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The q-version of a theorem of Bochner

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Abstract

Askey and Wilson (1985) found a family of orthogonal polynomials in the variable

$$s(k) = \tfrac{1}{2}(k + 1/k)$$

that satisfy a q-difference equation of the form

$$a(k)(p_n(s(qk)) - p_n(s(k))) + b(k)(p_n(s(k/q)) - p_n(s(k))) = \theta_n p_n(s(k)), \quad n = 0, 1, \dots .$$

We show here that this property characterizes the Askey–Wilson polynomials. The proof is based on an "operator identity" of independent interest. This identity can be adapted to prove other characterization results. Indeed it was used in (Grünbaum and Haine, 1996) to give a new derivation of the result of Bochner alluded to in the title of this paper. We give the appropriate identity for the case of difference equations (leading to the Wilson polynomials), but pursue the consequences only in the case of q-difference equations leading to the Askey–Wilson and big q-Jacobi polynomials. This approach also works in the discrete case and should yield the results in (Leonard, 1982).

Keywords: Askey–Wilson polynomials; Bispectral property

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1. Introduction

Bochner [3] proved that the only families of orthogonal polynomials $p_n(k)$ satisfying

$$L(p_0, p_1, p_2, \dots)^t = k(p_0, p_1, p_2, \dots)^t \tag{1}$$

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and

$$A\left(k,\frac{\mathrm{d}}{\mathrm{d}k}\right)p_n(k)=\theta_n p_n(k), \tag{2}$$

with L a tridiagonal semi-infinite matrix and A a second-order differential operator, are the Jacobi, Hermite, Laguerre and Bessel polynomials. The purpose of this paper is to consider the same question when A is replaced by a second-order q-difference operator

$$(Af)(k)=a(k)(f(qk)-f(k))+b(k)(f(q^{-1}k)-f(k)).$$

The case considered by Bochner is included here by passing to the limit $q=1$. An independent derivation of Bochner's result is given in [5] using a (very simplified) version of the "operator identity" to be derived below.

In the case $q=1$, the choice of the spectral parameter k is immaterial. If k is replaced by $s(k)$ in (1), the p_n's are polynomials in $s(k)$ and, using the chain rule, would still produce an operator $A(s,\mathrm{d}/\mathrm{d}s)$ satisfying (2). In the case $q\neq 1$, the change of spectral parameter would change the form of the operator A, and thus we are led to pose our problem in the following form.

Determine all instances of polynomials p_n in $s(k)$ normalized by $p_{-1}=0$, $p_0=1$, satisfying

$$p_n(s(k))=(s(k)-b_n)p_{n-1}(s(k))-a_{n-1}p_{n-2}(s(k)),\quad n\geqslant 1, \tag{1$'$}$$

and

$$a(k)(p_n(s(qk))-p_n(s(k)))+b(k)(p_n(s(q^{-1}k))-p_n(s(k)))=\theta_n p_n(s(k)),\quad n\geqslant 0. \tag{2$'$}$$

The existence of nontrivial solutions to (1′) and (2′) imposes a set of conditions on $s(k)$. A full discussion of these conditions is beyond the scope of this paper; we just remark that related issues have been considered in [13, 14]. For our purposes it suffices to observe that the two cases

(1) $s(k)=c_jk^j+c_{-j}k^{-j}$,

(2) $s(k)=c_j(\ln k)^j+c_{2j}(\ln k)^{2j}$

for arbitrary *fixed* integer j and arbitrary constants c_j, c_{-j}, c_{2j}, satisfy our conditions.

Note 1. The second case can be written in the form $c_jk^j+c_{2j}k^{2j}$ if the operator in (2′) is rewritten in the more standard "additive form" where $s(qk)$ becomes $s(k+\Delta)$ and $s(q^{-1}k)$ becomes $s(k-\Delta)$ with $\Delta=\ln q$.

In this paper we focus on case (1) but indicate in Notes 1, 2 and 5 the changes that are needed to deal with case (2). By renaming k^j and q^j, respectively, as k and q, we can then reduce $s(k)$ in case (1) to the form

$$s(k)=c\left(k+\frac{\varepsilon}{k}\right), \tag{3}$$

with arbitrary constants c and ε. By further rescaling, it is enough to consider the cases $\varepsilon=0$ and $c=1$ or $\varepsilon=1$ and $c=\frac{1}{2}$. This brings us in line with the standard choices

$$s(k)=k\quad\text{or}\quad s(k)=\frac{1}{2}\left(k+\frac{1}{k}\right).$$

Note 2. It is clear that the same type of reduction will give, in the "additive case", the standard form

$$s(k) = ck^2 + dk.$$

We can now state our main result:

Theorem 3. *The only solutions of* (1′) *and* (2′) *are given by the big q-Jacobi polynomials when* $s(k) = k$ *and by the Askey–Wilson polynomials when* $s(k) = \frac{1}{2}(k + 1/k)$.

In the case of $s(k) = k$ this result was obtained "in essence" by Hahn [8]. For a nice survey on these matters see [10].

These two families of orthogonal polynomials include by appropriate specializations and limiting procedures many other families. In particular, the q-Laguerre polynomials that feature in Section 5 of this paper can be obtained this way [9].

In [6] we have worked out an extension of Bochner's original problem (1) and (2) (in the case $q = 1$), by allowing L in (1) to be a tridiagonal *doubly* infinite matrix. In this case the solutions are no longer necessarily polynomials, and in fact the generic case involves any solution of the *hypergeometric* equation. It is also seen in [6] that the solutions of this extended Bochner's problem are intimately related to the Virasoro algebra. It is possible to work out a similar extension in the context of q-special functions and the q-Virasoro algebra. This will be reported elsewhere [7].

The paper is organized as follows. In Section 2 we state and prove a basic lemma, which we use in Section 3 to pin down the sequences a_n, b_n and θ_n in (1′) and (2′) above. In Section 4 we prove the theorem above. Finally, in Section 5, we present some examples of orthogonal polynomials satisfying higher order q-difference equations, which reduce to those investigated by Krall [11], when $q = 1$. In the spirit of [5], we see that these examples can be obtained from some of the classical orthogonal polynomials in the sense of [1], by an application of the Darboux transformation.

2. An operator identity

Rewrite (1′) and (2′) as

$$Lp = s(k)p, \quad p = (p_0 = 1, p_1(s(k)), p_2(s(k)), \dots)^t, \tag{4}$$

$$Ap = \Theta p, \tag{5}$$

with

$$L = \begin{pmatrix} b_1 & 1 & & & \\ a_1 & b_2 & 1 & & \\ & a_2 & b_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{6}$$

$s(k)$ as in (3) and the diagonal matrix $\Theta = \operatorname{diag}(\theta_0, \theta_1, \theta_2, \dots)$. The next lemma provides the q-version of a lemma used in [5] in the case $q = 1$, following a basic observation in [4].

Lemma 4. *Any solution of* (4) *and* (5) *satisfies the matrix identity*

$$(L^3\Theta - \Theta L^3) + x(L^2\Theta L - L\Theta L^2) + y(L\Theta - \Theta L) = 0, \tag{7}$$

with

$$x = -\frac{1+q+q^2}{q}, \qquad y = \varepsilon c^2 \frac{(q-1)^2(q+1)^2}{q^2}.$$

Proof. From (4) and (5) one obtains immediately that

$$\begin{aligned}
&(L^3\Theta - \Theta L^3)p + x(L^2\Theta L - L\Theta L^2)p + y(L\Theta - \Theta L)p \\
&\quad = (A(s^3p) - s^3Ap) + x(sA(s^2p) - s^2A(sp)) + y(A(sp) - sAp) \\
&\quad = a(k)(s(qk) - s(k))[s^2(k) + s^2(qk) + (1+x)s(k)s(qk) + y]p(s(qk)) \\
&\qquad + b(k)(s(q^{-1}k) - s(k))[s^2(k) + s^2(q^{-1}k) + (1+x)s(k)s(q^{-1}k) + y]p(s(q^{-1}k)).
\end{aligned}$$

Notice that the choice of $s(k)$ given in (3) cancels the two terms in the square brackets. As k varies, the $p(s(k))$ are linearly independent vectors, so that the *finite band* operator $(L^3\Theta - \Theta L^3) + x(L^2\Theta L - L\Theta L^2) + y(L\Theta - \Theta L)$ has infinite-dimensional kernel; hence it must vanish identically, proving our lemma. □

Note 5. If $s(k) = c(\ln k)^2 + d\ln k$, putting $\Delta = \ln q$, the operator identity becomes

$$(L^3\Theta - 3L^2\Theta L + 3L\Theta L^2 - \Theta L^3) - 2c\Delta^2(L^2\Theta - \Theta L^2) + \Delta^2(c^2\Delta^2 - d^2)(L\Theta - \Theta L) = 0.$$

3. Solving for L and Θ

We are now ready to exploit the lemma above to derive *necessary* conditions that L and Θ should satisfy if (4) and (5) are to hold. We restrict ourselves to the case where all a_n's are nonzero. The same requirement is needed to derive the classical result of Bochner.

In the sequel we shall denote by

$$[\alpha] = \frac{q^\alpha - 1}{q-1},$$

the q-analogue of α.

In order to solve for L and Θ, we proceed along the lines of [5]. Equating the diagonals of (7) to zero, starting with the upper one, we obtain at the $(n, n+3)$th entry the equations

$$\theta_{n+2} - \frac{[3]}{q}\theta_{n+1} + \frac{[3]}{q}\theta_n - \theta_{n-1} = 0, \quad n = 1, 2, \dots, \tag{8}$$

whose general solution is given by

$$\theta_n = q^{1-n}[n]([n]u + v) + w, \tag{9}$$

with u, v, w as free parameters. Since we can shift the θ_n's by an arbitrary constant, we may always assume that $w = 0$. For our purposes it is clear that only the ratio u/v (or v/u) plays an important role in (7). Equating the $(n, n+2)$th and $(n, n+1)$th entries to zero, we obtain, for $n \geqslant 1$,

$$(\theta_{n+1} - \theta_{n+2})b_{n+2} + (\theta_{n+1} - \theta_{n-1})b_{n+1} + (\theta_{n-2} - \theta_{n-1})b_n = 0, \tag{10}$$

and

$$(\theta_n - \theta_{n+2})a_{n+1} + (\theta_{n+1} + \theta_n - \theta_{n-1} - \theta_{n-2})a_n + (\theta_{n-3} - \theta_{n-1})a_{n-1} + (\theta_n - \theta_{n-1}) \times \frac{(b_{n+1}q - b_n)(b_{n+1} - b_n q)}{q} + \varepsilon c^2 \frac{(q-1)^2(q+1)^2}{q^2}(\theta_n - \theta_{n-1}) = 0, \tag{11}$$

where by convention $a_0 := 0$ and the θ_n's, $n \in \mathbb{Z}$, are given by (9).

Using (9) we see that the general solution of (10) depends on two free parameters b_1 and $r = b_2 - b_1$, explicitly:

$$b_n = b_1 + \frac{[n-1]z_{n-1}}{z_{2n-3}z_{2n-1}}(rq^{n-2}([3]u + v) + b_1(q^{n-2} - 1)((1-q)v + (1-q^n)u)), \tag{12}$$

with

$$z_n = v + [n]u.$$

Going now into Eq. (11) one sees, after some labor, that the general solution for a_n depends on one free parameter a_1 and is given by

$$a_n = \frac{q^{n-1}[n-1][n]z_{n-1}z_n\tilde{a}_n\tilde{\tilde{a}}_n}{(q+1)^2 z_{2n-2}z_{2n-1}^2 z_{2n}} + a_1\frac{q^{n-1}[n](v + [2]u)z_{n-1}}{z_{2n-2}z_{2n}} + \varepsilon c^2 \frac{(q-1)^2[n-1][n]z_{n-1}z_n}{z_{2n-2}z_{2n}}, \tag{13}$$

with

$$\tilde{a}_n = -r(v + [3]u) + b_1 v(q-1) + b_1 u(q^{n+1} + q^n - q^2 - 1),$$
$$\tilde{\tilde{a}}_n = -q^{n-1}r(v + [3]u) + b_1 v(q^n - q^{n-1} - q^2 + 1) + b_1 u(1 + q - q^{n-1} - q^{n+1}).$$

In summary, Θ is given by (9) and L is determined by (12) and (13).

4. Proof of the main theorem

In this section we identify the solutions of our problem with polynomials obtained from some of the basic hypergeometric series. We follow the notations in [9]. The general solution given by (9), (12) and (13) depends on four free parameters: the ratio v/u or (u/v), b_1, $r = b_2 - b_1$ and a_1. There will be two basic cases depending on whether $\varepsilon = 0$ or $\varepsilon = 1$.

The *basic hypergeometric series* (or q-hypergeometric series) ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}\middle| q; k\right) = \sum_{j=0}^{\infty} \frac{(a_1, \dots, a_r; q)_j}{(b_1, \dots, b_s; q)_j}(-1)^{(1+s-r)j} q^{(1+s-r)j(j-1)/2} \frac{k^j}{(q;q)_j},$$

where $(a;q)_j$ denote the q-shifted factorials

$$(a;q)_0 = 1, \qquad (a;q)_j = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{j-1}), \quad j = 1,2,3,\ldots,$$

and

$$(a_1,\ldots,a_r;q)_j = (a_1;q)_j(a_2;q)_j\cdots(a_r;q)_j.$$

The *Askey–Wilson polynomials* [2] are defined in [9] by

$$\tilde{p}_n(s;a,b,c,d\,|\,q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ak, ak^{-1} \\ ab, ac, ad \end{matrix}\,\middle|\, q;q\right), \quad s(k) = \frac{1}{2}\left(k + \frac{1}{k}\right).$$

They satisfy the recurrence relation

$$s\tilde{p}_n(s) = A_n\tilde{p}_{n+1}(s) + \left[\frac{a}{2} + \frac{1}{2a} - (A_n + C_n)\right]\tilde{p}_n(s) + C_n\tilde{p}_{n-1}(s),$$

with

$$\begin{aligned} A_n &= \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{2a(1-abcdq^{2n-1})(1-abcdq^{2n})}, \\ C_n &= \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{2(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \end{aligned} \tag{14}$$

They are known to satisfy the following second-order q-difference equation:

$$a(k)(\tilde{p}_n(s(qk)) - \tilde{p}_n(s(k))) + a(k^{-1})(\tilde{p}_n(s(q^{-1}k)) - \tilde{p}_n(s(k))) = \theta_n\tilde{p}_n(s(k)),$$

with

$$a(k) = \frac{(1-ak)(1-bk)(1-ck)(1-dk)}{(1-k^2)(1-qk^2)}, \qquad \theta_n = (q^{-n}-1)(1-abcdq^{n-1}).$$

The *big q-Jacobi polynomials* were introduced in [1]. Here we adopt the normalization given in [10] (see also [9, p. 58]):

$$\tilde{p}_n(k;a,b,c,d\,|\,q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, qak/c \\ qa, -qad/c \end{matrix}\,\middle|\, q;q\right).$$

Their monic version $p_n(k;a,b,c,d\,|\,q)$ (in short $p_n(k)$) satisfies the recurrence relation

$$p_n(k) = (k - b_n)p_{n-1}(k) - a_{n-1}p_{n-2}(k), \quad n \geqslant 1,$$

with

$$\begin{aligned} a_n &= \frac{q^{n-1}(1-q^n)(1-q^na)(1-q^nb)(1-q^nab)(d+q^nbc)(c+q^nad)}{(1-q^{2n-1}ab)(1-q^{2n}ab)^2(1-q^{2n+1}ab)}, \\ b_n &= \frac{q^{2(n-1)}(1+q)(ab(d-c)+ad-bc) + q^{n-1}(1+abq^{2n-1})(c-d+bc-ad)}{(1-abq^{2n})(1-abq^{2n-2})} \end{aligned} \tag{15}$$

and the following second-order q-difference equation

$$a(k)(p_n(qk) - p_n(k)) + b(k)(p_n(q^{-1}k - p_n(k)) = \theta_n p_n(k),$$

with

$$a(k) = \frac{(akq - c)(bkq + d)}{k^2 q}, \qquad b(k) = \frac{(k - c)(k + d)}{k^2},$$

$$\theta_n = (q^{-n} - 1)(1 - abq^{n+1}).$$

With these preliminaries, we can now prove Theorem 3 stated in the Introduction.

Proof of Theorem 3. We consider the two cases:

Case a: $\varepsilon = 0$ *and* $c = 1$. Substituting

$$u = x_1(q-1)^2, \qquad v = \frac{(q-1)(x_1 q - 1)}{q},$$

$$b_1 = \frac{x_2 + x_3 q}{1 - q^2 x_1},$$

$$r = \frac{(q-1)(x_1 x_3 q^4 + 2x_1 x_2 q^3 + (x_1 x_2 + x_3) q^2 + 2x_3 q + x_2)}{(x_1 q^2 - 1)(x_1 q^4 - 1)},$$

$$a_1 = \frac{(1-q)(x_1^2 x_4 q^4 - x_1 x_2 x_3 q^3 - (x_1(x_2^2 + 2x_4) + x_3^2) q^2 - x_2 x_3 q + x_4)}{(1 - x_1 q^2)^2 (1 - x_1 q^3)},$$

with $x_1 = ab$, $x_2 = c - d$, $x_3 = ad - bc$, $x_4 = cd$, into (12) and (13) leads to the recursion relation (15) satisfied by the monic big q-Jacobi polynomials. The above formulas show that the set of parameters v/u (or u/v), b_1, r and a_1 *is equivalent to the set of parameters* x_1, x_2, x_3 *and* x_4. Since the big q-Jacobi polynomials depend only on x_1, x_2, x_3 and x_4, this case corresponds exactly to the big q-Jacobi polynomials.

Case b: $\varepsilon = 1$ *and* $c = \frac{1}{2}$. Let $x_1 = a + b + c + d$, $x_2 = ab + ac + ad + bc + bd + cd$, $x_3 = abc + abd + acd + bcd$ and $x_4 = abcd$ be the symmetric functions of a, b, c, d. Then, substituting

$$u = \frac{(q-1)^2 x_4}{q^2}, \qquad v = \frac{(1-q)(q - x_4)}{q^2},$$

$$b_1 = \frac{x_3 - x_1}{2(x_4 - 1)},$$

$$r = \frac{(1-q)}{2(x_4 - 1)(x_4 q^2 - 1)} [x_3(2 + (1 + x_4)q) - x_1(1 + x_4(2q + 1))],$$

$$a_1 = \frac{(1-q)(x_4^3 - x_2 x_4^2 + (x_1 x_3 - x_4)x_4 - (x_1^2 - 2x_2)x_4 - x_4 - x_3^2 + x_1 x_3 - x_2 + 1)}{4(1 - x_4)^2 (1 - x_4 q)},$$

into (12) and (13) leads to the recursion relations of the monic Askey–Wilson polynomials, that is $a_n = A_{n-1} C_n$ and $b_n = \frac{1}{2}a + (1/2a) - (A_{n-1} + C_{n-1})$, with A_n, C_n given by (14). Since the

Askey–Wilson polynomials just depend on the symmetric functions of a, b, c, d *and one can solve* the above relations uniquely for x_1, x_2, x_3 and x_4 in terms of v/u (or u/v), b_1, r and a_1, the proof of the theorem is complete. □

5. Some examples of order 4

The results above show that as long as one considers orthogonal polynomials satisfying second-order q-difference equations, one cannot get away from the big q-Jacobi and the Askey–Wilson polynomials, i.e., the "classical orthogonal polynomials" in the sense of [1]. In the case $q = 1$, Krall [11] found orthogonal polynomials satisfying fourth-order differential equations. In [5] we have shown that his examples can be obtained by application of the "Darboux process" to some (carefully chosen) classical orthogonal polynomials. Here we exhibit some examples of orthogonal polynomials satisfying fourth-order q-difference equations produced by an application of the same method and which reduce as $q \to 1$ to some of the examples found by Krall.

First we recall a convenient formulation of the Darboux process in the context of semi-infinite tridiagonal matrices.

Consider the tridiagonal matrix L defined in (6), and attempt to factorize it as a product of an upper and lower tridiagonal matrix

$$L = A \cdot B, \tag{16}$$

where the first factor is

$$A = \begin{pmatrix} \alpha_1 & 1 & & & & \\ 0 & \alpha_2 & 1 & & & \\ & 0 & \alpha_3 & 1 & & \\ & & 0 & \alpha_4 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

and the second one is

$$B = \begin{pmatrix} 1 & 0 & & & & \\ \beta_1 & 1 & 0 & & & \\ & \beta_2 & 1 & 0 & & \\ & & \beta_3 & 1 & 0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Other ways of "normalizing" the matrices L, A and B are possible of course. The convention adopted here leads to monic orthogonal polynomials.

Eq. (16) amounts to

$$b_n = \alpha_n + \beta_n, \quad a_n = \alpha_{n+1}\beta_n, \qquad n = 1, 2, \ldots,$$

from which we see that one can solve recursively for $\beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \ldots$, in terms of α_1 and the entries of L:

$$\beta_1 = b_1 - \alpha_1, \qquad \alpha_2 = \frac{a_1}{b_1 - \alpha_1},$$

$$\beta_2 = b_2 - \frac{a_1}{b_1 - \alpha_1}, \qquad \alpha_3 = \frac{a_2}{b_2 - \dfrac{a_1}{b_1 - \alpha_1}}, \text{ etc.}$$

The crucial observation here is that α_1 is a *free parameter*. Suppose now we form the product in the reversed order,

$$\tilde{L} = B \cdot A,$$

then $\tilde{L}$ is a new tridiagonal matrix of the form (6). We shall call $\tilde{L}$ the *Darboux transform* of L. With the convention that $\beta_0 = 0$, its entries $\tilde{b}_n, \tilde{a}_n$ are given by:

$$\begin{aligned} &\tilde{b}_n = \beta_{n-1} + \alpha_n = b_n + \beta_{n-1} - \beta_n, \quad n = 1, 2, \ldots, \\ &\tilde{a}_1 = \alpha_1 \beta_1, \qquad \tilde{a}_n = \beta_n \alpha_n = a_{n-1} \frac{\beta_n}{\beta_{n-1}}, \quad n = 2, 3, \ldots, \end{aligned} \tag{17}$$

We are now ready to tackle two examples to illustrate the method.

5.1. The q-Krall Laguerre polynomials

The *q-Laguerre polynomials* are defined in [9] by

$$L_n^{(\alpha)}(x; q) = \frac{1}{(q; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix} \middle| q; q^{n+\alpha+1} \right).$$

They satisfy the recurrence relation

$$\begin{aligned} -q^{2n+\alpha+1} x L_n^{(\alpha)}(x; q) = (1 - q^{n+1}) L_{n+1}^{(\alpha)}(x; q) - [(1 - q^{n+1}) + q(1 - q^{n+\alpha})] L_n^{(\alpha)}(x; q) \\ + q(1 - q^{n+\alpha}) L_{n-1}^{(\alpha)}(x; q), \end{aligned}$$

and the q-difference equation

$$\frac{q^\alpha(1 + x)}{x} (L_n^{(\alpha)}(qx; q) - L_n^{(\alpha)}(x; q)) + \frac{1}{x} (L_n^{(\alpha)}(x/q; q) - L_n^{(\alpha)}(x; q)) = q^\alpha(q^n - 1) L_n^{(\alpha)}(x; q).$$

Putting $x = (1 - q)k$, we obtain that the monic q-Laguerre polynomials $p_n^{(\alpha)}(k)$ satisfy

$$k p_n^{(\alpha)} = p_{n+1}^{(\alpha)} + b_{n+1} p_n^{(\alpha)} + a_n p_{n-1}^{(\alpha)},$$

with

$$a_n = \frac{[n][n + \alpha]}{q^{4n+2\alpha-1}}, \qquad b_n = \frac{[n] + q[n - 1 + \alpha]}{q^{2n+\alpha-1}}.$$

Consider now the case $\alpha = 1$ and pick

$$\alpha_1 = \frac{[R]}{[1+R]}.$$

We obtain for the factorization of L (see (16)):

$$\alpha_n = \frac{[n]([R+1] + q[n-2])}{q^{2n-1}([R+1] + q[n-1])},$$

$$\beta_n = \frac{[n]([R+1] + q[n])}{q^{2n}([R+1] + q[n-1])}.$$

Now permuting the factors in (16) gives $\tilde{L} = B \cdot A$, with entries $\tilde{a}_n$ and $\tilde{b}_n$ computed according to (17). When $q \to 1$, these can be seen to be precisely the coefficients of the recursion relations for the monic Krall–Laguerre polynomials (see [5, 11]). One can see that the (monic) polynomials defined from the $\tilde{L}$ alluded to above satisfy the following fourth-order q-difference equation:

$$\sum_{j=-2}^{+2} A_j(k)(p_n(q^j k) - p_n(k)) = \theta_n p_n(k), \tag{18}$$

with

$$A_2(k) = \frac{(kq-k-1)(kq(q-1)-1)}{k^2(q+1)(q-1)^4},$$

$$A_1(k) = \frac{(kq-k-1)(kq^3(q^R-1) - kq(q^{R+1}-1) + q^2+1)}{k^2 q(q-1)^4},$$

$$A_{-1}(k) = -\frac{kq^2(q^R-1) - k(q^{R+1}-1) + q^2+1}{k^2(q-1)^4},$$

$$A_{-2}(k) = \frac{q^2}{k^2(q-1)^4(q+1)},$$

$$\theta_n = [n]\left(\frac{[n+1]}{q+1} + q[R]\right).$$

An appropriate limit of (18) gives the equation found by Krall [11].

5.2. *The q-Krall Jacobi polynomials*

The monic *little q-Jacobi polynomials* $p_n(k; a, b \mid q)$ can be defined in terms of the monic big q-Jacobi polynomials $p_n(k; a, b, c, d \mid q)$ by the formula (see [10])

$$p_n(k; a, b \mid q) = p_n(k; b, a, 1, 0 \mid q).$$

They satisfy the recursion relation given by

$$a_n = A_{n-1}C_n, \qquad b_n = A_{n-1} + C_{n-1},$$

with

$$A_n = q^n \frac{(1 - aq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})},$$

$$C_n = aq^n \frac{(1 - q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}.$$

Consider now the special case

$$a = q, \qquad b = q^A,$$

and define

$$\alpha_1 = \frac{[R]}{[A+2][R+A+1]}.$$

We obtain for the factorization of L:

$$\alpha_n = \frac{q^n[n][n+A]\tau(n-1)}{[2n+A][2n+A-1]\tau(n)},$$

$$\beta_n = \frac{q^{n-1}[n][n+A]\tau(n+1)}{[2n+A][2n+A+1]\tau(n)},$$

with

$$\tau(n) = q^R[n-1][n+A] + q^{n-1}[n+A+R],$$

which defines the Darboux transform $\tilde{L}$ of L via (17). When $q \to 1$, the resulting formula for $\tilde{L}$ reduces to the coefficients of the recursion relations of the Krall–Jacobi polynomials (see [5, 11]). One can see that the (monic) polynomials defined from $\tilde{L}$ satisfy the fourth-order q-difference equation of the type (18) with

$$A_2(k) = \frac{(kq^{A+1} - 1)(kq^{A+2} - 1)}{q^{2A+3}k^2},$$

$$A_1(k) = -\frac{(q+1)(kq^{A+1} - 1)[k(q^{R+A+2} + q^{R+1} + q^2 - q) - q^R(1 + q^2)]}{q^{R+2A+4}k^2},$$

$$A_{-1}(k) = -\frac{(q+1)(k-1)[k(q^{R+A+1} + q^R + q - 1) - q^R(1 + q^2)]}{q^{R+2A+3}k^2},$$

$$A_{-2}(k) = \frac{(k-1)(k-q)}{q^{2(A+1)}k^2},$$

$$\begin{aligned}\theta_n = {} & q^{-(2n+2A+R+3)}(q^n - 1)(q^{n+A+1} - 1) \\ & \times (q^{R+2n+A+2} - q^{R+n+A+1} - q^{R+n} + q^{R+1} - q^{n+2} + q^n).\end{aligned}$$

An appropriate limit of this equation gives back the equation found by Krall [11].

Remark. We thank Mourad Ismail for pointing out that our two examples of orthogonal polynomials satisfying fourth-order q-difference equations are built by starting from special, or limiting, instances of the *little* q-Jacobi polynomials. It is a challenge to find examples built from the *big* q-Jacobi polynomials or from the Askey–Wilson polynomials.

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