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The Fourth Order Difference Equation for the Laguerre-Hahn Polynomials Orthogonal on Special Non-uniform Lattices

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Abstract. We firstly establish the fourth order difference equation satisfied by the Laguerre-Hahn polynomials orthogonal on special non-uniform lattices in general case, secondly give it explicitly for the cases of polynomials r-associated to the classical polynomials orthogonal on linear, q-linear and q-nonlinear (Askey-Wilson) lattices, and thirdly give it "semi-explicitly" for the class one Laguerre-Hahn polynomials orthogonal on linear lattice.

Key words: Laguerre-Hahn orthogonal on special non-uniform lattices polynomials, r-associated polynomials

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1. Introduction

Laguerre-Hahn orthogonal polynomials are generally presented as orthogonal polynomials for which the corresponding Stieltes function satisfies some Riccati equation. The polynomials r-associated to classical classes are then perceived as their simplest non-trivial (to mean here, non-semi-classical) realizations. Laguerre-Hahn orthogonal polynomials are on the other side generally expected to satisfy a fourth order differential (difference) equation.

Continuous Laguerre-Hahn orthogonal polynomials were introduced in [7]. The corresponding fourth order differential equation has been established in [11]. The approach adopted there has been extended as well to the discrete and q-Laguerre-Hahn orthogonal polynomials in [4, 5] respectively. In [11] as in [4, 5], the equations were written explicitly for the cases of polynomials r-associated to the corresponding classical situations, that is Jacobi polynomials and specializations in [11], Hahn, big q-Jacobi polynomials and specializations in [4, 5] respectively.

Laguerre-Hahn orthogonal on special non-uniform lattices (snul) polynomials were introduced in [8]. Essential difference-recurrence relations were also established in [8]. In this work, starting at those difference-recurrence relations, we establish the corresponding fourth order difference equation. This will be done in the third section. In the last section,

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we firstly give explicitly that equation for the cases of polynomials r-associated to all classical classes that is, classical polynomials orthogonal on linear lattices (result equivalent for example to that from [4]), classical polynomials orthogonal on q-linear lattices (result equivalent for example to that from [5]) and Askey-Wilson class, and secondly give it "semi-explicitly" (i.e. up to an explicit system of non-linear difference equations satisfied by the coefficients in the three-term recurrence relations) for the class one Laguerre-Hahn polynomials orthogonal on linear lattice. In the following section, we recall the concepts needed here.

2. The Laguerre-Hahn orthogonal on snul polynomials

Searching for two functions $\eta_2(x)$ and $\eta_1(x)$ such that the difference operator

$$(\mathcal{D}f)(x) = \frac{f(\eta_2(x)) - f(\eta_1(x))}{\eta_2(x) - \eta_1(x)} \tag{1}$$

leaves a polynomial of degree $n-1$ when applied to a polynomial of degree n, one finds that $\eta_2(x)$ and $\eta_1(x)$ must be two roots y of some quadratic equation [8, 10]:

$$F(x, y) := c_0y^2 + 2c_1xy + c_2x^2 + 2c_3y + 2c_4x + c_5 = 0.$$

Searching next for a parametrization $x(s)$, $y(s)$ such that $\eta_2(x(s)) = y(s+1)$, $\eta_1(x(s)) = y(s)$, one is led to [8, 10]:

$$\begin{aligned} x(s) &= \tilde{c}_1q^s + \tilde{c}_2q^{-s} + \tilde{c}_3, \\ y(s) &= \tilde{c}_4x\left(s - \frac{1}{2}\right) + \tilde{c}_5, \end{aligned} \tag{2}$$

the so-called special non-uniform lattice (snul).

Let $P_n(y(s))$ be a sequence of orthogonal on snul lattice polynomials with the orthogonality measure $d\varepsilon$, and

$$S(y) := \int_{Supp.\varepsilon} \frac{d\varepsilon(\tau)}{y - \tau},$$

the corresponding Stieltjes function. The polynomials $P_n(y(s))$ are called (class ι) Laguerre-Hahn orthogonal on snul polynomials (LHP) iff the Stieltjes function $S(y(s))$ satisfies the Riccati equation [8]:

$$\begin{aligned} &\mathcal{A}(x(s))\frac{S(y(s+1)) - S(y(s))}{y(s+1) - y(s)} \\ &\quad = \mathcal{B}(x(s))S(y(s+1))S(y(s)) + \mathcal{C}(x(s))\frac{S(y(s+1)) + S(y(s))}{2} + \mathcal{D}(x(s)) \end{aligned} \tag{3}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are polynomials of degrees $\leq \iota + 2$, $\iota + 2$, $\iota + 1$ and ι, respectively. Nowadays, most of the known orthogonal polynomials belong to this class. The subclass of semi-classical orthogonal polynomials [8, 10], corresponds to the case $\mathcal{B} = 0$. The classical polynomials appear then as the semi-classical of class $\imath = 0$.

Next, let $P^{(m)}_{n-m}(y(s))$, $m \in Z^+$ be the m-associated polynomials of $P_n(y(s))$, i.e.

$$\begin{aligned} a_{n+1}P^{(m)}_{n+1-m} + (b_n - y(s))P^{(m)}_{n-m} + a_n P^{(m)}_{n-1-m} = 0, \\ a_{n+1}P_{n+1} + (b_n - y(s))P_n + a_n P_{n-1} = 0 \end{aligned} \tag{4}$$

$(n, m = 0, 1, 2, \ldots)$ and let $S_m(y(s))$ be the corresponding Stieltjes functions. It can be proved that (see [8]) $S_m(y(s))$ also satisfies a Riccati equation similar to (3):

$$\begin{aligned} \mathcal{A}_m(x(s))\frac{S_m(y(s+1)) - S_m(y(s))}{y(s+1) - y(s)} &= \mathcal{B}_m(x(s))S_m(y(s+1))S_m(y(s)) \\ + \mathcal{C}_m(x(s))\frac{S_m(y(s+1)) + S_m(y(s))}{2} &+ \mathcal{D}_m(x(s)), \quad m = 0, 1, 2, \ldots \end{aligned} \tag{5}$$

where $\mathcal{A}_m, \mathcal{B}_m, \mathcal{C}_m$ and $\mathcal{D}_m(\mathcal{A}_0 = \mathcal{A},\ \mathcal{B}_0 = \mathcal{B},\ \mathcal{C}_0 = \mathcal{C},\ \mathcal{D}_0 = \mathcal{D})$ are as well polynomials of degree $\leq \imath + 2, \imath + 2, \imath + 1$ and $\imath$, respectively. In other words, the class of LHP is invariant under the operation of passage to associated polynomials (see another approach in [12]).

Let us write (5) in the homographic form:

$$S_m(y(s+1)) = \frac{\left(\frac{\mathcal{A}_m(x(s))}{y(s+1)-y(s)} + \frac{\mathcal{C}_m(x(s))}{2}\right)S_m(y(s)) + \mathcal{D}_m(x(s))}{\frac{\mathcal{A}_m(x(s))}{y(s+1)-y(s)} - \frac{\mathcal{C}_m(x(s))}{2} - \mathcal{B}_m(x(s))S_m(y(s))}. \tag{6}$$

The coefficients of this transformation iterate as follows [8]:

$$\begin{aligned} &\frac{\mathcal{A}_{m+1}(x(s))}{y(s+1) - y(s)} + \frac{\mathcal{C}_{m+1}(x(s))}{2} \\ &\quad = \frac{\mathcal{A}_m(x(s))}{y(s+1) - y(s)} - \frac{\mathcal{C}_m(x(s))}{2} - (y(s+1) - b_m)\mathcal{D}_m(x(s)); \\ &\frac{\mathcal{A}_{m+1}(x(s))}{y(s+1) - y(s)} - \frac{\mathcal{C}_{m+1}(x(s))}{2} \\ &\quad = \frac{\mathcal{A}_m(x(s))}{y(s+1) - y(s)} + \frac{\mathcal{C}_m(x(s))}{2} + (y(s) - b_m)\mathcal{D}_m(x(s)); \\ &\left(\frac{\mathcal{A}^2_{m+1}(x(s))}{(y(s+1) - y(s))^2} - \frac{\mathcal{C}^2_{m+1}(x(s))}{4}\right) - \left(\frac{\mathcal{A}^2_m(x(s))}{(y(s+1) - y(s))^2} - \frac{\mathcal{C}^2_m(x(s))}{4}\right) \\ &\quad = a^2_m \mathcal{D}_m \mathcal{D}_{m-1} - a^2_{m+1}\mathcal{D}_{m+1}\mathcal{D}_m \end{aligned} \tag{7}$$

with

$$\mathcal{B}_m(x(s)) := a^2_m \mathcal{D}_{m-1}(x(s)), \quad m := 0, 1, 2, \ldots. \tag{8}$$

Consider next the following notations (letting from now on $m \equiv n$):

$$\alpha_n(x(s)) = -\left(\frac{\mathcal{A}_n(x(s))}{y(s+1)-y(s)} - \frac{\mathcal{C}_n(x(s))}{2}\right);$$
$$\beta_n(x) = \frac{\mathcal{A}_n(x(s))}{y(s+1)-y(s)} + \frac{\mathcal{C}_n(x(s))}{2}; \ \gamma_n(x(s)) = \frac{\mathcal{B}_n(x(s))}{\mu_0};$$
$$\delta_n(x(s)) = \mu_0 \mathcal{D}_n(x(s)), \tag{9}$$

μ_n being the moments of $d\varepsilon$.

Using the relations (7), (8) and (4), it has been established in [8] that the LHP satisfy the following difference-recurrence relations:

$$\beta_n(x(s))P_n(y(s)) + a_n\mathcal{D}_n(x(s))P_{n-1}(y(s))$$
$$= \beta_0(x(s))P_n(y(s+1)) + \gamma_0(x(s))P^{(1)}_{n-1}(y(s+1)), \tag{10}$$
$$\alpha_n(x(s))P_n(y(s+1)) + a_n\mathcal{D}_n(x(s))P_{n-1}(y(s+1))$$
$$= \alpha_0(x(s))P_n(y(s)) + \gamma_0(x(s))P^{(1)}_{n-1}(y(s)), \tag{11}$$
$$\beta_{n+1}(x(s))P^{(1)}_{n-1}(y(s+1)) + a_{n+1}\mathcal{D}_n(x(s))P^{(1)}_n(y(s+1))$$
$$= \beta_0(x(s))P^{(1)}_{n-1}(y(s)) + \delta_0(x(s))P_n(y(s)), \tag{12}$$
$$\alpha_{n+1}(x(s))P^{(1)}_{n-1}(y(s)) + a_{n+1}\mathcal{D}_n(x(s))P^{(1)}_n(y(s))$$
$$= \alpha_0(x(s))P^{(1)}_{n-1}(y(s+1)) + \delta_0(x(s))P_n(y(s+1)). \tag{13}$$

Remark 2.1. Considering the relations in (7) and (8) for $m := r, r+1, \ldots$, one finds that the difference-recurrence relations for the polynomials $P^{(r)}_{n-r}$, r-associated to P_n, are found from the preceding ones by shifting the initial value for n, from $n := 0$ to $n := r$. One then obtains in place of (10)–(13):

$$\beta_n(x(s))P^{(r)}_{n-r}(y(s)) + a_n\mathcal{D}_n(x(s))P^{(r)}_{n-1-r}(y(s))$$
$$= \beta_r(x(s))P^{(r)}_{n-r}(y(s+1)) + \gamma_r(x(s))P^{(1+r)}_{n-1-r}(y(s+1)), \tag{14}$$
$$\alpha_n(x(s))P^{(r)}_{n-r}(y(s+1)) + a_n\mathcal{D}_n(x(s))P^{(r)}_{n-1-r}(y(s+1))$$
$$= \alpha_r(x(s))P^{(r)}_{n-r}(y(s)) + \gamma_r(x(s))P^{(1+r)}_{n-1-r}(y(s)), \tag{15}$$
$$\beta_{n+1}(x(s))P^{(1+r)}_{n-1-r}(y(s+1)) + a_{n+1}\mathcal{D}_n(x(s))P^{(1+r)}_{n-r}(y(s+1))$$
$$= \beta_r(x(s))P^{(1+r)}_{n-1-r}(y(s)) + \delta_r(x(s))P^{(r)}_{n-r}(y(s)), \tag{16}$$
$$\alpha_{n+1}(x(s))P^{(1+r)}_{n-1-r}(y(s)) + a_{n+1}\mathcal{D}_n(x(s))P^{(1+r)}_{n-r}(y(s))$$
$$= \alpha_r(x(s))P^{(1+r)}_{n-1-r}(y(s+1)) + \delta_r(x(s))P^{(r)}_{n-r}(y(s+1)). \tag{17}$$

Remark 2.2. In the semi-classical situation, $\gamma_0(x) = 0$ and it was proven in [10] that the class of semi-classical on snul lattices polynomials is fully characterized by Eqs. (10) and

(11). A similar question remains open for the LHP. Morever, in the semi-classical case, the second order difference equation is obtained directly by a combination of Eqs. (10) and (11). The result reads [8]:

$$\begin{aligned}&\mathcal{D}_n(x(s-1))\beta_0(x(s))P_n(y(s+1))\\&\quad-[\mathcal{D}_n(x(s-1))\beta_n(x(s))-\mathcal{D}_n(x(s))\alpha_n(x(s-1))]P_n(y(s))\\&\quad-\mathcal{D}_n(x(s))\alpha_0(x(s-1))P_n(y(s-1))=0.\end{aligned}\tag{18}$$

In the classical case, $\mathcal{D}_n(x(s))$ is a constant (in x) and (18) becomes:

$$\begin{aligned}&\beta_0(x(s))P_n(y(s+1))-[\beta_n(x(s))-\alpha_n(x(s-1))]P_n(y(s))\\&\quad-\alpha_0(x(s-1))P_n(y(s-1))=0.\end{aligned}\tag{19}$$

3. The fourth order difference equation

Let us then combine together Eqs. (10) and (11) on the one side and Eqs. (12) and (13), on the other side. As a result, we obtain:

$$\begin{aligned}&\sigma_n(x(s))P_n(y(s+2))+\zeta_n(x(s))P_n(y(s+1))+\nu_n(x(s))P_n(y(s))\\&\quad+t_n(x(s))P^{(1)}_{n-1}(y(s+2))+z_n(x(s))P^{(1)}_{n-1}(y(s))=0,\end{aligned}\tag{20}$$

$$\begin{aligned}&f_n(x(s))P_n(y(s+2))+g_n(x(s))P_n(y(s))+h_n(x(s))P^{(1)}_{n-1}(y(s+2))\\&\quad+v_n(x(s))P^{(1)}_{n-1}(y(s+1))+w_n(x(s))P^{(1)}_{n-1}(y(s))=0,\end{aligned}\tag{21}$$

where

$$\begin{aligned}\sigma_n(x(s))&:=\frac{\beta_0(x(s+1)}{\mathcal{D}_n(x(s+1))};\ \nu_n(x(s)):=-\frac{\alpha_0(x(s))}{\mathcal{D}_n(x(s))};\ t_n(x(s)):=\frac{\gamma_0(x(s+1))}{\mathcal{D}_n(x(s+1))}\\z_n(x(s))&:=-\frac{\gamma_0(x(s))}{\mathcal{D}_n(x(s))};\ f_n(x(s)):=-\frac{\delta_0(x(s+1))}{\mathcal{D}_n(x(s+1))};\ g_n(x(s)):=\frac{\delta_0(x(s))}{\mathcal{D}_n(x(s))}\\h_n(x(s))&:=-\frac{\alpha_0(x(s+1))}{\mathcal{D}_n(x(s+1))};\ w_n(x(s)):=\frac{\beta_0(x(s))}{\mathcal{D}_n(x(s)}\\\zeta_n(x(s))&:=\frac{\mathcal{D}_n(x(s+1))\alpha_n(x(s))-\mathcal{D}_n(x(s))\beta_n(x(s+1))}{\mathcal{D}_n(x(s+1))\mathcal{D}_n(x(s))}\\v_n(x(s))&:=\frac{\mathcal{D}_n(x(s))\alpha_{n+1}(x(s+1))-\mathcal{D}_n(x(s+1))\beta_{n+1}(x(s))}{\mathcal{D}_n(x(s+1))\mathcal{D}_n(x(s))}.\end{aligned}\tag{22}$$

Solving Eqs. (20) and (21) relatively to $P^{(1)}_{n-1}(y(s+2))$, $P^{(1)}_{n-1}(y(s+1))$ and $P^{(1)}_{n-1}(y(s))$, as linear combinations of $P_n(y(s+3))$, $P_n(y(s+2))$, $P_n(y(s+1))$ and $P_n(y(s)))$, with coefficients depending on $x(s)$ and n and taking into account the fact that for example $P^{(1)}_{n-1}(y(s+1))$ is a shift of $P^{(1)}_{n-1}(y(s))$, we obtain the expected fourth order difference equation:

$$
\begin{aligned}
&\left[\frac{A_n(x(s+1))N_n(x(s+1))}{G_n(x(s+1))N_n(x(s+1)) - F_n(x(s+1))R_n(x(s+1))}\right] P_n(y(s+2))\\
&\quad+\left[\frac{A_n(x(s))R_n(x(s))}{G_n(x(s))N_n(x(s)) - F_n(x(s))R_n(x(s))}\right.\\
&\quad+\left.\frac{B_n(x(s+1))N_n(x(s+1)) - F_n(x(s+1))K_n(x(s+1))}{G_n(x(s+1))N_n(x(s+1)) - F_n(x(s+1))R_n(x(s+1))}\right] P_n(y(s+1))\\
&\quad+\left[\frac{R_n(x(s))B_n(x(s)) - K_n(x(s))G_n(x(s))}{G_n(x(s))N_n(x(s)) - F_n(x(s))R_n(x(s))}\right.\\
&\quad+\left.\frac{C_n(x(s+1))N_n(x(s+1)) - F_n(x(s+1))L_n(x(s+1))}{G_n(x(s+1))N_n(x(s+1)) - F_n(x(s+1))R_n(x(s+1))}\right] P_n(y(s))\\
&\quad+\left[\frac{C_n(x(s))R_n(x(s)) - L_n(x(s))G_n(x(s))}{G_n(x(s))N_n(x(s)) - F_n(x(s))R_n(x(s))}\right.\\
&\quad+\left.\frac{E_n(x(s+1))N_n(x(s+1)) - F_n(x(s+1))M_n(x(s+1))}{G_n(x(s+1))N_n(x(s+1)) - F_n(x(s+1))R_n(x(s+1))}\right] P_n(y(s-1))\\
&\quad+\left[\frac{R_n(x(s))E_n(x(s)) - M_n(x(s))G_n(x(s))}{G_n(x(s))N_n(x(s)) - F_n(x(s))R_n(x(s))}\right] \quad P_n(y(x(s-2)) = 0, \qquad (23)
\end{aligned}
$$

where (having in mind (22) and (9)):

$$
\begin{aligned}
A_n(x(s)) &:= \frac{t_n(x(s-2))(h_n(x(s-1))\sigma_n(x(s-1)) - f_n(x(s-1))t_n(x(s-1)))}{\upsilon_n(x(s-1))t_n(x(s-1))};\\
B_n(x(s)) &:= \sigma_n(x(s-2)) + \frac{t_n(x(s-2))h_n(x(s-1))\zeta_n(x(s-1))}{\upsilon_n(x(s-1))t_n(x(s-1))};\\
C_n(x(s)) &:= \zeta_n(x(s-2))\\
&\qquad + \frac{t_n(x(s-2))(h_n(x(s-1))\upsilon_n(x(s-1)) - t_n(x(s-1))g_n(x(s-1)))}{\upsilon_n(x(s-1))t_n(x(s-1))};\\
E_n(x(s)) &:= \upsilon_n(x(s-2));\ G_n(x(s)) := z_n(x(s-2));\\
K_n(x(s)) &:= \frac{f_n(x(s-2))t_n(x(s-2)) - h_n(x(s-2))\sigma_n(x(s-2))}{t_n(x(s-2))};\\
L_n(x(s)) &:= -\frac{h_n(x(s-2))\zeta_n(x(s-2))}{t_n(x(s-2))};\ N_n(x(s)) := \upsilon_n(x(s-2));\\
M_n(x(s)) &:= \frac{t_n(x(s-2))g_n(x(s-2)) - h_n(x(s-2))\upsilon_n(x(s-2))}{t_n(x(s-2))};
\end{aligned}
$$

$$
\begin{aligned}
R_n(x(s)) &:= \frac{t_n(x(s-2))w_n(x(s-2)) - z_n(x(s-2))h_n(x(s-2))}{t_n(x(s-2))};\\
F_n(x(s)) &:= \frac{t_n(x(s-2))(h_n(x(s-1))z_n(x(s-1)) - w_n(x(s-1))t_n(x(s-1)))}{v_n(x(s-1))t_n(x(s-1))}.
\end{aligned} \tag{24}
$$

Remark 3.1. Considering Remark 2.1, we clearly note that the fourth order difference equation for $P_{n-r}^{(r)}$ is obtained from the preceding one satisfied by P_n, by shifting the initial value for n, from $n := 0$ to $n := r$. More precisely, one needs to replace in (22) $\alpha_0,\ \beta_0,\ \gamma_0$ and δ_0 respectively by α_r, β_r, γ_r and δ_r.

Remark 3.2. We have already in this section used the solutions of the system of equations (20) and (21), solved relatively to the $P_{n-1}^{(1)}$, as functions of the P_n. Adding a solution of that system now relatively to the P_n as functions of the $P_{n-1}^{(1)}$, we obtain the following inverse difference relations:

$$P_{n-1}^{(1)}(y(s)) := \Phi_n(x(s))[P_n(y(s))]; \tag{25}$$

$$P_n(y(s)) := \Psi_n(x(s))\big[P_{n-1}^{(1)}(y(s))\big]; \tag{26}$$

where

$$
\begin{aligned}
&\Phi_n(x(s))[\theta_n(y(s))]\\
&\quad := \left[\frac{A_n(x(s+2))N_n(x(s+2))}{F_n(x(s+2))R_n(x(s+2)) - G_n(x(s+2))N_n(x(s+2))}\right]\theta_n(y(s+3))\\
&\quad + \left[\frac{B_n(x(s+2))N_n(x(s+2)) - K_n(x(s+2))F_n(x(s+2))}{F_n(x(s+2))R_n(x(s+2)) - G_n(x(s+2))N_n(x(s+2))}\right]\theta_n(y(s+2))\\
&\quad + \left[\frac{C_n(x(s+2))N_n(x(s+2)) - F_n(x(s+2))L_n(x(s+2))}{F_n(x(s+2))R_n(x(s+2)) - G_n(x(s+2))N_n(x(s+2))}\right]\theta_n(y(s+1))\\
&\quad + \left[\frac{E_n(x(s+2))N_n(x(s+2)) - F_n(x(s+2))M_n(x(s+2))}{F_n(x(s+2))R_n(x(s+2)) - G_n(x(s+2))N_n(x(s+2))}\right]\theta_n(y(s)),
\end{aligned} \tag{27}
$$

and

$$
\begin{aligned}
&\Psi_n(x(s))[\theta_n(y(s))]\\
&\quad := \left[\frac{\hat{A}_n(x(s+2))\hat{N}_n(x(s+2))}{\hat{F}_n(x(s+2))\hat{R}_n(x(s+2)) - \hat{G}_n(x(s+2))\hat{N}_n(x(s+2))}\right]\theta_n(y(s+3))\\
&\quad + \left[\frac{\hat{B}_n(x(s+2))\hat{N}_n(x(s+2)) - \hat{K}_n(x(s+2))\hat{F}_n(x(s+2))}{\hat{F}_n(x(s+2))\hat{R}_n(x(s+2)) - \hat{G}_n(x(s+2))\hat{N}_n(x(s+2))}\right]\theta_n(y(s+2))\\
&\quad + \left[\frac{\hat{C}_n(x(s+2))\hat{N}_n(x(s+2)) - \hat{F}_n(x(s+2))\hat{L}_n(x(s+2))}{\hat{F}_n(x(s+2))\hat{R}_n(x(s+2)) - \hat{G}_n(x(s+2))\hat{N}_n(x(s+2))}\right]\theta_n(y(s+1))\\
&\quad + \left[\frac{\hat{E}_n(x(s+2))\hat{N}_n(x(s+2)) - \hat{F}_n(x(s+2))\hat{M}_n(x(s+2))}{\hat{F}_n(x(s+2))\hat{R}_n(x(s+2)) - \hat{G}_n(x(s+2))\hat{N}_n(x(s+2))}\right]\theta_n(y(s)),
\end{aligned} \tag{28}
$$

where the hat has the meaning that $\hat{X}$ is obtained from X by replacing σ_n, ζ_n, ν_n, t_n, z_n, f_n, g_n, h_n, v_n, w_n by h_n, v_n, w_n, f_n, g_n, t_n, z_n, σ_n, ζ_n, ν_n *respectively* (thanks to the symmetric form of the system (20)–(21)).

Shifting in Φ_n and Ψ_n the initial value for n from $n := 0$ to $n := r$, it becomes clear that the obtained operators play the role of "raising" and "lowering" operators within the sequence of (normalized) polynomials satisfying the following three-term recurrence relation:

$$P_n^{(r)}(y(s)) = (y(s) - b_r)P_{n-1}^{(r+1)}(y(s)) - a_{r+1}^2 P_{n-2}^{(r+2)}(y(s)). \tag{29}$$

4. Explicit examples

4.1. The fourth order difference equation for the polynomials r-associated to the classical polynomials

We now go further and calculate explicitly the coefficients in the fourth order difference equation satisfied by the polynomials $P_{n-r}^{(r)}$ r-associated to the classical orthogonal polynomials (up to the Askey-Wilson polynomials). Let us remark first that the problem consists essentially in solving the system (7) (in the classical case i.e. $\iota = 0$, $\mathcal{B}_0 = 0$). The coefficients in the fourth order difference equation are then obtained directly using (22) and (24) and considering Remark 3.1.

In the following, the cases of linear, q-linear and Askey-Wilson lattices are treated in detail separately. For simplicity, canonical forms of lattices are choosen.

4.1.1. The linear case: $y(s) := s$, $x(s) = s$. Considering (9), one notes that $\alpha_n(x(s))$ and $\beta_n(x(s))$ need to be searched under the forms (recall that we are searching classical solutions, so $\iota = 0$ in (3)):

$$\begin{aligned} \alpha_n(x(s)) &:= -\left(\alpha_n^0 + \alpha_n^1 s + \alpha_n^2 s^2\right), \\ \beta_n(x(s)) &:= \beta_n^0 + \beta_n^1 s + \alpha_n^2 s^2. \end{aligned} \tag{30}$$

Those expressions satisfy the system (7) iff (here and in all that follows, x_i, y_i, $i := 0, 1, \ldots,$ are arbitrary constants):

$$\begin{aligned} \boldsymbol{\alpha}_{\mathbf{n}}(\mathbf{x}(\mathbf{s})) := \Big\{&\big[2x_2^2 n + x_2 y_1 - x_2 x_1\big]s^2 + \big[-2x_2^2 n^2 + (-x_2 y_1 + 3x_2 x_1)n \\ &+ y_1 x_1 - x_1^2\big]s + \big[x_2^2 n^3 + (x_2 y_1 - 2x_2 x_1)n^2 \\ &+ \big(x_2 x_0 - y_1 x_1 + x_1^2 + x_2 y_0\big)n - x_0 x_1 + x_0 y_1\big]\Big\}\Big/\{-2x_2 n - y_1 + x_1\}; \end{aligned} \tag{31}$$

$$\begin{aligned} \boldsymbol{\beta}_{\mathbf{n}}(\mathbf{x}(\mathbf{s})) := \Big\{&\big[2x_2^2 n + x_2 y_1 - x_2 x_1\big]s^2 + \big[2x_2^2 n^2 + (-x_2 x_1 + 3x_2 y_1)n \\ &+ y_1^2 - y_1 x_1\big]s + \big[x_2^2 n^3 + (-x_2 x_1 + 2x_2 y_1)n^2 \\ &+ \big(x_2 x_0 + x_2 y_0 - y_1 x_1 + y_1^2\big)n + y_0 y_1 - y_0 x_1\big]\Big\}\Big/\{2x_2 n + y_1 - x_1\}; \end{aligned} \tag{32}$$

$$\mathcal{D}_{\mathbf{n}}(\mathbf{x}(\mathbf{s})) := -2x_2 n - x_2 - y_1 + x_1; \tag{33}$$

$$\begin{aligned}\mathbf{b_n} := \big\{&\big(2x_2^2 - x_2x_1 - x_2y_1\big)n^2 + \big(2x_2^2 + x_2y_1 - 3x_2x_1 + x_1^2 - y_1^2\big)n \\ &+ \big(x_2y_1 - x_2x_1 + x_0y_1 - x_0x_1 + x_1^2 - y_0y_1 + y_0x_1 - y_1x_1\big)\big\}\big/\big\{4x_2^2n^2 \\ &+\big(4x_2^2 + 4x_2y_1 - 4x_2x_1\big)n + 2x_2y_1 - 2x_2x_1 + y_1^2 - 2y_1x_1 + x_1^2\big\};\end{aligned} \tag{34}$$

$$\begin{aligned}\mathbf{a_n^2} := \big(&-x_2^4n^6 + \big(3x_2^3x_1 - 3x_2^3y_1\big)n^5 + \big(-3x_2^2y_1^2 - 2x_2^3x_0 - 2x_2^3y_0 \\ &+ 7x_2^2y_1x_1 - 3x_2^2x_1^2\big)n^4 + \big(-5x_2x_1^2y_1 + 5x_2y_1^2x_1 + 4x_2^2y_0x_1 - x_2y_1^3 \\ &- 4x_2^2x_0y_1 + 4x_2^2x_0x_1 + x_2x_1^3 - 4x_2^2y_0y_1\big)n^3 + \big(-x_2^2y_0^2 - x_2^2x_0^2 \\ &+ x_1^3y_1 + 2x_2^2y_0x_0 - 3x_2x_0y_1^2 + y^3x_1 - 2x_2y_0y_1^2 + 5x_2y_0y_1x_1 \\ &+ 5x_2x_0y_1x_1 - 2x_2x_0x_1^2 - 3x_2y_0x_1^2 - 2x_1^2y_1^2\big)n^2 + (-2x_2y_0x_0x_1 \\ &- 2x_1^2y_1y_0 + 2y_1^2x_1x_0 + y_1^2x_1y_0 - x_1^2y_1x_0 + x_2x_0^2x_1 - x_2y_0^2y_1 \\ &+ x_2y_0^2x_1 - x_2x_0^2y_1 + x_1^3y_0 + 2x_2y_0x_0y_1 - y_1^3x_0\big)n\big\}\big/\big\{16x_2^4n^4 \\ &+ \big(32x_2^3y_1 - 32x_2^3x_1\big)n^3 + \big(-48x_2^2y_1x_1 + 24x_2^2x_1^2 - 4x_2^4 + 24x_2^2y_1^2\big)n^2 \\ &+ \big(-24x_2y_1^2x_1 + 24x_2x_1^2y_1 - 8x_2x_1^3 - 4x_2^3y_1 + 4x_2^3x_1 + 8x_2y_1^3\big)n \\ &+ \big(-x_2^2y_1^2 + y_1^4 - 4y_1^3x_1 + 6x_1^2y_1^2 + 2x_2^2y_1x_1 - 4x_1^3y_1 - x_2^2x_1^2 + x_1^4\big)\big\};\end{aligned} \tag{35}$$

4.1.2. The q-linear case: $y(s) := q^s, x(s) := q^s$. Now, $\alpha_n(x(s))$ and $\beta_n(x(s))$ need to be searched under the forms:

$$\begin{aligned}\alpha_n(x(s)) &:= -\big(\alpha_n^0q^{-s} + \alpha_n^1 + \alpha_n^2q^s\big), \\ \beta_n(x(s)) &:= \alpha_n^0q^{-s} + \beta_n^1 + \beta_n^2q^s.\end{aligned} \tag{36}$$

Those expressions satisfy the system (7) iff :

$$\begin{aligned}\boldsymbol{\alpha}_{\mathbf{n}}(\mathbf{x}(\mathbf{s})) := \big\{&\big[y_2qx_2q^{2n} - x_2^2\big]q^s + [(y_1x_2q + qy_2x_1)q^{2n} + (-y_1x_2q \\ &- x_2x_1)q^n] + [x_0qq^{3n}y_2 - x_0x_2q^n]q^{-s}\big\}\big/\{qx_2q^n - q^2y_2q^{3n}\};\end{aligned} \tag{37}$$

$$\begin{aligned}\beta_{\mathbf{n}}(\mathbf{x}(\mathbf{s})) := \big\{&\big[y_2^2q^2q^{3n} - y_2qx_2q^n\big]q^s + [(y_1y_2q^2 + qy_2x_1)q^{2n} + (-y_1x_2q \\ &- qy_2x_1)q^n] + [x_0y_2qq^{2n} - x_0x_2]q^{-s}\big\}\big/\{q^2y_2q^{2n} - qx_2\};\end{aligned} \tag{38}$$

$$\mathcal{D}_{\mathrm{n}}(\mathbf{x}(\mathbf{s})) := x_2q^{-2}q^{-n} - y_2q^n; \tag{39}$$

$$\begin{aligned}\mathbf{b_n} := \big\{&-(q^3y_2x_1 + y_2q^4y_1)q^{3n} + (q^3y_2x_1 + q^2y_2x_1 + q^3y_1x_2 \\ &+y_1x_2q^2)q^{2n} - (y_1x_2q^2 + x_2x_1q)q^n\big\}\big/\big\{q^4y_2^2q^{4n} \\ &+(-y_2qx_2 - x_2y_2q^3)q^{2n} + x_2^2\big\};\end{aligned} \tag{40}$$

$$
\begin{aligned}
\mathbf{a_n^2} := \big\{ & -y_2^3 q^4 x_0 q^{7n} + \big(y_2^3 q^4 x_0 + y_2^2 q^4 y_1 x_1 + q^3 y_2^2 x_0 x_2\big) q^{6n} \\
& - \big(y_2^2 q^4 y_1 x_1 + x_2 q^3 y_1 y_2 x_1 + q^3 y_2^2 x_1^2 - q^3 y_2^2 x_0 x_2 + q^4 y_1^2 y_2 x_2\big) q^{5n} \\
& - \big(-q^4 y_1^2 y_2 x_2 - 2x_2 q^3 y_1 y_2 x_1 - q^3 y_2^2 x_1^2 - x_2 q^2 y_2 x_1^2 - x_2^2 q^3 y_1^2 \\
& + 2q^3 y_2^2 x_0 x_2 + 2y_2 x_0 x_2^2 q^2\big) q^{4n} - \big(-y_2 x_0 x_2^2 q^2 + x_2 q^3 y_1 y_2 x_1 \\
& + x_2^2 q^3 y_1^2 + x_2 q^2 y_2 x_1^2 + y_1 x_1 x_2^2 q^2\big) q^{3n} \\
& + \big(q x_0 x_2^3 + y_2 x_0 x_2^2 q^2 + y_1 x_1 x_2^2 q^2\big) q^{2n} - q x_0 x_2^3 q^n \big\} \big/ \big\{ q^4 y_2^4 q^{8n} \\
& - \big(y_2^3 x_2 q^2 + 2y_2^3 x_2 q^3 + y_2^3 x_2 q^4\big) q^{6n} + \big(2x_2^2 y_2^2 q^2 + 2x_2^2 y_2^2 q^3 \\
& + 2x_2^2 y_2^2 q\big) q^{4n} - \big(x_2^3 y_2 q^2 + x_2^3 y_2 + 2x_2^3 y_2 q\big) q^{2n} + x_2^4 \big\}; \qquad (41)
\end{aligned}
$$

4.1.3. The Askey-Wilson case: $y(s) := \frac{q^s+q^{-s}}{2}$, $x(s) := qq^s + q^{-s}$. In that case, one can also verify from (9) that $\alpha_n(x(s))$ and $\beta_n(x(s))$ need to be searched under the forms:

$$
\begin{aligned}
\alpha_n(x(s)) &:= \frac{\alpha_n^0 q^{-2s} + \alpha_n^1 q^{-s} + \beta_n^2 + q\beta_n^1 q^s + q^2\beta_n^0 q^{2s}}{q^{-s} - qq^s}, \\
\beta_n(x(s)) &:= \frac{\beta_n^0 q^{-2s} + \beta_n^1 q^{-s} + \beta_n^2 + q\alpha_n^1 q^s + q^2\alpha_n^0 q^{2s}}{qq^s - q^{-s}}.
\end{aligned} \qquad (42)
$$

Those expressions satisfy the system (7) iff (here as above, we exclude, of course, trivial solutions):

$$
\begin{aligned}
\boldsymbol{\alpha}_\mathbf{n}(\mathbf{x}(\mathbf{s})) := \big\{ & \big[q^2 y_0 x_0 q^{2n} - q^2 y_0^2\big] q^{2s} + [(y_0 x_1 q + y_1 x_0 q) q^{2n} + (-y_0 q y_1 - y_0 x_1 q) q^n] q^s \\
& + \big[-q x_0^2 q^{4n} + \big(q x_0^2 + y_2 x_0 + q x_0 y_0\big) q^{3n} \\
& + \big(-q y_0^2 - q x_0 y_0 - y_0 y_2\big) q^n + y_0^2 q\big] + [(y_1 x_0 + x_1 x_0) q^{3n} \\
& + (-y_0 x_1 - y_1 x_0) q^{2n}] q^{-s} + \big[q^{4n} x_0^2 - q^{2n} x_0 y_0\big] q^{-2s} \big\} \big/ \\
& \{[-q x_0 q^{3n} + q y_0 q^n] q^s + [x_0 q^{3n} - y_0 q^n] q^{-s}\}; \qquad (43) \\
\boldsymbol{\beta}_\mathbf{n}(\mathbf{x}(\mathbf{s})) := \big\{ & \big[q^2 q^{4n} x_0^2 - q^2 y_0 x_0 q^{2n}\big] q^{2s} + [(y_1 x_0 q + x_1 q x_0) q^{3n} \\
& + (-y_0 x_1 q - y_1 x_0 q) q^{2n}] q^s + \big[-q x_0^2 q^{4n} + \big(q x_0^2 + y_2 x_0 + q x_0 y_0\big) q^{3n} \\
& + \big(-q y_0^2 - q x_0 y_0 - y_0 y_2\big) q^n + y_0^2 q\big] + [(y_0 x_1 + y_1 x_0) q^{2n} \\
& + (-y_1 y_0 - y_0 x_1) q^n] q^{-s} + \big[q^{2n} x_0 y_0 - y_0^2\big] q^{-2s} \big\} \big/ \\
& \big\{\big[q x_0 q^{3n} - q y_0 q^n\big] q^s + \big[-x_0 q^{3n} + y_0 q^n\big] q^{-s}\big\}; \qquad (44) \\
\boldsymbol{\mathcal{D}}_\mathbf{n}(\mathbf{x}(\mathbf{s})) := {} & 2y_0 q^{-n} - 2x_0 q q^n; \qquad (45) \\
\mathbf{b_n} := {} & \{(-y_1 x_0 q - x_1 q x_0) q^{3n} + (y_1 x_0 + y_0 x_1 + y_1 x_0 q + y_0 x_1 q) q^{2n} \\
& + (-y_1 y_0 - y_0 x_1) q^n\} \big/ \big\{2q^2 x_0^2 q^{4n} + 2(-y_0 x_0 - q^2 x_0 y_0) q^{2n} + 2y_0^2\big\}; \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a_n^2} :={}& \{qx_0^4q^{8n} + \left(-x_0^3y_0q - x_0^3y_2 - qx_0^4\right)q^{7n} + \left(y_2x_0^2y_0 + x_0^3y_2 + x_0^2y_1x_1\right)q^{6n} \\
&+ \left(-y_0x_1y_1x_0 - y_1^2x_0^2 - y_0x_1^2x_0 + qx_0^2y_0^2 - x_0^2y_1x_1 + y_2x_0^2y_0 + x_0^3y_0q\right)q^{5n} \\
&+ \left(-2qx_0^2y_0^2 + g_0^2x_1^2 - 2y_2x_0^2y_0 + y_1^2y_0x_0 + y_1^2x_0^2 + y_0x_1^2x_0 - 2y_0^2y_2x_0\right. \\
&\left.+ 2y_0x_1y_1x_0\right)q^{4n} + \left(-y_0x_1y_1x_0 - y_0^2x_1^2 - g_0^2x_1y_1 + qx_0y_0^3\right. \\
&\left.+ y_0^2y_2x_0 + qx_0^2y_0^2 - y_1^2y_0x_0\right)q^{3n} + \left(y_2y_0^3 + y_0^2y_2x_0 + y_0^2x_1y_1\right)q^{2n} \\
&+ \left(-y_2y_0^3 - y_0^4q - qx_0y_0^3\right)q^n + \left[qy_0^4\right]\}/\{qx_0^4q^{8n} \\
&+ \left(-x_0^3y_0 - 2x_0^3y_0q - x_0^3y_0q^2\right)q^{6n} + \left(2qx_0^2y_0^2 + 2x_0^2y_0^2q^2\right. \\
&\left.+ 2x_0^2y_0^2\right)q^{4n} + \left(-q^2x_0y_0^3 - x_0y_0^3 - 2qx_0y_0^3\right)q^{2n} + \left[qy_0^4\right]\}.
\end{aligned}
\tag{47}
$$

Let us remark that if $\mathcal{L}(z, n)$ is the Askey-Wilson second order q-difference operator (in its canonical form) [6]:

$$
\mathcal{L}(z, n) = v(z)\mathbf{E}_q - (v(z) + v(z^{-1})) + v(z^{-1})\mathbf{E}_q^{-1} - \lambda(n) \tag{48}
$$

where

$$
\begin{aligned}
v(z) &= \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}, \\
\mathbf{E}_q^i(\mathcal{P}_n(\chi(z))) &= \mathcal{P}_n(\chi(q^iz)), \quad i \in Z, \\
\chi(z) &= \frac{z+z^{-1}}{2}, \\
\lambda(n) &= -(1-q^{-n})(1-abcdq^{n-1});
\end{aligned}
\tag{49}
$$

and $H(z, n)$ the second order difference operator given in left hand side of (19), with $\alpha_n(x(s))$ and $\beta_n(x(s))$ given in Section 4.1.3 and with q^s replaced by z, then one can verify that,

$$
H(z, n) = (z - z^{-1})\mathcal{L}(z, n) \tag{50}
$$

where the following correspondence needs to be performed:

$$
\begin{aligned}
&y_0 := -1;\ y_1 := a+b+c+d;\ y_2 := -(ab+ac+ad+bc+bd+cd); \\
&x_1 := (abc+abd+bcd+acd)q^{-1};\ x_0 := -abcdq^{-2}
\end{aligned}
\tag{51}
$$

4.2. *The fourth order difference equation for the class one LHP*

We saw above that in the classical situation ($\iota = 0,\ \mathcal{B}_0 = 0$), one is able to solve the system (7) in terms of elementary functions. We can not expect to do this in non-classical situations, as we know that in the continuous case, the coefficients in the three-term recurrence relations are related to Painlevé trancendents (see [9] in semi-classical differential situation). Below we give solutions of (7) for the case of class $\iota = 1$ ($\mathcal{B}_0$ not necessary zero) and for simplicity in the case of linear lattice $y(s) = x(s) = s$, up to explicit non-linear difference equations satisfied by the coefficients in the three-term recurrence relations. This furnishes naturally "semi-explicit" fourth order difference equations for the corresponding polynomials by the formula (23) and naturally for the polynomials r-associated to them according to Remark 3.1.

We have ($\iota = 1$, $y(s) = x(s) = s$): (c_1, c_2, c_3, c_4, arbitrary parameters; x_n, y_n of course others than in 4.1)

$$\begin{aligned} \alpha_n(x(s)) &:= -\left(\alpha_n^0 + \alpha_n^1 s + \alpha_n^2 s^2 + c_3 s^3\right), \\ \beta_n(x(s)) &:= \beta_n^0 + \beta_n^1 s + \beta_n^2 s^2 + c_3 s^3, \end{aligned} \tag{52}$$

with

$$\begin{aligned} \boldsymbol{\alpha}_\mathbf{n}^\mathbf{0} := \big[&4x_n y_n c_3^2 - 4c_2c_3c_4 + 8c_2c_1c_3 - 4y_{n+1}x_{n+1}c_3^2 + 4c_3^2c_2n + 8c_2^2c_3n \\ &- 2c_3^2c_4n - 3y_{n+1}y_n^2 - y_{n+1}^3 - (4c_3n + 8c_2 + 2c_3 - 2c_4)y_ny_{n+1} \\ &- (-2c_3n - 4c_2 + c_4)y_n^2 - (-2c_3c_4n + 4c_2c_3 + c_3^2 + 2c_3^2n^2 + 2c_3^2n - 2c_3c_4 \\ &- 4c_4c_2 + 8c_3c_2n + 4c_2^2 + 4c_1c_3)y_{n+1} - \big(2c_3c_4n + 4c_4c_2 - 8c_2c_3n - 2c_3^2n^2 \\ &- 4c_2^2 + c_3^2 + 2c_3c_4 - 4c_1c_3 - 2c_3^2n\big)y_n - \big(-8c_3^3 + 8c_3^3n - 32c_3^2c_2 \\ &- 16c_2c_3c_4 - 4c_3^2c_4 + 32c_2c_3^2n\big)x_n - (-2c_3 - 4c_2 + c_4 - 2c_3n)y_{n+1}^2 \\ &- \big(-32c_2c_3^2n - 32c_2c_3^2 - 8c_3^3n + 16c_2c_3c_4 + 4c_3^2c_4 - 8c_3^3\big)x_{n+1} \\ &- \big(4c_3c_4 - 8c_3^2 - 8c_3^2n\big)y_nx_{n+1} - \big(8c_3^2n + 12c_3^2 - 4c_3c_4\big)y_{n+2}x_{n+1} \\ &+ 4c_2c_3^2n^2 + y_n^3 + 3y_ny_{n+1}^2 - 4c_2c_3c_4n - c_3^2c_4 + 2c_3^3n^2 + 4c_1c_3^2 - 4c_2^2c_4 \\ &- \big(4c_3c_4 - 8c_3^2n + 8c_3^2\big)x_ny_{n+1} - \big(8c_3^2n - 12c_3^2 - 4c_3c_4\big)x_ny_{n-1}\big]\big/\big(8c_3^2\big); \end{aligned}$$

$$\boldsymbol{\alpha}_\mathbf{n}^\mathbf{1} := c_1 - \frac{1}{2}(c_3 + c_4)n + \frac{1}{2}c_3n^2 - \frac{1}{2}y_n; \quad \boldsymbol{\alpha}_\mathbf{n}^\mathbf{2} := c_2 + \frac{1}{2}c_3 + \frac{1}{2}c_4 - c_3n;$$

$$\begin{aligned} \boldsymbol{\beta}_\mathbf{n}^\mathbf{0} := \big[&4c_3^2x_ny_n - 4c_2c_3c_4 + 8c_1c_2c_3 - 4c_3^2x_{n+1}y_{n+1} + 4c_2c_3^2n - 8c_2^2c_3n \\ &+ \big(-4c_3c_4 + 8c_3^2n - 8c_3^2\big)y_{n+1}x_n + \big(-8c_3^2n + 4c_3c_4 - 12c_3^2\big)y_{n+2}x_{n+1} \\ &+ \big(-4c_3c_4 + 8c_3^2 + 8c_3^2n\big)y_nx_{n+1} + \big(-8c_3^2n + 12c_3^2 + 4c_3c_4\big)y_{n-1}x_n \\ &+ 2c_3^2c_4n - 3y_{n+1}y_n^2 - y_{n+1}^3 + 4c_2c_3^2n^2 + y_n^3 + 3y_ny_{n+1}^2 - 4c_2c_3c_4n + (4c_2 \\ &+ c_4 - 2c_3n)y_n^2 + \big(32c_3^2c_2 + 32c_3^2c_2n - 8c_3^3n - 16c_2c_3c_4 - 8c_3^3 + 4c_3^2c_4\big)x_{n+1} \\ &+ \big(-4c_3^2c_4 - 8c_3^3 - 32c_3^2c_2n + 16c_2c_3c_4 + 32c_3^2c_2 + 8c_3^3n\big)x_n + (-2c_3n \\ &- 2c_3 + c_4 + 4c_2)y_{n+1}^2 + \big(4c_4c_2 + 4c_2^2 - 2c_3c_4 - c_3^2 - 8c_3c_2n - 2c_3c_4n \\ &+ 4c_3c_1 + 2c_3^2n^2 + 2c_3^2n\big)y_n + \big(8c_2c_3n - 2c_3^2n^2 - 4c_2^2 + 2c_3c_4n + 2c_3c_4 \end{aligned}$$

$$
\begin{aligned}
&+4c_2c_3 - c_3^2 - 4c_4c_2 - 4c_1c_3 - 2c_3^2 n\big)y_{n+1} + (-2c_4 + 2c_3 - 8c_2 \\
&+4c_3 n)y_n y_{n+1} + c_3^2 c_4 - 2c_3^3 n^2 - 4c_1 c_3^2 + 4c_2^2 c_4\big]/\big(8c_3^2\big); \\
\boldsymbol{\beta}_{\mathbf{n}}^{\mathbf{1}} &:= \frac{1}{2}y_n + c_1 - \frac{1}{2}(c_3+c_4)n + \frac{1}{2}c_3 n^2; \quad \boldsymbol{\beta}_{\mathbf{n}}^{\mathbf{2}} := c_2 - \frac{1}{2}c_3 - \frac{1}{2}c_4 + c_3 n; \\
\mathbf{D}_n(x(s)) &:= (c_4 - 2c_3 n)s + [(2c_3 n - c_4 - c_3)y_n + (c_4 - c_3 - 2c_3 n)y_{n+1} \\
&+4c_3c_2 n - 2c_4c_2]/(2c_3);
\end{aligned}
$$

where x_n and y_n given by

$$
\begin{aligned}
a_n^2 &:= x_n \\
b_n &:= (y_{n+1} - y_n - 2c_2 + c_3)/(2c_3);
\end{aligned} \tag{53}
$$

are required to satisfy the following non-linear difference system

$$
\begin{aligned}
&-8c_2 y_{n+1}y_{n+2} - 4c_3^2 y_{n+2}x_{n+2} - y_{n+2}^3 + 8c_3^2 c_2 + 4c_2 y_{n+2}^2 - 3y_{n+2}y_{n+1}^2 \\
&+3y_{n+1}y_{n+2}^2 - y_n^3 + 2y_{n+1}^3 + 8c_3^2 y_{n+1}x_{n+1} - 8c_2c_3c_4 - 4c_3^2 y_n x_n + 16c_3^2 c_2 n \\
&+8c_2 y_n y_{n+1} + 3y_{n+1}y_n^2 - 3y_n y_{n+1}^2 - 4c_2 y_n^2 + \big(32c_2c_3c_4 - 32c_3^2c_2 - 64c_3^2c_2 n\big)x_{n+1} \\
&+\big(-8c_3^2 n - 20c_3^2 + 4c_3c_4\big)x_{n+2}y_{n+3} + \big(8c_3c_4 - 4c_3^2 - 16c_3^2 n\big)x_{n+1}y_n \\
&+\big(4c_3c_4 - 8c_3^2 n + 8c_3^2\big)x_n y_{n+1} + \big(8c_3^2 n - 12c_3^2 - 4c_3c_4\big)x_n y_{n-1} \\
&+\big(-4c_3c_4 + 16c_3^2 + 8c_3^2 n\big)x_{n+2}y_{n+1} + \big(-8c_3c_4 + 16c_3^2 n + 12c_3^2\big)x_{n+1}y_{n+2} \\
&+\big(-2c_3^2 n^2 - c_3^2 - 4c_2^2 + 2c_3c_4 n - 4c_1c_3 + 2c_3^2 n\big)y_n \\
&+\big(-2c_3^2 n^2 - 4c_2^2 + 2c_3c_4 n - 4c_1c_3 + 4c_3c_4 - 6c_3^2 n - 5c_3^2\big)y_{n+2} \\
&+\big(64c_3^2c_2 + 32c_3^2c_2 n - 16c_2c_3c_4\big)x_{n+2} + \big(4c_3^2 n^2 + 2c_3^2 - 4c_3c_4 + 8c_2^2 - 4c_3c_4 n \\
&+8c_1c_3 + 4c_3^2 n\big)y_{n+1} + \big(-16c_2c_3c_4 - 32c_3^2c_2 + 32c_3^2c_2 n\big)\,x_n = 0,
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
&-2c_3^3 - 8c_1c_3^2 - 8c_2^2c_3 - 4c_3^3 n + 4c_3^2c_4 n + 4c_3^2c_4 - 4c_3^3 n^2 - 2c_3 y_{n+1}^2 \\
&-8c_3^3 x_{n+1} + (c_4 - 2c_3 n - 4c_3)y_{n+2}^2 + (8c_3c_2 n - 4c_2c_3 - 4c_4c_2)y_n + (12c_2c_3 \\
&-4c_4c_2 + 8c_3c_2 n)y_{n+2} + \big(-16c_3^3 - 8c_3^3 n + 4c_3^2c_4\big)x_{n+2} + (8c_4c_2 - 8c_2c_3 \\
&-16c_3c_2 n)y_{n+1} + (-2c_3 + 2c_3 n - c_4)y_n^2 + \big(-8c_3^3 + 8c_3^3 n - 4c_3^2c_4\big)x_n \\
&+(6c_3 - 2c_4 + 4c_3 n)y_{n+1}y_{n+2} + (-4c_3 n + 2c_4 + 2c_3)y_n y_{n+1} = 0.
\end{aligned} \tag{55}
$$

Equations (54) and (55) are the most general (i.e. without loss of degree of freedom in parameters) connecting the coefficients in the three-term recurrence relations for the class one LHP orthogonal on linear lattice. They contain for example the ones obtained in [3]. Let us remark that Eqs. (54) and (55) can be written in explicit form in rapport with the highest differences (i.e. x_{n+4} and y_{n+4}). It can be seen also that loosing one degree of freedom (in parameters c_i) allows one to write (54)–(55) explicitly not only in function of a_n^2 but also in function of b_n. Let us remark finally that in principle, in spite of the fact that the present system is related to Painlevé transcendents (see [9] for the continuous semi-classical case), some of its particular cases have particular rational solutions as predicted in [2] (see also [1]). A deep analysis of that system requires however a specific consecration.

Summary

While the formula (23) established in Section 3 gives the fourth order difference equation for the general LHP, the expressions for α_n, β_n, a_n^2 and $\mathcal{D}_n$ calculated in Sections 4.1.1, 4.1.2 and 4.1.3 lead directly to (considering Remark 3.1) explicit expressions of the fourth order difference equations for the polynomials $P_{n-r}^{(r)}(y(s))$, r-associated to $P_n(y(s))$, the classical polynomials orthogonal on linear, q-linear and q-nonlinear (Askey-Wilson) lattices and finally, the expressions for α_n, β_n and $\mathcal{D}_n$ calculated in Section 4.2 lead directly to "semi-eplicit" expressions of the fourth order difference equations for the class one LHP orthogonal on linear lattice and naturally for the polynomials r-associated to them (considering Remark 3.1 as well).

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