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$w'(x)/w(x)=Q(x)/P(x)$, consider $I_{n,k}=\int(p_n(x)p_{n-k}(x))'P(x)w(x)dx$ and replace the derivative in terms of the orthogonal polynomials themselves ( $(p_n)'=\frac{n}{a_n}p_{n-1}+\frac{b_0+\ldots b_{n-2}-(n-1)b_{n-1}}{a_n a_{n-1}}p_{n-2}+\ldots$ ); on the other hand , integrate by parts , using $(Pw)'=(Q+P')w$ . This produces the equations , knowing that , if f is a polynomial , $\int p_n(x)p_m(x)f(x)w(x)dx$ is the (n,m)-entry of f(A) , where A is the Jacobi matrix $\begin{bmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \end{bmatrix}$

For $w(x)=|x|^{\rho}\exp{-x^{2m}}$ , the equations are ($b_n=0$)

(5) $$F_n(a)=2ma_n(A^{2m-1})_{n-1,n}=n+\rho\,\mathrm{odd}(n) \quad ,n=1,2,\ldots$$

Examples : m=1 $F_n(a)=2a_n^2$

m=2 $F_n(a)=4a_n^2(a_{n-1}^2+a_n^2+a_{n+1}^2)$

m=3 $F_n(a)=6a_n^2(a_{n-2}^2a_{n-1}^2 \quad +a_{n-1}^4+2a_{n-1}^2a_n^2 \quad +a_n^4+2a_n^2a_{n+1}^2+a_{n-1}^2a_{n+1}^2+a_{n+1}^4 +a_{n+1}^2a_{n+2}^2)$.

One sees that (4) is clearly a consequence of (3) which must still be proved . Freud used fixed-point iteration by solving (5) for $a_n$ [gives $a_n$ as a function of $a_{n\pm1}$ ,$a_{n\pm2}$ ,...] , but the method diverges when m>3 . The proof presented here is motivated by the Newton-Raphson iteration [for m=2 see Lew & Quarles , J.Approx.Th. 38(1983)357-379] Considering $F_n(a)$ as a function of the logarithms of the $a_k$'s , one finds that the Jacobian matrix (Fréchet derivative) $J(a)=[\partial F_n(a)/\partial \log a_k]$ is symmetric and positive definite $\sum_{nk}\sum u_n u_k \partial F_n(a)/\partial \log a_k \geq 4m^2\sum_n a_n^{2m} u_n^2$ . Now ,if $a'_n$ is a positive solution of (5) and if $a''_n=[n/C(2m)]^{1/2m}$ , so that $F_n(a'')-F_n(a')=o(n)$ , one can estimate how a' and a" are close together by

$$F_n(a'')-F_n(a')=\int_0^1\sum_k(\partial F_n(a)/\partial \log a_k)u_k\,dt \quad (u_k=\log a''_k-\log a'_k \quad ; \quad \log a_k=\log a_k'+tu_k).$$

$$\sum_n(F_n(a'')-F_n(a'))u_n = \int_0^1\sum_{nk}\sum u_n u_k \partial F_n(a)/\partial \log a_k\,dt \geq 4m^2\sum_n u_n^2\int_0^1 a_n^{2m}\,dt$$

resulting (after some technicalities) in $u_n \xrightarrow[n\to\infty]{} 0$ and establishing common asymptotic behaviour of $a'_n$ and $a''_n$ : Q.E.D.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 11.0226 | 12.0207 | 13.0192 | 14.0178 | 15.0166 | 16.0156 | 17.0147 | 18.0139 | 19.0131 | 20.0125 |
| 3 | 11.0076 | 12.0069 | 13.0064 | 14.0059 | 15.0055 | 16.0052 | 17.0049 | 18.0046 | 19.0044 | 20.0042 |
| 7 | 11.0058 | 12.0018 | 13.0046 | 14.0018 | 15.0038 | 16.0017 | 17.0033 | 18.0016 | 19.0028 | 20.0015 |
| 3 | 10.9813 | 12.0109 | 12.9828 | 14.0106 | 14.9840 | 16.0103 | 16.9849 | 18.0101 | 18.9856 | 20.0098 |
| 1 | 10.9921 | 11.9967 | 12.9934 | 13.9972 | 14.9942 | 15.9976 | 16.9949 | 17.9978 | 18.9955 | 19.9980 |
| 7 | 10.9934 | 11.9939 | 12.9944 | 13.9948 | 14.9951 | 15.9954 | 16.9957 | 17.9959 | 18.9962 | 19.9964 |
| .5 | | | | | | | | | | |
| 2 | 13.1503 | 10.6189 | 15.1261 | 12.6032 | 17.1086 | 14.5912 | 19.0953 | 16.5817 | 21.0850 | 18.5739 |
| 8 | 12.0273 | 11.5196 | 14.0227 | 13.5172 | 16.0194 | 15.5153 | 18.0170 | 17.5137 | 20.0151 | 19.5125 |
| 8 | 11.2716 | 12.2208 | 13.2720 | 14.2215 | 15.2723 | 16.2221 | 17.2724 | 18.2226 | 19.2725 | 20.2230 |
| .5 | | | | | | | | | | |
| 6 | 9.1342 | 13.6434 | 11.1145 | 15.6212 | 13.0999 | 17.6049 | 15.0886 | 19.5925 | 17.0796 | 21.5827 |
| 1 | 10.0219 | 12.5260 | 12.0189 | 14.5218 | 14.0166 | 16.5188 | 16.0148 | 18.5165 | 18.0134 | 20.5147 |
| 9 | 10.6968 | 11.7939 | 12.6992 | 13.7929 | 14.7011 | 15.7921 | 16.7026 | 17.7915 | 18.7039 | 19.7908 |
| 4 | 10.7424 | 11.7462 | 12.7436 | 13.7468 | 14.7445 | 15.7472 | 16.7451 | 17.7475 | 18.7457 | 19.7478 |