The simplest examples are [5]

$m = 1 \quad F_n(a) = 2\, a_n^2$

$m = 2 \quad F_n(a) = 4\, a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2)$ [3], [28] eq. (42)

$m = 3 \quad F_n(a) = 6a_n^2\,(a_{n-2}^2 a_{n-1}^2 + a_{n-1}^4 + 2a_{n-1}^2 a_n^2 + 2a_n^2 a_{n+1}^2 + a_{n+1}^2 a_{n-1}^2 + a_{n+1}^4 + a_{n+1}^2 a_{n+2}^2 + a_n^4)$

for m = 4 and m = 5, the expressions of $F_n(a)$ contain respectively 20 and 48 terms (problem : show that this number is $(m+1)2^{m-2}$).

The production of equations for the recurrence coefficients can obviously be extended to other weight functions, at least to exponentials of polynomials [16]. One can continue up to weight functions satisfying $w'(x)/w(x)$ = rational function. Linear 2^d order differential equations for the orthogonal polynomials can also be constructed in theses cases [1] [7] [8] [9][15] [22] [28] .

Let us propose now an explicit form of (6) :

$$(7) \quad F_n(a) = 2m\, a_n^2 \sum_{i_1=-1}^{m-1} a_{n+i_1}^2 \sum_{i_2=i_1-1}^{m-2} a_{n+i_2}^2 \cdots \sum_{i_{m-1}=i_{m-2}-1}^{1} a_{n+i_{m-1}}^2 .$$

Indeed, by accumulating sums of products in the upper half of powers of the matrix A, one obtains

$$(A^r)_{n,n+r-2p} = a_n a_{n+1} \cdots a_{n+r-2p-1} \sum_{i_1=-1}^{r-p-1} a_{n+i_1}^2 \sum_{i_2=i_1-1}^{r-p-2} a_{n+i_2}^2 \cdots \sum_{i_p=i_{p-1}-1}^{r-2p} a_{n+i_p}^2$$

$$0 \leqslant 2p < r$$

readily checked by induction on p.

PROPERTIES OF THE POSITIVE SOLUTION.

First, we look for bounds : as we are investigating solution(s) with positive a_n^2 of (6), and as $F_n(a) > 2m\, a_n^{2m}$ (take $i_1 = \ldots = i_{m-1} = 0$ in (7)), one has $a_n < [\,(n + \rho\ \mathrm{odd}(n))/2m\,]^{1/2\,m}$. With this upper bound for $a_{n\pm1}, a_{n\pm2}, \ldots$, one solves for a_n in (6). As $F_n(a)$ is a polynomial in a_n with positive coefficients, one obtains a *lower* bound for a_n, also behaving like $c^t\, n^{1/2\,m}$ for large n, so that

$$(8) \qquad (n/C_1)^{1/2\,m} < a_n < (n/C_2)^{1/2\,m},\ n \geqslant 1.$$