A PROOF OF FREUD'S CONJECTURE ABOUT THE ORTHOGONAL POI

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If

(1) $$a_{n+1}p_{n+1}(x)=(x-b_n)p_n(x)-a_np_{n-1}(x)$$

is the recurrence relation of the orthonormal polynomials related to a weight function $w(x)$ on the whole real line , (i.e., $\int_{-\infty}^{\infty} p_n(x)p_m(x)w(x)dx = \delta_{n,m}$) , one expects that

(2) $$\max_{a_n,b_n} \quad \log a_n + \frac{1}{2n\pi}\int_{-1}^{1} (1-x^2)^{-1/2}\log w(b_n+2a_nx)dx$$

gives good asymptotic estimates for a_n and b_n ([Mhaskar & Saff Construct.Approx.] corresponds to approximately equioscillating [equal ripple] $(w(x))^{1/2}p_n(x)$ between b_n-2a_n and b_n+2a_n) can also be derived from the Ansatz

(3) $$\lim a_{n+1}/a_n = 1 \quad \lim b_n/a_n \text{ exists when } n \to \infty .$$

Also , the main asymptotic behaviour of p_n is $\log p_n(z) = \sum_{k=1}^{n} \log \frac{z-b_k+[(z-b_k)^2-4a_k{}^2]^{1/2}}{2a_k} + o(n)$, z nonreal , with consequences on the density of zeros [Nevai & Dehesa , Ullman.]See Nevai ISNM for more <\$100.00>.

One hopes a complete and general proof very soon [Lubinsky , Mhaskar & Saff ,Nevai , Rakhmanov ,Ullman].

For the Freud's weight $|x|^\rho\exp-|x|^\alpha$, one has $\log a_n - \frac{C(\alpha)}{n\alpha}(a_n)^\alpha + O(\rho/n)$ maximized by $[n/C(\alpha)]^{1/\alpha}$, where $C(\alpha)=2\Gamma(\alpha)/[\Gamma(\alpha/2)]^2$.

The precise form of the Freud's conjecture is :

(4) $$\lim \frac{a_n}{[n/C(\alpha)]^{1/\alpha}}=1 \quad \text{when } n \to \infty , \text{ for any } \rho > -1 , \alpha>0 ,$$

General techniques , such as inequalities on Freud-Christoffel functions [Freud ,Nevai ,Lubinsky] , or potential theory [Mhaskar & Saff , Rakhmanov ,Ullman] , do not seem to be able to reach (3) : the inequalities are not sharp enough , or only average behaviour is established (however , Rakhmanov [Math. USSR Sb. 47(1984) 155-193] proved that the largest zero of p_n satisfies $\lim X_n/[n/C(\alpha)]^{1/\alpha} = 2$, $n\to\infty$, if $\alpha > 1$).

Freud [Proc.R.Irish Acad. 76A (1976) 1-6] succeeded in proving (4) for some even integer values of α by the study of explicit nonlinear equations for the a_n's. The construction of these equations , going back to Laguerre (1885 Oeuvres II 685-711) , asks basically that $w'(x)/w(x)$ must be a rational function [Shohat Duke Math. J. 5(1939) 401-417 ; Askey , Hahn , Hendriksen & van Rossum , McCabe with emphasis on the differential equation aspect]. If

values of $C(\alpha)(a_n)^\alpha$

α										
										$\rho=0$
8.0	2.7440	1.9951	2.9972	4.0880	5.0531	6.0360	7.0365	8.0311	9.0274	10.0249
4.0	1.3708	1.9362	3.0616	4.0098	5.0197	6.0128	7.0121	8.0103	9.0093	10.0083
3.0	1.1615	1.9621	3.0441	3.9928	5.0193	5.9995	7.0112	8.0012	9.0077	10.0017
1.0	0.9003	2.0132	2.9587	4.0129	4.9706	6.0123	6.9760	8.0118	8.9792	10.0113
0.5	0.8926	1.9797	2.9704	3.9901	4.9825	5.9934	6.9876	7.9951	8.9904	9.9961
.25	0.9104	1.9577	2.9757	3.9812	4.9854	5.9877	6.9896	7.9908	8.9919	9.9927
										$\rho=0.$
8.0	8.0642	0.9154	5.5749	2.8221	7.3509	4.7160	9.2456	6.6707	11.1860	8.6402
4.0	2.5357	1.5216	4.1539	3.5351	6.0715	5.5324	8.0459	7.5271	10.0342	9.5228
1.0	1.2328	2.2053	3.2617	4.2132	5.2676	6.2166	7.2699	8.2185	9.2710	10.2198
										$\rho=-0.$
8.0	0.3289	5.0695	1.3555	6.0579	3.2854	7.8099	5.2046	9.7270	7.1618	11.6756
4.0	0.4351	2.6068	2.0796	4.5976	4.0409	6.5606	6.0316	8.5421	8.0260	10.5321
1.0	0.5513	1.8006	2.6632	3.7998	4.6807	5.7979	6.6887	7.7963	8.6935	9.7949
0.5	0.5969	1.7203	2.7199	3.7374	4.7328	5.7420	6.7379	7.7441	8.7407	9.7454