

The transformation of polynomial eigenfunctions of linear second-order difference operators: a special case of Meixner polynomials.

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Abstract. As polynomial eigenfunctions of a linear second-order difference operator, a special case of the Meixner polynomials, $M_n^{(2,c)}(x+1)$, is transformed explicitly, into new complete sequences of non-classical orthogonal polynomials.

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1. INTRODUCTION.

Consider the factorization chain

$$(1) \quad \begin{aligned} H_j(x) - \mu_j &= L_j(x)R_j(x) \\ H_{j+1}(x) - \mu_j &= R_j(x)L_j(x), \quad j \in \mathbb{Z}, \quad x \in \mathcal{C} \end{aligned}$$

where H_j is a second-order linear operator, and L_j and R_j are first order linear operators. Such a factorization chain can be used to generate "exactly solvable" (in complete sequences of orthogonal polynomial solutions) second-order linear operator. For that, three kinds of methods can be applied.

The first method consists in imposing to the factorization chain "quasi-periodicity" conditions (see [27]). Such a method has been used to generate the Charlier, Kravchuk and Meixner polynomials (see [4]) as well as their q -versions (see [27]). It has been used also to generate a non-classical sequence of orthogonal polynomials related to the Hermite polynomials (see [7]).

The second method consists in imposing to the factorization chain, another kind of self-similarity, a particular case of the so-called "shape-invariance" (see f.e. [15]). More precisely, one demands that the variable j in Eq. (1) acts not simply as an index but as a full independent variable. The method has been applied in [20] to generate the Charlier, Kravchuk, Meixner and Hahn polynomials. Also, the method has been applied to the cases of hypergeometric polynomials on linear lattices (see [2]) and on q -nonlinear lattices (see [3]). It can also be applied to generate special cases of the so-called semiclassical or more generally Laguerre-Hahn polynomials [17, 18], as it can be seen from [4, 5]. Note that the shape-invariance method was used in [16] to generate a non-classical sequence of orthogonal polynomials related to the Jacobi polynomials.

The third method (in which we are interested here) consists in imposing to the factorization chain that two successive links belong to totally different families. In that case, an operator (Hamiltonian) in a given position in the chain is seen as a transformation of the operator lying in the preceding one. The method was used in [26] to modify the Hermite polynomials into non-classical sequences of orthogonal polynomials (including the above cited ones from [7]). In this work, the method is used to modify efficiently, the special case of the Meixner polynomials, $M_n^{(2,c)}(x+1)$, as polynomial eigenfunctions of a linear second-order difference operator, into new complete sequences of non-classical orthogonal polynomials.

In the following section, we give a detailed discussion of the third method when applied to a general second-order linear difference operator and especially when applied to the second-order difference operator of hypergeometric type (see Eq. (15)). One will note for example the link between the method and the commonly used Christoffel and Geronimus transformations (see f.e. [10, 11, 28]). The third section, the main part of this work, is devoted to the evoked explicit transformation of the special Meixner $M_n^{(2,c)}(x+1)$ polynomials. The last section is essentially a concluding one. Some outlooks are also given there. Lastly, an appendix is given. It contains, for some instances of new polynomials, a collection in tables of a few important data such as intervals of orthogonality, weight functions, the first four polynomials.

2. GENERALITIES.

Let us first consider a general situation. The studied second-order difference operator H ,

$$(2) \quad H(x) = u(x)\mathbf{E}_x + v(x) + w(x)\mathbf{E}_x^{-1},$$

$\mathbf{E}_x^i[h(x)] = h(x+i)$, $i \in \mathbb{Z}$, is in the first place, supposed to act in a linear space of functions, say \mathcal{L} , and to admit a non-empty set of eigenfunctions. One can pick $(\Psi_{\hat{\alpha}}, \lambda_{\hat{\alpha}})$ as one of its pairs of eigenfunctions and corresponding eigenvalues. In that case, one can factorize H and then define \tilde{H} as follows

$$(3) \quad \begin{aligned} H - \lambda_{\hat{\alpha}} &= L_{\hat{\alpha}}R_{\hat{\alpha}} \\ \tilde{H} - \lambda_{\hat{\alpha}} &= R_{\hat{\alpha}}L_{\hat{\alpha}} \end{aligned}$$

where

$$(4) \quad \begin{aligned} R_{\hat{\alpha}} &= 1 + f_{\hat{\alpha}}(x)\mathbf{E}_x^{-1}; & L_{\hat{\alpha}} &= u(x)\mathbf{E}_x + g_{\hat{\alpha}}(x); \\ f_{\hat{\alpha}}(x) &= -\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)}; & g_{\hat{\alpha}}(x) &= -w(x)\frac{\Psi_{\hat{\alpha}}(x-1)}{\Psi_{\hat{\alpha}}(x)}. \end{aligned}$$

As it can be easily seen, the similarity reductions

$$(5) \quad \rho H \rho^{-1}; \quad \tilde{\rho} \tilde{H} \tilde{\rho}^{-1}$$

where

$$(6) \quad \frac{\rho^2(x+1)}{\rho^2(x)} = \frac{u(x)}{f_{\hat{\alpha}}(x+1)g_{\hat{\alpha}}(x+1)} = \frac{u(x)}{w(x+1)}; \quad \frac{\tilde{\rho}^2(x+1)}{\tilde{\rho}^2(x)} = \frac{u(x)}{f_{\hat{\alpha}}(x+1)g_{\hat{\alpha}}(x)},$$

allow to transform H and \tilde{H} into their *formal* symmetric form (i.e. like $A(x+1)\mathbf{E}_x + B(x) + A(x)\mathbf{E}_x^{-1}$). Denote by $\ell^2(a, b; \rho^2)$ the linear space of vectorial functions $(\psi(a), \psi(a+1), \dots, \psi(b))$, in which is defined a discrete-weighted inner product

$$(7) \quad (\psi, \phi)_{\rho} = \sum_a^b \psi(x)\phi(x)\rho^2(x), \quad -\infty < a < b \leq +\infty$$

(clearly, such a summation is performed on $a, a+1, \dots, b$). Such defined $\ell^2(a, b; \rho^2)$ is well known to be a separable Hilbert space (see for example [1], for a general theory).

The similar space for $\tilde{\rho}^2$ will be denoted by $\tilde{\ell}^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$.

Letting $\{\Psi_{\alpha}, \lambda_{\alpha}\}$, $\alpha \in N_{\alpha}$ (a denumerable index set, not containing $\hat{\alpha}$), be a set of eigenfunctions and corresponding mutually different eigenvalues of H , one easily finds that the set $(\tilde{\Psi}_{\hat{\alpha}}, \lambda_{\hat{\alpha}})$, $\{\tilde{\Psi}_{\alpha}, \lambda_{\alpha}\}$, $\alpha \in N_{\alpha}$ where

$$(8) \quad \begin{aligned} \tilde{\Psi}_{\alpha} &= R_{\hat{\alpha}}\Psi_{\alpha} \\ L_{\hat{\alpha}}\tilde{\Psi}_{\alpha} &= 0, \end{aligned}$$

is a resulting set of eigenfunctions and corresponding mutually different eigenvalues of \tilde{H} . Moreover, the completeness of $\tilde{\Psi}_{\hat{\alpha}}, \tilde{\Psi}_{\alpha}$, $\alpha \in N_{\alpha}$ in $\tilde{\ell}^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$ can be deduced from that of Ψ_{α} , $\alpha \in N_{\alpha}$ in $\ell^2(a, b; \rho^2)$ in the sense of the following proposition.

Proposition 2.1. *Supposing that*

(c1) $L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}} \in \ell^2(a, +\infty; \rho^2)$, and $\{R_{\hat{\alpha}}\Theta_{\alpha}\} \in \tilde{\ell}^2(a+1, +\infty; \tilde{\rho}^2)$, for $\{\Theta_{\alpha}\}$ and $\tilde{\Theta}_{\hat{\alpha}}$ belonging to the first and second space respectively.

(c2) $\Theta_{\alpha}(x) = \tilde{\Theta}_{\hat{\alpha}}(x) = 0$, $x < a+1$.

(c3), $\alpha \in N_{\alpha}$, $\{\Theta_{\alpha}\}$ is complete in $\ell^2(a+1, +\infty; \rho^2)$.

Then

(r1) The operators $L_{\hat{\alpha}}$ and $R_{\hat{\alpha}}$ are $(\rho^2, \tilde{\rho}^2)$ -mutually adjoint, in the sense that

$$(9) \quad (\tilde{\Theta}_{\hat{\alpha}}, R_{\hat{\alpha}}\Theta_{\alpha})_{\tilde{\rho}^2} = (L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}}, \Theta_{\alpha})_{\rho^2}$$

(r2) $\tilde{\Theta}_{\hat{\alpha}}^0$ and $\{R_{\hat{\alpha}}\Theta_{\alpha}\}$, $\alpha \in N_{\alpha}$, are complete in $\tilde{\ell}^2(a+1, +\infty; \tilde{\rho}^2)$, where

$$\tilde{\Theta}_{\hat{\alpha}}^0(x) = \begin{cases} \tilde{Y}_{\hat{\alpha}}(x), & x \geq a+1 \\ 0, & x < a+1; \end{cases}$$

$\tilde{Y}_{\hat{\alpha}}(x)$ being defined by

$$(10) \quad L_{\hat{\alpha}}\tilde{Y}_{\hat{\alpha}}(x) = 0.$$

Proof: Simple summation by parts gives

$$\begin{aligned} (L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}}, \Theta_{\alpha})_{\rho^2} &= \sum_a^{+\infty} u(x)\tilde{\Theta}_{\hat{\alpha}}(x+1)\Theta_{\alpha}(x)\rho^2(x) \\ &\quad + \sum_a^{+\infty} g_{\hat{\alpha}}(x)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_{\alpha}(x)\rho^2(x) \\ &= \sum_{a+1}^{+\infty} u(x-1)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_{\alpha}(x-1)\rho^2(x-1) \\ &\quad + \sum_a^{+\infty} g_{\hat{\alpha}}(x)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_{\alpha}(x)\rho^2(x) \\ &= \sum_{a+1}^{+\infty} u(x-1)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_{\alpha}(x-1)\rho^2(x-1) \\ (11) \quad &\quad + \sum_{a+1}^{+\infty} g_{\hat{\alpha}}(x)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_{\alpha}(x)\rho^2(x), \end{aligned}$$

where we used (c2) for the last equality. On the other side

$$(12) \quad \begin{aligned} (\tilde{\Theta}_{\hat{\alpha}}, R_{\hat{\alpha}}\Theta_{\alpha})_{\tilde{\rho}^2} &= \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x)[\Theta_{\alpha}(x) + f_{\hat{\alpha}}(x)\Theta_{\alpha}(x-1)]\tilde{\rho}^2(x) \\ &= \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x)\Theta_{\alpha}(x)\tilde{\rho}^2(x) + \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x)f_{\hat{\alpha}}(x)\Theta_{\alpha}(x-1)\tilde{\rho}^2(x). \end{aligned}$$

Remarking that, the equality of Eq. (11) and Eq. (12) requires the conditions set by the definitions in Eq. (6), this proves (r1).

Remark next that for proving the completeness of the system in (r2) in $\tilde{\ell}^2(a+1, +\infty; \tilde{\rho}^2)$, it suffices to prove its closure in that space seen that the latter is separable and Hilbertian, while the equivalence between the completeness and the closure of systems in such spaces is a well-known result (see f.e. [13], theorem 4, parag.4, chap.3).

Let $\tilde{\Theta}_{\hat{\alpha}}$ be an element satisfying (c1). One can suppose without contradicting any hypothesis that it satisfies (c2) as well. Suppose next that, that element is orthogonal to the full set $\{R_{\hat{\alpha}}\Theta_{\alpha}\}$. So by the already proved (r1) and using (c3) and (c2), one obtains that the unique possibly non-vanishing coordinate of $L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}}$ is the one in the a^{th} place. Clearly, this coordinate reads $u(a)\tilde{\Theta}_{\hat{\alpha}}(a+1)$. But as it is easily seen, up to a multiplication by a constant, this is exactly the structure of the semi-infinite vector $L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}}^0$. In other words, $\tilde{\Theta}_{\hat{\alpha}}^0$ is essentially the unique element orthogonal to the whole set $\{R_{\hat{\alpha}}\Theta_{\alpha}\}$ which proves (r2) and the proposition is completely proved.

The adjointness such as that in Eq. (9) was intensively used in [12, 21] to give simplest proofs of the orthogonality relations for most of the classical polynomials (including the Askey-Wilson class). In those works, the role of $L_{\hat{\alpha}}$ and $R_{\hat{\alpha}}$ were played by the usual difference relations (considered there as the starting point) lowering and raising the degree of the polynomials and "perturbing" the parameters. Thus, ρ^2 and $\tilde{\rho}^2$ differed only by a "perturbation" of parameters.

Here, ρ^2 and $\tilde{\rho}^2$ will differ more than in the unique shape: We expect $\tilde{\rho}^2$ to be "non-classical" for given "classical" ρ^2 . This will be done efficiently on the special case of Meixner polynomials $M_n^{(2,c)}(x+1)$.

For treating the question of orthogonality for the new functions $\tilde{\Psi}_{\hat{\alpha}}$, $\tilde{\Psi}_{\alpha}$, $\alpha \in N_{\alpha}$, in $\tilde{\ell}^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$, one can adopt the following way. Firstly, certify, using the reasoning of the preceding proposition, the orthogonality of $\tilde{\Psi}_{\hat{\alpha}}$ and every $\tilde{\Psi}_{\alpha}$, $\alpha \in N_{\alpha}$ (provided that $\tilde{a} = a+1; \tilde{b} = b + \infty$). Secondly, deduce the orthogonality of

$\tilde{\Psi}_\alpha$ and $\tilde{\Psi}_\beta$ for $\alpha \neq \beta$, $\alpha, \beta \in N_\alpha$, as that of eigenfunctions corresponding to mutually different eigenvalues for a symmetric operator \tilde{H} , provided that (obtained using simple summation by parts)

$$(13) \quad u(x)\tilde{\rho}^2(x) (\tilde{\Psi}_\alpha\Delta\tilde{\Psi}_\beta - \tilde{\Psi}_\beta\Delta\tilde{\Psi}_\alpha) \Big|_{\tilde{a}-1}^{\tilde{b}} = 0,$$

where $\Delta = \mathbf{E}_x - 1$. It is this way that will be followed later to verify the orthogonality relations for the functions (polynomials) modifying the already evoked special case of Meixner polynomials $M_n^{(2,c)}(x+1)$.

Thus, from a Sturm-Liouville operator for which the eigenfunction expansion is known, we can generate a new (in the sense that the operators must belong to different families) Sturm-Liouville operator and the corresponding eigenfunction expansion.

But as we are "transforming" polynomials, the situation is that, the "transformation function" $\Psi_{\hat{\alpha}}$ can't be generally taken as a polynomial, while the "transformed functions", $\tilde{\Psi}_{\hat{\alpha}}$, $\tilde{\Psi}_\alpha$, $\alpha \in N_\alpha$ need to be polynomials or rational functions satisfying additional constraints. So, for a given Sturm-Liouville difference operator, a copious reservoir of "good" transformation functions can't be expected.

Remark that, in what precedes, the transformation function was deliberately chosen outside of the "transformable system" Ψ_α , $\alpha \in N_\alpha$. Supposing that the transformable functions $\Psi_\alpha(x)$ are, not only explicit functions of x , but also of α , so it has also a sense to chose as the transformation function, a function Ψ_γ for which $\gamma \in N_\alpha$. In that case, the new functions need to be searched as

$$(14) \quad \tilde{\Psi}_\alpha = \frac{R_\gamma\Psi_\alpha}{\lambda_\alpha - \lambda_\gamma},$$

so that the transformed $\tilde{\Psi}_\gamma$ must be understood as a limit (Hospital's rule) of the right hand side of Eq. (14) for $\alpha \rightarrow \gamma$. In orthogonal polynomials theory, transformations as in Eq. (14) (we mean here, when the acting functions are explicit functions of the independent variable α) are referred to as the *Christoffel transformations* having as inverse, the so-called *Geronimus transformations*. Much interesting studies and applications of those transformations can be found in [28, 8] and references therein.

The second-order difference operator in which we are interested here is the hypergeometric difference operator on linear lattices [22]

$$(15) \quad H = \sigma\Delta\nabla + \tau\Delta = (\sigma + \tau)\mathbf{E}_x - (2\sigma + \tau) + \sigma\mathbf{E}_x^{-1}$$

σ and τ being polynomials in x of second (at most) and first degree respectively.

Let $\lambda_n = n\tau' + \frac{1}{2}\sigma''n(n-1)$; $P_n(x)$, $n = 0, 1, 2, \dots$, be its eigenvalues and the corresponding polynomial eigenfunctions(see for ex. [22]). According to the above discussions, we can transform the polynomials as

$$(16) \quad \tilde{P}_n(x) = P_n(x) + f_{\hat{\alpha}}(x)P_n(x-1) = (1 + f_{\hat{\alpha}}(x)\mathbf{E}_x^{-1})P_n(x)$$

where

$$(17) \quad f_{\hat{\alpha}}(x) = -\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)},$$

provided that the function $\Psi_{\hat{\alpha}}(x)$ does not belong to the sequence $P_n(x)$, $n = 0, 1, 2, \dots$. Moreover, since the hypergeometric polynomials on linear lattices are (if taken in their canonical forms), not only polynomials in x but also in n (called dual polynomials), we can also use the Christoffel transformation (we here shifted x comparatively to Eq. (14), in order that the obtained polynomials (in n) be of degree exactly equal to x):

$$(18) \quad \tilde{P}_x^{(\gamma)}(n) = \frac{P_n(x+1) - \frac{P_\gamma(x+1)}{P_\gamma(x)}P_n(x)}{\lambda_n - \lambda_\gamma}.$$

Thus by Eq. (18), one can transform the polynomials dual to the hypergeometric polynomials (recall that for the Charlier, Meixner and Kravchuk polynomials, there exists self-duality) into non-classical polynomials with rational coefficients in the three-term recurrence relation (in [8], one can find detailed discussions

concerning the applications of the Christoffel transformation to any family of orthogonal polynomials admitting a dual sequence of orthogonal polynomials i.e. q -Racah polynomials and specializations (in (18), the situation is similar to that of [8], the variables x and n having interchanged their roles)).

Here, the transformation interesting us is the one given by Eqs. (16)-(17). In this case, for already explained reasons, the transformation function $\Psi_{\hat{\alpha}}(x)$ would lie outside the set of transformable polynomials $P_n(x)$. Moreover as it can be easily seen, if we expect that the transformed polynomials be of different degree (and so be independent), we must avoid the choice of $\Psi_{\hat{\alpha}}(x)$ as a polynomial. However, for evident reasons, the ratio $\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)}$ must be a rational function, say $\frac{N_{\hat{\alpha}}(x)}{D_{\hat{\alpha}}(x)}$. Thus, if for example the new ground state (provided it belongs to $\tilde{\ell}^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$) appears in the form $\tilde{\Psi}_{\hat{\alpha}(x)} = \frac{\hat{N}_0(x)}{D_{\hat{\alpha}}(x)}$, the new polynomials should be checked inside the set constituted of $\hat{N}_0(x)$ and the different numerators of the fractions

$$(19) \quad \tilde{P}_n(x) = P_n(x) - \frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)} P_n(x-1), \quad n = 0, 1, \dots$$

where

$$(20) \quad \frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)} = \frac{N_{\hat{\alpha}}(x)}{D_{\hat{\alpha}}(x)},$$

while the actual new weight should read $D_{\hat{\alpha}}^{-2}(x)\tilde{\rho}^2(x)$. The work of Samsonov-Ovcharov (see [26]), on the "modification" of the Hermite polynomials, is a typical example (unique in the literature, to our best knowledge) of a realization of this scheme. For the polynomials on a lattice, this scheme will be realized below only for a special case of the Meixner polynomials, namely the $M_n^{(2,c)}(x+1)$, the question remaining open for higher polynomials (i.e. Hahn, Askey-Wilson, ...).

The last remark, before starting with the calculation, concerns the degrees of the polynomials (in x) to be obtained by Eqs. (19)-(20). Writing the operator $\tilde{H}[\frac{1}{D_{\hat{\alpha}}(x)}]$ in the form

$$(21) \quad \tilde{\sigma}\Delta\nabla + \tilde{\tau}\Delta + \tilde{v},$$

one easily notes that the functions $\tilde{\sigma}$, $\tilde{\tau}$ and \tilde{v} are not necessarily of degree 2, 1 and 0 respectively. In other words, $\tilde{H}[\frac{1}{D_{\hat{\alpha}}(x)}]$ is not necessarily of hypergeometric type. Consequently, as in the case of the Hermite polynomials "modified" by Samsonov-Ovcharov (see also the polynomials generated in [7] by "quasi-periodicity" methods), we can't expect that the corresponding polynomials ($\hat{N}_0(x)$ and numerators in Eq. (19)) be of degrees exactly equal $n = 0, 1, 2, 3, \dots$. There is clearly no matter to be worried about this since that the polynomials conserve most of other properties common to usual orthogonal polynomials (completeness, orthogonality, difference and recurrence relations, difference (eigenvalue) equations, duality,...). The question of global study of orthogonal polynomials of such a category have been already raised in [8].

3. THE SPECIAL MEIXNER $\mathbf{M}_n^{(2,c)}(\mathbf{x}+1)$ CASE.

Recall that the Meixner polynomials $M_n^{(\delta,c)}(x)$ are a special case of general polynomial eigenfunctions of the hypergeometric difference operator in Eq. (15) (see for ex. [22]). The polynomials to be transformed here are Meixner's $\mathbf{M}_n^{(2,c)}(\mathbf{x}+1)$. They satisfy the second-order eigenvalue difference equation

$$(22) \quad [u(x)\mathbf{E}_x + v(x) + w(x)\mathbf{E}_x^{-1}] M_n^{(2,c)}(x+1) = (c-1)nM_n^{(2,c)}(x+1)$$

where

$$(23) \quad u(x) = c(x+3); \quad v(x) = -[(c+1)x + 3c + 1]; \quad w(x) = x+1,$$

as well as the usual recurrence relations

$$(24) \quad P_{n+1} + b_n P_n + a_n^2 P_{n-1} = x P_n$$

with

$$(25) \quad b_n = -\frac{(c+1)n+3c-1}{c-1}; \quad a_n^2 = \frac{n(n+1)c}{(c-1)^2}$$

and the difference relations

$$(26) \quad \begin{aligned} (c-1)a_{n+1}M_{n+1}^{(2,c)}(x+1) &= [u(x)\mathbf{E}_x + k(x;n)]M_n^{(2,c)}(x+1), \\ -(c-1)a_nM_{n-1}^{(2,c)}(x+1) &= [u(x)\mathbf{E}_x + l(x;n)]M_n^{(2,c)}(x+1), \end{aligned}$$

where

$$(27) \quad \begin{aligned} k(x;n) &= -x+n-1, \\ l(x;n) &= -c(x+3+n). \end{aligned}$$

They are orthogonal on $(-1, +\infty)$ with weight $\rho^2(x) = c^x(x+2)$.

The transformation functions are chosen as $c^{-x}\hat{M}_n^{(c)}(x)$, eigenfunctions of the same operator as the Meixner $M_n^{(2,c)}(x+1)$, but with eigenvalues $-(c-1)(n+2)$. Precisely, $\hat{M}_n^{(c)}(x) = M_n^{(2,c)}(-x-3)$.

The transformed rational type functions are $\tilde{\Psi}_{\gamma,0}(x)$, $\tilde{\Psi}_{\gamma,n+1}(x)$, $n, \gamma = 0, 1, 2, \dots$

$$(28) \quad \begin{aligned} [c(x+3)\mathbf{E}_x + g_\gamma(x)]\tilde{\Psi}_{\gamma,0}(x) &= 0, \\ \tilde{\Psi}_{\gamma,n+1}(x) &= [1 + f_\gamma(x)\mathbf{E}_x^{-1}]M_n^{(2,c)}(x+1) \end{aligned}$$

where

$$f_\gamma(x) = -\frac{\hat{M}_\gamma^{(c)}(x)}{c\hat{M}_\gamma^{(c)}(x-1)}; \quad g_\gamma(x) = v(x) - u(x)f_\gamma(x+1) + (c-1)(\gamma+2).$$

They satisfy the second-order eigenvalue difference equation

$$(29) \quad \begin{aligned} [u(x)\mathbf{E}_x + \tilde{v}_\gamma(x) + \tilde{w}_\gamma(x)\mathbf{E}_x^{-1}]\tilde{\Psi}_{\gamma,0}(x) &= -(c-1)(\gamma+2)\tilde{\Psi}_{\gamma,0}(x), \\ [u(x)\mathbf{E}_x + \tilde{v}_\gamma(x) + \tilde{w}_\gamma(x)\mathbf{E}_x^{-1}]\tilde{\Psi}_{\gamma,n+1}(x) &= (c-1)n\tilde{\Psi}_{\gamma,n+1}(x), \\ n, \gamma &= 0, 1, 2, \dots \end{aligned}$$

where

$$(30) \quad \begin{aligned} \tilde{v}_\gamma(x) &= v(x) + f_\gamma(x)u(x-1) - u(x)f_\gamma(x+1), \\ \tilde{w}_\gamma(x) &= f_\gamma(x)\frac{w(x-1)}{f_\gamma(x-1)}. \end{aligned}$$

The case $\gamma = 0$ leads to classical polynomials. The cases $\gamma \geq 1$ lead to rational functions orthogonal with non-classical weights and from which one can extract non-classical orthogonal polynomials. Below, the computations, results and various discussions are given for the cases of $\gamma = 1$, $\gamma = 2$, and $\gamma = 3$.

Second-order difference eigenvalue equations, eigenfunctions.

In this subsection, in each case, are written explicitly: The coefficients in the second-order difference eigenvalue equations and the first four eigenfunctions.

The case $\gamma = 1$.

The second order difference eigenvalue equation reads as in Eq. (29) with $u(x)$ from Eq. (23),

$$\tilde{v}_1(x) = \{ -((c+1)(c-1)^2x^3 + (c-1)(4c^2 - 7c - 5)x^2 + (6 - 20c^2 + 19c + 3c^3)x + 16c - 4c^2) \}$$

$$\begin{aligned} & / \{((c-1)x-2)((c-1)x+c-3)\}; \\ \tilde{w}_1(x) &= \frac{((c-1)x+c-3)x((c-1)x-1-c)}{((c-1)x-2)^2}; \end{aligned}$$

The new eigenfunctions are

$$\begin{aligned} \tilde{\Psi}_{1,0}(x) &= \frac{1}{(x+2)(x+1)((c-1)x-2)}; \\ \tilde{\Psi}_{1,n}(x) &= \frac{\mathcal{P}_{1,n}(x)}{\mathcal{Q}_1(x)}, n = 1, 2, 3, \dots \end{aligned}$$

where $\mathcal{Q}_1(x) = (c-1)x-2$ and $\mathcal{P}_{1,n}(x), n = 1, 2, 3, \dots$ are some polynomials of degree 1, 2, 3, ... respectively.

The case $\gamma = 2$.

The second order difference eigenvalue equation reads as in Eq. (29) with $u(x)$ from Eq. (23),

$$\begin{aligned} \tilde{v}_2(x) &= \{ -((c+1)(c-1)^4x^5 + 3(c^2-5c-4)(c-1)^3x^4 - (c^3+41c^2-85c-53)(c-1)^2x^3 - 3(c-1)(c^4-c^3-63c^2+77c+34)x^2 + (72-348c^2+312c+24c^4+12c^3)x + 180c-72c^2) \} \\ & / \{ ((c-1)^2x^2 - (c+5)(c-1)x+6)((c-1)^2x^2 + (c-1)(c-7)x-6c+12) \}; \\ \tilde{w}_2(x) &= \{ ((c-1)^2x^2 + (c-1)(c-7)x-6c+12)x((c-1)^2x^2 - 3(c-1)(c+1)x+2c+2c^2+2) \} \\ & / \{ ((c-1)^2x^2 - (c+5)(c-1)x+6)^2 \}; \end{aligned}$$

The new eigenfunctions are

$$\begin{aligned} \tilde{\Psi}_{2,0}(x) &= \{1\} / \{ (x+2)(x+1)(2(c-1)^2x-c^2 + ((c^2+10c+1)(c-1)^2)^{1/2} - 4c+5)(2(c-1)^2x-c^2 - ((c^2+10c+1)(c-1)^2)^{1/2} - 4c+5) \}; \\ \tilde{\Psi}_{2,n}(x) &= \frac{\mathcal{P}_{2,n}(x)}{\mathcal{Q}_2(x)}, n = 1, 2, 3, \dots \end{aligned}$$

where $\mathcal{Q}_2(x) = (c-1)^2x^2 - (c+5)(c-1)x+6$ and $\mathcal{P}_{2,n}(x), n = 1, 2, 3, \dots$ are some polynomials of degree 2, 3, 4, ... respectively.

The case $\gamma = 3$.

In order to avoid a fourth order algebraic equation, we have fixed the parameter c taking $c = 1/2$.

The second order difference eigenvalue equation reads as in Eq. (29) with $u(x)$ from Eq. (23),

$$\begin{aligned} \tilde{v}_3(x) &= \{ -3(x+3)(37536x+709x^4+5621x^3+21946x^2+43x^5+x^6+18432) \} \\ & / \{ 2(x+6)(x^2+15x+32)(x+7)(x^2+17x+48) \}; \\ \tilde{w}_3(x) &= \frac{(x+7)(x^2+17x+48)x(x+5)(x^2+13x+18)}{(x+6)^2(x^2+15x+32)^2}; \end{aligned}$$

The new eigenfunctions are

$$\tilde{\Psi}_{3,0}(x) = \frac{1}{(x+1)(x+2)(x+6)(2x+15-97^{1/2})(2x+15+97^{1/2})};$$

$$\tilde{\Psi}_{3,n}(x) = \frac{\mathcal{P}_{3,n}(x)}{\mathcal{Q}_3(x)}, n = 1, 2, 3, \dots$$

where $\mathcal{Q}_3(x) = (x+6)(x^2+15x+32)$ and $\mathcal{P}_{3,n}(x), n = 1, 2, 3, \dots$ are some polynomials of degree 3, 4, 5, ... respectively.

Orthogonality, weight functions.

We refer literally to the results of the discussions in the first section, particularly those related to the proposition 2.1. Here, $a = -1$, and the interval of orthogonality is $(0, +\infty)$. On the other side, various solutions of Eq. (6) for $\tilde{\rho}_\gamma$ are

$$\tilde{\rho}_1^2(x) = \frac{\tilde{\rho}_1^2(0)(3-c)(cx-x-2)(x+2)(x+1)c^x}{4(cx+c-x-3)};$$

$$\begin{aligned} \tilde{\rho}_2^2(x) = & \{c^x(x+1)(x+2)(2xc^2-4cx+2x+5-4c-c^2+((c^2+10c+1)(c-1)^2)^{1/2})(2xc^2-4cx+2x+ \\ & 5-4c-c^2-((c^2+10c+1)(c-1)^2)^{1/2})(c^2-8c+7-((c^2+10c+1)(c-1)^2)^{1/2})(c^2-8c+7+((c^2+ \\ & 10c+1)(c-1)^2)^{1/2})\tilde{\rho}_2^2(0)\} \\ & / \{2(-5+4c+c^2-((c^2+10c+1)(c-1)^2)^{1/2})(-5+4c+c^2+((c^2+10c+1)(c-1)^2)^{1/2})(2xc^2- \\ & 4cx+2x+c^2+7-8c+((c^2+10c+1)(c-1)^2)^{1/2})(2xc^2-4cx+2x-((c^2+10c+1)(c-1)^2)^{1/2}+c^2+ \\ & 7-8c)\}; \end{aligned}$$

$$\begin{aligned} \tilde{\rho}_3^2(x) = & \{2^{-x}7(x+1)(x+2)(x+6)(2x+15-97^{1/2})(2x+15+97^{1/2})(-17+97^{1/2})(17+97^{1/2})\tilde{\rho}_3^2(0)\} \\ & / \{12(x+7)(2x-97^{1/2}+17)(2x+17+97^{1/2})(15+97^{1/2})(-15+97^{1/2})\}; \end{aligned}$$

Recalling that the bottom function $\tilde{\Psi}_{\gamma,0}(x)$ is the solution of the equation (see Eq. (10))

$$(31) \quad L_{\tilde{\alpha}}\tilde{Y}(x) = 0,$$

one directly deduces from the proposition 2.1, the orthogonality, on the interval $(0, +\infty)$, of the bottom function $\tilde{\Psi}_{\gamma,0}(x)$ with each of the elements from the higher ladder $\tilde{\Psi}_{\gamma,n}(x), n = 1, 2, \dots$, relatively to the weights $\tilde{\rho}_\gamma^2(x)$. On the other side, as one can easily verify, the equation Eq. (13) is verified for $\alpha, \beta = 1, 2, \dots, \alpha \neq \beta$. Consequently, the transformed operator \tilde{H}_γ (in l.h.s. of Eq. (29)) is symmetric in the subspace of $\tilde{\ell}^2(0, +\infty; \tilde{\rho}_\gamma^2(x))$ generated by the higher ladder $\tilde{\Psi}_{\gamma,n}(x), n = 1, 2, \dots$. Hence, the functions from this ladder are there mutually orthogonal. Thus, we have obtained that all the new functions $\tilde{\Psi}_{\gamma,n}(x), n = 0, 1, 2, \dots$ are mutually orthogonal on the interval $(0, +\infty)$ relatively to the weights $\tilde{\rho}_\gamma^2(x)$ given above.

A direct consequence of this, are the orthogonality relations between polynomials related to the $\tilde{\Psi}$ -functions. Namely:

The polynomials

$$\bar{\mathcal{P}}_{1,0}(x) = 1;$$

$$\bar{\mathcal{P}}_{1,n}(x) = (x+2)(x+1)\mathcal{P}_{1,n}(x), n = 1, 2, 3, \dots$$

are orthogonal on the same interval as $\tilde{\Psi}_{1,n}(x)$ but now with the weight

$$\tilde{\rho}_1^2(x) = [(x+1)(x+2)\mathcal{Q}_1(x)]^{-2} \tilde{\rho}_1^2(x).$$

Identically, the polynomials

$$\bar{\mathcal{P}}_{2,0}(x) = \mathcal{Q}_2(x);$$

$$\bar{\mathcal{P}}_{2,n}(x) = [\tilde{\Psi}_{2,0}(x)]^{-1} \mathcal{P}_{2,n}(x), n = 1, 2, 3, \dots$$

are orthogonal on the same interval as $\tilde{\Psi}_{2,n}(x)$ but now with the weight

$$\tilde{\rho}_2^2(x) = [[\tilde{\Psi}_{2,0}(x)]^{-1} \mathcal{Q}_2(x)]^{-2} \tilde{\rho}_2^2(x).$$

Finally, the polynomials

$$\bar{\mathcal{P}}_{3,0}(x) = (x+6)^{-1} \mathcal{Q}_3(x);$$

$$\bar{\mathcal{P}}_{3,n}(x) = [(x+6)\tilde{\Psi}_{3,0}(x)]^{-1} \mathcal{P}_{3,n}(x), n = 1, 2, 3, \dots$$

are orthogonal on the same interval as $\tilde{\Psi}_{3,n}(x)$ but now with the weight

$$\tilde{\rho}_3^2(x) = [[(x+6)\tilde{\Psi}_{3,0}(x)]^{-1} \mathcal{Q}_3(x)]^{-2} \tilde{\rho}_3^2(x).$$

Completion.

The completion of the functions $\tilde{\Psi}_{\gamma,n}(x)$, $n = 0, 1, 2, \dots$, in $\tilde{\ell}^2(0, +\infty; \tilde{\rho}_\gamma^2(x))$ follows from the one of the Meixner $M_n^{(2,c)}(x+1)$ in $\ell^2(-1, +\infty; \rho^2(x))$. This is a direct consequence of the proposition 2.1. In particular here, $a = -1$, so that $\tilde{\Theta}_\alpha$ and Θ_α considered in the proposition 2.1 are obtained respectively from $\tilde{\Psi}_{\gamma,0}(x)$ and $M_\alpha^{(2,c)}(x+1)$ by sending coordinates corresponding to negative x , to 0.

On the other side, the completion of the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x)$, $n = 0, 1, 2, \dots$ in $\tilde{\ell}^2(0, +\infty; \tilde{\rho}_\gamma^2(x))$ follows from that of $\tilde{\Psi}_{\gamma,n}(x)$, $n = 0, 1, 2, \dots$ in $\tilde{\ell}^2(0, +\infty; \tilde{\rho}_\gamma^2(x))$.

This completion holds well in spite of the fact that, in each of the constructed sequences of polynomials, there exists at least one number $n \in \mathbb{Z}^+$ such that, no polynomial from that sequence has degree exactly equal n .

Difference and recurrence relations.

As other deducible properties of the new functions, one finds from Eq. (28) and the recurrence relations for the Meixner $M_n^{(2,c)}(x+1)$ (see Eq. (25)) polynomials, the following five-term recurrence relations:

$$\tilde{\Psi}_{\gamma,n+2}(x) + [b_{n+1} + b_n - 2x + 1] \tilde{\Psi}_{\gamma,n+1}(x) + [a_{n+1}^2 + a_n^2 + (b_n - x + 1)(b_n - x)] \tilde{\Psi}_{\gamma,n}(x) + [a_n^2(b_n + b_{n-1} - 2x + 1)] \tilde{\Psi}_{\gamma,n-1}(x) + a_n^2 a_{n-1}^2 \tilde{\Psi}_{\gamma,n-2}(x) = 0, n = 3, 4, \dots$$

But as easily seen, these recurrence relations are satisfied by any linear combination (with coefficients depending of x and not of n) of Meixner $M_n^{(2,c)}(x+1)$, and $M_n^{(2,c)}(x)$. In other words, the relations do not depend of $f_\gamma(x)$.

More characteristic recurrence relations for these functions are the following three-term recurrence relations that one can establish from Eq. (28), the difference eigenvalue Eq. (22) and the difference relations Eqs. (26)-(27) satisfied by the Meixner $M_n^{(2,c)}(x+1)$ polynomials:

$$[(c-1)a_{n+1}(f_\gamma(x)l(x-1;n) - w(x) - f_\gamma(x)l(x;n) + f_\gamma(x)(v(x) - \lambda_n) - f_\gamma^2(x)u(x-1))] \tilde{\Psi}_{\gamma,n+1}(x) + [f_\gamma(x)k(x-1;n)l(x;n) - f_\gamma(x)k(x-1;n)(v(x) - \lambda_n) + f_\gamma^2(x)k(x-1;n)u(x-1) - w(x)l(x;n) - f_\gamma(x)k(x;n)l(x-1;n) + k(x;n)w(x) + f_\gamma(x)l(x-1;n)(v(x) - \lambda_n) - f_\gamma^2(x)u(x-1)l(x-1;n)] \tilde{\Psi}_{\gamma,n}(x) + [(c-1)a_n(f_\gamma(x)(v(x) - \lambda_n) -$$

$$f_\gamma(x)k(x;n) - f_\gamma^2(x)u(x-1) + f_\gamma(x)k(x-1;n) - w(x)]\tilde{\Psi}_{\gamma,n-1}(x) = 0, n = 2, 3, \dots$$

The obtained functions satisfy also the following third-order difference relations, which follow clearly from the difference relations Eqs. (26)-(27) satisfied by the Meixner $M_n^{(2,c)}(x+1)$ polynomials, together with Eq. (28) and its inverse:

$$-\lambda_\gamma(c-1)a_{n+1}\tilde{\Psi}_{\gamma,n+1}(x) = \{[1 + f_\gamma(x)\mathbf{E}_x^{-1}][u(x)\mathbf{E}_x + k(x;n)][u(x)\mathbf{E}_x + g_\gamma(x)]\}\tilde{\Psi}_{\gamma,n}(x)$$

$$\begin{aligned} & \lambda_\gamma(c-1)a_n\tilde{\Psi}_{\gamma,n-1}(x) \\ &= \{[1 + f_\gamma(x)\mathbf{E}_x^{-1}][u(x)\mathbf{E}_x + l(x;n)][u(x)\mathbf{E}_x + g_\gamma(x)]\}\tilde{\Psi}_{\gamma,n}(x) \quad n = 2, 3, \dots \end{aligned}$$

Using the second-order difference eigenvalue equation Eq. (29) satisfied by the new functions and the preceding relations, one can reach, if necessary, the following first-order difference relations

$$-\lambda_\gamma(c-1)a_{n+1}\tilde{\Psi}_{\gamma,n+1}(x) = \{-u(x)(v(x+1) - \lambda_n) + u(x)g_\gamma(x+1) + k(x;n)u(x) + f_\gamma(x)u(x-1)u(x) - u(x)f_\gamma(x)k(x-1;n)g_\gamma(x-1)/w(x)]\mathbf{E}_x + [-u(x)w(x+1) + k(x;n)g_\gamma(x) + f_\gamma(x)g_\gamma(x)u(x-1) + f_\gamma(x)k(x-1;n)u(x-1) - f_\gamma(x)g_\gamma(x-1)k(x-1;n)(v(x) - \lambda_n)/w(x)]\}\tilde{\Psi}_{\gamma,n}(x),$$

$$\begin{aligned} & \lambda_\gamma(c-1)a_n\tilde{\Psi}_{\gamma,n-1}(x) = \{-u(x)(v(x+1) - \lambda_n) + u(x)g_\gamma(x+1) + l(x;n)u(x) + f_\gamma(x)u(x-1)u(x) - u(x)f_\gamma(x)l(x-1;n)g_\gamma(x-1)/w(x)]\mathbf{E}_x + [-u(x)w(x+1) + l(x;n)g_\gamma(x) + f_\gamma(x)g_\gamma(x)u(x-1) + f_\gamma(x)l(x-1;n)u(x-1) - f_\gamma(x)g_\gamma(x-1)l(x-1;n)(v(x) - \lambda_n)/w(x)]\}\tilde{\Psi}_{\gamma,n}(x) \quad n = 2, 3, \dots \end{aligned}$$

It is clear that the polynomials $\bar{P}_{\gamma,n}(x)$ satisfy the same recurrence relations as the functions $\tilde{\Psi}_{\gamma,n}(x)$. Their difference relations are also obviously deduced from those of $\tilde{\Psi}_{\gamma,n}(x)$.

Duality.

Parallel to the above results for the functions $\tilde{\Psi}_{\gamma,n}(x)$ and the polynomials $\bar{P}_{\gamma,n}(x)$, one can of course deduce the corresponding results for the polynomials dual to the polynomials $\bar{P}_{\gamma,n}(x)$ (more precisely, dual to the functions $\tilde{\Psi}_{\gamma,n}(x)$), $n = 1, 2, \dots$, thanks to the existence of dual (self-dual) polynomials for the Meixner polynomials. We will denote $\mathcal{D}_{\gamma,x}(n) = \check{\rho}(x)\tilde{\Psi}_{\gamma,n+1}(x)$, $n, x = 0, 1, 2, \dots$, $\check{\rho}(x) = c^x\Gamma(x+3)$.

$\mathcal{D}_{\gamma,x}(n)$ are polynomials in n , of degree exactly equal $x+1$ and satisfy the usual three-term recurrence relations (in x)

$$(32) \quad P_{x+1} + b_{\gamma,x}P_x + a_{\gamma,x}^2P_{x-1} = nP_x$$

with (see Eq. (29)):

$$(33) \quad b_{\gamma,x} = \check{v}_\gamma(x); \quad a_{\gamma,x}^2 = \check{w}_\gamma(x)u(x-1)/(c-1)^2.$$

The polynomials in n

$$(34) \quad \check{\mathcal{D}}_{\gamma,0} = 1; \quad \check{\mathcal{D}}_{\gamma,1} = (c-1)n - b_{\gamma,0};$$

$$(35) \quad \check{\mathcal{D}}_{\gamma,x+1} = ((c-1)n - b_{\gamma,x})\check{\mathcal{D}}_{\gamma,x} - a_{\gamma,x}^2\check{\mathcal{D}}_{\gamma,x-1}, \quad x = 1, 2, \dots$$

are for $x = 0, 1, 2, \dots$ of degree exactly equal x and are clearly orthogonal, the condition required in the well known Favard theorem ($b_{\gamma,x}$ reel and $a_{\gamma,x}^2 > 0$, $x \geq 1$), being satisfied. On the other side, $\mathcal{D}_{\gamma,x}(n) = \check{\mathcal{D}}_{\gamma,x}(n)\check{\rho}(0)\tilde{\Psi}_{\gamma,n+1}(0)$, where from Eq. (28), $\tilde{\Psi}_{\gamma,n+1}(0) = (\gamma)_n(\frac{c}{c-1})^n[f_\gamma(0) + 1 + \frac{1}{2}n(1 - \frac{1}{c})]$, $(\gamma)_0 = 1$, $(\gamma)_n = \gamma(\gamma+1) \dots (\gamma+n-1)$, $n = 1, 2, \dots$. Consequently,

$$(36) \quad \tilde{\Psi}_{\gamma,n+1}(x) = \frac{(\gamma)_n(\frac{c}{c-1})^n}{c^x\Gamma(x+3)} \cdot [2f_\gamma(0) + 2 + n(1 - \frac{1}{c})]\check{\mathcal{D}}_{\gamma,x}(n),$$

$$x, n = 0, 1, \dots$$

4. CONCLUDING REMARKS AND OUTLOOKS.

Concluding this work, let us first remark that the obtained polynomials and functions can, on their turn, be transformed in a similar manner. Here, the reservoir of transformation functions is composed of

$$(37) \quad \Phi_{\hat{\alpha},n}(x) = [1 + f_{\hat{\alpha}} \mathbf{E}_x^{-1}] c^{-x} \hat{M}_n(x), \quad n, \hat{\alpha} \in \mathcal{N}, \quad n \neq \hat{\alpha},$$

and clearly, the process can, in principle, be repeated infinitely.

The existence of other efficiently transformable particular cases of the hypergeometric difference operator H in Eq. (15), "higher" to the special Meixner $M_n^{(2,c)}(x+1)$ case treated here, is of course not precluded. As noted in [6], it is neither precluded that such solvable Hamiltonians could be generated by self-similarity or shape-invariance (symmetry) techniques. The fact that any two of the three methods evoked in the introduction may lead to the same result is illustrated by the already noted fact that the non-classical polynomials in ([7], Eq. (4.13)) related to the Hermite polynomials generated there by "quasi-periodicity" methods have been rediscovered in [26], using the "modification method" on the Hermite polynomials (see the case $m = 2$ in [26]).

It is worth remarking that no efficient "modification" of a fourth-order Sturm-Liouville difference (or differential) operator (using for example one of the formulas from the section 2.2.2 of the second chapter in [5]) is known in the literature.

Remark next that if the constraint of being polynomials is rejected as well for the "transformable" functions as for the "transformed" ones (retaining only the constraints of integrability and completeness), the situation becomes much easier. Thus, nowadays, the used here method is one of the main tools for generating new exactly solvable Hamiltonians in quantum mechanics (see f.e. [19, 23, 24, 25, 30]).

In [29], transformations of orthogonal polynomials were studied whose weights are obtained by multiplying a *rational* function to the original ones. Let us note although the weights $\tilde{\rho}_\gamma^2(x)$, $\gamma = 1, 2, 3$ are obtained by multiplying rational functions to $\rho_\gamma^2(x)$, $\gamma = 1, 2, 3$, respectively, the situation here is generally that the new weights are obtained from the old ones by multiplying to them not by rational functions but a ratio of products of Gamma or may be exponential functions. So that the polynomials studied here are not in principle of the same kind than those from [29]. Another remarkable difference resides in that the polynomials studied in [29] are "ordinary" orthogonal polynomials satisfying a three-term recurrence relation and have degrees exactly equal $0, 1, 2, \dots$, while the orthogonal polynomials studied here do not satisfy the usual three-term recurrence relation and are not of degrees exactly equal $0, 1, 2, \dots$ (see above).

As already noted the polynomials studied here conserve most of the properties of "ordinary" discrete orthogonal polynomials (completeness, orthogonality, difference and recurrence relations, difference (eigenvalue) equations, duality,...). A global study of orthogonal polynomials of such a category (other explicit examples are found in [26, 7, 8]) is of a certain interest.

APPENDIX.

In this appendix, we give for the concrete cases of $\gamma = 1$, $\gamma = 2$ and $\gamma = 3$ (see Tables 1, 2 and 3 respectively), the transformed interval of orthogonality, the transformed weight function and the first four transformed polynomials.

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TABLE 1. The case $\gamma = 1$.

(a, b)	$(0, +\infty)$
$\tilde{\rho}_1^2(x)$	$\frac{\tilde{\rho}_1^2(0)4(3-c)c^x}{(cx-x-2)(x+2)(x+1)(cx+c-x-3)}$
$\mathcal{P}_{1,0}(x)$	1
$\mathcal{P}_{1,1}(x)$	$(x+1)(x+2)((c-1)^2x-3c+3)$
$\mathcal{P}_{1,2}(x)$	$(c-1)(x+1)(x+2)((c-1)^2x^2+3(c-1)^2x-8c)$
$\mathcal{P}_{1,3}(x)$	$(c-1)((c-1)^3x^3+(7c-2)(c-1)^2x^2+(c-1)(12c^2-19c-3)x-30c^2)$ $(x+1)(x+2)$

TABLE 2. The case $\gamma = 2$.

(a, b)	$(0, +\infty)$
$\tilde{\rho}_2^2(x)$	$\frac{\{72c^x(((c-1)(c+5))^2-(c^2+10c+1)(c-1)^2)$ $((c-1)(c-7))^2-(c^2+10c+1)(c-1)^2\}\tilde{\rho}_2^2(0)\}$ $/\{(2x(c-1)^2-(c-1)(c+5))^2-(c^2+10c+1)(c-1)^2)$ $((c-1)^2x^2-(c+5)(c-1)x+6)^2(x+1)(x+2)$ $((2x(c-1)^2+(c-1)(c-7))^2-(c^2+10c+1)(c-1)^2)\}$
$\mathcal{P}_{2,0}(x)$	$(c-1)^2x^2-(c+5)(c-1)x+6$
$\mathcal{P}_{2,1}(x)$	$(c-1)((c-1)^2x^2-(c+7)(c-1)x+12)$ $(x+1)(x+2)((2c-1)^2x-(c+5)(c-1))^2-(c^2+10c+1)(c-1)^2)$
$\mathcal{P}_{2,2}(x)$	$(c-1)((c-1)^3x^3+(2c-7)(c-1)^2x^2-(c-1)(3c^2+19c-12)x+30c)$ $(x+1)(x+2)((2c-1)^2x-(c+5)(c-1))^2-(c^2+10c+1)(c-1)^2)$
$\mathcal{P}_{2,3}(x)$	$(x+1)(x+2)((2c-1)^2x-(c+5)(c-1))^2-(c^2+10c+1)(c-1)^2)$ $(c-1)((c-1)^4x^4+6(c-1)^4x^3+(5c^2-46c+5)(c-1)^2x^2$ $-12(c^2+7c+1)(c-1)^2x+108c^2)$

TABLE 3. The case $\gamma = 3$.

(a, b)	$(0, +\infty)$
$\tilde{\rho}_3^2(x)$	$\frac{\{2^{-x}32^42016\tilde{\rho}_3^2(0)\}}{\{(x+1)(x+2)(x+6)((2x+15)^2-97)(x^2+15x+32)^2$ $(x+7)((2x+17)^2-97)\}}$
$\mathcal{P}_{3,0}(x)$	$x^2+15x+32$
$\mathcal{P}_{3,1}(x)$	$-(x+1)(x+2)((2x+15)^2-97)(x+15)(x+4)(x+8)$
$\mathcal{P}_{3,2}(x)$	$\frac{1}{5}(x+1)(x+2)((2x+15)^2-97)(x^4+24x^3+137x^2-66x-1152)$
$\mathcal{P}_{3,3}(x)$	$-\frac{1}{4}(x+1)(x+2)((2x+15)^2-97)(x^5+18x^4-7x^3-960x^2-1740x$ $+4032)$

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