Such bounds were already established by other means in 1973 [20] for any $\alpha \geq 1$; the upper bound can be deduced from the Freud-Lubinsky theory ([11] lemma 7.2).

What are the best bounds that can be obtained in this way ? From a lower (resp. upper) bound a'_n behaving like $c'n^{1/2m}$ for large n, used for $a_{n\pm1}, a_{n\pm2}, \ldots$ in (6), one solves for a_n and find an upper (resp. lower) bound behaving like $c''n^{1/2m}$, and the process can be iterated. Concentrating on the relation between c' and c", one finds a polynomial equation ($4c''^2(2c'^2 + c''^2) = 1$ for m = 2, $6c''^2(5c'^4+4c'^2c''^2+c''^4)=1$ for m = 3,...) that can be written $c'' = f(c')$. The iteration will converge if $-1 < f'(c) < 0$ at the fixed point (f is decreasing), which happens for m=1,2 and 3 (the values are 0, -1/2 and -7/8), allowing a proof of the Freud's conjecture in these cases [5]. Unfortunately, this argument breaks down for $m > 3$: the values of f' for m = 4 and 5 are -19/16 and -187/128... (the formula seems to be $\sum_{k=1}^{m-1} (-1)^k \binom{-1/2}{k}$).

Next, one tries an expression a^*_n that we hope to be close to the solution, at least for large n : $F_n(a^*) - n - \rho\ \mathrm{odd}(n) = o(n)$. Returning to (4) as guiding principle, one has, if $a^*_{n+i} \sim a^*_n$, $F_n(a^*) \sim C(2m)(a^*_n)^{2m}$, where C(2m) is the function of (1), i.e., $2m\binom{2m-1}{m}$, found by using (6) and knowing that the elements of a power of a matrix A with equal elements (Toeplitz matrix) are the coefficients of the expansion of the same power of $az^{-1} + az$. With $a^*_n = (n/C(2m))^{1/2m}$, it is easy to show that $F_n(a^*) - n - \rho\ \mathrm{odd}(n) = O(1)$, using $\left(\frac{n+i}{n}\right)^{1/2m} \sim 1 + i/(2mn)$. One can go further, and build an asymptotic series satisfying formally the equations [17] : for our problem, the two first terms are $a_n \sim \left(\frac{n}{C(2m)}\right)^{1/2m} \left[1 + \frac{\rho-(-1)^n(2m-1)\rho}{4mn} \right]$.

Finally, we must show how such an a* is actually close to the true solution a. Indeed, the preceding manipulations show only that F(a*) and F(a) are close together, which is not conclusive. Should F be a linear operator, say F(a) = Xa, the relation would be $a^* - a = X^{-1}(F(a^*) - F(a))$: one should investigate the bounded invertibility