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A PROOF OF FREUD'S CONJECTURE ABOUT THE ORTHOGONAL POLYNOMIALS RELATED TO $|x|^\rho \exp(-x^{2m})$, FOR INTEGER m.

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Orthonormal pol. $p_n(x)=\gamma_n x^n+\cdots$, $\int_{-\infty}^{\infty} p_n(x)p_k(x)w(x)\,dx=\delta_{n,k}$ w even: $xp_{n-1}=a_n p_n+a_{n-1}p_{n-2}$ $a_0=0$ $a_n=\gamma_{n-1}/\gamma_n$

$$x\begin{bmatrix}p_0\\p_1\\\vdots\end{bmatrix}=\begin{bmatrix}0&a_1&&\\a_1&0&a_2&\\&a_2&0&\ddots\\&&\ddots&\ddots\end{bmatrix}\begin{bmatrix}p_0\\p_1\\\vdots\end{bmatrix}\quad xp=Ap\quad A \text{ symmetric tridiagonal}\quad x^n p=A^n p\quad \int F(x)p_ip_j w=(F(A))_{i+1,j+1}\ \text{if } F \text{ polynomial}$$

with $w(x)=|x|^\rho e^{-x^{2m}}$

$$\int (p_n p_{n-1})' w = \begin{cases}\int p_n' p_{n-1} w = n/a_n \quad \text{[FREUD 1976]}\\ -\int p_n p_{n-1} w' = \int p_n p_{n-1}\left(2mx^{2m-1}-\frac{\rho}{x}\right)w = 2m\left(A^{2m-1}\right)_{n,n+1} - \frac{\rho}{a_n}\,\mathrm{odd}(n)\end{cases}$$

if m integer, polynomial in $a_{n-m+1},\dots,a_{n+m-1}$

$$\text{from } (A^r)_{n,n+r-2p} = a_n a_{n+1}\cdots a_{n+r-2p-1}\sum_{i_1=-1}^{r-p-1} a^2_{n+i_1}\sum_{i_2=i_1-1}^{r-p-2} a^2_{n+i_2}\cdots\sum_{i_p=i_{p-1}-1}^{r-2p} a^2_{n+i_p}\quad 0\le 2p<r \quad \text{(induction)}$$

Nonlinear eq. (*):
$$\boxed{F_n(a)=a_n\left(A^{2m-1}\right)_{n,n+1}=a_n^2\sum_{i_1=-1}^{m-1}a^2_{n+i_1}\sum_{i_2=i_1-1}^{m-2}a^2_{n+i_2}\cdots\sum_{i_{m-1}=i_{m-2}-1}^{1}a^2_{n+i_{m-1}}=\frac{n+\rho\,\mathrm{odd}(n)}{2m}.}$$

$m=2$: LEW & QUARLES 1983

Proposition: (*) has only one positive solution; $\dfrac{a_n}{(Cn)^{\frac{1}{2m}}}\xrightarrow[n\to\infty]{}1$, with $C=\dfrac{m!\,(m-1)!}{(2m)!}$.

$C_1^{\frac{1}{2m}}\le \lim_{n\to\infty}{}^{\sup}_{\inf}\dfrac{a_n}{n^{1/2m}}\le C_2^{\frac{1}{2m}}$, $0<C_1\le C_2$ already established for any real $m\ge 1/2$ [NEVAI 1973].

$C_1=C_2=C$ for $m=1,2,3$ [FREUD 1976].

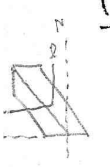
N.B. a_n expected near ξ_n, abscissa of maximum of $|x|^n w(x)$ [LUBINSKY & SHARIF 1983, LUBINSKY], here, $\xi_n=\left(\frac{n+\rho\,\mathrm{odd}(n)}{2m}\right)^{\frac{1}{2m}}$

Lemma 1: the matrix $\left[a_\ell\dfrac{\partial F_n(a)}{\partial a_\ell}\right]_{n,\ell=1,2,\dots}$ is symmetric, positive definite, if $a_n>0$, $n=1,2,\dots$
$$\boxed{\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}u_n u_\ell a_\ell\frac{\partial F_n(a)}{\partial a_\ell}\ge 2m\sum_{n=1}^{\infty}a_n^{2m}u_n^2}$$
for any finite real sequence $\{u_n\}_{n=1,2,\dots}$

Lemma 2: if $\{S_N\},\{\alpha_N\}$ are positive increasing sequences, $\dfrac{\alpha_{N+1}}{\alpha_N}\xrightarrow[N\to\infty]{}1$, $S_N\le\sqrt{\alpha_N S_N}+\frac{1}{\beta}(S_{N+m-1}-S_N)$, $\beta>0$, then, $\lim_{n\to\infty}\sup\dfrac{S_N}{\alpha_N}\le 1$, or S_N grows exponentially.

Proof of proposition, using lemma 1 & 2: let $\{a_n'\}$ be a positive solution (we know there is at least one, with $C_1'n\le a_n'^{2m}\le C_2'n$, $0<C_1'\le C_1\le C_2\le C_2'$), and $\{a_n''\}$ a guess satisfying $F_n(a'')-\frac{n+\rho\,\mathrm{odd}(n)}{2m}=O(n^{1/2-\varepsilon})$ ($(Cn)^{\frac{1}{2m}}$ is O.K.). Consider $\log a_n=\log a_n'+t(\log a_n''-\log a_n')$ $0\le t\le 1 \Rightarrow C_1'n\le a_n^{2m}\le C_2'n$.

Then, $dF_n(a)=\sum_\ell\frac{\partial F_n}{\partial a_\ell}a_\ell\delta_\ell\,dt$, $\delta_\ell=\log a_\ell''-\log a_\ell'$, $\boxed{F_n(a'')-F_n(a')=\int_0^1\sum_{\ell=n-m+1}^{n+m-1}a_\ell\frac{\partial F_n(a)}{\partial a_\ell}\delta_\ell\,dt}$



$$\sum_{n=1}^{N}(F_n(a'')-F_n(a'))\delta_n=\int_0^1\sum_{n=1}^{N}\sum_{\ell=n-m+1}^{\max(n+m-1,N)}a_\ell\frac{\partial F_n}{\partial a_\ell}\delta_\ell\delta_n\,dt+\int_0^1\sum_{n=N-m+2}^{N}\sum_{\ell=N+1}^{n+m-1}a_\ell\frac{\partial F_n}{\partial a_\ell}\delta_\ell\delta_n\,dt$$

Schwarz: $\le\sqrt{\sum_{n=1}^{N}\frac{[F_n(a'')-F_n(a')]^2}{n}}\sqrt{\sum_{n=1}^{N}n\delta_n^2}$ lemma 1: $\ge c^t\sum_{n=1}^{N}n\delta_n^2$ quadratic form in $\delta_{N-m+2},\dots,\delta_{N+m-1}$ $\le c^t\left[(N+1)\delta^2_{N+1}+\cdots+(N+m-1)\delta^2_{N+m-1}\right]$

$[\delta_{N-m+2},\dots,\delta_j]\begin{bmatrix}\delta_{N+1}\\ \vdots\\ \delta_{N+m-1}\end{bmatrix}$

lemma 2 with $S_N=\sum_{n=1}^{N}n\delta_n^2$, $\alpha_N=c^t N^{1-2\varepsilon}$ $\Rightarrow \delta_N\xrightarrow[N\to\infty]{}0$ **Q.E.D.**

Proof of lemma 1:
$$\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}a_\ell\frac{\partial F_n(a)}{\partial a_\ell}u_\ell u_n=2\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}a_\ell^2\frac{\partial F_n}{\partial a_\ell^2}=2\sum_{n=1}^{\infty}u_n\sum_{i_1,\dots,i_{m-1}}\underbrace{\sum_\ell u_\ell a_\ell^2\frac{\partial}{\partial a_\ell^2}\left(a_n^2a^2_{n+i_1}\cdots a^2_{n+i_{m-1}}\right)}_{a_n^2\cdots a^2_{n+i_{m-1}}(u_n+\cdots+u_{n+i_{m-1}})}$$

rearranging in $a^2_{j_1}\cdots a^2_{j_m}$
$$=2\sum_{\{j_1,\dots,j_m\}}a^2_{j_1}\cdots a^2_{j_m}(u_{j_1}+\cdots+u_{j_m})\left(\sum{}^*u_n\right)$$

$\sum^* u_n$ is performed on the values of n such that, for some permutation $\begin{pmatrix}1&\cdots&m\\k_1&\cdots&k_m\end{pmatrix}$, $j_{k_1}=n$, $j_{k_2}=n+i_1,\dots,$ $j_{k_m}=n+i_{m-1}\iff j_{k_{s-1}}-1\le j_{k_s}$, $s=2,\dots,m$, $j_{k_m}-1\le j_{k_1}$ j_{k_2} j_{k_1} j_{k_m}: if $\begin{pmatrix}1&\cdots&m\\k_1&\cdots&k_m\end{pmatrix}$ acceptable, so are the circular permutations $\begin{pmatrix}1&\cdots&m-1&m\\k_2&\cdots&k_m&k_1\end{pmatrix},\dots,$ until the sequence $\{j_{k_1},\dots,j_{k_m}\}$ is repeated, which takes m/p steps if $(j_{k_1},\dots,j_{k_m})$ has p periods: $\sum^* u_n=(u_{j_1}+\cdots+u_{j_m})\sum\limits_{\substack{\text{~~independent~~ non equivalent}\\\text{permutations}}}\frac{1}{p}$

$j_1=\cdots=j_m=n$ gives $m\,u_n^2$.

Zeros: ULLMAN 1980 RAHMANOV 1982: largest zero $\sim 2(Cn)^{1/2m}$, any real $m>1/2$

Asymptotics of $p_n(x)$ $m=2$ NEVAI 1983 ($\leftarrow$ Laguerre differential eq.) $m=3$ SHEEN

conjectures for real $m\ge 1/2$.