We are interested in sequences satisfying (8), as they are the only candidates for positive solution. Therefore, the sequence $\{\delta_n\}$ is *bounded*. Using the lower bound of (8) and the lemma, with $\delta_1,\ldots,\delta_N$ as the finite sequence u, the first integral is *larger* than $S_N = c^t \sum_{n=1}^{N} n\delta_n^2$. The second integral is a quadratic form in $\delta_{N-m+2},\ldots,\delta_{N+m-1}$, with coefficients bounded by c^t N, and is therefore also bounded by $c^t \sum_{N-m+2}^{N+m-1} n\delta_n^2 = c^t(S_{N+m-1} - S_{N-m+1})$. If a' and a" are both positive solutions of (6), $S_N \leqslant c^t\ (S_{N+m-1} - S_{N-m+1})$, or $S_{N+m-1}/S_{N-m+1} >$ a constant > 1 : a subsequence of $\{S_n\}$ would increase exponentially , which is impossible (the δ_n's being bounded, S_N is bounded by N^2), establishing unicity. With a' the positive solution of (6), and a" the expected asymptotic estimate $\{[n/C(2m)]^{1/2m}\}$, we recall that $F_n(a") - F_n(a') = O(1)$, so that the left-hand side of (9) is bounded by $c^t \sqrt{\log N}\sqrt{S_N}$ (Schwarz inequality), and one has $S_N \leqslant c^t \sqrt{\log N}\sqrt{S_N} + c^t(S_{N+m-1} - S_{N-m+1})$. In order to avoid exponential increase of a subsequence of $\{S_n\}$, one must have $S_N \leqslant c^t \log N$, implying $\delta_N \leqslant c^t(\log N/N)^{1/2} \xrightarrow[N\to\infty]{} 0$, or $a'_N/a''_N \xrightarrow[N\to\infty]{} 1$: Q.E.D.

The form of the Newton's algorithm that takes full advantage of the theorem is :

1) solve $J(a^*)\ \delta^* = F(a^*) - F(a)$ through the Cholesky factorisation of $J(a^*)$

2) the new estimate of a_n is $a_n^* \exp(-\delta_n^*)$, n=1,2,... . One remarks that any positive estimate of the solution produces an new estimate that is still positive.

<u>Proof of the lemma</u>. Symmetry of J(a) : $\frac{\partial A^{2m-1}}{\partial a_k} = \sum_{i=0}^{2m-2} A^i \frac{\partial A}{\partial a_k} A^{2m-2-i}$,

$a_k\ \partial F_n(a)/\partial a_k = 2^m a_k a_n \sum_{i=0}^{2m-2} (A^i)_{n,k+1} (A^{2m-2-i})_{k,n+1} + (A^i)_{n,k} (A^{2m-2-i})_{k+1,n+1}$,

$n \neq k$, using (6), the only two nonzero elements of $\partial A/\partial a_k$, and the symmetry of A itself, establishes the symmetry. Positive definiteness : one uses directly (7)

$$2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} u_n u_k a_k^2\ \partial F_n(a)/\partial a_k^2 = 4m \sum_{n=1}^{\infty} u_n \sum_{i_1,\ldots,i_{m-1}} \sum_{k=1}^{\infty} u_k a_k^2 \frac{\partial}{\partial a_k^2}(a_n^2 a_{n+i_1}^2 \ldots a_{n+i_{m-1}}^2)$$

which means that one keeps a product $a_n^2 \ldots a_{n+i_{m-1}}^2$ times u_k whenever k is one of the numbers $n, n+i_1,\ldots,n+i_{m-1}$, subject to the conditions in (7). Some product $a_{j_1}^2 \ldots a_{j_m}^2$,