''(z)\sim\dfrac{\nu_1 i}{\sqrt{abc^3}}z^{-3/2}, \mathcal{V}'(z)=-\pi i\mu'(z)\sim -2\dfrac{\nu_1 i}{\sqrt{abc^3}}z^{-1/2}, \mathcal{V}(z)\sim$ constant $-4\dfrac{\nu_1 i}{\sqrt{abc^3}}z^{1/2}$, and $\nu_1>0$.

Finally, $\mathcal{V}'$ and $\mathcal{V}$ are

$$\mathcal{V}'(z)=\int_\infty^z\frac{(\nu_0x+\nu_1)dx}{\sqrt{x^3(x-c)^3(x-a)(x-b)}},\tag{3.8}$$

$$\mathcal{V}(z)=\int_\infty^z\frac{(\nu_0x+\nu_1)(z-x)dx}{\sqrt{x^3(x-c)^3(x-a)(x-b)}}=z\mathcal{V}'(z)-\int_\infty^z\frac{(\nu_0x+\nu_1)dx}{\sqrt{x(x-c)^3(x-a)(x-b)}}.\tag{3.9}$$

Behaviour of $\mathcal{V}(z)$ and its derivatives near a and b: let $\dfrac{\nu_0b+\nu_1}{\sqrt{b^3(b-c)^3(b-a)}}=Re^{i\Phi}$, so $\mathcal{V}''(z)\sim Re^{i\Phi}(z-b)^{-1/2}$, near b, $\mathcal{V}'(z)\sim -1/2+2Re^{i\Phi}(z-b)^{1/2}$, as we know that $\mathcal{V}'(a)=\mathcal{V}'(b)=-1/2$, and $\mathcal{V}(z)\sim$ constant $-z/2+(4R/3)e^{i\Phi}(z-b)^{3/2}$, where the constant is $\mathcal{V}(b)+b/2$. As the square roots must remain continuous outside the cut F, we follow $z-b=|z-b|e^{i\theta}$ from $\theta=\theta_0$ on the right side of F, to $\theta=\theta_0+2\pi$ on the left side. The real part of $\mathcal{V}(z)+z/2-\mathcal{V}(b)-b/2$ must vanish at the starting point, so $\Phi+3\theta_0/2=\pi/2$ and the real part of $\exp(i\Phi+3\theta/2)$ is $\cos(\pi/2+3(\theta-\theta_0)/2)=-\sin(3(\theta-\theta_0)/2)$ is indeed negative for $\theta-\theta_0<2\pi/3$, positive between $2\pi/3$ and $4\pi/3$, and negative again between $4\pi/3$ and 2π, explaining the pattern of signs in fig. 3.

Some values are in table 1, their formulas will be established in section 4 of the present study.

3.2.4. *Second equation for the parameters.* Unit charges: on the sides of the cut F bearing a unit positive charge, we know that

$$\mathcal{V}'(z)=-1/2\pm\pi i\mu_p'(z),\tag{3.10}$$

(from (2.5), Sokhotskyi-Plemelj, again), or near any arc joining a and b not crossing $E=[0,c]$.

So, as seen above, the integral of $\mathcal{V}'(z)+1/2$ from a to b on the right side of the cut has a negative imaginary part, and

$$\mathcal{V}(b)-\mathcal{V}(a)=(a-b)/2-\pi i\tag{3.11}$$

on the right side of the cut F. From (3.9),

$\mathcal{V}(b)-\mathcal{V}(a)=\underbrace{b\mathcal{V}'(b)-a\mathcal{V}'(a)}_{(a-b)/2}-\displaystyle\int_a^b\frac{(\nu_0x+\nu_1)dx}{\sqrt{x(x-c)^3(x-a)(x-b)}}=-\pi i$, knowing that $\mathcal{V}'(a)=\mathcal{V}'(b)=-1/2$, ***our second equation*** is (3.4).

Remark that we could have taken in (3.9),
$\mathcal{V}(z) = (z-c)\mathcal{V}'(z) - \int_\infty^z \frac{(\nu_0 x+\nu_1)dx}{\sqrt{x^3(x-c)(x-a)(x-b)}}$, leading to
$\int_a^b \frac{(\nu_0 x+\nu_1)dx}{\sqrt{x^3(x-c)(x-a)(x-b)}} = -\pi i$ as well, so that one must have

$$\int_a^b \frac{(\nu_0 x+\nu_1)dx}{\sqrt{x^3(x-c)(x-a)(x-b)}} = \int_a^b \frac{(\nu_0 x+\nu_1)dx}{\sqrt{x(x-c)^3(x-a)(x-b)}}, \tag{3.12}$$

and $\mathcal{V}'(a) = \mathcal{V}'(b)$ means $\int_a^b \frac{(\nu_0 x+\nu_1)dx}{\sqrt{x^3(x-c)^3(x-a)(x-b)}} = 0$, but is a consequence of (3.12): with $R(x) = \sqrt{x(x-c)(x-a)(x-b)}$ as before,
$\int_a^b \frac{\nu_0 x+\nu_1}{xR(x)}dx - \int_a^b \frac{\nu_0 x+\nu_1}{(x-c)R(x)}dx = 0 = -c\int_a^b \frac{\nu_0 x+\nu_1}{x(x-c)R(x)}dx = 0.$

When $c \to +0$, the second part of (3.4) tends to
$\frac{1}{4\pi i}\int_a^{-a} \frac{t}{\sqrt{t^2-a^2}}\,\frac{-2a^2 x}{t}\,dt = a^2x/2$; and the first part is the half of the contour integral about the cut F of $(\nu_0/x+\nu_1/x^2)/\sqrt{(x-a)(x-b)}$ leaving πi times the residue $\nu_0/\sqrt{ab}$ at $x=0$, so, $\nu_0 \sim -\sqrt{ab} = -|a|$. This leaves $\nu_0 \sim a^2/2 \sim -|a| \to -2$ when $c \to 0$, in agreement with numerical tests.

Rate of decrease of error norm is (3.5).

4. **Elliptic integrals of first, second, and third kind.**

4.1. **Change of variables.**

We follow Byrd & Friedman [20, p.133]:
A convenient transformation sending the four branchpoints $z = 0$, c, a, and $b = \overline{a}$ on and from a symmetric set $\{\mp 1, \mp ik'/k\}$, is

$$v = \operatorname{cn} u = \frac{Az+B(z-c)}{Az-B(z-c)} \Leftrightarrow z = -\frac{Bc}{A-B} + \frac{2ABc/(A-B)}{A+B-(A-B)v} = \frac{Bc(1+v)}{A+B-(A-B)v},$$

where A and B are the absolute values $|a-c|$ and $|a|$. So,

$$v = \frac{\dfrac{z}{|a|} + \dfrac{z-c}{|a-c|}}{\dfrac{z}{|a|} - \dfrac{z-c}{|a-c|}}. \tag{4.1}$$

In particular, at $z = a$ and b, $v = [\exp(\pm i\arg a)+\exp(\pm i\arg(a-c))]/[\exp(\pm i\arg a)-\exp(\pm i\arg(a-c))] = \mp i\cot[\arg(a/(a-c))/2] = \mp i\cot(\theta/2)$, where θ is the angle at a or b in the figure 8. These values for $v = \operatorname{cn}\, u$ must be $\pm ik'/k$, so that $s = \operatorname{sn}\, u = \pm\sqrt{1+k'^2/k^2} = \pm 1/k$ there. We also have $e^{i\theta/2} = k' + ik$.

As neither a nor b is known, we may as well take k and $\zeta = \dfrac{A-B}{A+B}$.
$A = (A+B)(1+\zeta)/2, B = (A+B)(1-\zeta)/2,$
$c^2 = A^2+B^2-2AB\cos\theta = (A+B)^2[1+\zeta^2-(1-\zeta^2)\cos\theta]/2$
$= (A+B)^2(k^2+\zeta^2k'^2).$
$a,b = \dfrac{(1-\zeta)c(1\mp ik'/k)/2}{1+\pm i\zeta k'/k}, a+b = \dfrac{(1-\zeta)c(k^2-k'^2\zeta)}{k^2+\zeta^2k'^2}, c-a-b = \dfrac{c\zeta}{k^2+\zeta^2k'^2},$
$ab = \dfrac{(1-\zeta)^2c^2}{4(k^2+\zeta^2k'^2)}$

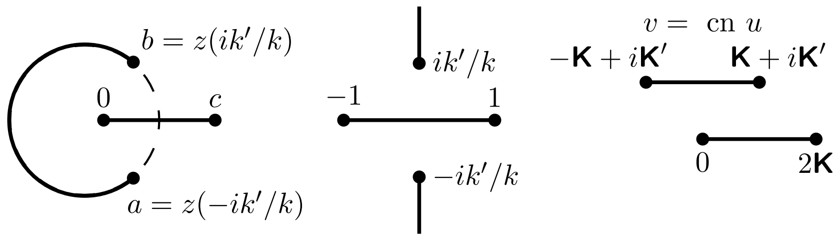

FIGURE 4. The $z-$ plane, the $v-$plane, and the $u-$plane.

Also, $a, b = \dfrac{c}{1 - \dfrac{A(k' \pm ik}{B(k' \mp ik}} = \dfrac{c}{1 - \dfrac{|a-c|}{|a|}e^{\pm i\theta}}$.

$z = \dfrac{(1-\zeta)c(v+1)}{2(1-\zeta v)} = \dfrac{(1-\zeta^{-1})c}{2} + \dfrac{(1-\zeta^2)c}{2\zeta(1-\zeta v)}, z - c = \dfrac{(1+\zeta)c(v-1)}{2(1-\zeta v)}, z - a, b = \dfrac{(1-\zeta^2)c(kv \pm i\zeta k')}{2(1-\zeta v)(k \pm i\zeta k')}$,
$(z-a)(z-b) = \dfrac{(1-\zeta^2)^2 c^2 (k^2 v^2 + k'^2)}{4(1-\zeta v)^2(k^2 + \zeta^2 k'^2)}$ [20, 361.54 p.215].

4.2. **Theorem.** *The best rational approximation problem of* $\exp(-(n+\nu)z)$ *involves the sets* $E = [0, c]$ *and* $F = [a, b]$ *, which are mapped by* (4.1) *on* $[-1, 1]$ *and an arc joining* $-ik'/k$ *to* ik'/k *avoiding* $[-1, 1]$*, see fig.* 4.

A first equation for the modulus k *and* $\zeta = \dfrac{|a-c| - |a|}{|a-c| + |a|}$ *is*

$$\mathbf{E} + (k^2 - \alpha^2)\frac{\mathbf{K} - \Pi}{\alpha^2} = 0, \tag{4.2}$$

where $k' = \sqrt{1-k^2}, \alpha^2 = -k^2(1-\zeta^2)/\zeta^2, \zeta^2 = k^2/(k^2 - \alpha^2)$ *and*

$$\begin{aligned} \mathbf{K} &= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \qquad \mathbf{E} = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi, \\ \Pi &= \int_0^{\pi/2} \frac{d\varphi}{(1 - \alpha^2 \sin^2 \varphi)\sqrt{1 - k^2 \sin^2 \varphi}} = \int_0^{\mathbf{K}} \frac{du}{1 - \alpha^2 \operatorname{sn}^2 u} \end{aligned} \tag{4.3}$$

are the complete elliptic integrals of first, second, and third[4] *kind.*
The second equation for k *and* α *(or* k *and* ζ*) is*

$$\mathbf{E}(\mathbf{K} - \mathbf{E}) = \frac{\pi^2 \zeta}{c(1-\zeta^2)} = \frac{\pi^2 k \sqrt{k^2 - \alpha^2}}{-\alpha^2 c}. \tag{4.4}$$

Finally, the rate of decrease is given by

$$\log \rho = \frac{\pi}{1+\zeta}\left[-\frac{\zeta \mathbf{E}'}{\mathbf{K} - \mathbf{E}} + \frac{\mathbf{E}' - \mathbf{K}'}{\mathbf{E}}\right] \tag{4.5}$$

4.3. **Proof.**

[4] α^2 is often written n, or $-n$, or $1-p$ in the literature.

k	name	α^2	$2\mathbf{K}/\pi$	$2\mathbf{E}/\pi$	$2\Pi/\pi$
0	Padé	$-\infty$	1	1	0
$k \to 0$		$-4/k^4$	$1+k^2/4$	$1-k^2/4$	$k^2/2$
0.25		-972.90	1.016199	0.984187	0.032076
0.50		-49.130939	1.073182	0.934215	0.143696
0.75		-4.78774	1.216574	0.839365	0.465456
$k_\infty = 0.908909\ldots,$	'1/9'	0	1.477626	0.738813	1.477626
0.95		0.55037	1.648852	0.702014	2.746061
0.99		0.928447	2.13688	0.654748	13.9059

TABLE 2. $\alpha^2, \mathbf{K}, \mathbf{E}$, and Π as functions of k from (4.2).

4.3.1. *First equation.*

The integral (3.6) is a constant times $\displaystyle\int_{ik'/k}^{-ik'/k} \sqrt{\frac{1-v^2}{k^2v^2+k'^2}}\frac{dv}{(1-\zeta v)^2}$, integrating from ik'/k to $-ik'/k$ through $\pm i\infty$, or, with $v =$ cn$u$,

$$\int_{-\mathbf{K}+i\mathbf{K}'}^{\mathbf{K}+i\mathbf{K}'} \frac{\mathrm{sn}^2 u}{(1-\zeta\ \mathrm{cn} u)^2}du = \int_{-\mathbf{K}}^{\mathbf{K}} \frac{1}{(k\ \mathrm{sn}\ u - i\zeta\ \mathrm{dn}\ u)^2}du$$

from $\mathrm{sn}(u+i\mathbf{K}') = 1/(k\ \mathrm{sn}u)$, $\mathrm{cn}(u+i\mathbf{K}') = \mathrm{dn}u/(ik\ \mathrm{sn}u)$, (Jahnke & Emde [42, VI A 4.2] $= 2\int_0^{\mathbf{K}} \dfrac{\zeta^2\ \mathrm{dn}^2\ u - k^2\ \mathrm{sn}^2\ u}{(k^2\ \mathrm{sn}^2\ u + \zeta^2\ \mathrm{dn}^2\ u)^2}du = 2\int_0^{\mathbf{K}} \dfrac{\zeta^2 - k^2(1+\zeta^2)\ \mathrm{sn}^2\ u}{(\zeta^2 + k^2(1-\zeta^2)\ \mathrm{sn}^2\ u)^2}du$ = a constant times (4.2), see [20, 362.15 & 16, also 410.07 & 08].

The equation (4.2) has exactly one root $\alpha^2 \in (-\infty, k^2)$ when $-1 < k < 1$, as the left-hand side of (4.2) is $\mathbf{E} - \displaystyle\int_0^{\pi/2} \frac{k^2-\alpha^2}{1-\alpha^2\sin^2\varphi}\frac{\sin^2\varphi\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$ which is an increasing function of α^2, starting with $\mathbf{E}-\mathbf{K} < 0$ at $\alpha^2 = -\infty$, and reaching $\mathbf{E} > 0$ at $\alpha^2 = k^2$.

Incidentally (!), at $\alpha = 0$, we have $\mathbf{E} - \displaystyle\int_0^{\pi/2} \frac{k^2\sin^2\varphi\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \mathbf{E} - (\mathbf{K}-\mathbf{E}) = 2\mathbf{E}-\mathbf{K}$, known to vanish when $c = \infty$, the '1/9' case ... The value of the modulus is then $k_\infty = 0.9089085575485414782361189087447935049010139693404...$ [53].

When c is small, (4.2) is $\alpha^2\pi/2$ times $(2/\pi)(\mathbf{E}-\mathbf{K}) + 2\Pi/\pi \sim -k^2/2 + 1/|\alpha|$ [42, V.C.1] and $\int du/(1-\alpha^2\ \mathrm{sn}^2\ u) \sim \int_0^\infty du/(1-\alpha^2u^2) = \pi/(2|\alpha|)$ for α^2 strongly negative), so, the left hand side of (4.2) is about $\alpha^2\pi/2$ times $-k^2/2 + 1/|\alpha|$ whence $\alpha^2 \sim -4/k^4$, or $\zeta \sim k^3/2$.

We could not resist looking "beyond ∞", when $\alpha^2 > 0$ in table 2.

4.3.2. *Second equation.*

We now keep track of all the constants in (3.4).

From (3.3) with $t = \dfrac{c(1-\zeta)(v+1)}{2(1-\zeta v)} = \dfrac{(1-\zeta^{-1})c}{2} + \dfrac{(1-\zeta^2)c}{2\zeta(1-\zeta v)}$,

$$\nu_0 x + \nu_1 = \frac{1}{4\pi i}\int_{ik'/k}^{-ik'/k} \sqrt{\frac{(v^2-1)(k^2+\zeta^2k'^2)}{(1-\zeta^2)(k^2v^2+k'^2)}}$$
$$\left[\frac{2ab(1-\zeta v)(x-c)}{(1-\zeta)c(v+1)} + \frac{2(a-c)(b-c)(1-\zeta v)x}{(1+\zeta)c(v-1)}\right]\frac{(1-\zeta^2)c\,dv}{2(1-\zeta v)^2},$$

integrating from ik'/k to $-ik'/k$ through $\pm i\infty$, or, with $v=\text{ cn}u$, and with $\chi=\dfrac{-1}{4\pi}\sqrt{\dfrac{k^2+\zeta^2k'^2}{1-\zeta^2}}$,

$$\nu_0x+\nu_1=\chi\int_{-\mathsf{K}+i\mathsf{K}'}^{\mathsf{K}+i\mathsf{K}'}[(1+\zeta)ab(x-c)(1-\text{ cn}u)-(1-\zeta)(a-c)(b-c)x(1+\text{ cn}u)]\frac{du}{1-\zeta\text{ cn}u},$$
$$=\chi\int_{-\mathsf{K}}^{\mathsf{K}}\frac{(1+\zeta)ab(x-c)(ik\text{ sn}u-\text{ dn}u)-(1-\zeta)(a-c)(b-c)x(ik\text{ sn}u+\text{ dn}u)}{ik\text{ sn}u-\zeta\text{ dn}u}du$$
$$=2\chi\int_0^{\mathsf{K}}\frac{(1+\zeta)ab(x-c)(k^2\text{ sn}^2u+\zeta\text{ dn}^2u)-(1-\zeta)(a-c)(b-c)x(k^2\text{ sn}^2-\zeta\text{ dn}^2u)}{k^2\text{ sn}^2u+\zeta^2\text{ dn}^2u=\zeta^2(1-\alpha^2\text{ sn}^2u)}du$$
$$=2\chi\left[(1+\zeta)ab(x-c)\left(\frac{\mathsf{K}}{1+\zeta}+\frac{\Pi}{\zeta(1+\zeta)}\right)-(1-\zeta)(a-c)(b-c)x\left(\frac{\mathsf{K}}{1-\zeta}-\frac{\Pi}{\zeta(1-\zeta)}\right)\right]$$
$$=\frac{-c^2}{8\pi\sqrt{(1-\zeta^2)(k^2+\zeta^2k'^2)}}\left[(1-\zeta)^2(x-c)\left(\mathsf{K}+\frac{\Pi}{\zeta}\right)-(1+\zeta)^2x\left(\mathsf{K}-\frac{\Pi}{\zeta}\right)\right]$$
$$=\frac{-c^2}{8\pi\zeta}\sqrt{\frac{1-\zeta^2}{k^2+\zeta^2k'^2}}\,[(1-\zeta)(x-c)(\mathsf{K}-(1-\zeta)\mathsf{E})+(1+\zeta)x\,(\mathsf{K}-(1+\zeta)\mathsf{E})]$$

using $ab=\dfrac{(1-\zeta)^2c^2}{4(k^2+\zeta^2k'^2)},(a-c)(b-c)=\dfrac{(1+\zeta)^2c^2}{4(k^2+\zeta^2k'^2)}$ from above, and (4.2) in the form $\Pi=\mathsf{K}-(1-\zeta^2)\mathsf{E}$, so

$$\nu_0=\frac{-c^2}{4\pi\zeta}\sqrt{\frac{1-\zeta^2}{k^2+\zeta^2k'^2}}\left[\mathsf{K}-(1+\zeta^2)\mathsf{E}\right],\nu_1=\frac{c^3}{8\pi\zeta}\sqrt{\frac{1-\zeta^2}{k^2+\zeta^2k'^2}}(1-\zeta)\left[\mathsf{K}-(1-\zeta)\mathsf{E}\right].\qquad(4.6)$$

We now need (3.4) and (3.12)

$$\pi i=\int_a^b\frac{(\nu_0+\nu_1/x)dx}{\sqrt{x(x-c)(x-a)(x-b)}}=\int_{ik'/k}^{-ik'/k}\frac{\left[\nu_0+\nu_1\dfrac{2(1-\zeta v)}{(1-\zeta)c(v+1)}\right]\dfrac{(1-\zeta^2)c\,dv}{2(1-\zeta v)^2}}{\sqrt{\dfrac{(1-\zeta^2)^3c^4(v^2-1)(k^2v^2+k'^2)}{16(1-\zeta v)^4(k^2+\zeta^2k'^2)}}}$$

$$\pi=\quad 2\sqrt{\frac{k^2+\zeta^2k'^2}{1-\zeta^2}}\;\int_{ik'/k}^{-ik'/k}\frac{\left[\nu_0+\nu_1\dfrac{2(1-\zeta v)}{(1-\zeta)c(v+1)}\right](\;dv=-\text{ sn}\,u\text{ dn}\,u\,du)}{c\sqrt{(1-v^2)(k^2v^2+k'^2)}}$$
$$=\frac{-2}{c}\sqrt{\frac{k^2+\zeta^2k'^2}{1-\zeta^2}}\;\int_{-\mathsf{K}+i\mathsf{K}'}^{\mathsf{K}+i\mathsf{K}'}\left[\nu_0+\nu_1\frac{2(1-\zeta\text{ cn}\,u)}{(1-\zeta)c(1+\text{ cn}\,u)}\right]du$$
$$=\frac{-2}{c}\sqrt{\frac{k^2+\zeta^2k'^2}{1-\zeta^2}}\;\int_{-\mathsf{K}}^{\mathsf{K}}\left[\nu_0+\nu_1\frac{2(k\text{ sn}\,u+i\zeta\text{ dn}\,u)}{(1-\zeta)c(k\text{ sn}\,u-i\text{ dn}\,u)}\right]du$$
$$=\frac{-2}{c}\int_0^{\mathsf{K}}\left[\frac{-c^2}{2\pi\zeta}(\mathsf{K}-(1+\zeta^2)\mathsf{E})+\frac{c^2}{2\pi\zeta}(\mathsf{K}-(1-\zeta)\mathsf{E})(-\zeta+(1+\zeta)k^2\text{ sn}^2u)\right]du$$
$$=\frac{c}{\pi\zeta}\;[\mathsf{K}(\mathsf{K}-(1+\zeta^2)\mathsf{E})-(\mathsf{K}-(1-\zeta)\mathsf{E})(-\zeta\mathsf{K}+(1+\zeta)(\mathsf{K}-\mathsf{E})]$$

$=\dfrac{c}{\pi\zeta}\;(1-\zeta^2)\mathsf{E}(\mathsf{K}-\mathsf{E})$, using $\displaystyle\int_0^{\mathsf{K}}\text{ sn}^2u\,du=(\mathsf{K}-\mathsf{E})/k^2$ [1, 16.26.1] and (4.4) follows, completing with (4.2) the two equations for the two real unknowns k and α^2 (or k and ζ). Table 1 is done with numerical solutions of equations (4.2) and (4.4) (and (4.8) for ρ).

As c is a continuous function of k and ζ, existence of solution follows. It is even possible to look at $k>0.9089...$, where $c<0$, the meaning of this region being a small enigma (§8).

It is also strongly suspected, but without proof here, that c is an increasing function of k and ζ, implying unicity of solution.

When $c\to 0$, we expect k and $\zeta\to 0$, $\mathsf{K}-\mathsf{E}$ very small $\sim\pi k^2/4$, and we know $\zeta\sim k^3/2$, so, $c\sim 4k$.

A curious consequence of (4.4) is the quadratic relation

$$\frac{(\nu_0 c+\nu_1)^2}{(c-a)(c-b)} - \frac{\nu_1^2}{ab} = -\frac{c^3}{4}. \tag{4.7}$$

Indeed, from $ab = \dfrac{(1-\zeta)^2c^2}{4(k^2+\zeta^2k'^2)}, (a-c)(b-c) = \dfrac{(1+\zeta)^2c^2}{4(k^2+\zeta^2k'^2)}$ and (4.6), $\dfrac{(\nu_0c+\nu_1)^2}{(c-a)(c-b)} = \dfrac{c^4(1-\zeta^2)}{16\pi^2\zeta^2}[-\mathbf{K}+(1+\zeta)\mathbf{E}]^2$, $\dfrac{\nu_1^2}{ab} = \dfrac{c^4(1-\zeta^2)}{16\pi^2\zeta^2}[\mathbf{K}-(1-\zeta)\mathbf{E}]^2$, so that the difference is $\dfrac{c^4(1-\zeta^2)}{16\pi^2\zeta^2}4\zeta\mathbf{E}(\mathbf{E}-\mathbf{K})$ and the result follows from (4.4).

4.3.3. *Rate of error norm decrease.* From (3.5)

$$\log\rho = 2\ \mathrm{Re}\int_0^a \frac{(\nu_0 t+\nu_1)dt}{\sqrt{t(t-c)^3(t-a)(t-b)}}$$

$$= 2\ \mathrm{Re}\ \frac{4(1-\zeta)}{c^2}\sqrt{\frac{k^2+\zeta^2k'^2}{(1-\zeta^2)^3}}\int_{-1}^{-ik'/k}\frac{(1-\zeta v)\left[\nu_0\dfrac{(1-\zeta)c(v+1)}{2(1-\zeta v)}+\nu_1\right]dv}{(v-1)\sqrt{(v^2-1)(k^2v^2+k'^2)}}$$

$$= 2\ \mathrm{Re}\ \frac{4(1-\zeta)}{c^2}\sqrt{\frac{k^2+\zeta^2k'^2}{(1-\zeta^2)^3}}\int_{-1}^{-ik'/k}\frac{2(1-\zeta)(\nu_0c+\nu_1)+((1-\zeta)\nu_0c-2\zeta\nu_1)(v-1)}{2(v-1)\sqrt{(v^2-1)(k^2v^2+k'^2)}}dv$$

$$= 2\ \mathrm{Re}\ \frac{4i(1-\zeta)}{c^2}\left[\int_{2\mathbf{K}}^{-\mathbf{K}+i\mathbf{K}'}\frac{2c^3((1+\zeta)\mathbf{E}-\mathbf{K})}{8\pi\zeta(\ \mathrm{cn}\ u-1)}du - \frac{c^3(\mathbf{K}-\mathbf{E})}{4\pi\zeta}(-3\mathbf{K}+i\mathbf{K}')\right]$$

$$= 2\ \mathrm{Re}\ 4ic(1-\zeta)\left[-\frac{(1+\zeta)\mathbf{E}-\mathbf{K}}{4\pi\zeta}\underbrace{[\Delta(u-E(u)]_{2\mathbf{K}}^{-\mathbf{K}+i\mathbf{K}'}}_{-3\mathbf{K}+i\mathbf{K}'+3\mathbf{E}-i(\mathbf{K}'-\mathbf{E}')} - \frac{\mathbf{K}-\mathbf{E}}{4\pi\zeta}(-3\mathbf{K}+i\mathbf{K}')\right]$$

$$\log\rho = -\frac{c(1-\zeta)}{\pi\zeta}[((1+\zeta)\mathbf{E}-\mathbf{K})\mathbf{E}'+(\mathbf{K}-\mathbf{E})\mathbf{K}'], \tag{4.8}$$

from [20, 119.02, p.26 Fig. 12, 361.51] and [5, p.81, 82]. and (4.5) follows.

When c is small, $k\sim c/4$, $\mathbf{E}$ and $\mathbf{K}\sim\pi/2$, $\mathbf{K}-\mathbf{E}\sim\pi k^2/4\sim\pi c^2/64$, $\zeta\sim k^3/2\sim c^3/128$, $\mathbf{E}'\sim 1$, and $\mathbf{K}'\sim\log(4/k)\sim\log(16/c)$, so, $\log\rho\sim -c[1/2-(\pi c^2/64)/(\pi c^3/128)[1-\log(16/c)]] = -c/2+2-2\log(16/c)$, same as Meinardus-Braess in § 1.2.

And when $c\to\infty$, the '1/9' case, it is better to use (4.4),

$$\log\rho = \frac{\pi}{1+\zeta}\left[\frac{\mathbf{K}-(1+\zeta)\mathbf{E}}{\mathbf{E}(\mathbf{K}-\mathbf{E})}\mathbf{E}' - \frac{\mathbf{K}'}{\mathbf{E}}\right]$$

If $c\to\infty, \zeta\to 1$, and we know that $\mathbf{K}-2\mathbf{E}\to 0$, so, $\log\rho\sim -\pi\mathbf{K}'/\mathbf{K}$ remains!

5. Strong asymptotics.

5.1. **Introduction.** We considered expressions like $X_n = A_ne^{B_n}$, where A_n is convergent, and where B_n increases linearly, say, $B_n = nB^*+O(1)$. The root asymptotics deals only with e^{B^*} = the limit of $X_n^{1/n}$. We now try a more accurate asymptotic expression $X_n\sim A_\infty e^{nB^*+B^{**}}$ meaning $\lim_{n\to\infty}\dfrac{X_n}{A_\infty e^{nB^*+B^{**}}} = 1$, where A_∞ and B^{**} are the limits of A_n and B_n-nB^*, if these limits exist.

Consider $f_n(z) = \displaystyle\int_C\frac{\rho_n(t)\,dt}{z-t}$, with $\rho_n(t) = \exp(-(n+\nu)t)$, so that, as in (2.1), $f_n(z) = -2\pi i\exp(-(n+\nu)z)$ for z inside the contour C containing the $F-$cut.

We look for more than just the main behaviour with respect to n of the rational approximation, that's why the parameter ν must not be neglected.

We proceed with accurate asymptotic descriptions of p_n, q_n, $p_n f_n - q_n$, and $\|f_n - q_n/p_n\|_E/\rho^n$.

5.2. **Theorem.** *The best rational approximation q_n/p_n of degree n to $\exp(-(n+\nu)x)$ on $[0,c]$ satisfies*

$$\begin{aligned} p_n(z) &\sim [(z-a)(z-b)]^{-1/4}\exp((n+\nu)\mathcal{V}_p(z) - (\nu-1/2)\mathcal{V}_{*,p}(z)), \\ p_n(z)\exp(-(n+\nu)z) - q_n(z) &\sim C_n[(z-a)(z-b)]^{-1/4} \\ &\exp[(n+\nu)(2\mathcal{V}_z(z) - \mathcal{V}_p(z)) - (\nu-1/2)(2\mathcal{V}_{*,z}(z) - \mathcal{V}_{*,p}(z))], \end{aligned} \tag{5.1}$$

when $n\to\infty$, and where $\mathcal{V}$ is the complex potential introduced in § 3, and $\mathcal{V}_* = \mathcal{V}_{*,z} - \mathcal{V}_{*,p}$ is the auxiliary complex potential

$$\mathcal{V}_*(z) = C\int_\infty^z \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}, \quad C = \sqrt{\frac{1-\zeta^2}{k^2+\zeta^2 k'^2}}\,\frac{\pi c}{4\mathbf{K}}. \tag{5.2}$$

and with $C_n = \exp(-(n+\nu)(2\mathcal{V}(a)+a) + (2\nu-1)\mathcal{V}_*(a))$.

On F, one must add the contributions from the two sides of F:
$p_n(z) \sim [(z-a)(z-b)]^{-1/4} e^{(n+\nu)\mathcal{V}_z(z)-(\nu-1/2)\mathcal{V}_{*,z}(z)}$
$\{\exp[-(n+\nu)\mathcal{V}_+(z) + (\nu-1/2)\mathcal{V}_{*,+}(z)] + \exp[-(n+\nu)\mathcal{V}_-(z) + (\nu-1/2)\mathcal{V}_{*,-}(z)]\},\ z\in F.$

Finally, the error norm

$$\|\exp(-(n+\nu(z)) - q_n(z)/p_n(z)\|_E \sim 2\rho^{n+\nu}\rho_*^{1/2-\nu}, \tag{5.3}$$

where $\rho = \exp(-2V(a) - a + 2V(0))$ is the main rate of decrease discussed in (4.5)-(4.8)*; and*

$$\rho_* = \exp(2(V_1(0) - V_1(a)) = \exp\left(-\frac{\pi\mathbf{K}'}{\mathbf{K}}\right). \tag{5.4}$$

5.3. **Lines of poles.** We have a combination of $e^{nB_+^*(z)+B_+^{**}(z)}$ and $e^{nB_-^*(z)+B_-^{**}(z)}$ in the region of interest for $P_n(z)$. Poles of the rational approximant, i.e., zeros of P_n occur when the two exponentials have a common absolute value, so when $nB_+^*(z)+B_+^{**}(z)$ and $nB_-^*(z)+B_-^{**}(z)$ have the same real part. When $n\to\infty$, the limit locus is given by $B_+^*(z) - B_-^*(z)$ pure imaginary. This limit locus is reached especially fast if $B_+^{**}(z) - B_-^{**}(z)$ is pure imaginary too on the limit locus. This happens obviously in (5.5) and just above when $\nu = 1/2$. This explains the $n+1/2$ phenomenon of Fig. 1 when $c=0$. The same conclusion holds for $c\neq 0$ too, as seen from the second formula of (5.1).

5.4. **Proof.**

5.4.1. *Padé case.* Consider first the limit case $c=0$.

The Padé approximant of degree n to $\exp(-(n+\nu)z)$ is known to be $\dfrac{{}_1F_1(-n,-2n,-(n+\nu)z)}{{}_1F_1(-n,-2n,(n+\nu)z)}$ [9, (5.39)] [69, §75], where the ${}_1F_1$ expansions are limited to their $n+1$ first terms.

The monic denominator is

$$P_n(z) = \frac{(2n)!}{n!(n+\nu)^n}\,{}_1F_1(-n,-2n,(n+\nu)z) = z^n y_n(2/((n+\nu)z)),$$

where $y_n(u) = {}_2F_0(-n,n+1;-u/2)$ is a Bessel polynomial and satisfies

$u^2\dfrac{d^2y_n(u)}{du^2} + 2(u+1)\dfrac{dy_n(u)}{du} - n(n+1)y_n(u) = 0$, [67, §18.34 with $a=2$], or

$\dfrac{d^2[ue^{-1/u}y_n(u)]}{du^2} + \left(-\dfrac{1}{u^4} - \dfrac{n(n+1)}{u^2}\right)ue^{-1/u}y_n(u) = 0$, $\dfrac{d^2[xe^{-(n+\nu)/(2x)}y_n(2x/(n+\nu))]}{dx^2}$

$-\left(\dfrac{(n+\nu)^2}{4x^4}+\dfrac{n(n+1)}{x^2}\right) xe^{-(n+\nu)/(2x)}y_n(2x/(n+\nu))=0$, with $x=(n+\nu)u/2=1/z$, a differential equation with a large parameter
$d^2Y(x)/dx^2+q(x,n)Y(x)=0$ [26, §4.2] $q(x,n)=-(1/x^2+1/(4x^4))n^2$
$-(1/x^2+\nu/(2x^4))n+\cdots$ with turning points near $x=\pm i/2$ or $z=\pm 2i$.

The first approximation of the solution of this Liouville-Green-Steklov-WKB problem is $(-q(x,n))^{-1/4}$ times a combination of the two exponential functions $\exp\left(\pm\int^x\sqrt{-q(\xi,n)}d\xi\right)$ [11, §10.1] amounting for $y_n(u)$ to be a combination of
$x^{-1}e^{(n+\nu)/(2x)}(1/x^2+1/(4x^4))^{-1/4}\exp\left(\pm\int^x\sqrt{n(n+1)/\xi^2+(n+\nu)^2/(4\xi^4)}d\xi\right)$,
or $z^n e^{(n+\nu)z/2}(1/4+1/z^2)^{-1/4}\exp\left(\pm\left[\int^z\sqrt{n(n+1)\eta^2+(n+\nu)^2\eta^4/4}\,\dfrac{d\eta}{\eta^2}\right]\right)$ for $P_n(z)$ at last.
The main behaviour with respect to n is $z^n\exp[n(z/2\pm(\mathcal{V}(z)+z/2)]$ from $\mathcal{V}'(z)=-\dfrac{1}{2}+\dfrac{\sqrt{z^2+4}}{2z}$ used in (3.7). From the z^n behaviour of $P_n(z)$ for large z, we keep only the minus sign, so for the monic denominator,
$P_n(z)\sim z^n(1+4/z^2)^{-1/4}\exp\left(\int_\infty^z\left[(n+\nu)/2-\sqrt{n(n+1)/\eta^2+(n+\nu)^2/4}\right]\,d\eta\right)$.

$$P_n(z)\sim z^n(1+4/z^2)^{-1/4}\exp\left(\frac{n+\nu}{2}\left[z-\sqrt{z^2+4}+2\log\left(\frac{\sqrt{z^2+4}+2}{z}\right)\right]\right.$$
$$\left.-(\nu-1/2)\log\left(\frac{\sqrt{z^2+4}+2}{z}\right)\right) \quad (5.5)$$

From (3.7), we recognize the exponential of

$$-(n+\nu)\mathcal{V}(z)-(\nu-1/2)\log\left(\frac{\sqrt{z^2+4}+2}{z}\right)$$
$$=-(n+1/2)\mathcal{V}(z)+(\nu-1/2)[z-\sqrt{z^2+4}]/2.$$

Remark that the coefficients of z^n, z^{n-1}, z^{n-2} of the monic denominator are

$$\begin{aligned}P_n(z)&=z^n y_n(2/((n+\nu)z))=z^n{}_2F_0(-n,n+1;-1/((n+\nu)z))\\&=z^n+\frac{n(n+1)z^{n-1}}{n+\nu}+\frac{(n-1)n(n+1)(n+2)z^{n-2}}{2(n+\nu)^2}+\cdots\end{aligned} \quad (5.6)$$

When z is on the $F-$cut between $-2i$ and $2i$, one must consider the two possible square roots in (5.5). This will be discussed after a second method.

Wong & Zhang [92] use a generating function yielding here

$$y_n\left(\frac{2}{(n+\nu)z}\right)=\frac{-4^n\;n!}{\pi i(n+\nu)^n z^n}\oint\exp\left(\frac{(n+\nu)(1-\zeta)z}{2}-(n+1)\log(1-\zeta^2)\right)d\zeta$$

on a contour enclosing $\zeta=1$ but excluding $\zeta=0$. An accurate asymptotic estimate can then be achieved with saddle points analysis when z is not close to $\pm 2i$, as

$$y_n\left(\frac{2}{(n+\nu)z}\right)\sim-\frac{n!}{\pi i}\frac{4^n}{(n+\nu)^n z^n}$$
$$\times[\sqrt{-2\pi/f''(\zeta_+)}\,\exp(f(\zeta_+))+\sqrt{-2\pi/f''(\zeta_-)}\,\exp(f(\zeta_-))],$$

with $f(\zeta)=\dfrac{(n+\nu)(1-\zeta)z}{2}-(n+1)\log(1-\zeta^2)$, the saddlepoints $\zeta_\pm$ are the two roots of $f'(\zeta)=0$, so of $\zeta^2+\dfrac{4(n+1)}{(n+\nu)z}\zeta-1=0$,

$\zeta_\pm=\dfrac{-2(n+1)\pm\sqrt{4(n+1)^2+(n+\nu)^2z^2}}{(n+\nu)z}\sim\dfrac{-2\pm\sqrt{z^2+4}}{z}+2(\nu-1)\dfrac{1\mp 2/\sqrt{z^2+4}}{(n+\nu)z}$

$=\zeta_{\pm,\infty}\left[1\mp\dfrac{2(\nu-1)}{n\sqrt{z^2+4}}\right]$, where $\zeta_{\pm,\infty}=\dfrac{-2\pm\sqrt{z^2+4}}{z}$ are the limits when $n\to\infty$; $f(\zeta_\pm)=f(\zeta_{\pm,\infty})+o(1)$, from $f'(\zeta_\pm)=0$, so

$$\begin{aligned}f(\zeta_\pm)&\sim(n+\nu)(1-\zeta_{\pm,\infty})z/2-(n+1)\log(1-\zeta_{\pm,\infty}^2)\\&=(n+\nu)[(1-\zeta_{\pm,\infty})z/2-\log(4\zeta_{\pm,\infty}/z)]+(\nu-1)\log(4\zeta_{\pm,\infty}/z),\end{aligned}$$

using $1-\zeta_{\pm,\infty}^2=4\zeta_{\pm,\infty}/z$, $f''(\zeta_\pm)=2(n+1)\dfrac{1+\zeta_\pm^2}{(1-\zeta_\pm^2)^2}=\dfrac{(n+\nu)^2z^2(\zeta_\pm^{-1}+\zeta_\pm)}{8(n+1)\zeta_\pm}$

$=\pm\dfrac{(n+\nu)z\sqrt{(n+\nu)^2z^2+4(n+1)^2}}{4(n+1)\zeta_\pm}\sim\pm\dfrac{nz\sqrt{z^2+4}}{4\zeta_{\pm,\infty}}$ follows.

This leaves for the accurate asymptotic behaviour of $P_n(z)=z^ny_n(2/((n+\nu)z))$ at most two terms of the form

$$\begin{aligned}&-\frac{n!}{\pi i}\frac{4^n}{(n+\nu)^n}\sqrt{-2\pi\frac{4\zeta_{\pm,\infty}}{nz\sqrt{z^2+4}}}\exp\{(n+\nu)[(1-\zeta_{\pm,\infty})z/2-\log(4\zeta_{\pm,\infty}/z)]\\&\quad+(\nu-1)\log(4\zeta_{\pm,\infty}/z)\}\sim 2^{2n+1}e^{-n-\nu}(z^2+4)^{-1/4}\exp\{(n+\nu)[(1-\zeta_{\pm,\infty})z/2\\&\quad-\log(4\zeta_{\pm,\infty}/z)]+(\nu-1/2)\log(4\zeta_{\pm,\infty}/z)\}\sim e^{-n-\nu}(1+4/z^2)^{-1/4}z^n\\&\quad\exp\{(n+1/2)[(1-\zeta_{\pm,\infty})z/2-\log(\zeta_{\pm,\infty})]+(\nu-1/2)(1-\zeta_{\pm,\infty})z/2\},\end{aligned}$$

confirming (5.5).

When z is large, we only keep the term with $\zeta_{+,\infty}=\dfrac{\sqrt{z^2+4}-2}{z}=1-2/z+\cdots$. In other regions, the two terms must be considered, as carefully established by Wong & Zhang [92, Thm A]. The neighbourhood of $\pm 2i$ is also considered by the same authors [92, Thm B].

The asymptotic estimate is also $(1+4/z^2)^{-1/4}z^n\exp[-(n+\nu)\mathcal{V}_\pm(z)+(\nu-1/2)\log((-2\pm\sqrt{z^2+4})/z)]$ from the formula for the complex potential $\mathcal{V}$ in (3.7). The potential $\mathcal{V}$ is created by a unit positive charge spread on F joining $-2i$ to $2i$, a negative unit charge concentrated at the origin, and satisfies $\mathcal{V}_+(z)+\mathcal{V}_-(z)=-z+\pi i$. The function $\mathcal{V}_0(z)=\log[(-2+\sqrt{z^2+4})/z]$ is a potential function too, created by the same charges, and satisfying $\mathcal{V}_{0,+}(z)+\mathcal{V}_{0,-}(z)=\pi i$ on the two sides of any arc joining $-2i$ to $2i$ avoiding the origin, therefore a potential without "external field". This will be discussed in a more general setting in (5.8).

5.4.2. *Formal orthogonal polynomials.* We do not expect to be able to use differential equations in general, and turn to the formal orthogonality property of §2.2.

We follow A. Aptekarev [6, §1.3]:

We add to the hypotheses already given the symmetry conditions with respect to the real axis, F made of a single connected arc of endpoints a and b with a positive density of poles in the interior of F, then we get an strong asymptotic of the denominators p_n as orthogonal polynomials on F.

First, we consider the Joukowsky map $z=\dfrac{a+b}{2}+(b-a)\dfrac{\Phi_0(z)+\Phi_0(z)^{-1}}{2}$ defining the algebraic function $\Phi_0(z)=[2z-a-b+2\sqrt{(z-a)(z-b)}]/(b-a)$. As long as we consider symmetric expressions in Φ_0 and Φ_0^{-1}, we must not worry about the sign of the square root, but the current convention is to take $\Phi_0(z) \sim 4z/(b-a)$ for large z, and Φ_0 continuous outside the $F-$cut joining a and b. One often writes $\Phi_0 = e^{i\theta}$, where θ is not limited to real values.

For a useful relation with integrals on arcs, let us defined as above $f(z) = [(z-a)(z-b)]^{-1/2}$ as continuous outside F and such that $\lim zf(z) = 1$ when $z \to \infty$. Then the Cauchy formula gives $f(z)$ outside a contour containing $[a,b]$ as the integral of $[2\pi i(z-t)]^{-1} f(t)dt$ on the contour, which we shrink on the two sides of the cut, so it becomes the integral from a to b of $(2\pi i)^{-1}[f_-(t) - f_+(t)]$, which is here $[(t-a)(b-t)]^{-1/2}/\pi$, the Chebyshev weight:

$$\text{Chebyshev example:} \qquad \frac{1}{\sqrt{(z-a)(z-b)}} = \frac{1}{\pi}\int_a^b \frac{dt}{(z-t)\sqrt{(t-a)(b-t)}},$$
$$\fint_a^b \frac{dt}{(z-t)\sqrt{(t-a)(b-t)}} = 0, \quad z \in (a,b). \tag{5.7}$$

The latter vanishing principal value follows from the opposite values of $f = \Phi_0'/\Phi_0 = \mathcal{V}_0'$ on the two sides of $[a,b]$.

Next, let p_n be a polynomial of degree n orthogonal with respect to a possibly complex-valued weight function w on an arc joining a to b. Actually, we need w to be analytic in a convenient region, so that several arcs joining a to b may be considered as support of w. This also requires formal orthogonality $\langle f,g\rangle = \int_a^b f(t)g(t)w(t)dt$, where no complex conjugate appears. The polynomial $p_n(z)$ is also the denominator of the n,n ***Padé*** approximation to the Stieltjes-type, or Markov-type, function $\int_a^b \frac{w(t)\,dt}{z-t}$ [35, 49–51, 58, 66].

Then, the Bernstein-Szegő estimate, extended to a general arc $[a,b]$, is an accurate asymptotic formula for p_n involving $\Phi_0^{\pm n}$ and factors whose product reconstructs w^{-1}. These ideas will be used in (5.8).

This is already seriously at variance with the classical Markov-Bernstein-Szegő theory associated to positive weight functions on real supports.

The extension to complex weights has a long history, summarized in [6, §2.1.] , see also [55, 58].

Here, p_n is not orthogonal with respect to a weight function independent of n, but with respect to $\dfrac{\rho_n(t)}{(t-z_1^{(n)})\cdots(t-z_{2n+1}^{(n)})} \sim \rho_n(t)\exp(-(2n+1)\mathcal{V}_z(t))$ depending on n, as seen in §2.2. Actually, the interpolation points $z_1^{(n)},\ldots,z_{2n+1}^{(n)}$ will have to be described more accurately than through their limit distribution μ_z on E. Let $(z-z_1^{(n)})\cdots(z-z_{2n+1}^{(n)}) \sim \exp(\mathcal{V}_{z,n}(z))$, with $\mathcal{V}_{z,n}(z) - 2n\mathcal{V}_z(z)$ bounded but left undefined for the moment.

We return to the settings of § 2.4, the best rational approximation to $f_n(z) = (2\pi i)^{-1}\oint \frac{\rho_n(t)dt}{z-t}$, with $\rho_n(t) = \exp(n\Phi(t) + \Psi(t))$. The denominator p_n must be orthogonal to all polynomials of degree $< n$ with respect to $\rho_n(t)\exp(-(2n+1)\mathcal{V}_z(t))$. As in § 2.4, we try

$$p_n(t) = A_+(t)\exp(n(\mathcal{V}_z(t) - \Phi(t)/2 + i\theta_q(t))) + A_-(t)\exp(n(\mathcal{V}_z(t) - \Phi(t)/2 - i\theta_q(t)))$$

(from [6, §2.2, 2.3]), and check the orthogonality of p_n and $p_n(t)/(t-p)$, amounting essentially to the principal value of the integral on an arc (a,b) of $2A_+(t)A_-(t)\exp(\Psi(t) - \mathcal{V}_z(t))/(t-p)$. This latter principal value must vanish for any $p \in F$.

We consider now a new complex potential $\mathcal{V}_\Psi$ reproducing (I), (II), (III), and (V) and (VI) of § 2.4, but with 2Ψ instead of Φ.

This also means that $\mathcal{V}'_{\Psi,+}(z)+\mathcal{V}'_{\Psi,-}(z) = 2\Psi'(z)$, so, each $A_\pm(t)$ will contain a $\exp(-\Psi_\pm(t)/2)$ factor. What remains must be a vanishing principal value, we use (5.7).

In summary, if q_n/p_n is the best rational approximation of degree n to $f_n(z) = A\displaystyle\int_C \exp(n\Phi(t)+\Psi(t))\frac{dt}{z-t}$ on $E=[0,c]$, with Φ and Ψ analytic, then,

$$\begin{aligned} p_n(z) &\sim [(z-a)(z-b)]^{-1/4}\exp[\mathcal{V}_{z,n}(z)/2 - n\mathcal{V}(z) - \mathcal{V}_\Psi(z)/2], \\ f_n(z)p_n(z) - q_n(z) &\sim 2\pi i A C_n [(z-a)(z-b)]^{-1/4}\exp[\mathcal{V}_{z,n}(z)/2 + n\mathcal{V}(z) + \mathcal{V}_\Psi(z)/2] \end{aligned} \tag{5.8}$$

where $\mathcal{V}$ and $\mathcal{V}_\Psi$ are the complex potentials related to a unit negative charge on E, and a unit positive charge on F = an arc joining a to b, with a constant real part on E, and real parts of $-\Phi/2$ and $-\Psi$ on F, and where $C_n = \exp(-n(2\mathcal{V}(a)-\Phi(a)) - \mathcal{V}_\Psi(a)+\Psi(a))$, and where $\mathcal{V}_{z,n} = 2n\mathcal{V}_z + \mathcal{V}_{\Psi,z}$ (see the end of the present section).

On F, we must sum the contributions of the two sides:
$p_n(z) \sim [(z-a)(z-b)]^{-1/4}e^{(n+1/2)\mathcal{V}_z(z)}\{\exp[-n\mathcal{V}_+(z) - \mathcal{V}_{\Psi,+}(z)/2]$
$-\exp[-n\mathcal{V}_-(z) - \mathcal{V}_{\Psi,-}(z)/2]\}$, with $\mathcal{V}_+(z)+\mathcal{V}_-(z) = \Phi(z) +$ constant $= \Phi(z) + 2\mathcal{V}(a) - \Phi(a)$, $\mathcal{V}_{\Psi,+}(z)+\mathcal{V}_{\Psi,-}(z) = 2\Psi(z) +$ constant $= 2\Psi(z) + 2\mathcal{V}_\Psi(a) - 2\Psi(a)$.

The last line of (5.8) follows from (2.3) where we perform the integral of
$p_n^2(t)\exp(n\Phi(t)+\Psi(t)-(2n+1)\mathcal{V}_z(t))/(z-t)$ dominated as above on F by
$2((t-a)(t-b))^{-1/2}\exp(n\Phi(t)+\Psi(t) - n[\mathcal{V}_+(t)+\mathcal{V}_-(t)] - [\mathcal{V}_{\Psi,+}(t)+\mathcal{V}_{\Psi,-}(t)]/2)/(z-t)$
$= \exp(-n(2\mathcal{V}(a)-\Phi(a)) - \mathcal{V}_\Psi(a)+\Psi(a))/[(z-t)\sqrt{(t-a)(t-b)}]$, there is no principal value now, as z in not on F, we apply the first formula of (5.7).

Aptekarev's great paper [6, §2.2, 2.3] contains accurate statements on the meaning of "$\sim$": it may be uniform convergence on compacts in the interior of F, but sometimes in the whole of F, when the endpoints singularities of $[(z-a)(z-b)]^{-1/4}$ and $\mathcal{V}(z)$ cancel each other...

Why this play with 2Ψ and $\mathcal{V}_\Psi/2$? Because all the complex potential functions $\mathcal{V}, \mathcal{V}_\Psi$ considered here are related to unit charges on E and F, $[(z-a)(z-b)]^{-1/4}$ represents a charge 1/2 on F (principle of argument of the logarithm on a contour circling F), which is cancelled by $-\mathcal{V}_\Psi(z)/2$ in (5.8); the $2n+1$ interpolation points on E correspond to a charge $-(2n+1)$ thanks to the $\mathcal{V}_\Psi(z)/2$ term in the second line of (5.8).

Even if $\Psi(t)\equiv 0$, $\mathcal{V}_\Psi(t)\equiv 0$ would be wrong, as the condition on the charges would not be fulfilled. We then need $\mathcal{V}_*(z)$, the potential of (E,F) without external field. When E and F are the two arcs $[0,c]$ and $[a,b]$, we have $\mathcal{V}_*(z) = C\displaystyle\int_\infty^z \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}$, as encountered in Zolotarev problems [4, p.319, appendix E english edition] [12, 25, 30, 84], as in investigations on orthogonal polynomials on two intervals [3], [5, chap. 10]. The constant C is precisely such that E and F carry unit charges, that $\mathcal{V}_*(b)-\mathcal{V}_*(a) = \pi i$, so $C = \dfrac{\pi i}{\displaystyle\int_a^b [t(t-c)(t-a)(t-b)]^{-1/2}dt} =$
$\dfrac{(a-c)(b-c)c}{4(\nu_0+\nu_1/x^*)}$ from (3.3) with $1/x^* = [(a-c)(b-c)+ab]/(abc)$.

With elliptic integrals notations (4.6), $1/x^* = \dfrac{(1+\zeta)^2}{(1-\zeta)^2 c} + \dfrac{1}{c} = 2\dfrac{1+\zeta^2}{(1-\zeta)^2 c}$, and $C = \sqrt{\dfrac{k^2+\zeta^2 k'^2}{1-\zeta^2}}\,\dfrac{\pi(a-c)(b-}{c^2(1+\zeta}$
$= \sqrt{\dfrac{1-\zeta^2}{k^2+\zeta^2k'^2}}\,\dfrac{\pi c}{4\mathbf{K}}$, and (5.2) follows.

We will also need $\mathcal{V}_{*,p}(z) = \displaystyle\int_F \log(z-t)\,d\mu_{*,p}(t) = \int_F \frac{\mathcal{V}'_{*,+}(t)-\mathcal{V}'_{*,-}(t)}{2\pi i}\log(z-t)dt$
$= \displaystyle\int_F \frac{C\,\log(z-t)dt}{\pi i\sqrt{t(t-c)(t-a)(t-b)}} = \log(z) - \frac{\omega}{z} + o(1/z)$, where

$$\omega = \int_a^b \frac{Ctdt}{\pi i\sqrt{t(t-c)(t-a)(t-b)}} = \frac{\displaystyle\int_a^b \frac{tdt}{\sqrt{t(t-c)(t-a)(t-b)}}}{\displaystyle\int_a^b \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}} = \frac{\nu_0 c+\nu_1}{\nu_0+\nu_1/x^*}, \text{ so}$$

$$\mathcal{V}_{*,p}(z) = \log(z) + c(1-\zeta)\frac{\mathbf{K}-(1+\zeta)\mathbf{E}}{2\zeta\mathbf{K}z} + o(1/z), \tag{5.9}$$

from (3.3), (4.6).

When z is not close to F, we keep in the first line of (5.8) $\mathcal{V}(z)$ and $\mathcal{V}_\Psi(z)$ remaining bounded for large z, whereas $\mathcal{V}_z(z)$ (which is regular on F) behaves like $\log z$ for large z, so that the product $[(z-a)(z-b)]^{-1/4}e^{(n+1/2)\mathcal{V}_z(z)}$ behave as z^n, as it should!

Finally, on any $z \in E$, $f_n(z) - q_n(z)/p_n(z) \sim 2\pi i C_n \exp[2n\mathcal{V}(z) + \mathcal{V}_\Psi(z)]$, where the two potential functions have a constant real part, and opposite imaginary parts. On E, we sum the contributions from the two sides:

$f_n(z) - q_n(z)/p_n(z) \sim 2\pi i C_n\{\exp[2n\mathcal{V}_+(z) + \mathcal{V}_{\Psi,+}(z)] + \exp[2n\mathcal{V}_-(z) + \mathcal{V}_{\Psi,-}(z)]\}$.

The common real part yields the strong asymptotics of the error norm

$$\|f_n - q_n/p_n\|_E \sim 4\pi A \exp[n(2V(0) - 2V(a) + \Phi(a)) \ + V_\Psi(0) - V_\Psi(a) + \Psi(a)]. \tag{5.10}$$

For the imaginary parts, remember that $\mathcal{V}$ and $\mathcal{V}_\Psi$ have opposite pure imaginary derivatives $\pm\pi\mu_z'$ and $\pm\pi\mu_{z,\Psi}'$ on the two sides of $E = [0, c]$, so that the error function oscillates like $\cos(2n\pi\mu_z + \pi\mu_{z,\Psi})$ on E. The corrected formula for the measure of the $2n+1$ interpolation points is therefore $\mu_{z,n} = 2n\mu_z + \mu_{z,\Psi}$ instead of $(2n+1)\mu_z$, and the corresponding potential is

$$\mathcal{V}_{z,n} = 2n\mathcal{V}_z + \mathcal{V}_{z,\Psi}. \tag{5.11}$$

5.4.3. *Return to the approximation to the exponential function.* With our problem of rational interpolation to $f_n(z) = \exp(-(n+\nu)z) = -\dfrac{1}{2\pi i}\displaystyle\int_C \exp(n\Phi(t)+\Psi(t))\frac{dt}{z-t}$, $\Phi(t) = -t, \Psi(t) = -\nu t$, the discontinuity of $\mathcal{V}_\Psi$ is the same as for $2\nu\mathcal{V}$, and we recover a unit charge by the combination $\mathcal{V}_\Psi = 2\nu\mathcal{V} + (1-2\nu)\mathcal{V}_*$ [62, eq. (10)]. We have now

$$\begin{aligned} p_n(z) &\sim [(z-a)(z-b)]^{-1/4}\exp[\mathcal{V}_{z,n}(z)/2 - (n+\nu)\mathcal{V}(z) + (\nu-1/2)\mathcal{V}_*(z)], \\ &\sim [(z-a)(z-b)]^{-1/4}\exp((n+\nu)\mathcal{V}_p(z) - (\nu-1/2)\mathcal{V}_{*,p}(z)), \\ p_n(z)\exp(-(n+\nu)z) - q_n(z) &\sim C_n[(z-a)(z-b)]^{-1/4} \\ &\quad \exp[\mathcal{V}_{z,n}(z)/2 + (n+\nu)\mathcal{V}(z) - (\nu-1/2)\mathcal{V}_*(z)], \end{aligned}$$

with $C_n = \exp(-(n+\nu)(2\mathcal{V}(a)+a) + (2\nu-1)\mathcal{V}_*(a))$, and $\mathcal{V}_{z,n} = 2(n+\nu)\mathcal{V}_z - (2\nu-1)\mathcal{V}_{*,z}$ from (5.11) and $\mathcal{V}_{z,\Psi} = 2\nu\mathcal{V}_z + (1-2\nu)\mathcal{V}_{*,z}$.

On F, one must add the contributions from the two sides of F:

$p_n(z) \sim [(z-a)(z-b)]^{-1/4}e^{\mathcal{V}_{z,n}(z)/2}\{\exp[-(n+\nu)\mathcal{V}_+(z)+(\nu-1/2)\mathcal{V}_{*,+}(z)]+\exp[-(n+\nu)\mathcal{V}_-(z)+(\nu-1/2)\mathcal{V}_{*,-}(z)]\}$, $z \in F$.

Error norm.

$\|\exp(-(n+\nu(z))-q_n(z)/p_n(z)\|_E \sim 2\exp[-2(n+\nu)(V(a)+a/2-V(0))-(2\nu-1)(V_*(a)-V_*(0)]$, so, (5.3) follows

$$\|\exp(-(n+\nu(z)) - q_n(z)/p_n(z)\|_E \sim 2\rho^{n+\nu}\rho_*^{1/2-\nu},$$

where $\rho = \exp(-2V(a) - a + 2V(0))$ is the main rate of decrease discussed in (4.8)-(4.5); and $\rho_* = \exp(2(V_*(0) - V_*(a)))$.

From $\mathcal{V}_*(z) = C\int_\infty^z \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}} = \pi i \frac{\int_\infty^z \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}}{\int_a^b \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}}$, as seen above,

$\rho_* = \exp\left(2\pi i \frac{\int_a^0 \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}}{\int_a^b \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}}\right) = \exp\left(-\frac{\pi \mathsf{K}'}{\mathsf{K}}\right)$, K' being related to the integral from 0 to a, and $2\mathsf{K}$ to the integral from a to b.

End of proof!

5.5. **Some numerical checks.**

We now proceed with various numerical checks of the validity of (5.1):

5.5.1. *Real pole position.* We look at the real pole of the approximation when n is odd. From (5.1), poles occur when $-(n+\nu)\mathcal{V}_+(z) + (\nu - 1/2)\mathcal{V}_{*,+}(z)$ and $-(n+\nu)\mathcal{V}_-(z) + (\nu - 1/2)\mathcal{V}_{*,-}(z)$ have the same real part. Let α be the real root of $\mathrm{Re}(\mathcal{V}_+(z) - \mathcal{V}_-(z)) = 0$. We test the $O(n^{-1})$ correction by the Newton step

$$\alpha + \ \mathrm{Re}\ \frac{(\nu - 1/2)[\mathcal{V}_{*,+}(\alpha) - \mathcal{V}_{*,-}(\alpha)]}{(2\mathcal{V}'(\alpha) + 1)n}, \tag{5.12}$$

where we used $\mathcal{V}'_-(z) = -1 - \mathcal{V}'(z)$ if $\mathcal{V}_+$ is taken as $\mathcal{V}$.

1. When $c = 0$, the real zeros of the Padé denominators ${}_1F_1(-n, -2n, z)$ to e^{-z} are $-2, -4.644$, $-7.293, -9.944, -12.594$ for $n = 1, 3, \dots, 9$, soon very close to $(n+1/2)\alpha$, with $\alpha = -1.325\dots$, so the empirical formula for the n^{th} degree approximation real pole to $\exp(-(n+\nu)z)$ is $\frac{(n+1/2)\alpha}{n+\nu} = \alpha - \frac{(\nu - 1/2)\alpha}{n} + o(1/n)$. Does it fit with (5.12)? From (3.7), $2\mathcal{V}'(z) + 1 = \sqrt{z^2+4}/z$ and $\mathcal{V}_*(z) = \int_\infty^z \frac{2dt}{t\sqrt{t^2+4}} = -\log[(\sqrt{z^2+4}+2)/z] = \mathcal{V}(z) + z/2 - \sqrt{z^2+4}/2$, and (5.12) follows, knowing that α is the real root of $\mathcal{V}_+(z) - \mathcal{V}_-(z) = 0$.

2. When $c = 1$, the real pole is very stable when $\nu = 1/2$ and the value $\alpha = -0.9315$ follows. For various values of ν and n, one finds the empirical formula for the real pole $\alpha + \frac{1.2(\nu - 1/2)}{n+\nu}$. One has $\mathcal{V}'(\alpha) = 0.4326$ found by integrating $\mathcal{V}''$ in (3.2) from $-\infty$ to α; $\mathcal{V}_{*,\pm}(\alpha)$ needed in (5.12) is obtained by integrating (5.2) between a and α on the half of F, the value is $\pm 1.1468 + \pi i/2$ (the parameter C in (5.2) is precisely such that the imaginary part is $\pi/2$, here, $C = 2.0019$). One then finds 1.230 in (5.12).

3. When $c = 5$, we find $\alpha = -0.5045$, the empirical formula for the real pole is $\alpha + \frac{0.63(\nu - 1/2)}{n+\nu}$, $2\mathcal{V}'(\alpha) + 1 = 2.295$ and $\mathcal{V}_{*,\pm}(\alpha) = \pm 0.735 + \pi i/2$, ($C = 2.3416$), so $\mathrm{Re}\frac{\mathcal{V}_{*,+}(\alpha) - \mathcal{V}_{*,-}(\alpha)}{(2\mathcal{V}'(\alpha)+1)} = 0.6405$.

5.5.2. *Second coefficient of the monic denominator.* Let $p_n(z) = z^n + \alpha_n z^{n-1} + \cdots$. What is α_n? From the second line of (5.1), one must find the two first terms of the expansions of $\mathcal{V}_p$ and $\mathcal{V}_{*,p}$. It will be seen in (6.9) that $\exp((n+\nu)\mathcal{V}_p(z)) = [z - c(1-\lambda) + O(1/z)]^{n+\nu} = z^{n+\nu}[1 - (n+\nu)c(1-\lambda)/z + \cdots]$, with $\lambda = 4\nu_1^2/(abc^3)$, and from (5.9), $\mathcal{V}_{*,p}(z) = \log(z) - \omega/z + o(1/z)$, with $\omega = -c(1-\zeta)\frac{\mathsf{K} - (1+\zeta)\mathsf{E}}{2\zeta \mathsf{K}}$, so $\exp(-(\nu - 1/2)\mathcal{V}_{*,p}(z)) = z^{1/2-\nu}[1 + (\nu - 1/2)\omega/z + o(1/z)]$. Finally, $[(z-a)(z-b)]^{-1/4} = z^{-1/2}[1 + (a+b)/(4z) + o(1/z)]$, and $p_n(z) = z^n[1 + (-(n+\nu)(1 - \lambda) + (\nu - 1/2)\omega + (a+b)/4)/z + o(1/z)$:

$$\alpha_n = (n+\nu)\beta + (\nu - 1/2)\gamma + \delta, \beta = -c(1-\lambda) = -c\left(1 - 4\frac{\nu_1^2}{abc^3}\right),$$

$$\gamma = \omega = -c(1-\zeta)\frac{\mathbf{K} - (1+\zeta)\mathbf{E}}{2\zeta\mathbf{K}}, \delta = (a+b)/4.$$

1. For instance, when $c = 0$, $\alpha_n = \dfrac{n(n+1)}{n+\nu} = n+1-\nu+O(1/n)$ from (5.6), so $\beta = 1, \gamma = -2, \delta = 0$. We also check $\mathcal{V}_z(z) = \log z$, so, from (3.7), $\mathcal{V}_p(z) = \mathcal{V}_z(z) - \mathcal{V}(z) = z/2 - \sqrt{z^2+4}/2 + \log(\sqrt{z^2+4}+2) = \log z + 1/z + o(1/z)$ for large z, whence $\beta = 1$ again; $\mathcal{V}_*(z) = -\log[(\sqrt{z^2+4}+2)/z]$ as seen above, $\mathcal{V}_{*,p}(z) = \log z - \mathcal{V}_*(z) = \log(\sqrt{z^2+4}+2) = \log z + 2/z + o(1/z)$, confirming $\gamma = -2$. Finally, the limits when $c \to 0$ of $-c(1-\lambda) = 4\nu_1^2/(abc^2)$ and $c\dfrac{\mathbf{K}-\mathbf{E}}{2\zeta\mathbf{K}}$ are 1 and $\lim \dfrac{ck^2 = c^3/16}{2\zeta = 2c^3/64} = 2$, from table 1, as they should be.

2. Numerical approximations of β, γ, δ from actually computed denominators for $c = 1$ are 0.59, -1.51, and 0.19, whereas $\beta = -c(1 - 4\nu_1^2/(abc^3)) = 0.591401$ from (6.8), (6.9), and table 1, $\zeta = 0.0076133, 2\mathbf{K}/\pi = 1.015644, 2\mathbf{E}/\pi = 0.984716 : \gamma = -1.50380, (a+b)/4 = 0.218$ there (table 1);

3.$(\beta, \gamma, \delta) = (0.11, -0.37, 0.50)$ when $c = 5$, instead of $\beta = 0.109136, \zeta = 0.36530, 2\mathbf{K}/\pi = 1.23593, 2\mathbf{E}/\pi = 0.829238 : \gamma = -0.36471, (a+b)/4 = 0.544$ there, a slightly satisfactory match.

4. From the very accurate data of Carpenter & al. [21] for the approximation of e^{-z} when $c = \infty$, $p_{10}(z) = z^{10} + 5.9426z^9 + \cdots, p_{20}(z) = z^{20} + 11.9158z^{19} + \cdots, p_{30}(z) = z^{30} + 17.8898z^{29} + \cdots$, so, after division by n needed by translation to $f_n(z) = e^{-nz}$, $\beta = \gamma = 0, \delta = 0.597$ indeed very close to $(a+b)/4$ of table 1.

Exercise. Show that $\beta = c\left(4\dfrac{\nu_1^2}{abc^3} - 1\right)$ and $\gamma = -c(1-\zeta)\dfrac{\mathbf{K}-(1+\zeta)\mathbf{E}}{2\zeta\mathbf{K}} \to 0$ when $c \to \infty$.

Indeed, $ab = \dfrac{(1-\zeta)^2c^2}{4(k^2+\zeta^2k'^2)}$ and from (4.6), $\dfrac{\nu_1}{c^{3/2}\sqrt{ab}} = \dfrac{\sqrt{c(1-\zeta^2)}}{4\pi\zeta}[\mathbf{K} - (1-\zeta)\mathbf{E}]$, so, $\zeta \to 1$ and $c \to \infty$ in such a way that $c(1-\zeta) \to 2\sqrt{ab} = 2|a|$ at $c = \infty$. Then, $\dfrac{\nu_1}{c^{3/2}\sqrt{ab}} \to \dfrac{\sqrt{4|a|}}{4\pi}\mathbf{K} = 1/2$ from the "1/9" theory $\sqrt{|a|} = \pi/\mathbf{K}$, table 2 of [53] ($\xi_1$ and ξ_2 of [53] are $\sqrt{-a}$ and $\sqrt{-\overline{a}}$), also $|a|\omega = \pi$ [54, eq. (34)]. Finally, use (3.3) as $\dfrac{\nu_1}{c^{3/2}\sqrt{ab}} =$

$$-\frac{\sqrt{ab}}{4\pi i}\int_a^b \sqrt{\frac{t/c-1}{t(t-a)(t-b)}}dt = \frac{\sqrt{ab}}{4\pi}\int_a^b \frac{1 - \dfrac{t}{2c} + O(1/c^2)}{\sqrt{t(t-a)(t-b)}}dt = \frac{\sqrt{ab}}{4\pi}\int_a^b \frac{1 - \dfrac{\sqrt{t^2(1-t/c)}}{2c} + O(1/c^2)}{\sqrt{t(t-a)(t-b)}}dt =$$

$1/2 + O(1/c^2)$, using the limit 1/2 just found, and (3.6) to get rid of the $O(1/c)$. The result for γ is mush easier to get, as we know that $c(1-\zeta)$ remains bounded, we have the limit of $\mathbf{K} - 2\mathbf{E}$ known to vanish at $c = \infty$.

5.5.3. *Error norm.* Check of (5.3-5.4).

For instance, at $c = 5$, $n = 5$, $\nu = 0, 1/2, 1$, the error norms are $[1.528\ 10^{-6}, 1.877\ 10^{-6}, 2.214\ 10^{-6}] = 2\rho^{n+\nu} \times [0.227, 0.982, 4.083]$; at $n = 10$, $[5.248\ 10^{-12}, 6.369\ 10^{-12}, 7.570\ 10^{-12}] = 2\rho^{10+\nu} \times [0.231, 0.989, 4.142]$. Indeed, $\sqrt{\rho_*} = \exp(-\pi\mathbf{K}'/(2\mathbf{K})) = 0.237 = 1/4.226$ (table 3).

When $c = \infty$, it has been remarked in (4.5) that $\rho = \exp(-\pi\mathbf{K}'/\mathbf{K})$ too, so that $\|\exp(-(n+\nu)z) - r_n(z)\|_E \sim 2\rho^{n+1/2}$ then, a conjecture in [52, 53], proved, as a very small by-product, in [6] !

For small c, $k \sim c/4$, $\rho \sim (c^2/256)\exp(2-c/2)$ as seen at the end of § 4.3.3, $\mathsf{K}' \sim \log(4/k)$, $\mathsf{K} \sim \pi/2$, so, $\rho_* \sim k^2/16 \sim c^2/256$, the error norm
$\sim 2(c^2/256)^{n+1/2}\exp((2-c/2)(n+\nu))$.

Now, the Meinardus-Braess estimate of § 1.2, adapted to $\exp(-(n+\nu)x)$ on $[0,c]$, so, $L = (n+\nu)c$, error norm $\sim 2\exp(-(n+\nu)c/2)\left(\dfrac{(n+\nu)ce}{16n+8}\right)^{2n+1}$
$\sim 2(c^2/256)^{n+1/2}\exp[-(n+\nu)c/2 + 2n+1 + (2n+1)\underbrace{\log((n+\nu)/(n+1/2))}_{\sim(\nu-1/2)/n}]$.

6. AAK

6.1. **Rational approximation through Chebyshev expansions.** Consider first the Chebyshev expansion of a function which is here

$$F(t) = \exp(-(n+\nu)c(t+1)/2) = c_0/2 + \sum_1^\infty c_k T_k(t) \tag{6.1}$$

where $x = c(t+1)/2$ sends $t \in [-1,1]$ to $x \in [0,c]$. This does not work when $c = \infty$, $x = \alpha_n(1+t)/(1-t)$, with any $\alpha_n > 0$, is then used [21, p.392]. It is known that the coefficients in (6.1) are $c_k = 2\exp(-(n+\nu)c/2)I_k(-(n+\nu)c/2)$, where I_k is the k^{th} modified Bessel function [68] [75, chap.3 §4].

The recurrence relation satisfied by the c_ks can be found from Bessel functions identities, or also from the differential equation satisfied by F in (6.1), in the form $F(t) = -((n+\nu)c/2)\int F(t)dt +$ constant, using formulas for the integral of Chebyshev polynomials (Fox & Parker [27, §5.7]).

$$c_k + (n+\nu)c\frac{c_{k-1}-c_{k+1}}{4k} = 0, k = 1, 2, \ldots \tag{6.2}$$

Check for small c: $F(t) = 1-(n+\nu)c(T_1(t)+1)/2 + (n+\nu)^2c^2(T_2(t)/2 + 2T_1(t) + 3/2)/8$, so, $c_0 = 2-(n+\nu)c+3(n+\nu)^2c^2/8+\cdots, c_1 = -(n+\nu)c/2+(n+\nu)^2c^2/4+\cdots, c_2 = (n+\nu)^2c^2/8+\cdots$

It is therefore extremely easy and cheap to compute the numerical values of a large sequence of the coefficients. However, stability requires the coefficients to be computed in a particular order Miller's algorithm, introduction §7 of Abr. & Stegun [1][5], also Fox & Parker [27, §5.10]).

One considers now the approximation of $\sum_1^\infty c_k z^k$ by meromorphic functions in $|z| > 1$ with exactly n poles in that region. Such functions can always be written in the form

$$\tilde{r}(z) = \frac{p(z)}{q(z)} = \frac{\displaystyle\sum_{-\infty}^{n} d_k z^k}{\displaystyle\sum_{0}^{n} e_k z^k}$$

where the n zeros of q must have modulus larger than 1. With respect to the supremum norm on the unit circle, the best approximation $\tilde{r}^*$ to $\sum_1^\infty c_k z^k$ in this class is characterized by the property that except in degenerate cases, the error function $\sum_1^\infty c_k z^k - \tilde{r}^*(z)$ describes an exact circle of winding number $2n+1$ centered at the origin, as z describes the unit circle [2, 60, 61, 82, 85]. Let σ_n be the radius of this circle. A consequence is that the error function can then be written

$$\sum_1^\infty c_k z^k - \tilde{r}^*(z) = b(z) = \frac{\sigma_n b_1(z)}{b_2(z)} = \sigma_n \frac{\overline{u_1} + \overline{u_2}z + \cdots}{u_1 z^{-1} + u_2 z^{-2} + \cdots}, \tag{6.3}$$

[5] This book is a masterpiece, and even the introduction of a masterpiece deserves to be read!

where the denominator b_2 of b is holomorphic in $|z| > 1$ and must have exactly n zeros in $1 < |z| < \infty$, which will be precisely the zeros of the denominator q of $\tilde{r}^*$:

$$b_2(z) = u_1 z^{-1} + u_2 z^{-2} + \cdots = (e_n + e_{n-1} z^{-1} + \cdots + e_0 z^{-n}) v(z) = z^{-n} q(z) v(z),$$

where v is still holomorphic (and without zeros in $1 < |z| < \infty$). Therefore, multiplying the two sides of (6.3) by $b_2(z)$,

$$\left(\sum_1^\infty c_k z^k\right) b_2(z) - z^{-n} p(z) v(z) = \sigma_n b_1(z)\ , \qquad \text{or}$$

$$\left(\sum_1^\infty c_k z^k\right)(u_1 z^{-1} + u_2 z^{-2} + \cdots) = \sigma_n(\overline{u_1} + \overline{u_2} z + \cdots) + \text{ negative powers of } z,$$

i.e.

$$\boldsymbol{H} U = \sigma_n \overline{U},$$

where U is the vector $[u_1, u_2, \dots]^T$, and $\boldsymbol{H}$ is the infinite Hankel matrix

$$\boldsymbol{H} = [c_{k+m-1}], k, m = 1, \dots \tag{6.4}$$

Numerically, one considers a large finite section $k, m = 1, 2, \dots, N$.

σ_n is the n^{th} *singular value*, or $s-$number, of $\boldsymbol{H}$ [60, § 4], (since $\overline{\boldsymbol{H}U} = \sigma_n U$, i.e., $\overline{\boldsymbol{H}}\boldsymbol{H}U = \sigma_n^2 U$). The fact that the n^{th} singular value of $\boldsymbol{H}$ ($\sigma_0 \geqslant \sigma_1 \geqslant \dots$) is indeed related to a vector U such that $u_1 z^{-1} + u_2 z^{-2} + \cdots$ has exactly n zeros in $1 < |z| < \infty$ requires a deeper understanding of Hankel matrices theory [2, 60, 61].

If the coefficients c_k happen to be real, then $\sigma_n = |\lambda_n|$, the absolute value of the n^{th} eigenvalue of $\boldsymbol{H}$ (starting at $n = 0$).

The negative powers add to $(u_1 z^{-1} + u_2 z^{-2} + \cdots) p(z)/q(z) = z^{-n} v(z) p(z)$ allowing to retrieve the numerator p.

For instance, with $\nu = 1/2, c = 1$, the coefficients c_k are $[0.51090, -0.40344, 0.21749,$ $-0.08709, 0.02749, -0.00713, 0.00157, -0.00030, 0.00005, \dots]$, and with $n = 5$, $\sigma_5 = 4.2433\ 10^{-10}$, $U = i[0.00014, 0.00307, 0.03144, 0.17926, 0.54605, 0.59020, -0.52349,$ $0.20747, -0.05596, 0.01172, -0.00203, 0.00030, \dots]$. The sixth eigenvalue λ_5 of $\boldsymbol{H}$ happens to be negative.

The last step in CF approximation is to project $\widetilde{R}(x) = (c_0 + \tilde{r}^*(z) + \tilde{r}^*(z^{-1}))/2$ onto a rational function of degree n of $x = (z + z^{-1})/2$. One naturally chooses $Q(x) = q(z)q(z^{-1})$ as the denominator and determines the numerator $P(x)$ such that the Chebyshev expansion of $R_{CF}(x) = P(x)/Q(x)$ and $\widetilde{R}(x)$ agree through the $T_n(x) = (z^n + z^{-n})/2$ term. This operation destroys the exact equioscillation of the error function, but the perturbation is usually very much smaller than σ_n [85].

It may be useful to recover a free parameter in the making of the function F, it is here

$$F(t) = \exp\left(\frac{(n+\nu)\alpha(t+1)}{t - 1 - 2\alpha/c}\right).$$

We now use $F'(t) = \dfrac{2(n+\nu)\alpha(1+\alpha/c)}{(t-1-2\alpha/c)^2} F(t)$ as $(t - 1 - 2\alpha/c)^2 F(t)$ $- \displaystyle\int 2(t - 1 - 2\alpha/c) F(t) dt = 2(n+\nu)\alpha(1+\alpha/c) \int F(t) dt +$ const., so $\dfrac{c_{k-2} + 2c_k + c_{k+2}}{4} - (1 + 2\alpha/c)(c_{k-1} + c_{k+1}) + (1 + 2\alpha/c)^2 c_k - \dfrac{c_{k-2} - c_{k+2}}{2k} + [1 + 2\alpha/c - (n + \nu)\alpha(1+\alpha/c)] \dfrac{c_{k-1} - c_{k+1}}{k} = 0, k = 1, 2, \dots,$ or

$$\frac{k-2}{4k}c_{k-2} - \frac{(k-1)(1+2\alpha/c)-(n+\nu)\alpha(1+\alpha/c)}{k}c_{k-1} + [1/2+(1+2\alpha/c)^2]c_k$$
$$- \frac{(k+1)(1+2\alpha/c)+(n+\nu)\alpha(1+\alpha/c)}{k}c_{k+1} + \frac{k+2}{4k}c_{k+2} = 0, k=1,2,\ldots, c_{-1}=c_1 \quad (6.5)$$

These recurrence relations are numerically solved by a "compact method" [27] producing three-terms relations $\xi_k c_k + \eta_k c_{k+1} + \sigma_k c_{k+2} = \tau_k c_0$ (upper triangular matrix relations), $k = 1, 2, \ldots$, themselves solved for $k = N, N-1, \ldots, 1$ with the approximation $c_{N+1} = c_{N+2} = 0$. The last unknown c_0 follows from $F(-1) = c_0/2 + \sum_1^N (-1)^k c_k = 1$.

The parameter α may be useful if one looks for a faster decrease of the $|c_k|$s with k. If we expect the ratios $\rho_k = c_{k+1}/c_k$ to be slowly variable with k (Poincaré-Perron [64]),
$(\rho+\rho^{-1})^2/4 - (1+2\alpha/c)(\rho+\rho^{-1}) - \alpha(1+\alpha/c)(n/k)(\rho-\rho^{-1}) + (1+2\alpha/c)^2 = 0$ follows from (6.5) for large k and n. We only look at the roots with $|\rho| < 1$. Most effective α should be such that ρ is minimal when k is close to $2N$. Numerical tests suggest that it happens in the case of a double root.

When $\alpha \to \infty$, $(n/k)(\rho^{-1} - \rho)/c + 4/c^2 = 0$ also from (6.2). If k is close to $2N$ and is much larger than nc, then ρ is close to $-nc/(8N)$.

This may also be discussed through a Fourier coefficient formula, or a contour integral:

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \exp\left(\frac{(n+\nu)\alpha(\cos\theta+1)}{\cos\theta - 1 - 2\alpha/c}\right) e^{-ik\theta} d\theta = \frac{1}{\pi i}\oint \exp\left(\frac{(n+\nu)\alpha(u+2+u^{-1})}{u-2-4\alpha/c+u^{-1}}\right)\frac{du}{u^{k+1}}$$

estimated at a saddle point

$$-\frac{4(n+\nu)\alpha(1+\alpha/c)}{(u-2-4\alpha/c+u^{-1})^2} - \frac{k+1}{u} = 0. \quad (6.6)$$

Change of variables: if $i\mu r = \dfrac{u^{1/2}+u^{-1/2}}{u^{1/2}-u^{-1/2}}$, $u + u^{-1} = 2\dfrac{\mu^2 r^2 - 1}{\mu^2 r^2 + 1}$,
$z = -\dfrac{\alpha(u+2+u^{-1})}{u-2-4\alpha/c+u^{-1}} = \dfrac{\alpha\mu^2 r^2}{1+\alpha/c+(\alpha/c)\mu^2 r^2} = \dfrac{cr^2}{1+r^2}$ if $\mu^2 = 1 + c/\alpha$.

6.2. **Image charges.** The change of variable suggested just above $z = \dfrac{cr^2}{1+r^2}$ $\Leftrightarrow r = \pm\sqrt{\dfrac{1}{c/z-1}}$ sending $z \in E = [0, c]$ to $r \in$ the whole real line $\mathbb{R}$ sheds new light on the relevant potential function. Indeed, $V\left(\dfrac{cr^2}{1+r^2}\right)$ is a harmonic function of two real variables which happens to be constant on the real line. This is achieved by any linear combination constant $+\displaystyle\sum_k w_k \log\left|\frac{r-r_k}{r-\overline{r_k}}\right|$, as $|r - r_k| = |r - \overline{r_k}|$ if, and only if, r is real. The potential is therefore created by charges $-w_k$ at the points r_k, and charges w_k at their images $\overline{r_k}$ = complex conjugate of r_k, [18, p. 485], [45, chap. IX].

There may be a relation with § 7.3.2 ('reflected sets') of [10].

Considering only points distributions where $-\overline{r_k}$ and r_k have the same charge, we have a combination of $\log\left|\dfrac{(r-r_k)(r+\overline{r_k})}{(r-\overline{r_k})(r+r_k)}\right|$, or, keeping an analytic form for the complex potential function, $\mathcal{V}\left(\dfrac{cr^2}{1+r^2}\right)$ = constant $+\displaystyle\int_\Gamma \log\left(\frac{r-s}{r+s}\right) d\tilde{\mu}(s)$, where Γ is the set of $r-$ values corresponding to the cut F in, say, the upper half $r-$plane. This complex potential $\mathcal{V}$ must be the same as encountered before in the $z-$plane outside $E = [0, c]$.

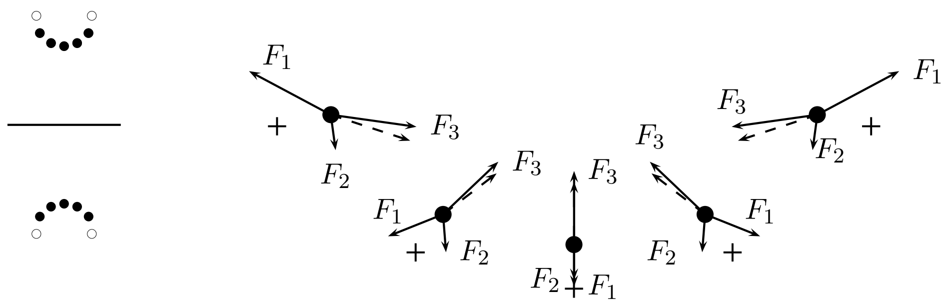

FIGURE 5. images of the poles in the $r = \pm\sqrt{\dfrac{1}{c/z-1}}-$ plane, and the forces acting on them. The dashed vector is $-F_1-F_2$, found to be close to F_3. An actual set of 5 points in equilibrium is shown by + marks.

So, $\dfrac{d\mathcal{V}(z)}{dz}+\dfrac{1}{2}$ taking opposite values on the two sides of F, the same holds for $\dfrac{d\mathcal{V}}{dr}+\dfrac{dz}{2dr}=\dfrac{d\mathcal{V}}{dr}+\dfrac{cr}{(1+r^2)^2}$ on the two sides of Γ. More precisely, as $\dfrac{d\mathcal{V}(z)}{dz}+\dfrac{1}{2}=\pm\pi i\dfrac{d\mu_p(z)}{dz}$ from (3.10), on the two sides of F, where $d\mu_p$ is the continuous limit measure of the poles, $\dfrac{d\mathcal{V}}{dr}+\dfrac{cr}{(1+r^2)^2}=\pm\pi i\dfrac{2cr}{(1+r^2)^2}\dfrac{d\mu_p(z)}{dz}=\pm\pi i\dfrac{d\mu_p\left(\dfrac{cr^2}{1+r^2}\right)}{dr}$ on the two sides of Γ. This means that $\dfrac{d\mathcal{V}\left(\dfrac{cr^2}{1+r^2}\right)}{dr}=\int_\Gamma\left(-\dfrac{1}{r-s}+\text{ reg.}\right)$ $d\mu_p\left(\dfrac{cs^2}{1+s^2}\right)$, where "reg." is regular in Γ. But we just saw that the conditions for $\mathcal{V}(z)$ on E are fulfilled by an integral of $\log((r-s)/(r+s))$ on Γ, so of $-1/(r-s)+1/(r+s)$ for the derivative[6], and

$$\frac{d\mathcal{V}\left(\dfrac{cr^2}{1+r^2}\right)}{dr}=\int_\Gamma\left(-\frac{1}{r-s}+\frac{1}{r+s}\right)d\mu_p\left(\frac{cs^2}{1+s^2}\right)$$

follows. Let $p_k, k=1,\dots,n$ be the poles of the n^{th} degree approximant. Then the integral on Γ is soundly discretized by $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\left(-\dfrac{1}{r-r_k}+\dfrac{1}{r+r_k}\right)$, where r_k is the root in the upper $r-$plane of $p_k=cr_k^2/(1+r_k^2)$.

$c=1, n=5$, poles $r_k=\pm0.695i, \pm0.116\pm0.721i, \pm0.215\pm0.809i$ (endpoints $a,b\mapsto\pm0.244\pm0.962i$)

Check: for large z, we know from (3.2) that $\mathcal{V}'\sim-\nu_0/(2z^2)$, so, near $r=i$, $d\mathcal{V}/dr\sim-\dfrac{\nu_0}{2z^2}\dfrac{dz}{dr}=-\dfrac{\nu_0(1+r^2)^2}{2c^2r^4}\dfrac{2cr}{(1+r^2)^2}\to-i\nu_0/c$. We compare with $\dfrac{2}{n}\displaystyle\sum_1^n\dfrac{r_k}{1+r_k^2}$ in some cases:

[6]For the signs: $-\log(r-s)$ and $-1/(r-s)$ for positive charges; $\log(r-s)$ and $1/(r-s)$ for negative charges.

c	$n=5$	$n=10$	$n=15$	ν_0/c
1	$2.2365\ i$	$2.1530\ i$	$2.1233\ i$	-2.0608
2	$1.1922\ i$	$1.1544\ i$	$1.1409\ i$	-1.1124
5	0.6085 i	$0.5980\ i$	$0.5943\ i$	-0.5863

Interesting confirmation, as the r_ks have been estimated by the Gutknecht & Trefethen's application of the AAK theory [85], and ν_0 has been computed with elliptic integrals in (4.6).

Also, as $\dfrac{d\mathcal{V}}{dr} + \dfrac{cr}{(1+r^2)^2}$ takes opposite values on the two sides of Γ, $\dfrac{d\mathcal{V}}{dr} =$

$$\fint_\Gamma \left(-\frac{1}{r-s} + \frac{1}{r+s}\right) d\mu_p\left(\frac{cs^2}{1+s^2}\right) = -\frac{cr}{(1+r^2)^2}$$

on Γ. Discretization at $r_k \in \Gamma$ is

$$\frac{1}{n}\sum_{\substack{j=1\\j\neq k}}^{n}\left(-\frac{1}{r_k-r_j} + \frac{1}{r_k+r_j}\right) = -\frac{cr_k}{(1+r_k^2)^2}, \qquad k=1,\dots,n,$$

which are equilibrium conditions on Γ, easily seen as the vanishing of the resultant of forces acting on a positive particle at r_k, which is repelled by its neighbours at r_j, $j \neq k$, attracted by the images $-r_j, j=1,\dots,n$, and submitted to the supplementary force $cr_k/(1+r_k^2)$. So, the particle at r_k is submitted to the three forces F_1, F_2 and F_3 which are[7] the complex conjugates of $\dfrac{1}{n}\displaystyle\sum_{\substack{j=1\\j\neq k}}^{n}\dfrac{1}{r_k-r_j}, -\dfrac{1}{n}\displaystyle\sum_{j=1}^{n}\dfrac{1}{r_k+r_j}$ and $-cr_k/(1+r_k^2)$. The right part of fig. 5 shows an example of these forces acting on the points corresponding to a set of poles, and how they are close, but not exactly, to equilibrium. A set of points in equilibrium is also shown nearby. This is an illustration of how the *limit distribution* of poles when $n\to\infty$ is the required equilibrium continuous distribution.

6.3. **Quadratic relations.** We return to the original variable:

$$\frac{d\mathcal{V}(z)}{dz} = \frac{dr}{dz}\int_a^b \frac{-2s}{r^2-s^2}d\mu_p(t), \quad \left(\text{with } r=\sqrt{\frac{1}{c/z-1}}, s=\sqrt{\frac{1}{c/t-1}}\right),$$

$$= \frac{c}{2z^2(c/z-1)^{3/2}}\int_a^b \frac{-2(z-c)(t-c)(c/t-1)^{-1/2}}{c(z-t)}d\mu_p(t), \text{ so,}$$

$$\frac{d\mathcal{V}(z)}{dz} = -\int_a^b \sqrt{\frac{t(t-c)}{z(z-c)}}\,\frac{d\mu_p(t)}{z-t}\,, \tag{6.7}$$

relating the total potential $\mathcal{V}$ to the limit distribution of poles μ_p only, whereas we remind that we have $\mathcal{V}=\mathcal{V}_z-\mathcal{V}_p$, and $\dfrac{d\mathcal{V}_p(z)}{dz} = \displaystyle\int_a^b \frac{d\mu_p(t)}{z-t}$.

We look now at what happens on the two sides of F: from (2.5), $\mathcal{V}'(z) = P_1(z) \pm \pi i\mu_p'(z)$, where P_1 is an appropriate principal value. Actually, $P_1(z)\equiv -1/2$ in our case. Of course, the same pattern holds for $\mathcal{V}_p$ itself: $\mathcal{V}_p'(z) = P_2(z) \mp \pi i\mu_p'(z)$, with $\mp$ instead of $\pm$ because $\mathcal{V}_p' = \mathcal{V}_z' - \mathcal{V}'$. The same discontinuity $\mp\pi i\mu_p'(z)$ appears in $\mathcal{V}'^2(z) = 1/4 \mp \pi i\mu_p'(z) - \pi^2(\mu_p'(z))^2$ on the two sides of F, so that $\mathcal{V}_p'(z) - (\mathcal{V}'(z))^2$ has no more discontinuity on F. Remark also that $\mathcal{V}_p'$ has no singularity on E, and the same is true of $\mathcal{V}'^2$ (as $\mathcal{V}'$ has opposite values on the two sides of E), so, $\mathcal{V}_p' - (\mathcal{V}')^2$ must be a meromorphic function in the whole complex plane. This function will be determines from its derivative.

[7] Remind that force = − gradient of potential = − complex conjugate of derivative of complex potential.

Derivate $\mathcal{V}'_p$ as $\mathcal{V}''_p(z) = -\int_a^b \frac{\mu'_p(t)dt}{(z-t)^2} = \int_a^b \frac{\mu''_p(t)dt}{z-t}$. Now, on F, $\mu''_p(t) = \mathcal{V}''(t)/(\pi i)$, so, from (3.2), $\mathcal{V}''_p(z) = \int_a^b \frac{(\nu_0 t+\nu_1)dt/(\pi i)}{(z-t)\sqrt{t^3(t-c)^3(t-a)(t-b)}}$, multiply by $z^2(z-c)^2 = t^2(t-c)^2 + (z-t)(z^3+z^2t+zt^2+t^3-2z^2c-2ztc-2t^2c+zc^2+tc^2)$:

$z^2(z-c)^2\mathcal{V}''_p(z) = P(z) + \int_a^b (\nu_0 t+\nu_1)\sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\frac{dt}{\pi i(z-t)}$, where P is the 3rd degree polynomial

$P(z) = \int_a^b \frac{(\nu_0 t+\nu_1)(z^3+z^2t+zt^2+t^3-2z^2c-2ztc-2t^2c+zc^2+tc^2)dt/(\pi i)}{\sqrt{t^3(t-c)^3(t-a)(t-b)}}$.

From (3.1), (3.6), $z^2(z-c)^2\mathcal{V}''_p(z) = P(z) + 2(\nu_0 z+\nu_1)\sqrt{\frac{z(z-c)}{(z-a)(z-b)}}\mathcal{V}'(z)$. A new division by $z^2(z-c)^2$ and integration yield at last

$$\mathcal{V}'_p(z) = (\mathcal{V}'(z))^2 + \int^z \frac{P(t)dt}{t^2(t-c)^2}.$$

A very painful derivation, inspired by, but surprisingly not using (6.7).

Let us look at P more closely. Let m_r be the moment $\int_a^b \frac{t^r dt/(\pi i)}{\sqrt{t^3(t-c)^3(t-a)(t-b)}}$. We have $P(z) = (\nu_0 m_1+\nu_1 m_0)z^3 + (\nu_0 m_2+\nu_1 m_1)z^2 + (\nu_0 m_3+\nu_1 m_2)z + \nu_0 m_4+\nu_1 m_3 - 2c(\nu_0 m_1+\nu_1 m_0)z^2 - 2c(\nu_0 m_2+\nu_1 m_1)z - 2c(\nu_0 m_3+\nu_1 m_2) + c^2(\nu_0 m_1+\nu_1 m_0)z + c^2(\nu_0 m_2+\nu_1 m_1)$. The actual degree of P is 2 instead of 3, as the coefficient of z^3 is $\nu_0 m_1+\nu_1 m_0 = \int_a^b \mathcal{V}''(t)dt/(\pi i) = (\mathcal{V}'(b)-\mathcal{V}'(a))/(\pi i) = 0$. The coefficient of z^2 is $\nu_0(m_2-2cm_1)+\nu_1(m_1-2cm_0) = \nu_0 m_2+\nu_1 m_1 = \int_a^b t\mathcal{V}''(t)dt/(\pi i) = (b\mathcal{V}'(b)-a\mathcal{V}'(a))/(\pi i) - \int_a^b \mathcal{V}'(t)dt/(\pi i) = -(b-a)/(2\pi i) + (\mathcal{V}(b)-\mathcal{V}(a))/(\pi i) = -1$ from (3.11), as it should, as $\mathcal{V}'_p(z) = 1/z + O(1/|z|^2)$ for large z. The last coefficients λ_1 and λ_0 of $P(t) = -t^2+\lambda_1 t+\lambda_0$ are the integrals of $(t-c)^2$ and $t(t-c)^2$, so, $\lambda_1 = \nu_0(m_3-2cm_2+c^2m_1)+\nu_1(m_2-2cm_1+c^2m_0)$ and $\lambda_0 = \nu_0(m_4-2cm_3+c^2m_2)+\nu_1(m_3-2cm_2+c^2m_1)$.

For m_3, the best combination is $m_3 - cm_2 = \int_a^b \frac{tdt}{\pi i\sqrt{t(t-c)(t-a)(t-b)}} = \frac{4(\nu_0 c+\nu_1)}{(a-c)(b-c)c}$, from (3.3) at $x=c$.

Next, $m_2 - cm_1$ is the complete integral of first kind $\frac{1}{\pi i}\int_a^b \frac{dt}{\sqrt{(t-a)(t-b)t(t-c)}}$ $= \frac{4(\nu_0+\nu_1/x^*)}{(a-c)(b-c)c}$ from (3.3) at $x = x^* = \frac{abc}{(a-c)(b-c)+ab}$ ensuring $ab(x^*-c) = -(a-c)(b-c)x^*$.

Note the beautiful $m_3 - 2cm_2 + c^2m_1 = \frac{4(1-c/x^*)\nu_1}{(a-c)(b-c)c} = -\frac{4\nu_1}{abc}$.

The even more beautiful $m_4 - 2cm_3 + c^2m_2 = 0$ follows from (3.6)!

$\lambda_1 = \nu_0(m_3-2cm_2+c^2m_1)+\nu_1(m_2-2cm_1+c^2m_0) = \nu_0(m_3-cm_2)$
$+\nu_1(m_2-cm_1)+c = 4\frac{\nu_0(\nu_0 c+\nu_1)+\nu_1(\nu_0+\nu_1/x^*)}{(a-c)(b-c)c}+c$

$$= \frac{4c\nu_0^2+8\nu_0\nu_1+4\frac{(c-a)(c-b)+ab}{abc}\nu_1^2}{(a-c)(b-c)c}+c = \frac{4(\nu_0 c+\nu_1)^2}{(a-c)(b-c)c^2}+4\frac{\nu_1^2}{abc^2}+c = \frac{8\nu_1^2}{abc^2} \text{ from (4.7).}$$

$\lambda_0 = \nu_0(m_4-2cm_3+c^2m_2)+\nu_1(m_3-2cm_2+c^2m_1) = -\frac{4\nu_1^2}{abc}$.

$$\frac{P(t) = -t^2 + \lambda_1 t + \lambda_0}{t^2(t-c)^2} = \frac{\lambda_0}{c^2t^2} + \frac{\lambda_0 + \lambda_1 c - c^2}{c^2(t-c)^2} - \frac{2\lambda_0 + c\lambda_1}{c^2 t(t-c)} = \frac{\lambda_0/c^2}{t^2} + \frac{-\lambda_0/c^2 - 1}{(t-c)^2}.$$

So that

$$\begin{aligned} \mathcal{V}_p'(z) &= (\mathcal{V}'(z))^2 + \frac{\lambda}{z} + \frac{1-\lambda}{z-c}, \\ \mathcal{V}_z'(z) &= (\mathcal{V}'(z))^2 + \mathcal{V}'(z) + \frac{\lambda}{z} + \frac{1-\lambda}{z-c}, \end{aligned} \tag{6.8}$$

with $\lambda = \dfrac{4\nu_1^2}{abc^3}$. For large z,

$$\exp(\mathcal{V}_p(z)) = z - c(1-\lambda) + O(1/z), \exp(\mathcal{V}_z(z)) = z + \nu_0/2 - c(1-\lambda) + O(1/z), \tag{6.9}$$

from $\mathcal{V}'(z) \sim -\nu_0/(2z^2)$ in (3.2).

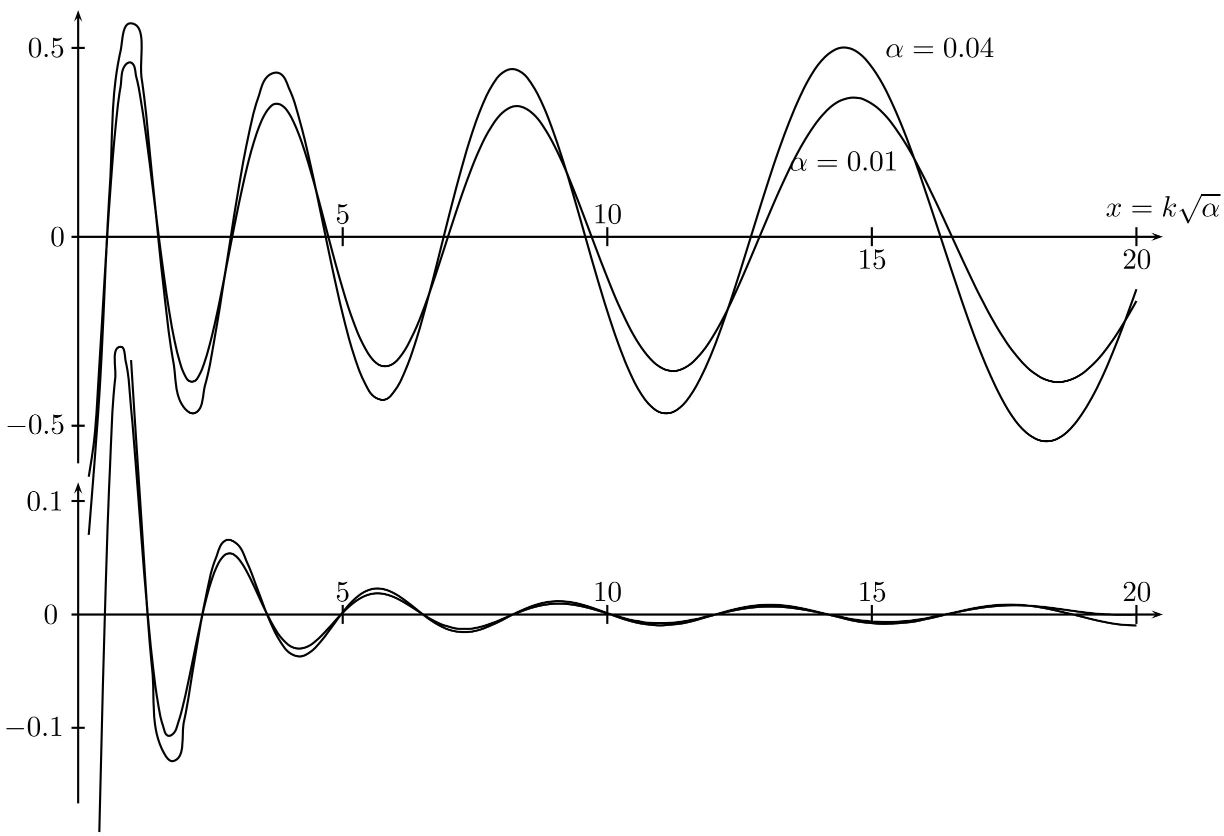

FIGURE 6. Chebyshev coefficients times $k\exp(2x/\sqrt{c})$ as functions of $x = k\sqrt{\alpha}$ for $c = 1$ and $c = 10$, $n + \nu = 10.5$.

6.4. **Integral Hankel operator.** When $\alpha \to 0$, the c_ks vary so slowly that $\boldsymbol{H}$ turns into an integral operator! Indeed, write (6.5) (times $4k$) as

$\Delta^4((k-2)c_{k-2}) - (\alpha/c)\Delta^2((k-1)c_{k-1}) - 4(n+\nu)\alpha(1+\alpha/c)(c_{k+1} - c_{k-1}) + 16(\alpha/c)^2 kc_k = 0.$

Consider $kc_k \sim$ a function $\gamma(k\sqrt{\alpha})$. Then,

$$\sum_{m=0}^{\infty} c_{k+m} f(y_m = m\sqrt{\alpha}) = \sum_{m=0}^{\infty} \underbrace{(k+m)c_{k+m}}_{\gamma(x+y)} \frac{f(y)[\sqrt{\alpha} = \Delta y]}{(k+m)\sqrt{\alpha} = x+y} \sim \int_0^{\infty} \frac{\gamma(x+y)}{x+y} f(y)dy.$$

Also, $(k+j)c_{k+j} \sim \gamma((k+j)\sqrt{\alpha}) = \gamma(k\sqrt{\alpha}) + j\sqrt{\alpha}\gamma'(k\sqrt{\alpha}) + (\alpha j^2/2)\gamma''(k\sqrt{\alpha}) + \cdots$, $\Delta^r kc_k \sim \alpha^{r/2} d^r\gamma(x)/dx^k$ at $x = k\sqrt{\alpha}$.

$$d^4\gamma(x)/dx^4 - \frac{1}{c}d^2\gamma(x)/dx^2 - 8(n+\nu)\frac{\alpha(1+\alpha/c)}{k\alpha^{3/2}=x\alpha}d\gamma(x)/dx + 16\frac{1}{c^2}\gamma(x) = 0.$$

Main behaviour: the Poincaré-Perron's ρ^k behaviour holds for small α with ρ such that $|\rho| < 1$ and $(\rho + 1/\rho)^2 - 4(1 + 2\alpha/c)(\rho + 1/\rho) + 4(1 + 2\alpha/c)^2 = 0$, so $\rho \sim 1 - 2\sqrt{\alpha/c}$, $\rho^k = (\rho^{1/\sqrt{\alpha}})^x \to \exp(-2x/\sqrt{c})$.

When α is small, the saddle-point equation (6.6) is solved by $u \approx 1$, as $u - 2 - 4\alpha/c + u^{-1} \sim \pm iA/\sqrt{k}$, with $A = \sqrt{4(n+\nu)\alpha(1+\alpha/c)}$, so $u \sim 1 + 2\alpha/c \pm iA/(2\sqrt{k}) + \sqrt{4\alpha/c \pm iA/\sqrt{k}} \sim 1 + 2\sqrt{\alpha/c}$ for large k. Then, the main behaviour of $c_k = \dfrac{1}{\pi i}\oint \exp\left(\dfrac{(n+\nu)\alpha(u+2+u^{-1})}{u-2-4\alpha/c+u^{-1}}\right)\dfrac{du}{u^{k+1}}$ is dominated by $\exp(\pm i4(n+\nu)\alpha\sqrt{k}/A)(1+2\sqrt{\alpha/c})^{-k} \sim \exp(\pm i4(n+\nu)\alpha\sqrt{k}/A - 2x/\sqrt{c})$.

Fig. 6 shows that a $\exp(-2x/\sqrt{c})$ envelope is an oversimplification for large c. There must be a quadratic term, as a $\exp(-(x+y)^2)$ kernel has been found when $c = \infty$ [53, § 3, Thm 1.].

7. Divided differences and B-splines

Denominator of rational interpolation can be interpreted as orthogonal polynomial with respect to a scalar product related to the interpolation points $z_1^{(n)}, \dots,$ $z_{2n+1}^{(n)}$, as in § 2.2 for functions defined by a contour integral.

The $z_j^{(n)}$ are unknowns here, their limit distribution μ_z and the related complex potential $\mathcal{V}_z$ were found so to fulfill the Gonchar-Rakhmanov-Stahl conditions of sections 2.3 and 2.4. Later on, a more precise estimate was needed for establishing strong asymptotics formulas, a modified potential called here $\mathcal{V}_{z,n}$ was introduced in (5.8) in § 5.4.2, by a kind of circular argument, assuming $\mathcal{V}_z$ as a first step. See here some loose considerations on the required orthogonality concept.

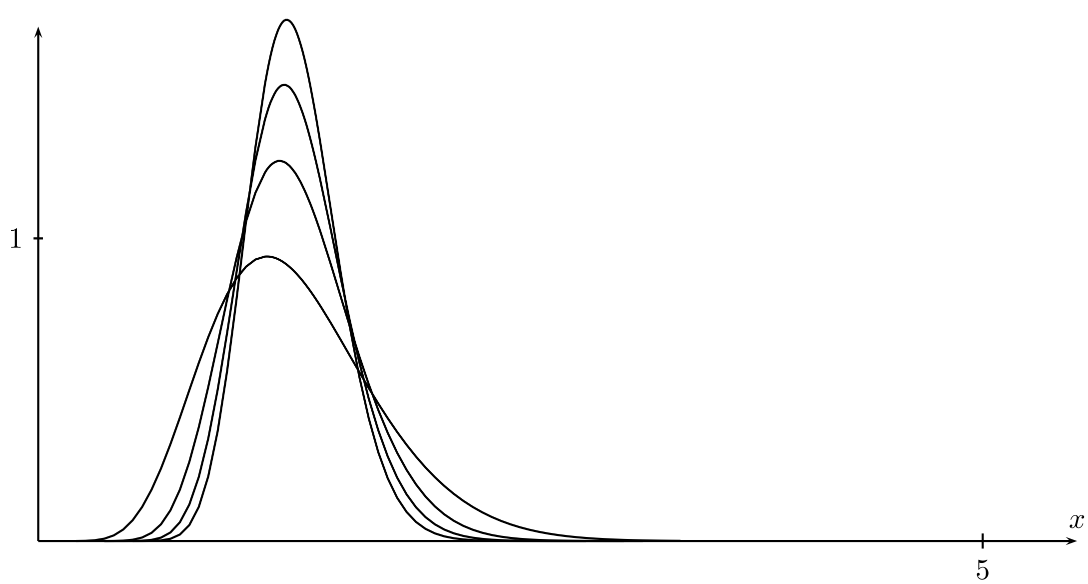

FIGURE 7. B-spline for $c = 5$, $n = 5, 10, 15, 20$.

From elementary polynomial interpolation theory, as the numerator q_n times any polynomial q of degree smaller than n interpolates p_nqf at the $2n+1$ points above, and as the degree of q_nq is still smaller than $2n$, the divided difference $[z_1^{(n)}, \dots, z_{2n+1}^{(n)}]_{p_nqf} = 0$, whence orthogonality of p_n

and q for the formal scalar product

$$\begin{aligned}\langle p,q\rangle_n &= [z_1^{(n)},\dots,z_{2n+1}^{(n)}]_{pqf}\\ &= \sum_{j=1}^{2n+1}\frac{p(z_j^{(n)})q(z_j^{(n)})f(z_j^{(n)})}{\prod\limits_{m\neq j}(z_j^{(n)}-z_m^{(n)})}\\ &= \frac{1}{2\pi i}\int_C \frac{p(t)q(t)f(t)\,dt}{(t-z_1^{(n)})\cdots(t-z_{2n+1}^{(n)})}\end{aligned}\tag{7.1}$$

as seen in (2.2).

Consider now the B-spline formula

$$\langle p,q\rangle_n = \int_{z_1^{(n)}}^{z_{2n+1}^{(n)}}\frac{B(x)}{(2n)!}\frac{d^{2n}}{dx^{2n}}(p(x)q(x)f(x))\,dx,\tag{7.2}$$

where $B(x)$ is actually the Curry-Schoenberg B-spline (deBoor [13, chap. IX], [14])

$$\begin{aligned}B(x) &= 2n[z_1^{(n)},\dots,z_{2n+1}^{(n)}]_{(u-x)_+^{2n-1}}\\ &= M(x;z_1^{(n)},\dots,z_{2n+1}^{(n)})\\ &= 2n\frac{B(x;z_1^{(n)},\dots,z_{2n+1}^{(n)})}{z_{2n+1}^{(n)}-z_1^{(n)}},\end{aligned}$$

where $[z_1^{(n)},\dots,z_{2n+1}^{(n)}]_{(u-x)_+^{2n-1}}$ means the divided difference at $u=z_1^{(n)},\dots,u=z_{2n+1}^{(n)}$ of the function of u whose value is $(u-x)^{2n-1}$ when $u>x$, and 0 when $u<x$.

See in fig. 7 some instances of $B(x)$ on the x_is of best approximants to $\exp(-(n+1/2)x)$, $n=5,10,15,20$.

Problem. Is there a clear limit behaviour for B? When the points $z_j^{(n)}$ are equidistant (*cardinal case*), the (scaled) limit is a Gaussian function [41, 87, 88].

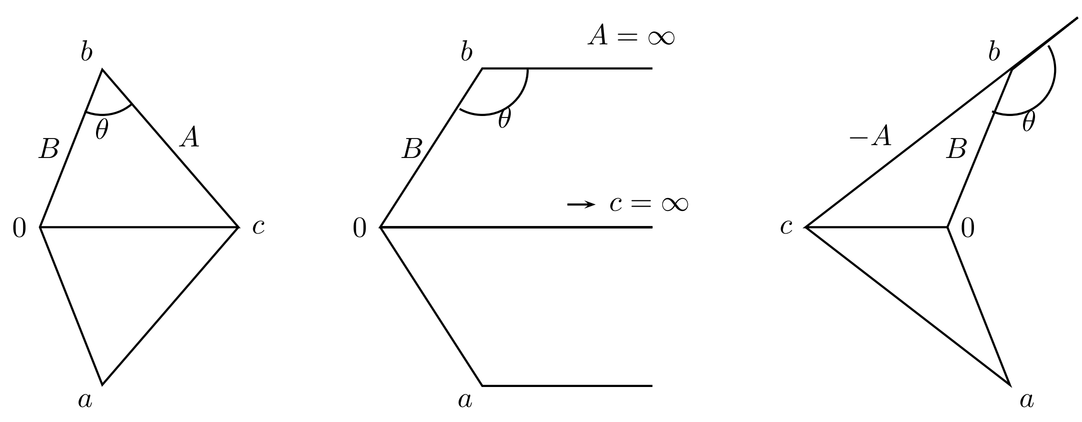

FIGURE 8. $\sin(\theta/2)=k<k_\infty=0.9089...$, $k=k_\infty$, $\sin(\theta/2)=k>k_\infty$.

8. **Beyond infinity? Exploring off-limits modulus.**

The approximation scheme stops at $k=k_\infty=0.9089...$ when $c=\infty$. Of course, we want to know what happens further, and the answer is simply $c<0$:

c	1/rho	a	k	alpha2	nu0	nu1	K	E	Pi	K’	E’
1.0	57.07	0.44-1.97i	0.247	-1043	-2.06	1.27	1.60	1.55	0.05	2.82	1.07
2.5	18.75	0.85-1.84i	0.550	-31.1	-2.33	4.22	1.72	1.44	0.28	2.07	1.24
3.3	15.14	0.97-1.76i	0.657	-11.8	-2.53	6.30	1.80	1.38	0.47	1.92	1.32
5.0	12.43	1.09-1.65i	0.769	-3.85	-2.93	11.1	1.94	1.30	0.81	1.78	1.40
10.0	10.55	1.16-1.51i	0.857	-1.01	-3.98	30.2	2.13	1.22	1.42	1.69	1.46
infty	9.29	1.19-1.39i	0.909	0	infty	infty	2.32	1.16	2.32	1.65	1.50
-10.0	8.41	1.20-1.29i	0.938	0.41	3.79i	27.9i	2.49	1.12	3.48	1.62	1.52
-5.0	7.77	1.20-1.21i	0.956	0.61	2.64i	9.6i	2.64	1.09	4.86	1.61	1.54
-3.33	7.29	1.19-1.15i	0.966	0.72	2.13i	5.08i	2.77	1.08	6.43	1.60	1.54
-2.5	6.91	1.17-1.10i	0.973	0.79	1.83i	3.23i	2.89	1.06	8.18	1.59	1.55
-1.0	5.61	1.10-0.92i	0.990	0.93	1.14i	0.76i	3.36	1.03	21.8	1.58	1.56
-0.5	4.72	1.02-0.79i	0.996	0.97	0.81i	0.26i	3.80	1.01	54.2	1.58	1.57

c	1	2.5	3.3	5	10	infty	-10	-5	-3.3	-2.5	-1	-0.5
nu0/nu1	-1.62	-0.55	-0.40	-0.26	-0.13	0	0.14	0.28	0.42	0.57	1.50	3.17
c^3/nu_1^2	0.62	0.88	0.93	1.01	1.10	1.19	1.28	1.36	1.43	1.50	1.74	1.92

TABLE 3. c, $1/\rho$, a, k, α^2, ν_0, $\nu_1, \mathbf{K}, \mathbf{E}, \Pi, \mathbf{K}'$, and $\mathbf{E}'$, followed by ν_0/ν_1 and c^3/ν_1^2

The parameter A is redefined as $|a-c|\text{sign}(c)$ in order to keep the formula $c^2 = A^2 + B^2 - 2AB\cos\theta$.

The parameters $c, \alpha^2, \zeta, \mathbf{K}, \mathbf{E}$, etc. vary smoothly when k becomes $> k_\infty$. For ν_0 and ν_1, the convenient combinations are ν_0/ν_1 and c^3/ν_1^2, see table 3.

So, we can still build the potential, starting with (3.2).

Curiously, $\mathcal{V}(z)$ is now pure imaginary on the real axis.

It is not even clear to know what the cut F may be. We have a solution without problem.

Problem. Find a scheme involving a modulus larger than k_∞.

9. Algorithms

Elliptic integrals of first and second kind are easily computed with Gauss-Landen transformations, also related to the arithmetic-geometric mean of two numbers
$M(a,b) = \lim_{n\to\infty} a_n$, where $a_0 = a, b_0 = b, a_{n+1} = (a_n + b_n)/2, b_{n+1} = \sqrt{a_n b_n}$,
then, $\mathbf{K} = \pi/(2M(1,k'))$ [1, chap.17] [15, p.14-18], [20, p.39], [46,67].

Indeed, we build an auxiliary sequence $c_n = \sqrt{a_n^2 - b_n^2}$ computed[8] for $n > 0$ as $c_n = c_{n-1}^2/(4a_n)$, then $\dfrac{2d\varphi_n/a_n}{\sqrt{1-(c_n/a_n)^2\sin^2\varphi_n}} = \dfrac{d\varphi_{n+1}/a_{n+1}}{\sqrt{1-(c_{n+1}/a_{n+1})^2\sin^2\varphi_{n+1}}}$ when $\varphi_0 = \varphi$,
$\tan(\varphi_{n+1} - \varphi_n) = (b_n/a_n)\tan\varphi_n$, or $\tan\varphi_{n+1} = \dfrac{2a_{n+1}\tan\varphi_n}{a_n - b_n\tan^2\varphi_n}$, or also
$\sin\varphi_{n+1} = a_{n+1}\sin 2\varphi_n/\sqrt{a_n^2 - c_n^2\sin^2\varphi_n}$. The new variable φ_{n+1} runs from 0 to π when φ_n runs from 0 to $\pi/2$ (complete integral), so that $K(c_n/a_n)/a_n = K(c_{n+1}/a_{n+1})/a_{n+1} = \cdots = K(0)/M = \pi/(2M)$.

For the complete integral of the second kind, $\mathbf{E}/\mathbf{K} = 1 - \sum_0^\infty 2^{n-1}c_n^2$, still with $a = 1, b = k'$. The limits are reached extremely fast (quadratically).

For the complete integral of the third kind (4.3), the algorithm is hardly longer (Bulirsch [17, I, II], Byrd & Friedman [20, §164.02]), we add three sequences d_n, h_n, p_n with $p_0 = \sqrt{1-\alpha^2}, d_0 = 1/p_0, h_0 = 1$, and $p_{n+1} = 4^n a_n b_n/p_n + p_n, h_{n+1} = d_n/p_n + h_n, d_{n+1} = 2(4^n a_n b_n h_n/p_n + d_n)$, then, $\Pi = \lim_{n\to\infty} \pi d_n/(2^{2n+1}a_n^2)$.

These sophisticated algorithms were used for the computation of the tables. The algorithms are extended to incomplete integrals as well [17, III], but simpler (and more robust) algorithms have been preferred when high accuracy is not needed.

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[8] Remark also that $c_{n+1}^2 = (a_n - b_n)^2/4$.

Most graphs (fig. 1, 2, 3, 5, 6, 7) used the incredibly efficient CF algorithm of Trefethen [86] building very fast excellent near-best rational approximations.

10. **Acknowledgements**

Many thanks to S.P. Suetin, the referees, and all the people who spent time on this issue of the *Mat. Sb.*

Just as Chebyshev benefited from lessons of Liouville, the first author had the privilege to receive lessons from A. Aptekarev on 14 Feb. 2001 in Louvain-la-Neuve and 28 May 2001 in Leuven. Spassiba!

Was elliptic functions such a dead subject (before Brent, Salamin, and the Borweins)? Not so in Leuven-Louvain: Georges Lemaître [48], of big bang fame, taught analytical mechanics with elliptic functions examples [38,44] (to often bewildered students), and Vitold Belevitch used them in filtering problems (see papers by J. Todd [84]) in MBLE research lab. The second author [63] was in both lines, so was J.P.Thiran [83]. And now, the subject is found in cryptography (J.J. Quisquater [43]).

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