The transformation of polynomial eigenfunctions of linear second-order q-difference operators: a special case of q-Jacobi polynomials.

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Abstract. A systematic discussion of the Mielnik-Ovcharov-Samsonov factorization method is given for the general linear second order q-difference equation. A special way for the factorization of some type of such equations is also shown. As an application, the set of the Little q-Jacobi $p_n(qx;c,q|q)$ or equivalently Big q-Jacobi $P_n(q^3x;q,c,0;q)$ polynomials is transformed into a non-classical one.

Keywords. q-Discrete factorization techniques, q-discrete orthogonal polynomials.

1. Introduction.

Consider the factorization chain

$$\begin{aligned} H_j(x)-\mu_j &= H_j^+(x)H_j^-(x) \\ H_{j+1}(x)-\mu_j &= H_j^-(x)H_j^+(x), \quad x\in\mathbf{R}, \quad j\in\mathbf{Z} \end{aligned} \tag{1}$$

where H is a differential or a difference operator. Including H in such a factorization chain may help either to solve it and to derive important properties for its solutions, or to produce from it other solvable operators not belonging to the same family as H itself. In the first case one may impose to the chain some self-similarity, demanding for example that the dressing variable j acts not just as an index but as a full independent variable. In the second case, one needs to have at his disposal a set (at least one) of transformation eigenfunctions for the transformable operator H, not belonging to the set of transformable eigenfunctions. In other words, a certain "bispectrality" is required for the operator H.
The first method here referred to as the Infeld-Hull-Miller (IHM) factorization method (FM) (see [5, 11, 12]), was already applied by Schrödinger in 1940 for solving the harmonic oscillator[19]. Nowadays, most of known exactly solvable second-order differential or difference operators can be solved by that technique. The second method, here referred to as the Mielnik-Samsonov-Ovcharov (MSO) factorization methods (FM) (see [13, 18]) appeared as a further to the first method by allowing to generate new exactly solvable Hamiltonians from already known (namely exactly solved by the first method) ones.
In this work, we will study the MSO factorization method for a general linear second-order q-difference equation

$$[u(x)E_q+v(x)+w(x)E_q^{-1}]y(x)=\lambda y(x), \tag{2}$$

where $c(x)=-(a(x)+b(x))$,$E_q^i f(x)=f(q^i x)$, $i\in\mathbf{Z}$ (here and in all that follows, $y(x)=S(h(x))$, for some function S, and $h(x)$ a certain first degree Laurent polynomial in x: $h(x)=k_1x+k_2x^{-1}+k_3$), and then apply the results to a special case of the hypergeometric q-difference equation [15] (h(x)=x):

$$[\sigma(x)D^qD_q+\tau(x)D_q]y(x)=\lambda y(x). \tag{3}$$

where $D_q f(x)=(E_q-1)f(x)/(qx-x)$, $D^q f(x)=(1-E_q^{-1})f(x)/(x-x/q)$, with $\sigma(x)=\sigma_0x^2+\sigma_1x+\sigma_2$, $\tau(x)=\tau_0x+\tau_1$, corresponding to a special of the Little q-Jacobi $p_n(qx;c,q|q)$ or equivalently Big q-Jacobi $P_n(q^3x;q,c,0;q)$ polynomials [9]. Non-hypergeometric type functions (associated to polynomial sequences) resulting from the transformation of the latter are particularly given.
Other works having common points with this one are mainly [1, 6, 16, 17]. The MSO FM is obviously also useful in the questions of quasi-exactly solvability [2, 3, 8, 7, 10].

2. MSO FM FOR THE SECOND ORDER q-DIFFERENCE EQUATION.

2.1. **The general case.**

Consider the general second-order q-difference operator

$$H(x) = u(x)E_q + v(x) + w(x)E_q^{-1}. \tag{4}$$

Suppose next that it is "bispectral" (not in the sense of [4]), in the sense that it admits two sequences of different systems of eigenelements say (λ_n, y_n) and (γ_n, z_n):

$$\begin{aligned} Hy_n(x) &= \lambda_n y_n(x) \\ Hz_n(x) &= \gamma_n z_n(x), \quad n = 0,1,\ldots. \end{aligned} \tag{5}$$

In that case, one can use one of the two eigenelements , say for example (γ_n, z_n), to transform H into another solvable operator $\tilde{H}$ in the following manner. Factorize H and define $\tilde{H}$ as follows,

$$\begin{aligned} H - \gamma_m &= L_m R_m \\ \tilde{H} - \gamma_m &= R_m L_m, \quad m = 0,1,\ldots. \end{aligned} \tag{6}$$

where

$$\begin{aligned} R_m = 1 + f(x,m)E_q^{-1} \qquad & L_m = u(x)E_q + g(x,m) \\ f(x,m) = -\tfrac{z_m(x)}{z_m(x/q)} \qquad & g(x,m) = -w(x)\tfrac{z_m(x/q)}{z_m(x)}. \end{aligned} \tag{7}$$

It follows from (6) that the functions $\tilde{y}_n(x,m)$, $m,n = 0,1,\ldots$ defined by

$$\begin{aligned} &[u(x)E_q + g(x,m)]\tilde{y}_0(x,m) = 0, \\ \tilde{y}_n(x,m) = [1 + &f(x,m)E_q^{-1}]y_{n-1}(x), \quad m = 0,1,\ldots, n = 1,2\ldots, \end{aligned} \tag{8}$$

are eigenfunctions of $\tilde{H}(x,m)$ corresponding to the eigenvalues γ_m, λ_n, for $m = 0,1,\ldots, n = 0,1,\ldots$ respectively. We will refer here to H and y_n as the *transformable* operator and functions respectively, z_n as the *transformation* functions and finally $\tilde{H}$ and $\tilde{y}_n$ as the *transformed* operator and functions respectively. The point here is that if

$$\tfrac{y_n(x)}{y_n(x/q)} \neq \tfrac{z_n(x)}{z_n(x/q)} \tag{9}$$

so, for a fixed m, the transformed functions $\tilde{y}_n(x,m)$, $n = 0,1,\ldots$ are non-trivial solutions of the transformed operator $\tilde{H}$ (in polynomial theory, when transforming polynomials into polynomials, one requires more conditions than (9) demanding for example that the z_n be the usual product of polynomials and an exponential type function while the y_n are polynomials). More ever under some additional conditions, the transformed functions $\tilde{y}_n$ admit most of the mathematical properties of the transformable y_n, such as difference eigenvalue equations, closure and orthogonality, difference and recurrence relations, duality, IHM and MSO factorization,...

2.1.1. *Difference equations.*

Clearly, the functions $\tilde{y}_n(x,m)$ satisfy the eigenvalue equation

$$\begin{aligned} \tilde{H}(x,m)\tilde{y}_0(x,m) &= \gamma_m \tilde{y}_0(x,m) \\ \tilde{H}(x,m)\tilde{y}_n(x,m) &= \lambda_{n-1}\tilde{y}_n(x,m), n = 1,2,\ldots \end{aligned} \tag{10}$$

for

$$\tilde{H}(x,m) = u(x)E_q + \tilde{v}(x,m) + \tilde{w}(x,m)E_q^{-1} \tag{11}$$

where

$$\begin{aligned} \tilde{v}(x,m) &= g(x,m) + f(x,m)u(x/q) + \gamma_m \\ &= v(x) + f(x,m)u(x/q) - u(x)f(qx,m) \\ \tilde{w}(x,m) &= f(x,m)g(x/q,m) = w(x)\tfrac{g(x/q,m)}{g(x,m)}. \end{aligned} \tag{12}$$

2.1.2. *Orthogonality, closure.* Consider the functions $\rho(x)$ and $\tilde{\rho}(x)$ defined by

$$\frac{\rho^2(qx)}{\rho^2(x)} = \frac{u(x)}{f(qx,m)g(qx,m)}\frac{h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}})}{h(xq^{\frac{1}{2}})-h(xq^{\frac{3}{2}})} = \frac{u(x)}{w(qx)}\frac{h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}})}{h(xq^{\frac{1}{2}})-h(xq^{\frac{3}{2}})};$$

$$\frac{\tilde{\rho}^2(qx,m)}{\tilde{\rho}^2(x,m)} = \frac{u(x)}{f(qx,m)g(x,m)}\frac{h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}})}{h(xq^{\frac{1}{2}})-h(xq^{\frac{3}{2}})} = \frac{u(x)g(qx,m)}{w(qx)g(x,m)}\frac{h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}})}{h(xq^{\frac{1}{2}})-h(xq^{\frac{3}{2}})}; \tag{13}$$

where h is a first order Laurent polynomial in x.
Interesting relations exist between $\rho(x)$, $\tilde{\rho}(x,m)$ and $\tilde{y}_0(x,m)$. One has

$$\begin{aligned} \tilde{\rho}^2(x,m) &= \rho^2(x)g(x,m); \\ \tilde{y}_0(x,m) &= \frac{1}{\rho^2(x)g(x,m)z_m(x)(h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}}))} \\ &= \frac{1}{\tilde{\rho}^2(x,m)z_m(x)(h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}}))} \end{aligned} \tag{14}$$

Next, as it is easily seen, the similarity reductions $\rho H\rho^{-1}$ and $\tilde{\rho}\tilde{H}\tilde{\rho}^{-1}$ send H and $\tilde{H}$ respectively, in their formal symmetric form , that is like

$$c(qx)\frac{h(xq^{\frac{1}{2}})-h(xq^{\frac{3}{2}})}{h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}})}E_q + b(x) + c(x)E_q^{-1} \tag{15}$$

or

$$a(x)E_q + b(x) + a(x/q)\frac{h(xq^{-\frac{3}{2}})-h(xq^{-\frac{1}{2}})}{h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}})}E_q^{-1}. \tag{16}$$

Denote by $\ell^2(q^\beta,q^\alpha;\rho^2)$ the linear space of q-discrete functions

$$\psi(x),\ x = q^\beta, q^{\beta-1},\ldots,q^\alpha;\ \alpha,\beta \in \mathbf{Z}\cup\{-\infty,\infty\} \tag{17}$$

in which is defined a discrete-weighted inner product

$$\begin{aligned} &(\psi,\phi)_{\rho^2} = \int_{q^\beta}^{q^\alpha}\psi(x)\phi(x)\rho^2(x)d_hx = \\ &\sum_\alpha^\beta \psi(q^{i+\frac{1}{2}})\phi(q^{i+\frac{1}{2}})\rho^2(q^{i+\frac{1}{2}})(h(q^i)-h(q^{i+1})) \end{aligned} \tag{18}$$

The similar space for $\tilde{\rho}^2$ will be denoted by $\tilde{\ell}^2(q^{\tilde{\beta}},q^{\tilde{\alpha}};\tilde{\rho}^2)$. Now let ℓ_1^2 and $\tilde{\ell}_1^2$ be the respective subspaces of ℓ^2 and $\tilde{\ell}^2$ containing those elements for which

$$\begin{aligned} &u(xq^{-\frac{1}{2}})\rho(xq^{-\frac{1}{2}})(h(x/q)-h(x))[\psi(xq^{\frac{1}{2}})\phi(xq^{-\frac{1}{2}})-\psi(xq^{-\frac{1}{2}})\phi(xq^{\frac{1}{2}})]\Big|_{q^\alpha}^{q^{\beta+1}} \\ &= 0 \end{aligned} \tag{19}$$

for ρ^2 equals ρ^2 and $\tilde{\rho}^2$ respectively (if one of the boundary point α or β is equal to ∞, so the vanishing of the expression in (19) is required only on the other extremal). Using summation by parts, one easily verifies that in ℓ_1^2 and $\tilde{\ell}_1^2$ respectively, the operators H and $\tilde{H}$ are symmetric, that is

$$(H\psi,\phi)_{\rho^2} = (\psi,H\phi)_{\rho^2}; (\tilde{H}\psi,\phi)_{\tilde{\rho}^2} = (\psi,\tilde{H}\phi)_{\tilde{\rho}^2}. \tag{20}$$

Let next ℓ_2^2 be the preimage of image of ℓ^2 in $\tilde{\ell}^2$ by R_m and $\tilde{\ell}_2^2$ be the preimage of image of $\tilde{\ell}^2$ in ℓ^2 by L_m both constrained to the conditions that

$$\begin{aligned} &u(xq^{-\frac{1}{2}})\rho(xq^{-\frac{1}{2}})(h(x/q)-h(x))\phi(xq^{\frac{1}{2}})\psi(xq^{-\frac{1}{2}})\Big|_{q^\alpha}^{q^{\beta+1}} \\ &= 0 \end{aligned} \tag{21}$$

for $\psi \in \ell^2$ and $\phi \in \tilde{\ell}^2$ (here also, if one of the boundary point α or β is equal to ∞, so the vanishing of the expression in (21) is required only on the other extremal). Also, using summation by parts, one proves that , defined in ℓ_2^2 and $\tilde{\ell}_2^2$ respectively, the operators R_m and L_m are $(\rho^2,\tilde{\rho}^2)$-mutually adjoint in the sense that

$$(\phi,R_m\psi)_{\tilde{\rho}^2} = (L_m\phi,\psi)_{\rho^2}. \tag{22}$$

Let us note directly that $\ell_2^2 \subset \ell_1^2 \subset \ell^2$ while $\tilde{\ell}_2^2 \subset \tilde{\ell}_1^2 \subset \tilde{\ell}^2$ (taking into account in particular, the first eq. in (14)). We are led to the following

Proposition 2.1. *Suppose that*
(c1) $y_n(x) \in \ell_2^2(q^\beta, q^\alpha; \rho^2),\ n = 0, 1.2, \ldots$
(c2) For m fixed, $\tilde{y}_n(x,m) \in \tilde{\ell}_2^2(q^{\tilde{\beta}}, q^{\tilde{\alpha}}; \tilde{\rho}^2),\ n = 0, 1, 2, \ldots,$
without the constraints of quadratic integrability for them.
(c3) The system $y_n(x)$ *is orthogonal and closed in* $\ell^2(q^\beta, q^\alpha; \rho^2)$.
Then, for m fixed, the system $\tilde{y}_n(x,m)$, $n = 0, 1, \ldots$ *or* $\tilde{y}_n(x,m)$, $n = 1, 2, \ldots$ *is orthogonal and closed in* $\tilde{\ell}_2^2(q^{\tilde{\beta}}, q^{\tilde{\alpha}}; \tilde{\rho}^2)$.

Proof The proof of the orthogonality of the new system is straightforward, considering the above considerations: It can be deduced from (20) or (22) under the condition (19) or (21), respectively. To prove its closure (i.e. no non-vanishing element in the space, that may be orthogonal to the system in its totality) , one uses (22) and the closure of the system $y_n(x)$ in $\ell^2(q^\beta, q^\alpha; \rho^2)$, to derive that $\tilde{y}_0(x,m)$ is the unique non-vanishing element orthogonal to the system $\tilde{y}_n(x,m) = R_m y_n(x)$. This means that the system $\tilde{y}_n(x,m)$, $n = 0, 1, 2, \ldots$ (or $\tilde{y}_n(x,m)$, $n = 1, 2, \ldots$ if $\tilde{y}_0(x,m)$ is not quadratically integrable) is closed in the space.

2.1.3. *Difference and recurrence relations.* Suppose that the transformable functions satisfy the difference relations

$$\begin{aligned} \alpha_n y_{n+1} &= H_n^- y_n \\ \beta_n y_n &= H_n^+ y_{n+1},\ \ n = 0, 1, \ldots \end{aligned} \tag{23}$$

On the other side, from (6), one has

$$\begin{aligned} \tilde{y}_{n+1} &= R_m y_n \\ (\lambda_n - \gamma_m) y_n &= L_m \tilde{y}_{n+1},\ \ n = 0, 1, \ldots \end{aligned} \tag{24}$$

A combination of (23) and (24) leads to the following three-term difference relations for $\tilde{y}_n,\ n = 1, 2, \ldots$

$$\begin{aligned} \alpha_n(\lambda_n - \gamma_m)\tilde{y}_{n+2} &= R_m H_n^- L_m \tilde{y}_{n+1} \\ \beta_n(\lambda_{n+1} - \gamma_m)\tilde{y}_{n+1} &= R_m H_n^+ L_m \tilde{y}_{n+2}. \end{aligned} \tag{25}$$

Using the difference eigenvalue equation satisfied by the $\tilde{y}_n$ (see (10)) and the preceding relations, one can naturally reach first order difference relations connecting $\tilde{y}_n$, $n = 1, 2, \ldots$ (in this connection, the concluding remark in [13] needs a clarification).
Suppose now that the transformable functions y_n satisfy a three-term recurrence relation

$$y_{n+1} + (b_n - h(x))y_n + a_n^2 y_{n-1} = 0, \tag{26}$$

so, using the first relation in (24), one shows that the transformed $\tilde{y}_n$, $n = 1, 2, \ldots$, satisfy the following five-term recurrence relation

$$\begin{aligned} &\tilde{y}_{n+4} + [b_{n+2} + b_{n+1} - h(x) - h(x/q)]\tilde{y}_{n+3} \\ &+[(b_{n+1} - h(x))(b_{n+1} - h(x/q)) + a_{n+1}^2 + a_{n+2}^2]\tilde{y}_{n+2} \\ &+a_{n+1}^2[b_{n+1} + b_n - h(x) - h(x/q)]\tilde{y}_{n+1} + a_{n+1}^2 a_n^2 \tilde{y}_n = 0. \end{aligned} \tag{27}$$

We remark however that the preceding relations do not include $\tilde{y}_0$. If for a_n^2 in (26), one has $a_0^2 = 0$, or if one suppose that $y_{-1} = 0$, so using the second relation in (24), one establishes the following difference-recurrence relations for the system of transformed functions $\tilde{y}_n$, $n = 0, 1, 2, \ldots$

$$\begin{aligned} &(\lambda_{n-1} - \gamma_m)(\lambda_n - \gamma_m)L_m \tilde{y}_{n+2} \\ &+(\lambda_{n-1} - \gamma_m)(\lambda_{n+1} - \gamma_m)(b_n - h(x))L_m \tilde{y}_{n+1} \\ &+(\lambda_{n+1} - \gamma_m)(\lambda_n - \gamma_m)L_m \tilde{y}_n = 0. \end{aligned} \tag{28}$$

2.1.4. *Duality.* Suppose that the transformable functions $y_n(x) = y_n(q^s)$ are also explicit functions of n. In that case, one can consider the functions $\theta_s(n) = \tilde{y}_n(q^s, m)$ dual to the transformed $\tilde{y}_n(x, m)$, $n = 0, 1, \ldots$ defining

$$\begin{aligned} \theta_s(0) &= \tilde{y}_0(q^s, m); \\ \theta_s(n) &= R_m y_n(q^s) = y_n(q^s) + f(q^s, m) y_n(q^{s-1}), \; n = 1, 2, \ldots \end{aligned} \tag{29}$$

From (6), one finds that the functions $\theta_s(n)$ satisfy the three-term recurrence relation

$$\begin{aligned} &\theta_{s+1}(n) + (\tilde{v}(q^s) - \delta_n)\theta_s(n) + \tilde{w}(q^s) u(q^{s-1})\theta_{s-1}(n) = 0, \\ &\delta_0 = \gamma_m, \; \delta_n = \lambda_{n-1}, \; n \geq 1. \end{aligned} \tag{30}$$

If $\tilde{w}(1)u(q^{-1}) = 0$, then the functions in (30) are up to a multiplication by $\theta_0(n)$, polynomials in δ_n of degree s. If $\tilde{v}(q^s)$ is real for $s \geq 0$ and $\tilde{w}(q^s)u(q^{s-1}) > 0$, $s > 0$, so the polynomials are naturally orthogonal with positive discrete weight (Favard theorem).

2.1.5. *IHM factorization.* Considering the formulas (25), and the second difference eigenvalue equation in (10)), one finds that for a fixed m, the operator

$$\tilde{\mathcal{H}}_n(x, m) = \tilde{H}(x, m) - \lambda_{n-1}, \;\; n = 1, 2, \ldots \tag{31}$$

admits the following factorization of IHM type:

$$\begin{aligned} r(x, n)\tilde{\mathcal{H}}_n - \mu_n &= B_n A_n \\ t(x, n)\tilde{\mathcal{H}}_{n+1} - \mu_n &= A_n B_n, \end{aligned} \tag{32}$$

for some first order difference operators A_n and B_n and some functions r and t, while $\mu_n = -\alpha_{n-1}\beta_{n-1}(\lambda_{n-1} - \gamma_m)(\lambda_n - \gamma_m)$. We will not evaluate the unknown expressions as we are only interested in principle here.

2.1.6. *MSO factorization.* Here also, the operator $\tilde{H}$ admits a factorization of MSO type. To be convinced in that, we need only to ensure that for fixed m, the operator $\tilde{H}(x, m)$ is also "bispectral". Indeed, we have

$$\begin{aligned} \tilde{H}(x, m)\tilde{y}_n(x, m) &= \tilde{\lambda}_n \tilde{y}_n(x, m), \; n = 0, 1, \ldots \\ \tilde{\lambda}_0 = \gamma_m; \;\; &\tilde{\lambda}_n = \lambda_{n-1}, \; n = 1, 2, \ldots \end{aligned} \tag{33}$$

and

$$\tilde{H}(x, m)\tilde{z}_k(x, m) = \gamma_k \tilde{z}_k(x, m), k = 0, 1, \ldots, k \neq m \tag{34}$$

where $\tilde{z}_k(x, m) = R_m z_k(x)$. Here, one can generalize (6) to obtain a chain like (1):

$$\begin{aligned} H^j(x, m_0, \ldots, m_{j-1}) - \gamma_{m_j} &= L_{m_j} R_{m_j} \\ H^{j+1}(x, m_0, \ldots, m_j) - \gamma_{m_j} &= R_{m_j} L_{m_j}, \end{aligned} \tag{35}$$

with

$$\begin{aligned} H^0(x, m_0) &= H(x) \\ z_n^0(x, m_0) &= z_n(x) \\ y_n^0(x, m_0) &= y_n(x) \\ R_{m_j} &= 1 - \frac{\tilde{z}^j_{m_j}(x, m_0, \ldots, m_{j-1})}{\tilde{z}^j_{m_j}(x/q, m_0, \ldots, m_{j-1})} E_q^{-1} \\ L_{m_j} \tilde{y}_0^{j+1}(x, m_0, \ldots, m_j) &= 0 \\ \tilde{y}_n^{j+1}(x, m_0, \ldots, m_j) &= R_{m_j} \tilde{y}_n^j(x, m_0, \ldots, m_{j-1}), n = 1, 2, \ldots \\ \tilde{z}_n^{j+1}(x, m_0, \ldots, m_j) &= R_{m_j} \tilde{z}_n^j(x, m_0, \ldots, m_{j-1}), n = 0, 1 \ldots, (n \neq m_j). \end{aligned} \tag{36}$$

2.2. **A special way.**

2.2.1. *The way.* Consider again the eigenvalue equation

$$[U(x)E_q + V(x) + W(x)E_q^{-1}]Q_n(x) = \Upsilon_n Q_n(x). \tag{37}$$

As already noted, it is not difficult to rewrite (37) under the form

$$[A(x)E_q + B(x) + C(x)E_q^{-1}]Y_n(x) = \Lambda_n Y_n(x). \tag{38}$$

where $B(x) = -(A(x)+C(x))$. Hence we will here consider (38) as a general starting equation. Consider the situation when $C(x)$ doesn't depend explicitly on q and $A(x) = cC(x)$, c, a constant (a similar reasoning should be developed considering that $A(x)$ do not depend explicitly on q and C(x)=c A(x)). In that case (38) reads

$$[cC(x)E_q - (cC(x)+C(x)) + C(x)E_q^{-1}]Y_n(x,q) = \Lambda_n(q)Y_n(x,q). \tag{39}$$

Substituting q by $1/q$ in (39), and performing a similarity reduction on the obtained operator in the left hand side, one gets

$$\begin{aligned} &[cC(x)E_q - (cC(x)+C(x)) + C(x)E_q^{-1}]\pi(x)Y_n(x,1/q) \\ &= \Lambda_n(1/q)\pi(x)Y_n(x,q). \end{aligned} \tag{40}$$

where

$$\pi(qx)/\pi(x) = 1/c. \tag{41}$$

This means that the operator in the left hand side of (39) and (40) is "bispectral" with two distinguished systems of eigenelements $(\lambda_n(q), Y_n(x,q))$ and $(\Lambda_n(1/q), Z_n(x,q))$ where $Z_n(x,q) = \pi(x)Y_n(x,1/q)$. Hence it can be transformed according to the scheme studied in the first subsection. But as one can see, if $A(x)$ is for example a polynomial, the functions $Y_n(x,q)$ are not in general orthogonal. That is why we rewrite (39) and (40) in a more convenient form for the transformation. For that, supposing that $\lambda_n(q) \neq 0$, for $n \geq 1$ (this is generally the case for polynomial type of solutions), we define the functions $y_n(x,q)$ by

$$\begin{aligned} &y_n(x,q) = \tfrac{1}{\Lambda_{n+1}(q)}[cE_q - (c+1) + E_q^{-1}]Y_{n+1}(x,q) \\ &n = 0, 1, \dots \end{aligned} \tag{42}$$

As one can verify, the functions $y_n(x,q)$ are given by

$$y_n(x,q) = \tfrac{Y_{n+1}(x,q)}{C(x)},\ n = 0, 1 \dots \tag{43}$$

and satisfy the eigenvalue equation

$$[u(x)E_q + v(x) + w(x)E_q^{-1}]y_n(x,q) = \lambda_n(q)y_n(x,q). \tag{44}$$

where

$$\begin{aligned} &u(x) = cC(qx); v(x) = -(c+1)C(x) - \Lambda_1(q); \\ &w(x) = C(x/q); \lambda_n(q) = \Lambda_{n+1}(q) - \Lambda_1(q). \end{aligned} \tag{45}$$

In particular, if $y_0(x,q) \equiv const$, then $v(x) = -(u(x)+w(x))$. Similarly, the functions

$$z_n(x,q) = \pi(x)y_n(x,1/q) \tag{46}$$

satisfy the equation

$$[u(x)E_q + v(x) + w(x)E_q^{-1}]z_n(x,q) = \gamma_n(q)z_n(x,q). \tag{47}$$

where $\gamma_n(q) = \Lambda_{n+1}(1/q) - \Lambda_1(q)$. Thus, the operator in the left hand side of (44) and (47) is "bispectral" and under additional boundary constraints, the functions $y_n(x,q)$ are orthogonal with the weight

$$\rho(x) = \tfrac{w(qx)}{x\pi(x)}. \tag{48}$$

Hence the considerations from the first subsection can be reported here.

2.2.2. *Example.*

The transformable and transformation functions. Applying the preceding considerations to the q-hypergeometric case (see(3)),

$$[A(x)E_q + B(x) + C(x)E_q^{-1}]Y_n(x) = \Lambda_n Y_n(x), \tag{49}$$

with

$$\begin{aligned} A(x) &= [(\sigma_0 + (1-1/q)\tau_0)x^2 + (\sigma_1 + (1-1/q)\tau_1)x + \sigma_2]/x^2; \\ C(x) &= [q(\sigma_0 x^2 + \sigma_1 x + \sigma_2)]/x^2; B(x) = -(A(x)+C(x)), \end{aligned} \tag{50}$$

one is led to the following simple "bispectral" situation

$$\begin{aligned} [u(x)E_q + v(x) + w(x)E_q^{-1}]y_n(x,q) &= \lambda_n(q)y_n(x,q) \\ [u(x)E_q + v(x) + w(x)E_q^{-1}]z_n(x,q) &= \gamma_n(q)z_n(x,q), \end{aligned} \tag{51}$$

where

$$u(x) = -c(q^3x-1)/x; w(x) = -(xq-1)/x; v(x) = -(u(x)+w(x)) \tag{52}$$

$$\lambda_n = q^{1-n}(1-q^n)(cq^{2+n}-1); \gamma_n = q^{1-n}(q^{2+n}-1)(c-q^n) \tag{53}$$

and the functions $y_n(x,q)$ are a special case of the Little q-Jacobi $p_n(qx;c,q|q)$ or equivalently Big q-Jacobi $P_n(q^3x;q,c,0;q)$ polynomials [9]. The transformation functions $z_n(x,q)$ being on the other side defined as in (41) and (46). For their use in the formulas (25) and (27), we give here for the polynomials $y_n(x,q)$ the difference relations (in literature, they are not given in this form) and recurrence ones. We have

$$\begin{aligned} c_1(n)y_{n+1}(x,q) &= [r(x)E_q + f_n(x)]y_n(x,q) \\ -a_{n+1}^2 c_1(n+1)c_1(n)y_n(x,q) &= [r(x)E_q + g_n(x)]y_{n+1}(x,q), \end{aligned} \tag{54}$$

$$y_{n+1}(x,q) + (b_n - x)y_n(x,q) + a_n^2 y_{n-1}(x,q) = 0 \tag{55}$$

where $(Q = q^n)$

$$c_1(n) = -\frac{cQ^2q^3+cQ^2q^2+Q^6c^3q^7-c^2Q^4q^6-c^2Q^4q^4-q-q^5c^2Q^4+q^4Q^2c}{Q(cQ^2q^3-1)(cQ^2q-1)} \tag{56}$$

$$b_n = \frac{(Q^2c^2q^2-Qcq^2+cQ^2q^2-2Qcq+c-Qc+1)Q}{(cQ^2q-1)(cQ^2q^3-1)q} \tag{57}$$

$$a_n^2 = \frac{Q^2(Qc-1)(Qcq-1)(-1+Q)(-1+qQ)c}{(-1+cQ^2)(cQ^2q-1)^2(-1+cQ^2q^2)q^3}; \;\; r(x) = -c(q^3x-1) \tag{58}$$

$$f_n(x) = \frac{qx}{Q} - \frac{-1-c+Qcq^2+Qcq}{cQ^2q^3-1}; \; g_n(x) = xcq^4Q - cqQ\frac{Qq^2+Qcq^2-q-1}{cQ^2q^3-1}. \tag{59}$$

Note finally that the polynomials $y_n(x)$ are orthogonal on the interval $[0,q^{-\frac{5}{2}}]$ with respect to the weight $\rho(x)$ given by (48) where $w(x)$ is given by (52) and $\pi(x)$ by (41). As the interval of orthogonality is finite, they are also closed in the corresponding inner product space.
The transformed functions. For a given m, the properties of the transformed functions $\tilde{y}_n(x,m)$, $n = 0,1,\ldots$ are those derived in the first subsection ("The general case") of the current section: They satisfy type (10) difference equations, type (25), (27) and (28) difference and recurrence relations. And since the conditions of orthogonality of the proposition (2.1) are satisfied, they are orthogonal in the inner product space $\tilde{\ell}_2^2(0,q^{-\frac{5}{2}};\tilde{\rho}^2)$ where $\tilde{\rho}^2$ is given by the formula in (14). For the closure, we have that the system $\tilde{y}_n(x,m)$, $n = 0,1,\ldots$ is closed in the space since the unique element $\tilde{y}_0(x,m)$ orthogonal to it in its totality is not quadratically integrable. On the other side, the transformed operator $\tilde{H}(x,m)$ admits IHM and MSO factorizations according to the scheme given in the first subsection. There is no however interesting duality relations.

How seem the transformed objects? Let us note that, using simple procedures in Maple V (see for ex [14]), allows to evaluate explicitly any one of them at least for no very higher m and n (as long as the

software and the computer capacities allow).
The case $m=1$ illustrates the first non-classical situation for the transformed objects. As only in this case, the required volume to display the main data is admissible, we consider only this case here. The main data are ($m=1$, $n=0,1,2$):

$$f(x,1)=-\frac{(cq-q^4)x-c+q}{((c-q^3)x-c+q)c};\ g(x,1)=\frac{(xq-1)c(xc-q^3x-c+q)}{(xcq-c-xq^4+q)x} \tag{60}$$

$$\begin{aligned}\tilde{v}(x,1)=[&(c^3q^4-2q^7c^2+c^2q^2+cq^{10}-2cq^5+q^8)x^3+(-c^3q^4-c^3q^3\\&-qc^3+q^7c^2+q^6c^2+2q^5c^2+c^2q^4-2qc^2-2cq^8+cq^5+2cq^4+q^3c\\&+cq^2-q^8-q^6-q^5)x^2+(c^3q^3+qc^3+c^3-3c^2q^4-c^2q^3+qc^2+cq^5\\&-q^3c-3cq^2+q^6+q^3+q^5)x-c^3+qc^2+cq^2-q^3]/\\&[((cq-q^4)x-c+q)((c-q^3)x-c+q)x]\end{aligned} \tag{61}$$

$$\tilde{w}(x,1)=-\frac{(xcq-c-xq^4+q)(x-1)(xc-q^3x-cq+q^2)}{(xc-q^3x-c+q)^2x} \tag{62}$$

$$\tilde{\rho}^2(x,1)=c_1\rho^2(x)g(x,1)=c_2\frac{q^2x-1}{x^3\pi(x)}\frac{(xq-1)(xc-q^3x-c+q)}{(xcq-c-xq^4+q)} \tag{63}$$

$$\begin{aligned}\tilde{y}_0(x,1)&=\frac{c_3}{\tilde{\rho}^2(x,1)\pi(x)y_1(x,1/q)(h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}}))}\\&=c_4\frac{(xq-1)(xc-q^3x-c+q)}{(q^2x-1)(xcq-c-xq^4+q)^2}\end{aligned} \tag{64}$$

where $c_i,\ i=1,\dots,4$ are some constants of integration,

$$\tilde{y}_1(x,1)=\frac{(-q+c)(xc-c-q^3x+1)}{(xc-q^3x-c+q)c} \tag{65}$$

$$\begin{aligned}\tilde{y}_2(x,1)=[&q(c^2q^2x+c-c^2q^2-c^2x^2q-q+xc^2q^4-xc^3q\\&+q^3c^2x+cq^7x^2-c^2q^4x^2+c^2q^5x+c^3q^4x^2-c^2q^7x^2-xc^3q^4\\&+x^2cq-q^5cx-xcq-q^3cx-xcq^4+cq^2+qc^3-qc^2+c^2x+xq^4\\&-q^4x^2+xq+xc^2q+cq^4x^2-c^2+cq-cq^2x-xc)]/\\&[(q^3c-1)(xc-q^3x-c+q)c]\end{aligned} \tag{66}$$

We will remark that if $w(x)\sim x^{\alpha}$ while $x\rightsquigarrow\infty$, so $\tilde{y}_0(x,m)\sim\frac{1}{x^{m+\alpha}}$ and $\tilde{y}_n(x,m)\sim x^{m+n-1}$, $n=1,2,\dots$ (in our particular case, $\alpha=0$ and $m=1$).
Let us note finally that similar transformation formulas for the special Meixner polynomials $M_n^{(2,c)}(x)$ can be found in [1].

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