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The factorization method for the general second-order q-difference equation and the Laguerre–Hahn polynomials on the general q-lattice

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Abstract

For the general linear second-order q-difference equation, we show the interconnection between the factorization method and the Laguerre–Hahn polynomials on the general q-lattice. Applications are then given in the cases of the hypergeometric and Askey–Wilson second-order q-difference equations.

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1. Introduction

Various aspects of factorization techniques had already been considered around the end of the nineteenth century in the works of Darboux [15], Burchnall and Chaundy [12, 13], Schrödinger [29] and other authors. Later, various generalizations and applications of the techniques were developed by many authors some of whom we will refer to throughout this work. In this paper we are concerned with the following considerations. Given a difference or differential operator $H(x)$, one can search for a second operator $\tilde{H}(x,\alpha)$ depending on a parameter α such that $\tilde{H}(x,\hat{\alpha}) = H(x)$ for a certain fixed value $\hat{\alpha}$ of α, and then consider the factorization chain (if it exists):

$$\begin{aligned} &\tilde{H}(x,\alpha) - \mu(\alpha) = H^{+}(x,\alpha)H^{-}(x,\alpha) \\ &\tilde{H}(x,\alpha+1) - \mu(\alpha) = H^{-}(x,\alpha)H^{+}(x,\alpha) \qquad x \in \mathbb{R}. \end{aligned} \tag{1}$$

If such a factorization exists, one can say that the operator $H(x)$ is *factorizable*. This property, once it exists for a given operator, is very fundamental for its characterization especially for its solvability. In particular, for differential or difference operators having polynomial solutions, it is well known that their solvability is closely related to their factorizability [1–4, 6–9, 11, 14, 20, 22–24, 29–32].

In this paper, we want to discuss another, slightly surprising, side of the factorization method, especially its interconnection with the so-called Laguerre–Hahn polynomials.

Recall that Laguerre–Hahn orthogonal polynomials on the general q-lattice $h(x) = k_1x + k_2x^{-1} + k_3$, $x = q^s$, $s \in \mathbb{Z}$, are defined as those for which the Stieltjes function $S_0(h(x))$ satisfies

$$A_0(x)\frac{S_0(h(qx)) - S_0(h(x))}{h(qx) - h(x)} = B_0(x)S_0(h(qx))S_0(h(x)) + C_0(x)(S_0(h(qx)) + S_0(h(x)))/2 + D_0(x) \qquad (2)$$

where $A_0(x), B_0(x), C_0(x)$ and $D_0(x)$ are Laurent polynomials of degree $\leqslant 2(k+2)$, $2(k+2), 2(k+1)$ and $2k$, respectively. The semi-classical subclass is characterized by $B_0(x) = 0$, while the classical one (up to the Askey–Wilson class) is obtained from the semi-classical by taking $k = 0$.

In this work, we will discuss the interconnection between the Laguerre–Hahn polynomials on the q-lattice $h(x)$ and the factorization method for a general linear second-order q-difference equation

$$\left[u(x)E_q + v(x) + w(x)E_q^{-1}\right] y(x) = \lambda y(x) \qquad (3)$$

where $E_q^i f(x) = f(q^i x)$, $i \in \mathbb{Z}$, $y(x) = S(h(x))$ for a certain function S, and then apply the results to the case of the hypergeometric second-order q-difference equation (here $h(x) = x$) [25]

$$[\sigma(x)D^q D_q + \tau(x)D_q]y(x) = \lambda y(x) \qquad (4)$$

where $\sigma(x) = \sigma_0x^2 + \sigma_1x + \sigma_2$, $\tau(x) = \tau_0x + \tau_1$, $\tau_0 \neq 0$, $D_q f(x) = (E_q - 1)f(x)/(qx - x)$, $D^q f(x) = \left(1 - E_q^{-1}\right)f(x)/(x - x/q)$, and the Askey–Wilson second-order q-difference equation

$$\left[\left[\frac{(q-1)^2}{4q}\frac{x^2-1}{x\omega(x)}\mathcal{D}_q\frac{x^2-1}{x}\upsilon\left(q^{-\frac{1}{2}}x\right)\omega\left(q^{-\frac{1}{2}}x\right)\right]\mathcal{D}_q\right]y(x) = \lambda y(x) \qquad (5)$$

where

$$\frac{\omega(qx)}{\omega(x)} = \frac{\upsilon(x)}{\upsilon((qx)^{-1})} \qquad \upsilon(x) = \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)} \qquad (6)$$

and

$$\mathcal{D}_q P(h(x)) = \frac{P\left(h\left(q^{\frac{1}{2}}x\right)\right) - P\left(h\left(q^{-\frac{1}{2}}x\right)\right)}{h\left(q^{\frac{1}{2}}x\right) - h\left(q^{-\frac{1}{2}}x\right)} \qquad h(x) = \frac{x + x^{-1}}{2}. \qquad (7)$$

We refer to equation (5) as the Askey–Wilson equation, due to the fact that the Askey–Wilson polynomials [5] constitute its complete sequence of polynomial solutions and the operator in (7) is the Askey–Wilson first-order divided difference operator [5].

2. The factorization method (FM) for the general second-order q-difference equation

2.1. The general case

Consider the general second-order q-difference eigenvalue equation

$$\left[A(x)E_q + B(x) + C(x)E_q^{-1}\right] Y_n(x) = \Lambda_n Y_n(x). \qquad (8)$$

It is not difficult to write it in the equivalent form

$$\left[\tilde{a}(x)E_q + \tilde{b}(x) + \tilde{c}(x)E_q^{-1}\right] y_n(x) = \lambda_n y_n(x) \qquad (9)$$

where $\tilde{b}(x) = -(\tilde{a}(x) + \tilde{c}(x))$. Hence we will consider (9) as the starting equation. More generally, we write (9) in the form

$$Ly_n(x) = \left[a(x)E_q + b(x) + c(x)E_q^{-1}\right] y_n(x) = \lambda(n)\theta(x)y_n(x) \tag{10}$$

where

$$a(x) = \theta(x)\tilde{a}(x) \qquad b(x) = \theta(x)\tilde{b}(x) \qquad c(x) = \theta(x)\tilde{c}(x) \tag{11}$$

for some $\theta(x) \neq 0$. Consider next the operator

$$H(x,n) = E_q[\rho(L - \lambda\theta)\rho^{-1}] = E_q^2 + (b(qx) - \lambda(n)\theta(qx))E_q + d(qx) \tag{12}$$

where

$$\rho(qx)/\rho(x) = a(x) \qquad d(x) = a(x/q)c(x). \tag{13}$$

So the eigenvalue equation (10) is 'equivalent' to

$$H(x,n)y_n(x) = 0 \tag{14}$$

in the sense that if $y_n(x)$ is a solution of (10), then $\rho(x)y_n(x)$ is a solution of (14) and conversely if $y_n(x)$ is a solution of (14), then $\rho^{-1}(x)y_n(x)$ is a solution of (10).

Consider now for H, the following type of factorization

$$\begin{aligned} H(x,n) - \mu(n) &= (E_q + g(x,n))(E_q + f(x,n)) \\ H(x,n+1) - \mu(n) &= (E_q + f(x,n))(E_q + g(x,n)) \end{aligned} \tag{15}$$

for some functions $f(x,n)$, $g(x,n)$ and constants (in x) $\lambda(n)$, $\mu(n)$. Consider next the eigenvalue equation

$$\tilde{L}y_n(x) = \left[g(x,-1)E_q - b(x) + f(x/q,-1)E_q^{-1}\right] y_n(x) = -\lambda(n)\theta(x)y_n(x) \tag{16}$$

and the operator

$$\tilde{H}(x,n) = E_q[\tilde{\rho}(\tilde{L} - \lambda\theta)\tilde{\rho}^{-1}] = E_q^2 + (b(qx) - \lambda(n)\theta(qx))E_q + \tilde{d}(qx) \tag{17}$$

where

$$\tilde{\rho}(qx)/\tilde{\rho}(x) = -g(x,-1) \qquad \tilde{d}(x) = g(x/q,-1)f(x/q,-1). \tag{18}$$

It is easily seen in this case also that the eigenvalue equation (16) is 'equivalent' to

$$\tilde{H}(x,n)y_n(x) = 0. \tag{19}$$

We can now give the main statement of this paper.

Proposition 2.1. *Suppose that there exist functions $f(x,n)$, $g(x,n)$, constants (in x) $\lambda(n)$, $\mu(n)$, for which H admits the factorization (15) with f and g such that*

$$f(x,n) - g(x,n-1) = c_1(n)h(x) + c_2(n) \tag{20}$$

$c_1(n) \neq 0, \infty$.

In that case, the following situations hold:

(i) If $\mu(-1) = 0$, then the solutions of the eigenvalue equation (16) are of classical polynomial type and satisfy the difference relations

$$\begin{aligned} y_{n+1}(x) &= (-g(x,-1)E_q + f(x,n))y_n(x) - \mu(n-1)y_{n-1}(x) = (-g(x,-1)E_q \\ &\quad + g(x,n-1))y_n(x) \qquad n = 0,1,2\ldots \end{aligned} \tag{21}$$

and the three-term recurrence relations (TTRR)

$$y_{n+1}(x) + \mu(n-1)y_{n-1}(x) = (c_1(n)h(x) + c_2(n))y_n(x) \tag{22}$$

$$y_0 = 1 \qquad y_1 = c_1(0)h(x) + c_2(0). \tag{23}$$

(ii) If $\mu(-1) \neq 0$, then the polynomials satisfying the TTRR (22) (but no longer the difference relations (21)), are no longer of classical type but purely (i.e. not semi-classical) of Laguerre–Hahn type.

(iii) If $\mu(-1) \neq 0$ but $\mu(-r-1) = 0$ for some non-vanishing number r (not necessarily an integer), then the polynomials satisfying the TTRR (22) are Laguerre–Hahn polynomials r-associated with classical polynomials.

(iv) If $\mu(-r-1) \neq 0$ for any number r, then the polynomials satisfying the TTRR (22) are Laguerre–Hahn polynomials not r-associated with polynomials of classical type.

Proof.

(i) Note first that from the relations in (15) follow, in particular, the equations

$$f(x,n)g(x,n) = d(qx) - \mu(n) \tag{24}$$

$$f(qx,n) + g(x,n) = b(qx) - \theta(qx)\lambda(n) \tag{25}$$

with

$$f(qx,n+1) + g(x,n+1) = f(x,n) + g(qx,n) \tag{26}$$

$$f(x,n+1)g(x,n+1) = f(x,n)g(x,n) + \mu(n) - \mu(n+1) \tag{27}$$

or equivalently the equations (24) and (25) together with the q-difference equation

$$\Delta_q(f(x,n) - g(x,n)) = (\lambda(n+1) - \lambda(n))\theta(x) \qquad \Delta_q = E_q - 1. \tag{28}$$

Next, substituting n with -1 in (24), one directly finds that $\mu(-1) = 0 \Rightarrow H = \tilde{H}$ (and conversely $H = \tilde{H} \Rightarrow \mu(-1) = 0$). Hence the operator $\tilde{H}$ admits, as H, a type (15) factorization. As a consequence, we have that if ψ_0 is such that

$$\tilde{H}(x,0)\psi_0 = 0 \tag{29}$$

it follows from (15) that the sequence of functions ψ_n given by

$$\begin{aligned} \psi_{n+1}(x) &= (E_q + f(x,n))\psi_n(x) \\ -\mu(n-1)\psi_{n-1}(x) &= (E_q + g(x,n-1))\psi_n(x) \qquad n = 0, 1, \ldots \end{aligned} \tag{30}$$

are solutions of (19). But from (25) ($n = 0$) and (26) ($n = -1$), it follows that $y_0 = 1$ is a solution of (16) with $n = 0$. Hence in (29), one can take $\psi_0 = \tilde{\rho}(x)$ (see (18)). This allows one to obtain (21) from (30). Finally, using (20) and (21), one obtains (22)–(23). To conclude, remark that the functions in (22) and (23) are of polynomial type. Moreover, since they are bispectral (they also satisfy (16)), they are necessarily of classical type (see [16, 17]).

(ii) To prove this, we need first of all to understand what represents $\mu(n)$ in the Laguerre–Hahn approach to orthogonal polynomials [21]. For this, consider the relations in (26) and (27), together with the relation in (20). Using (20) and translating n in (27), one finds that (20) and (26), (27) are equivalent to

$$\begin{aligned} f(x,n) &= g(x,n-1) + c_1(n)h(x) + c_2(n) \\ g(x,n) &= f(x,n-1) - (c_1(n)h(qx) + c_2(n)) \\ f(x,n)g(x,n) &= f(x,n-1)g(x,n-1) + \mu(n-1) - \mu(n). \end{aligned} \tag{31}$$

Now, suppose that some polynomials $y_n = y_n(h(x))$ satisfy the TTRR in (22). So, their normalized form $P_n = y_n(h(x))/\varrho(n)$ where

$$\varrho(n+1)/\varrho(n) = c_1(n) \tag{32}$$

satisfies

$$P_{n+1} + a_n^2 P_{n-1} = (h(x) - b_n) P_n \tag{33}$$

where

$$a_n^2 = \frac{\mu(n-1)}{c_1(n)c_1(n-1)} \qquad b_n = -c_2(n)/c_1(n). \tag{34}$$

Hence (31) may be written as

$$\begin{aligned} &f(x,n) = g(x,n-1) + (h(x) - b_n)c_1(n) \\ &g(x,n) = f(x,n-1) - (h(qx) - b_n)c_1(n)) \\ &f(x,n)g(x,n) = f(x,n-1)g(x,n-1) + a_n^2 c_1(n-1)c_1(n) - a_{n+1}^2 c_1(n)c_1(n+1). \end{aligned} \tag{35}$$

These are exactly the recurrence relations for the coefficients in the Ricatti equation for the Stieltjes function S_n of the polynomials n-associated with the polynomials in (33) (see [21]):

$$\begin{aligned} A_n(x)\frac{S_n(h(qx)) - S_n(h(x))}{h(qx) - h(x)} &= B_n(x)S_n(h(qx))S_n(h(x)) \\ &\quad + C_n(x)(S_n(h(qx)) + S_n(h(x)))/2 + c_1(n) \end{aligned} \tag{36}$$

with

$$\begin{aligned} f(x,n) &= \frac{A_{n+1}(x)}{h(qx) - h(x)} - \frac{C_{n+1}(x)}{2} \\ g(x,n) &= \frac{A_{n+1}(x)}{h(qx) - h(x)} + \frac{C_{n+1}(x)}{2} \\ B_n(x) &= a_n^2 c_1(n-1). \end{aligned} \tag{37}$$

Hence the Stieltjes function S_0 for the polynomials in (33) satisfies

$$\begin{aligned} A_0(x)\frac{S_0(h(qx)) - S_0(h(x))}{h(qx) - h(x)} &= B_0(x)S_0(h(qx))S_0(h(x)) \\ &\quad + C_0(x)(S_0(h(qx)) + S_0(h(x)))/2 + c_1(0) \end{aligned} \tag{38}$$

where

$$B_0(x) = \frac{\mu(-1)}{c_1(0)}. \tag{39}$$

Since by supposition $c_1(0) \neq \infty$, so the vanishing of $B_0(x)$ is exclusively linked to that of $\mu(-1)$. But the polynomials for which $B_0(x)$ does not vanish are of purely Laguerre–Hahn class (see [21]), which proves the non-classicality of the polynomials.

(iii) Suppose now that for some number r, $\mu(-r-1) = 0$. In this case, we can substitute n with $n-r$ in (15) and since $\tilde{\mu}(-1) = 0$, for $\tilde{\mu}(n) = \mu(n-r)$, we are led to the same conclusion of classical polynomial solutions satisfying (16), and (22) and (23) but with n replaced by $n-r$ (and 0 by $-r$) in the corresponding coefficients. Hence the polynomials satisfying (22) will be r-associated with classical polynomials.

(iv) This is a consequence of (ii) and (iii).

Remark 1. As can be verified (similarly to (ii)), equation (10) admits, under the conditions of the proposition, sequences of functional solutions satisfying difference relations as (30) and TTRR as (22). However, except in some special cases, for example when $a(x) = -g(x,-1)$, $c(x) = -f(x/q,-1)$ (in this case $\tilde{L} = -L$), these functional solutions are no longer of polynomial type. A question is that of knowing if there exists an interconnection between the functional solutions of (10) satisfying (30) and (22), and the orthogonal polynomials satisfying (22) (but not (21)). The answer is yes and it was shown in [21] that the polynomials satisfying (22) can be expressed as a combination of products of functions from two different sequences of solutions of (10) (considering (13) and (24) with $n = -1$) satisfying (22). We will note that polynomials r-associated with classical polynomials, such as any system of polynomials of Laguerre–Hahn type, satisfy not a second- but a fourth-degree difference equation (see [10]).

Remark 2. In [1], in particular the equivalence between the FM and the existence of raising and lowering operators for the hypergeometric difference equation on non-uniform lattices and some of its special forms was proved (see also [20] for the differential and discrete cases). However, in this paper we focus on the special form of the intertwining relations in (15), because of its interconnection with Laguerre–Hahn polynomials (as shown in proposition 2.1). On the other hand, we know that generic Laguerre–Hahn polynomials do not satisfy the usual three-term difference relations (i.e. no raising and lowering operators as for example in (21)) [21].

2.2. *The hypergeometric q-difference equation case*

The hypergeometric q-difference equation in (4) may be written as

$$\left[a(x)E_q + b(x) + c(x)E_q^{-1}\right] y_n(x) = \theta(x)\lambda(n)y_n(x) \tag{40}$$

with

$$a(x) = (\sigma_0 + (1-1/q)\tau_0)x + \sigma_1 + (1-1/q)\tau_1 + \sigma_2/x \qquad c(x) = q(\sigma_0 x + \sigma_1 + \sigma_2/x)$$
$$b(x) = -(a(x) + c(x)) \qquad \theta(x) = (1-1/q)x. \tag{41}$$

The results for the factorization of the type (15), for the operator

$$H(x,n) = E_q^2 + (b(qx) - \lambda(n)\theta(qx))E_q + d(qx) \tag{42}$$

$d(x) = a(x/q)c(x)$, read as follows

$$\begin{aligned} f(x,n) = {} & -\sigma_2/x - 1/2(-\tau_1 - qc_0(n) + \tau_1 q + q\sigma_1 + q^2\sigma_1)/q \\ & - (-\tau_0 + q^2\sigma_0 + q\sigma_0 + \tau_0 q + \lambda(n)q - \lambda(n+1))x/(1+q) \end{aligned}$$

$$\begin{aligned} g(x,n) = {} & -\sigma_2/x - 1/2(-\tau_1 - qc_0(n) + \tau_1 q + q\sigma_1 + q^2\sigma_1)/q - c_0(n) \\ & + (-(-\tau_0 + q^2\sigma_0 + q\sigma_0 + \tau_0 q + \lambda(n)q - \lambda(n+1))/(1+q) \\ & - \lambda(n+1) + \lambda(n))x \end{aligned} \tag{43}$$

$$\begin{aligned} \mu(n) = {} & 1/4\big(-q^6\sigma_1^2 + 2\tau_1^2 q^2 - 8q^4\sigma_2\sigma_0 + 4q^3\sigma_2\tau_0 + c_0^2(n)q^4 - q^2\sigma_1^2 \\ & + c_0^2(n)q^2 + 2\tau_1 q\sigma_1 - 4\tau_1 q^3\sigma_1 - \tau_1^2 - \tau_1^2 q^4 + 2q^4\sigma_1^2 + 2c_0^2(n)q^3 \\ & + 4q^2\sigma_2\lambda(n) + 4q^2\sigma_2\tau_0 - 4q\tau_0\sigma_2 + 4q^2\sigma_0\sigma_2 + 4q^2\lambda(n+1)\sigma_2 - 4q^4\sigma_2\tau_0 \\ & - 4q^4\sigma_2\lambda(n+1) - 4q^4\lambda(n)\sigma_2 + 2\tau_1 q^5\sigma_1 + 4q^6\sigma_2\sigma_0\big)\big/(q^2(1+q)^2) \end{aligned} \tag{44}$$

where

$$\begin{aligned} c_0(n) = {} & (-2\tau_1 q^4\sigma_0 + q^3\sigma_1\lambda(n) + q^3\sigma_1\lambda(n+1) + 2\tau_1 q^3\sigma_0 + q^2\lambda(n)\tau_1 \\ & + 2\tau_1 q^2\sigma_0 + \tau_1 q^2\lambda(n+1) + 2\tau_1 q^2\tau_0 - q\lambda(n+1)\sigma_1 - 2q\lambda(n+1)\tau_1 \\ & - 2\tau_1 q\lambda(n) - q\sigma_1\lambda(n) - 4\tau_1 q\tau_0 - 2\tau_1 q\sigma_0 + \lambda(n+1)\tau_1 \\ & + \tau_1\lambda(n) + 2\tau_1\tau_0)/(q(1+q)(\lambda(n+1) - \lambda(n))) \end{aligned} \tag{45}$$

and

$$\lambda(n) = ((1-q)q^{-n} + q^2\sigma_0/k)(q^n q\sigma_0 + q^n\tau_0 q - kq - q^n\tau_0 + k)(q-1)^{-2} \tag{46}$$

where k is a new free parameter. Equation (46) can equivalently be written as

$$\lambda(n) = -[1 - tq^{-n}]\left[\frac{q^2\sigma_0}{q-1} - \left(\frac{q\sigma_0}{q-1} + \tau_0\right)t^{-1}q^n\right] \tag{47}$$

where $t = \frac{q-1}{q^2\sigma_0}k$. We will note that all the functions of the variable n (f, g, μ) are explicit functions in $\lambda(n)$ and $\lambda(n+1)$ but implicit in n.
As can be verified, for $t = 1$, we have $\mu(-1) = \lambda(0) = 0$ (and $f(x/q, -1) = -c(x)$, $g(x, -1) = -a(x)$), and consequently the corresponding polynomials are of classical type.

If we let $t = q^{-r}$, the corresponding polynomials are Laguerre–Hahn polynomials r-associated with polynomials of classical type. It is clear that the representation of any real number t in the form $t = q^{-r}$ is not always possible unless we allow r to be complex. That is to say that if we allow r to be only real or integer number, the considered polynomials are Laguerre–Hahn polynomials not necessarily r-associated with polynomials of classical type.

Example 1 The q-Hahn case**.** In the q-Hahn case (see for example [19]), we have

$$\begin{aligned} a(x) &= \alpha(x-1)(x\beta q - q^{-N})/(x) \\ b(x) &= (x^2 - xq^{-N} - x\alpha q + q^{-N+1}\alpha)/x \end{aligned} \tag{48}$$

and the formulae for $f(x, n)$, $g(x, n)$, $\mu(n)$, $\lambda(n)$ for the factorization are obtained from those above by substituting

$$\begin{aligned} &\sigma_0 = 1/q \qquad \sigma_1 = -(q^{-N} + q\alpha)/q \qquad \sigma_2 = q^{-N}\alpha \qquad \tau_0 = (\alpha\beta q^2 - 1)/(q-1) \\ &\tau_1 = -(\alpha\beta q^2 + q^{-N+1}\alpha - q^{-N} - q\alpha)/(q-1). \end{aligned} \tag{49}$$

Example 2 The q-Big Jacobi case**.** In the q-Big Jacobi case (see for example [19]), we have

$$a(x) = aq(x-1)(bx-c)/x \qquad b(x) = (x - aq)(x - cq)/x \tag{50}$$

and the formulae for $f(x, n)$, $g(x, n)$, $\mu(n)$, $\lambda(n)$ for the factorization are obtained from those above by substituting

$$\begin{aligned} &\sigma_0 = 1/q \qquad \sigma_1 = -(a+c) \qquad \sigma_2 = aqc \\ &\tau_0 = (aq^2b - 1)/(q-1) \\ &\tau_1 = (q(a+c) - aq^2(b+c))/(q-1). \end{aligned} \tag{51}$$

The data above for the q-Hahn and q-Big Jacobi cases are clearly identical up to the correspondence: $a = \alpha$, $b = \beta$, $c = q^{-1-N}$.

2.3. *The Askey–Wilson second-order q-difference equation case*

Consider now the Askey–Wilson second-order q-difference equation in (5)–(7). The equation can also be written as

$$Ly_n(x) = \left[a(x)E_q - [a(x) + b(x)] + b(x)E_q^{-1}\right]y_n(x) = \lambda(n)\theta(x)y_n(x) \tag{52}$$

where

$$\begin{aligned} &a(x) = \frac{a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a_1x + a_2x^2}{qx - x^{-1}} \qquad b(x) = \frac{a_2x^{-2} + a_1x^{-1} + a_0 + a_{-1}x + a_{-2}x^2}{x - qx^{-1}} \\ &a_{-2} = 1 \qquad a_{-1} = -(a+b+c+d) \qquad a_0 = ab + ac + ad + bc + bd + cd \\ &a_1 = -(abc + abd + bcd + acd) \qquad a_2 = abcd \qquad \theta(x) = x - x^{-1}. \end{aligned} \tag{53}$$

The operator

$$H(x,n) = E_q^2 + (b(qx) - \lambda(n)\theta(qx))E_q + d(qx) \tag{54}$$

$d(x) = a(x/q)c(x)$, admits a factorization as the one in (15), with

$$\begin{aligned} f(x;n) &= \frac{f_{-2}x^{-2} + f_{-1}x^{-1} + f_0 + f_1 x + f_2 x^2}{qx - x^{-1}} \\ g(x;n) &= \frac{(f_{-2} - \beta_{-1})x^{-2} + (f_{-1} - \beta_0)x^{-1} + f_0 + (f_1 + \beta_0 q)x + (f_2 + \beta_1 q)x^2}{qx - x^{-1}} \end{aligned} \tag{55}$$

where

$$\begin{aligned} f_{-2}(n) &= \frac{\lambda(n) - q\lambda(n+1)}{q^2 - 1} - \frac{q + a_2}{q^2 + q} \qquad f_2(n) = \frac{\lambda(n)q - \lambda(n+1)}{1 - q^2}q^2 - \frac{q^2 + qa_2}{q+1} \\ \beta_0(n) &= \frac{1 - q}{(\lambda(n) - \lambda(n+1))q^3}\left\{\left(2\frac{\lambda(n)q - \lambda(n+1)}{1 - q^2}q^2 + \frac{\lambda(n+1) - \lambda(n)}{1 - q}q^2\right.\right. \\ &\qquad \left.\left. - 2\frac{q^2 + qa_2}{1 + q}\right)(a_1 + qa_{-1}) + (2a_1q^2 + 2a_2a_{-1}q)\right\} \\ \beta_{-1} &= \frac{\lambda(n+1) - \lambda(n)}{1 - q} \qquad \beta_1 = q\beta_{-1} \\ f_{-1}(n) &= \frac{\beta_0(n)}{2} - \frac{a_1 + qA_{-1}}{2q} \qquad f_1(n) = -\frac{q\beta_0(n)}{2} - \frac{a_1 + qa_{-1}}{2} \\ f_0(n) &= \frac{1}{q + q^2}\{q^2 - q^3 - a_0(q + q^2) + a_2(q - 1) + q^2(\lambda(n) + \lambda(n+1))\} \end{aligned} \tag{56}$$

while

$$\begin{aligned} \mu(n) &= a_0 + a_1a_{-1}q^{-1} + a_0a_2q^{-2} + f_0(n)\beta_{-1}(n) + f_{-1}(n)\beta_0(n) \\ &\qquad - 2f_{-2}(n)f_0(n) - f_{-1}^2(n) \end{aligned} \tag{57}$$

and

$$\lambda(n) = -(1 - tq^{-n})(1 - abcdt^{-1}q^{n-1}). \tag{58}$$

Here also, as one can verify, for $t = 1$, we have $\mu(-1) = \lambda(0) = 0$ (and $f(x/q, -1) = -c(x), g(x, -1) = -a(x)$), and the corresponding polynomials in (22) and (23) are of classical type. Taking $t = q^{-r}$, we obtain Laguerre–Hahn polynomials r-associated with classical polynomials. Otherwise (if such an exponential expression is not allowed for t), the corresponding polynomials are Laguerre–Hahn ones, not necessarily r-associated with classical polynomials.

An interesting outlook on which we are working is the extension of the FM to the fourth-order difference equation. It is in particular a natural expectation that the FM for such equations may allow one to derive what one can call Krall–Laguerre–Hahn polynomials.

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