A PROOF OF FREUD'S CONJECTURE ABOUT THE ORTHOGONAL POLYNOMIALS RELATED TO $|x|^\rho \exp(-x^{2m})$, FOR INTEGER m.

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ABSTRACT

Let $a_n p_n(x) = x p_{n-1}(x) - a_{n-1} p_{n-2}(x)$ be the recurrence relation of the orthonormal polynomials related to the weight function $|x|^\rho \exp(-|x|^\alpha)$, $\rho > -1$, $\alpha > 0$, on the whole real line. Freud's conjecture states that

$$(1) \quad \lim_{n\to\infty} \frac{a_n}{[n/C(\alpha)]^{1/\alpha}} = 1, \quad C(\alpha) = \frac{2\Gamma(\alpha)}{(\Gamma(\alpha/2))^2} = \frac{2^\alpha\,\Gamma((\alpha+1)/2)}{\sqrt{\pi}\,\Gamma(\alpha/2)} .$$

The proof for an even integer $\alpha = 2m$ uses nonlinear equations $F_n(a) = n + \rho\ \mathrm{odd}(n)$, considered by Freud himself. It is shown that $F_n(a^*) - n = o(n)$ when $n \to \infty$, where a_n^* is the expected asymptotically valid estimate $[n/C(\alpha)]^{1/\alpha}$. Bounds on $a_n - a_n^*$ are obtained through the invertibility properties of the matrix $[a_k\, \partial F_n(a)/\partial a_k]$, shown to be symmetric and positive definite. The numerical computation of the solution by Newton's method is considered.

INTRODUCTION

Important studies have been devoted recently to orthogonal polynomials related to weight functions whose support is the whole real line. If $\{p_n\}$ is the sequence of orthonormal polynomials related to w : $\int_{-\infty}^{\infty} p_n(x)p_m(x)w(x)dx = \delta_{n,m}$ $n,m = 0,1,\ldots,$ $w(x) \geq 0$, one tries to link the behaviour of $w(x)$ for large $|x|$, the behaviour of $p_n(x)$ for large n, including the distribution of the zeros $x_{1,n} < x_{2,n} < \ldots < x_{n,n}$ of p_n, and the behaviour for large n of the coefficients a_n and b_n of the recurrence relation

$$(2) \quad a_{n+1}\, p_{n+1}(x) = (x - b_n)p_n(x) - a_n\, p_{n-1}(x) \quad n \geq 0 \quad [a_0\, p_{-1} = 0].$$

Interesting applications occur in statistical physics [1][6] [30].

Here are some general results about the solution of this problem :