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Discrete factorization techniques for orthogonal polynomials on lattices

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To my late parents T. Nyabukeme [191x-1992], J. Barankana [190x-1985] and brother A. Kayoya [194x-1994].

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Contents

Introduction

In mathematics, the term "factorizability" rings first of all like "accessibility". Take for example a differential or a difference operator $H(x)$ of a given order. It is a known fact that the irreducible factorization of such an object is equivalent to the determination in quadratures of the corresponding kernels. The concept of the "*factorization technique*" to be met here consists in indirect exploitation of the phenomenon of "factorizability". Take the well known in linear algebra technique of similarity transformation of a $n \times n$ matrix A:

$$\tilde{A} = SAS^{-1}. \tag{1}$$

This transformation is equivalent to first factorize

$$A = RS \tag{2}$$

with non-singular S and then permute

$$\tilde{A} = SR. \tag{3}$$

The essential of merits of such a transformation is known to lie in its isospectrality. The operation should be repeated as long as necessary. In LR or QR-algorithm for example, the operation is repeated indefinitely.
Thus, the "*factorization technique*" to be met here can be seen as the technique "*factorize-permute*". Clearly, such a technique acquires most of non-triviality as much as the objects to deal with admit the non-commutativity property. As in the theory of matrices, this property is characteristic in the theory of differential or difference operators. In differential (difference) operators theory, the technique consists in sending

$$H(x) = L(x)R(x) + \mu \tag{4}$$

into

$$\tilde{H}(x) = R(x)L(x) + \mu \tag{5}$$

where H is a differential (difference) operator, μ a spectral parameter and L, R the result of the primordial factorization of $H - \mu$. Performing repeatedly the operation N times leads to the *factorization chain*

$$\begin{aligned} H_j(x) - \mu_j &= L_j(x)R_j(x) \\ H_{j+1}(x) - \mu_j &= R_j(x)L_j(x),\ j = 1, \ldots N. \end{aligned} \tag{6}$$

As far as we know, this technique was discovered and first used in [32]. There, the idea consisted in that if $\tilde{H}$ and $\tilde{G}$ are obtained by "transference" (4)-(5) respectively from H and G, so the pair $(\tilde{H}, \tilde{G})$ belongs to the same spectral curve (if any) as (H, G). The historical non-reference to [32] is probably linked to the fact that that work remained in oblivion for near a half-century, namely until the appearance in the seventies of a series of works [70, 71, 72, 73] rediscovering and extending a main result of [32, 33] (solution of commutation equations of differential (difference) operators).
It is clear that the operator R (L) transforms any eigenfunction of H ($\tilde{H}$) into an eigenfunction (if not zero) of $\tilde{H}$ (H). In the event where H is the Schrödinger operator, such a transformation was already applied by Darboux [38] and is currently credited to him. Let us remark that for this choice of H, there is equivalence between the factorization technique of Burchnall-Chaundy and the Darboux transformation. Moreover, a repeated application of the factorization technique N times on the Schrödinger operator is equivalent to a Darboux transformation of the same operator using an operator $R^{(N)}$ of order N.
The Darboux transformations for discrete Schrödinger operators were first studied in [84, 85], even if the factorization techniques for difference operators were already applied in [87]. Along the time, the factorization techniques, as well for differential as for difference operators, encountered many generalizations and applications in different areas of mathematics and mathematical physics: Quantum mechanics [105, 56, 39, 90, 17, 18, 37, 21, 93], special functions and orthogonal polynomials [29, 50, 118, 56, 57, 87, 116, 34, 35, 88, 89, 69, 60, 31, 17, 18, 107, 64, 104, 43, 44, 93, 115, 109, 110, 111, 112, 113, 114, 122], numerical calculus (as LR and QR algorithms) [119], bispectral problems [69, 42, 114], Lie algebras [86, 87, 88, 17, 18, 107, 93], commutation equations [32, 106, 109, 113], integrable systems [1, 92, 51, 84, 85].

Besides the orthogonal polynomials theory, the factorization techniques encountered much enthusiasm in quantum mechanics. It was used already in [105] solving the harmonic oscillator problem. Nowadays, most of known exactly solvable potentials can be obtained by this technique [105, 56, 39, 90, 17, 18, 37, 21, 93] which, on the other side, seems to offer still potentialities in the area [90, 98, 99, 100, 120].

The first major concern of this thesis consists in the applications of discrete factorization techniques in discrete polynomial eigenfunction problems. There is mainly questions of "generating", "solving" or "modifying" difference operators admitting complete sequences of polynomial eigenfunctions. More precisely, we are concerned in solving:

Problem 1. Find, for a given operator $H(s)$, a complete set of polynomial eigenfunctions.

Problem 2. Generate from a factorization chain an operator $H(s)$ having a complete set of polynomial eigenfunctions.

Problem 3. Generate from a given operator $H(s)$ having a complete set of polynomial eigenfunctions, another operator $\tilde{H}(s)$, having also a complete set of polynomial eigenfunctions but not belonging to the same family as the original one.

For solving such problems, by the help of the factorization chain (6), we resort to three kinds of methods. The first one consists in imposing to the factorization chain a "*quasi-periodicity*" behaviour. The second technique consists in imposing to the factorization chains a special "*shape-invariant*" behaviour (or *symmetry*). The third technique in the opposite consists in "*modifying*" polynomials: start from a known exactly solvable (in polynomials) difference operator and then generate a new exactly solvable (in polynomials) one. As it is easily seen, the third technique is convenient for solving type 3 problems, while the second and first methods can be applied on both type first or second problems.

In the discrete cases, those methods are applied inside two different types of factorization techniques. Writing a linear difference eigenvalue equation of order $2d$ in the form

$$H(s)v(s) = \left[\sum_{i=-d}^{d} A_i(s)\mathbf{E}_s^i \right] v(s) = \lambda v(s), \tag{7}$$

where $\mathbf{E}_s^i[h(s)] = h(s+i)$, $d \in \mathbf{Z}^+$, $i \in \mathbf{Z}$ and $A_i(s)$ are some scalar functions in s, so the first type of factorization techniques consists in factorizing exactly the operator $H(s) + c$, while the second type of factorization consists in factorizing exactly the operator $\mathbf{E}^d_s \circ [H(s)+c]$, for some constant (in s) c. For historical reasons, we will refer, in this thesis, to the first type of factorization as the *Spiridonov-Vinet-Zhedanov type of factorization*, and to the second type as the *Infeld-Hull-Miller type of factorization.*

Divided difference equation of hypergeometric type (or simply difference hypergeometric equation) reads [94]

$$\left[\tilde{\sigma}[y(s)]\frac{\Delta}{\Delta y(s-\frac{1}{2})}\left[\frac{\nabla}{\nabla y(s)}\right] + \frac{1}{2}\tilde{\tau}[y(s)]\left[\frac{\Delta}{\Delta y(s)} + \frac{\nabla}{\nabla y(s)}\right]\right] v(s) = \lambda v(s), \qquad (8)$$

where $\Delta f(s) = f(s+1) - f(s)$, $\nabla f(s) = f(s) - f(s-1)$, $\tilde{\sigma}(y)$ and $\tilde{\tau}(y)$ polynomials of at most second and first degree respectively and the lattice

$$y(s) = c_1 q^s + c_2 q^{-s} + c_3 = c_1(q^s + q^{-s-\mu}) + c_3 \qquad (9)$$

or particularly

$$y(s) = \tilde{c}_1 s^2 + \tilde{c}_2 s + \tilde{c}_3, \qquad (10)$$

or

$$y(s) = c_1 q^s + c_3, \qquad (11)$$

or

$$y(s) = \tilde{c}_2 s + \tilde{c}_3, \qquad (12)$$

for some constant (in s) μ, q, $c_i(q)$, $\tilde{c}_i$, λ. The equation (8) approximates, up to the second order of accuracy, the differential hypergeometric type equation

$$\tilde{\sigma}(x)v'' + \tilde{\tau}(x)v' = \lambda v. \qquad (13)$$

The polynomial solutions of Eq. (8) are called *discrete (difference) polynomials of hypergeometric type* or simply *hypergeometric polynomials on lattices.* In the event where the lattice is given by Eqs. (9), (10), (11), or (12), one says about *hypergeometric polynomials on q-nonlinear, nonlinear, q-linear,* or *linear lattice* respectively. We will refer also to the operator in the left hand side of Eq. (8) as the *difference hypergeometric operator* or equally *hypergeometric operator on lattice (q-nonlinear ...).* Solutions $v(s)$ of Eq.

(8) admits a fundamental property similar to the property of the differential equation (13): The difference derivative

$$v_1(s) = \frac{\Delta v(s)}{\Delta y(s)} \tag{14}$$

satisfies an equation that is analog to (8) with $y(s)$ replaced by $y_1(s) = y(s+\frac{1}{2})$ [94]. For this reason, orthogonal polynomial solutions of Eq. (8) are generally called "*classical orthogonal polynomials on lattices*", while those for Eq. (13) are referred to as the "very classical" ones. The most important classical orthogonal polynomials on lattices are: The Askey-Wilson, q-Racah and q-dual Hahn for the q-nonlinear lattices, Racah and dual Hahn for the nonlinear lattices, q-Hahn, q-Meixner, q-Kravchuk and q-Charlier for the q-linear lattices, Hahn, Meixner, Kravchuk and Charlier polynomials for the linear lattices.

Our first group of results consists mainly of solutions of the so-called problems $1, 2, 3$ relatively to the difference hypergeometric operator in Eq. (8), using the factorization techniques. For convenience, if a problem say x is solved for an operator admitting a sequence y of polynomial eigenfunctions, we will let us say that "the problem x is solved for the sequence y of polynomial eigenfunctions".

The factorization technique was applied to the problem 1 for the "very classical" orthogonal polynomials in [29]. The Hermite polynomials appeared by this technique in quantum mechanics already in [105]. The problem 2 was solved by this technique for the Charlier, Meixner, Kravchuk and (not as in the previous cases) the Hahn polynomials in [87]. The problem 3 was solved for the Hermite polynomials in [104]. The problem 2 was solved in [109] for the q-Charlier, the q-Meixner and q-Kravchuk polynomials. Also, the factorization technique helped implicitly to solve the equation (8) by its authors (see subsection 1.1.1, the factorization technique being applied here more explicitly than in the original works).

In this thesis:

Result 1. The (shape-invariant) Infeld-Hull-Miller factorization type is studied in general for the second-order and fourth-order eigenfunction equations (see the second parts in sections 2.2.1 and 2.2.2).

Result 2. For the Charlier , Meixner and Kravchuk polynomials, the problems 1 and 2 are solved (section 3.1), using the "quasi-periodicity" method.

Result 3. For the hypergeometric orthogonal polynomials on linear lattices and polynomials dual to them, the problem 1 is solved (section 3.2) as well as the problem 2 (section 3.2) (see also section 6.2 (including q-linear lattices)), using the "shape-invariance" method .

Result 4. For the hypergeometric orthogonal polynomials on q-nonlinear lattices and polynomials dual to them, the problem 1 is solved (section 3.3) as well as the problem 2 (section 3.3) (see also section 6.2), using the "shape-invariance" method.

Result 5. Difference hypergeometric functions generalizing the Askey-Wilson polynomials are given (subsection 3.3.2).

Result 6. For the hypergeometric polynomials on linear lattices, the problem 3 is discussed (chapter 4) and explicitly solved for the special $M_n^{(2,c)}(x+1)$ Meixner polynomials (chapter 4), using the "modification" method.

The polynomials $P_n(x)$ are called (class κ) *continuous Laguerre-Hahn orthogonal polynomials* iff the corresponding Stieltjes function $S(x)$ satisfies the Riccati equation [78]

$$A(x)S'(x) = B(x)S^2(x) + C(x)S(x) + D(x), \tag{15}$$

where A, B, C and D are polynomials of degrees $\kappa+2$, $\kappa+2$, $\kappa+1$ and κ, respectively. Most of nowadays known continuous orthogonal polynomials belong to this class. The subclass of *continuous semi-classical orthogonal polynomials* corresponds to the case $B=0$. The "very classical" polynomials appear then as the semi-classical of class $\kappa = 0$.

On the other side, the polynomials $P_n(y(s))$ are called (class κ) *Laguerre-Hahn orthogonal polynomials on special nonuniform lattice* iff the Stieltjes function $S(y(s))$ satisfies the Riccati equation [79]:

$$\begin{aligned}\mathcal{A}(x(s))\tfrac{S(y(s+1))-S(y(s))}{y(s+1)-y(s)} &= \mathcal{B}(x(s))S(y(s+1))S(y(s))\\ &+\mathcal{C}(x(s))\tfrac{S(y(s+1))+S(y(s))}{2} + \mathcal{D}(x(s))\end{aligned} \tag{16}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are polynomials of degrees $\leq \kappa+2$, $\kappa+2$, $\kappa+1$ and κ, respectively and

$$\begin{aligned} x(s) &= \tilde{c}_1 q^s + \tilde{c}_2 q^{-s} + \tilde{c}_3,\\ y(s) &= \tilde{c}_4 x(s-\frac{1}{2}) + \tilde{c}_5, \end{aligned} \tag{17}$$

the so-called special non-uniform lattice (snul) [79]. Most of nowadays known orthogonal polynomials belong to this class (in particular the continuous Laguerre-Hahn polynomials are included here by the limit $q \to 1$). The subclass of *semi-classical orthogonal polynomials* [79, 80], corresponds to the case $\mathcal{B} = 0$. The "classical" orthogonal polynomials (i.e. orthogonal polynomial solutions of (8)) appear then as the semi-classical of class $\kappa = 0$.

A curiosity question naturally arises: Does there exists a firm interconnection between the factorization techniques and the Laguerre-Hahn approach to orthogonal polynomials? The answer is yes and our "transition" result reads:
Result 7. We show the interconnection between the (shape-invariant) Infeld-Hull-Miller factorization technique and the Laguerre-Hahn approach to orthogonal polynomials. In passing, explicit examples of pure (i.e. non-semiclassical) Laguerre-Hahn polynomials are given. Among them, an example of pure Laguerre-Hahn polynomials "generable" from Infeld-Hull-Miller factorization chains (section 5.2).
This interconnection seems to be a positive indication for the possibilities of extensions of the Infeld-Hull-Miller factorization technique applied in this thesis to the second-order difference eigenvalue problem. Indeed, the Infeld-Hull-Miller factorization technique, once extended for example to the fourth order eigenvalue problem, is expected to be equivalent to a certain approach to orthogonal polynomials, extending the nowadays Laguerre-Hahn approach to orthogonal polynomials (see more precise comments after the systems (2.222)-(2.224)).

One of the fundamental questions in orthogonal polynomial theory is that of characterization of polynomials. Besides the "inevitable" recurrence relations, the polynomials are expected to satisfy difference-recurrence relations, linear difference equations (of eigenvalue type or not, of finite order or not), duality properties ... As, we will see, the classical (up to Askey-Wilson polynomials) for example are fully characterized by the difference hypergeometric eigenvalue equation (8). Also, a well known Leonard result [75], characterizes the q-Racah polynomials and their specializations as the unique orthogonal polynomials having orthogonal polynomials as dual sequences. A good survey on various characterization theorems for classical polynomials can be found in [75].
In the continuous case, a characterization theorem exists as well for the general semi-classical polynomials [83, 55]. Namely, a system of orthogonal polynomials are (continuous) semi-classical iff they satisfy a linear second-order differential equation with polynomial coefficients which degrees are bounded relatively to the degree of the polynomials. Such a characterization theorem does not exist for the semi-classical polynomials on lattices,

and considering the example in (5.45), there is no reason to expect it. Indeed, even if the corresponding moment functional is quasi-definite but not positive definite, there is an example of special non-semiclassical Laguerre-Hahn polynomials satisfying a linear second-order difference equation.
The largest characterization result for the semi-classical orthogonal polynomials on lattices can be found in [79, 80] and it reads: They satisfy a linear second-order difference equation with special coefficients and in the converse sense, any system of orthogonal polynomials satisfying a special difference-recurrence relation are necessarily semi-classical.
The "necessity" of some characterization theorems tells about the existence of difference (differential) equation that satisfies the given class of polynomials. But beyond that, there remains at least two unresolved questions: The one consists in establishing that equation formally, the other consists in establishing it explicitly . So practically, the approach to the "necessity" question divides in three tasks:
Problem 4. Prove the existence of a linear difference equation (not necessarily of eigenvalue type) of fixed order satisfied by the polynomials from a given family.
Problem 5. Establish the equation *formally.*
Problem 6. Establish it *explicitly.*
Clearly, a solution of the 6th problem leads to solutions of the 5th and 4th problems, while a solution of the 5th problem leads to that of the 4th.
The second major concern of this thesis consists in a contribution in this area for the so-called Laguerre-Hahn orthogonal polynomials on special non-uniform lattices (results 8, 9, 10 below).

The fourth order differential equation for the continuous Laguerre-Hahn polynomials has been established in [102]. The approach adopted there has been extended as well to the Laguerre-Hahn orthogonal polynomials on linear and q-linear lattices in [47] and [48] respectively. In [102] (see also [28, 101]) as in [47] (see also [103, 46]) and [48], the equations were written explicitly for the cases of polynomials r-associated to the corresponding classical situations, that is Jacobi polynomials and specializations in [102], Hahn, big q-Jacobi polynomials and specializations in [47] and [48] respectively.

The final group of results of this thesis:
Result 8. Starting at the difference-recurrence relations from [79], we establish (for the Laguerre-Hahn polynomials on special non-uniform lattices) the corresponding fourth order difference equation (in [79], an algorithm

(different than ours) for establishing such equation was given, but practical computations were not done)(section 6.1). This solves the "problem 5" for the polynomials.

Result 9. We give explicitly that equation for the cases of polynomials r-associated to all classical classes that is, classical orthogonal polynomials on linear lattices (result equivalent for example to that from [47]), classical orthogonal polynomials on q-linear lattices (result equivalent for example to that from [48]) and Askey-Wilson class (section 6.2). This solves the "problem 6" for the polynomials.

Result 10. Finally, we give it "semi-explicitly" (i.e. up to an explicit system of non-linear difference equations satisfied by the coefficients in the three-term recurrence relations) for the class one Laguerre-Hahn orthogonal polynomials on linear lattice (section 6.3). This solves (partially) the "problem 6" for the polynomials.

For avoiding any confusion, we will assume that the meaning assigned to a given symbolization is confined on the current chapter.

Except the section 5.1, the material of the chapters 3, 4, 5, 6 and the second part of the subsection 2.2.1 comes from our articles [22, 23, 24, 25, 26]. The results of the subsection 2.2.2 appear at first in this thesis.

Chapter 1

Hypergeometric polynomials on lattices

This chapter is essentially for recalling relevant concepts. For the reader not familiar with the subject matter, this recalling is however crucial in view of the fact that its content constitutes the basic tool of the whole chapters 3 and 4. The exposition encompasses the hypergeometric polynomials on linear, q-linear , nonlinear and q-nonlinear lattices as particular solutions of Eq. (8). More on the content as well as on the philosophy of the treated here matter can be found in a series of works essentially by Nikiforov-Suslov-Uvarov-Atakishiyev, in the early nineties. The monograph [94] and the articles [95, 19] are all together in general complete and self-contained. Globally, the material of this chapter will be implicitly referred to those works unless the contrary is specified.

1.1 The general case.

Consider the divided difference hypergeometric equation

$$\left[\tilde{\sigma}[y(s)]\frac{\Delta}{\Delta y(s-\frac{1}{2})}[\frac{\nabla}{\nabla y(s)}]+\frac{1}{2}\tilde{\tau}[y(s)][\frac{\Delta}{\Delta y(s)}+\frac{\nabla}{\nabla y(s)}]\right]v(s)=\lambda v(s), \quad (1.1)$$

where $\Delta f(s) = f(s+1) - f(s)$, $\nabla f(s) = f(s) - f(s-1)$, $\tilde{\sigma}(y)$ and $\tilde{\tau}(y)$ polynomials of at most second and first degree respectively and

$$y(s) = c_1 q^s + c_2 q^{-s} + c_3 = c_1(q^s + q^{-s-\mu}) + c_3 \quad (1.2)$$

for some constant (in s) μ, q, $c_i(q)$, λ. The equation (1.1) approximates, up to the second order of accuracy, the differential hypergeometric type equation

$$\tilde{\sigma}(x)v'' + \tilde{\tau}(x)v' = \lambda v. \tag{1.3}$$

We show that the equation admits polynomial solutions, give the Rodrigues type formula, the orthogonality relation and the hypergeometric representation.

1.1.1 The polynomial solutions.

For particular values of λ, say λ_n, the equation (1.1) admits polynomial solutions $P_n(y(s))$ of degree n. To see this, write first Eq. (1.1) in the more compact form:

$$\left[\sigma(s)\frac{\Delta}{\Delta y(s-\frac{1}{2})}\frac{\nabla}{\nabla y(s)} + \tau(s)\frac{\Delta}{\Delta y(s)}\right] v(s) = \lambda v(s), \tag{1.4}$$

where

$$\sigma(s) = \tilde{\sigma}[y(s))] - \frac{1}{2}\tilde{\tau}[y(s)]\Delta y(s-\frac{1}{2});\ \tau(s) = \tilde{\tau}[y(s)]. \tag{1.5}$$

Next, write Eq. (1.4) in a factorized form, translating the difference derivative operator in front

$$\left[[\sigma(s)\frac{\nabla}{\Delta y(s-\frac{1}{2})} + \tau(s)]\frac{\Delta}{\Delta y(s)}\right] v(s) = \lambda v(s), \tag{1.6}$$

and then permute the factorizing operators. We obtain that the function

$$v_1(s) = \frac{\Delta v(s)}{\Delta y(s)} \tag{1.7}$$

satisfies an equation that is analog to Eq. (1.4) with $y(s)$ replaced by $y_1(s) = y(s+\frac{1}{2})$:

$$\left[\sigma(s)\frac{\Delta}{\Delta y_1(s-\frac{1}{2})}\frac{\nabla}{\nabla y_1(s)} + \tau_1(s)\frac{\Delta}{\Delta y_1(s)}\right] v_1(s) = \mu_1 v_1(s), \tag{1.8}$$

where $\tau_1(s)$ is a certain polynomial of first degree in $y_1(s)$, $\mu_1 = \lambda - \frac{\Delta\tilde{\tau}[y(s)]}{\Delta y(s)}$. This is an extension of the related property of Eq. (1.3). Performing repeatedly the operation k times, one obtains that the function

$$v_k(s) = \prod_{j=0}^{k-1}\frac{\Delta}{\Delta y_{k-1-j}(s)}v(s),\ (k = 1, 2, \ldots) \tag{1.9}$$

$y_j(s) = y(s+\frac{j}{2})$, is a solution of

$$\left[\sigma(s)\frac{\Delta}{\Delta y_k(s-\frac{1}{2})}\frac{\nabla}{\nabla y_k(s)} + \tau_k(s)\frac{\Delta}{\Delta y_k(s)}\right] v_k(s) = \mu_k v_k(s), \tag{1.10}$$

where

$$\tau_k(s) = \frac{\sigma(s+k)-\sigma(s)+\tau(s+k)\Delta y(s+k-\frac{1}{2})}{\Delta y[s+\frac{k-1}{2}]};\, \mu_k = \lambda - \textstyle\sum_{j=0}^{k-1}\frac{\Delta\tau_j(s)}{\Delta y_j(s)}. \tag{1.11}$$

Conversely, if $v_k(s)$ is a solution of Eq. (1.10), then for non vanishing $\mu_0 = \lambda, \mu_1, \ldots, \mu_{k-1}$, the function

$$v_0(s) = \prod_{j=1}^{k}\frac{1}{\mu_{j-1}}\left[\sigma(s)\frac{\nabla}{\Delta y_{j-1}(s-\frac{1}{2})} + \tau_{j-1}(s)\right] v_k(s) \tag{1.12}$$

is a solution of Eq. (1.4). It can be shown that if $Q_n(y(s))$ is an arbitrary polynomial of degree n in $y(s)$, then

$$\frac{\Delta Q_n(y(s))}{\Delta y(s)} = r_{n-1}[y_1(s)], \tag{1.13}$$

where $r_{n-1}(y_1)$ is a polynomial of degree $n-1$ in y_1. Hence, for an arbitrary polynomial of degree n in $y(s)$ say $v(s)$, the function $v_k(s)$ given by Eq. (1.9) is a polynomial of degree $n-k$ in $y_k(s)$. Conversely, using Eq. (1.13) and the fact that for an arbitrary polynomial $Q_n(y(s)$,

$$\frac{Q_n[y(s+1)]+Q_n[y(s)]}{2} = t_n[y_1(s)] \tag{1.14}$$

where $t_n(y_1)$ is a polynomial of degree n in $y_1(s)$, one shows that if $v_k(s)$, solution of Eq. (1.10), is a polynomial of degree $n-k$ in $y_k(s)$, then $v_0(s)$, the solution of Eq. (1.4), given by Eq. (1.12), is a polynomial in $y(s)$ of degree n. Remarking that for $k=n$,

$$\begin{aligned}\lambda = \lambda_n &= \textstyle\sum_{j=0}^{n-1}\tilde{\tau}'_j \\ &= \tfrac{1}{2}\psi_q(n)\left[\left(q^{\frac{n-1}{2}} + q^{-\frac{n-1}{2}}\right)\tilde{\tau}' + \psi_q(n-1)\tilde{\sigma}''\right],\end{aligned} \tag{1.15}$$

the equation (1.10) admits a constant solution say v_n, we conclude from the preceding that the equation (1.4) admits as expected, a sequence of polynomial solutions $P_n(y(s))$, of degree n, $n \in \mathbf{Z}^+$,

$$P_n(y(s)) = \prod_{j=1}^{n}\frac{1}{\mu_{j-1}}\left[\sigma(s)\frac{\nabla}{\Delta y_{j-1}(s-\frac{1}{2})} + \tau_{j-1}(s)\right] v_n, \tag{1.16}$$

corresponding to $\lambda = \lambda_n$, provided $\lambda_n \neq \lambda_m$, $n \neq m$. As already noted, for convenience, we will call those polynomials *Hypergeometric polynomials on lattices.*

1.1.2 The Rodrigues type formula.

The formula (1.16) allows to calculate the polynomial solutions of Eq. (1.4). We below give an equivalent but more commonly applied formula. For that, remark first that for $\rho(s)$ and $\rho_k(s)$ satisfying

$$\frac{\Delta}{\Delta y(s-\frac{1}{2})}[\sigma(s)\rho(s)] = \tau(s)\rho(s); \frac{\Delta}{\Delta y_k(s-\frac{1}{2})}[\sigma(s)\rho_k(s)] = \tau_k(s)\rho_k(s) \quad (1.17)$$

respectively, the equations (1.4) and (1.10) can be written in the formal symmetric forms

$$\frac{\Delta}{\Delta y(s-\frac{1}{2})}[\sigma(s)\rho(s)\frac{\nabla v(x)}{\nabla y(s)}] = \lambda\rho(s)v(s) \quad (1.18)$$

and

$$\frac{\Delta}{\Delta y_k(s-\frac{1}{2})}[\sigma(s)\rho_k(s)\frac{\nabla v_k(x)}{\nabla y_k(s)}] = \mu_k\rho_k(s)v_k(s) \quad (1.19)$$

respectively. Using the relation

$$\rho_k(s) = \sigma(s+1)\rho_{k-1}(s+1); \qquad \rho_0(s) = \rho(s), \quad (1.20)$$

one easily deduces from Eq. (1.19) that

$$\rho_{k-1}(s)v_{k-1}(s) = \frac{1}{\mu_{k-1}}\frac{\nabla}{\nabla y_k(s)}[\rho_k(s)v_k(s)]. \quad (1.21)$$

As a consequence, the polynomial solutions $P_n(y(s))$ of Eq. (1.4) read

$$P_n(y(s)) = \frac{B_n}{\rho(s)}\nabla_n^{(n)}[\rho_n(s)]$$
$$= \frac{B_n}{\rho(s)}\frac{\nabla}{\nabla y_1(s)}\cdots\frac{\nabla}{\nabla y_{n-1}(s)}\frac{\nabla}{\nabla y_n(s)}[\rho_n(s)] \quad (1.22)$$

where B_n is a normalizing constant, known as the *difference analog of the Rodrigues type formula.* It is preferable to work with Eq. (1.22) rather than

Eq. (1.16) for example when searching the "hypergeometric representation" for the solutions (see below).

1.1.3 The orthogonality.

Multiplying the equation (1.18) for $P_n(s)$ by $P_m(s)$ and that of $P_m(s)$ by $P_n(s)$, subtracting members by members and summing over the values s_i, $a \le s_i \le b-1$, $s_{i+1} = s_i + 1$ one obtains

$$(\lambda_n - \lambda_m) \sum_{s_i=a}^{b-1} P_n(y(s_i)) P_m(y(s_i)) \rho(s_i) \Delta y(s_i - \tfrac{1}{2}) \\ = \sigma(s)\rho(s) \left[P_m(y(s)) \frac{\nabla P_n(y(s))}{\nabla y(s)} - P_n(y(s)) \frac{\nabla P_m(y(s))}{\nabla y(s)} \right] \Big|_a^b . \qquad (1.23)$$

Using Eq. (1.13) and Eq. (1.14), one easily shows that the expression in square brackets in Eq. (1.23) is a polynomial in $y(s-\frac{1}{2})$. Consequently, under the additional boundary and positivity conditions

$$\sigma(s)\rho(s) y^l(s - \tfrac{1}{2}) \Big|_{s=a,b} = 0, \;\; (l = 0, 1, \ldots), \qquad (1.24)$$

and

$$\rho(s_i)\Delta y(s_i - \frac{1}{2}) > 0, (a \le s_i \le b-1) \qquad (1.25)$$

respectively, we obtain the orthogonality relation on $[a, b-1]$ with weight $\rho(s)\Delta y(s-\frac{1}{2})$ for the polynomial solutions of Eq. (1.4):

$$\sum_{s_i=a}^{b-1} P_n(y(s_i)) P_m(y(s_i)) \rho(s_i) \Delta y(s_i - \tfrac{1}{2}) = \delta_{mn} d_n^2. \qquad (1.26)$$

It is easily seen that for a and b finite, $\rho(s_i) \neq 0$, $a \le \;\; s_i \le b-1$, the boundary conditions (1.24) can be reduced to

$$\sigma(a) = 0; \qquad \sigma(s) + \tau(s)\Delta y(s - \frac{1}{2}) \Big|_{s=b-1} = 0. \qquad (1.27)$$

1.1.4 The hypergeometric representation.

The (generalized) hypergeometric series are defined by

$${}_rF_s \left(\begin{array}{cccc} \alpha_1, & \alpha_2, & \ldots, & \alpha_r \\ \beta_1, & \beta_2, & \ldots, & \beta_s \end{array} \Bigg| \, z \right) \\ = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_r)_k}{(\beta_1)_k (\beta_2)_k \cdots (\beta_s)_k} \frac{z^k}{k!} . \qquad (1.28)$$

where $(a_1,\ldots,a_p)_k := (a_1)_k\ldots(a_p)_k$, $(a)_0=1$, $(a)_k = a(a{+}1)\ldots(a{+}k{-}1) = \frac{\Gamma(a+k)}{\Gamma(a)}$. Their q-versions, the basic hypergeometric series read

$$
\begin{aligned}
&{}_r\varphi_s\left(\begin{array}{cccc}\alpha_1, & \alpha_2, & \ldots, & \alpha_r\\ \beta_1, & \beta_2, & \ldots, & \beta_s\end{array}\middle|\, q;z\right)\\
&= \sum_{k=0}^{\infty}\frac{(\alpha_1;q)_k(\alpha_2;q)_k\cdots(\alpha_r;q)_k}{(\beta_1;q)_k(\beta_2;q)_k\cdots(\beta_s;q)_k}\left[(-1)^k q^{\frac{k(k-1)}{2}}\right]^{1+s-r}\frac{z^k}{(q;q)_k}.
\end{aligned}\qquad (1.29)
$$

where $(a_1,\ldots,a_p;q)_k := (a_1;q)_k\ldots(a_p;q)_k$, $(a;q)_0=1$, $(a;q)_k=(1-a)(1-aq)(1-aq^2)\ldots(1-aq^{k-1})$, $k=1,2\ldots$.
As well for the generalized hypergeometric series as for the basic ones , the radius of convergence is given by

$$
\rho_c = \begin{cases}\infty, & r<s+1\\ 1, & r=s+1\\ 0, & r>s+1.\end{cases}\qquad (1.30)
$$

Since $\lim_{q\to1}\frac{(q^a;q)_k}{(1-q)^k} = (a)_k$, the formulas in Eq. (1.28) and Eq. (1.29) are linked by

$$
\begin{aligned}
&\lim_{q\to1}{}_r\varphi_s\left(\begin{array}{cccc}q^{\alpha_1}, & q^{\alpha_2}, & \ldots, & q^{\alpha_r}\\ q^{\beta_1}, & q^{\beta_2}, & \ldots, & q^{\beta_s}\end{array}\middle|\, q;(q-1)^{1+s-r}z\right)\\
&\qquad = {}_rF_s\left(\begin{array}{cccc}\alpha_1, & \alpha_2, & \ldots, & \alpha_r\\ \beta_1, & \beta_2, & \ldots, & \beta_s\end{array}\middle|\, z\right).
\end{aligned}\qquad (1.31)
$$

One needs to remark the simplification of formulas (1.29) and (1.31), in the case $r=s+1$. The starting point for the search of the hypergeometric representation of polynomial solutions of Eq. (1.1) is the Rodrigues type formula in (1.22). By the general formula

$$
\nabla_n^{(n)}[f(s)] = \tilde\Gamma_q(n+1)\sum_{k=0}^{n}\frac{f(s-n+k)}{\prod_{l=0,l\neq k}^{n}\psi_q(k-l)\nabla y(s-\frac{n}{2}+\frac{k+l+1}{2})},\qquad (1.32)
$$

where

$$
\frac{\tilde\Gamma_q(s+1)}{\tilde\Gamma_q(s)} = \psi_q(s);\quad \psi_q(s)=\frac{1}{\kappa}(q^{\frac{s}{2}}-q^{-\frac{s}{2}})\ (\kappa = q^{\frac12}-q^{-\frac12})\qquad (1.33)
$$

for an arbitrary function $f(s)$, the formula in Eq. (1.22) is represented by a sum

$$
\begin{aligned}
P_n(y(s)) &= B_n\sum_{k=0}^{n}\frac{(-1)^{n-k}\tilde\Gamma_q(n+1)}{\tilde\Gamma_q(k+1)\tilde\Gamma_q(n-k+1)}\frac{\nabla y[s+k-\frac{n-1}{2}]}{\prod_{l=0}^{n}\nabla y[s+\frac{k-l+1}{2}]}\\
&\times\prod_{l=0}^{n-k-1}\sigma(s-l)\prod_{l=0}^{k-1}[\sigma(s+l)+\tau(s+l)\Delta y(s+l-\tfrac12)]
\end{aligned}\qquad (1.34)
$$

from which one searches to obtain type (1.29) form. After a set of computations, involving in particular the Watson and Sears transformations for the basic hypergeometric series, one attains the following final formula

$$\begin{aligned} P_n(s) = P_n(y(s)) &= \left(\tfrac{-A}{c_1 q^{\mu}\kappa^5}\right)^n B_n q^{-\frac{n}{2}\left[3s_1+s_2+s_3+s_4+\frac{3(n-1)}{2}\right]} \\ &\times \left(q^{s_1+s_2+\mu};q\right)_n \left(q^{s_1+s_3+\mu};q\right)_n \left(q^{s_1+s_4+\mu};q\right)_n \\ &\times {}_4\varphi_3\left[\begin{array}{c} q^{-n}, q^{\sum_{i=1}^4 s_i+2\mu+n-1}, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{array} \middle| q;q\right]. \end{aligned} \tag{1.35}$$

This is the hypergeometric representation of the general polynomial solutions of Eq. (1.1) on the general lattice (1.2). Specializing parameters in Eq. (1.35), which is equivalent to the specializations of parameters in $y(s)$, $\tilde{\tau}(s)$ and $\tilde{\sigma}(s)$, one obtains of course hypergeometric representations for the corresponding polynomials.

1.2 The classification.

Consider again the divided difference hypergeometric equation (1.1) (or (1.4)) on the lattice (1.2). The equation can also be written as

$$\hat{\sigma}(s)\frac{\Delta v(s)}{\Delta y(s)} - \sigma(s)\frac{\nabla v(s)}{\nabla y(s)} = \lambda \Delta y(s-\frac{1}{2})v(s) \tag{1.36}$$

where $\sigma(s)$ is given in Eq. (1.5) and

$$\hat{\sigma}(s) = \tilde{\sigma}[y(s))] + \frac{1}{2}\tilde{\tau}[y(s)]\Delta y(s-\frac{1}{2}) = \sigma(s) + \tau(s)\Delta y(s-\frac{1}{2}). \tag{1.37}$$

This means that the classification that we are searching for will depend on the lattice $y(s)$ in Eq. (1.2) and on the roots and the leading coefficients of $\sigma(s)$ and $\hat{\sigma}(s)$. Hence a little close analysis of the objects is necessary. The lattice $y(s)$ can be rewritten:

$$y(s) = c_1(q)\left(q^s + q^{-s-\mu}\right) + c_3(q) \tag{1.38}$$

$$= c_1(q)q^{-\frac{\mu}{2}}\kappa^2 \tfrac{q^{s+\frac{\mu}{2}}+q^{-(s+\frac{\mu}{2})}-2}{\kappa^2} + 2c_1(q)q^{-\frac{\mu}{2}} + c_3(q). \tag{1.39}$$

Hence, for

$$\begin{aligned} c_1(q)q^{-\frac{\mu}{2}}\kappa^2 &= \tilde{c}_1 = B \\ 2c_1(q)q^{-\frac{\mu}{2}} + c_3(q) &= -\tilde{c}_1\tfrac{q^{\frac{\mu}{2}}+q^{-\frac{\mu}{2}}-2}{\kappa^2} + \tilde{c}_3, \end{aligned} \tag{1.40}$$

and $q \to 1$, using the relation

$$\lim_{q\to 1} \frac{q^t + q^{-t} - 2}{\kappa^2} = \lim_{q\to 1} \psi_q^2(t) = t^2, \tag{1.41}$$

we have

$$\begin{aligned} y(s) &= \tilde{c}_1 \left(s + \tfrac{\mu}{2}\right)^2 - \tilde{c}_1 \left(\tfrac{\mu}{2}\right)^2 + \tilde{c}_3 = \tilde{c}_1 s^2 + \tilde{c}_2 s + \tilde{c}_3 \\ &= \tilde{c}_2 s \left(1 + \tfrac{s}{\mu}\right) + \tilde{c}_3, \quad \left(\tfrac{\tilde{c}_2}{\tilde{c}_1} = \mu\right). \end{aligned} \tag{1.42}$$

Clearly, for $\mu \to +\infty$, Eqs. (1.38) and (1.42) read respectively

$$y(s) = c_1 q^s + c_3 \tag{1.43}$$

$$y(s) = \tilde{c}_2 s + \tilde{c}_3. \tag{1.44}$$

Thus, a first global classification of the hypergeometric polynomials on lattices is obtained: Hypergeometric polynomials on q-nonlinear (Eq. (1.38)), nonlinear (Eq. (1.42)), q-linear (Eq. (1.43)) and linear (Eq. (1.44)) lattices. Those are the main classes of the polynomials. Each class needs next to be divided in subclasses obtained by specializing the "coefficients" in $\sigma(s)$ and $\hat{\sigma}(s)$. In the cases of q-nonlinear and nonlinear lattices, we remark that $y(s) = y(-s-\mu)$, from which one easily deduces that

$$\hat{\sigma}(s) = \sigma(-s-\mu). \tag{1.45}$$

On the other side, according to the Rodrigues formula (1.22), $P_1(y(s)) = B_1\tau(s)$. That is why, when specializing the coefficients in $\sigma(s)$, one needs to keep in mind that the ratio

$$\tau(s) = \frac{\hat{\sigma}(s) - \sigma(s)}{\Delta y(s - \frac{1}{2})} = \frac{\sigma(-s-\mu) - \sigma(s)}{\Delta y(s - \frac{1}{2})} \tag{1.46}$$

must be a polynomial of first degree (but not a constant). The equation (1.45) tells us also that in the evoked cases, the knowledge of "coefficients" of $\sigma(s)$ is sufficient for the knowledge of those of $\hat{\sigma}(s)$. This fact is also useful for the sequel. In the following paragraphs, only the cases of the most useful hypergeometric polynomials on lattices (Racah, dual Hahn, Hahn, Meixner, Kravchuk, Charlier and their q-versions) will be treated in details. Their main data (the pair (a, b) giving the interval of orthogonality $[a, b-1]$, the functions $\rho(s)$, $\sigma(s)$, $\hat{\sigma}(s)$, and B_n) will be displayed in tables. However, their positions in a global hierarchy of all hypergeometric polynomials on lattices will be given in an enclosed at the end of the chapter scheme: The *Nikiforov-Suslov-Uvarov Scheme.*

1.2.1 The case of q-nonlinear lattices $y(s) = c_1\left(q^s + q^{-s-\mu}\right) + c_3$.

In the general situation of q-nonlinear lattices, we have

$$\sigma(s) = q^{-2s}p_4(q^s) = Cq^{-2s}\prod_{i=1}^{4}(q^s - q^{s_i}) = A\prod_{i=1}^{4}\psi_q(s-s_i),$$
$$[q^s - q^{s_i} = \kappa q^{\frac{s+s_i}{2}}\psi_q(s-s_i); C = A\kappa^{-4}q^{-1/2\sum_{i=1}^{4}s_i}; A = const]. \quad (1.47)$$

The general situation corresponds to the so-called "q-Racah polynomials" (for $q \to 1$, they converge to the "Racah polynomials" (see Table 1.2)) and the "Askey-Wilson polynomials". One parameter in less corresponds to the so-called "q-dual Hahn polynomials" (for $q \to 1$, they converge to the "dual Hahn polynomials" (see Table 1.2)). Basic data for q-Racah on $y(s) = \bar{\mu}(s) = q^{-s} + \gamma\delta q^{s+1}$ (see [16, 62]) and q-dual Hahn polynomials on $y(s) = \frac{1}{2}(q^s + q^{-s})$ are displayed in Table 1.1.

In Table 1.1, $\tilde{\Gamma}_q(s)$ is given in Eq. (1.33) and

$$\tilde{\Pi}_q(s+1) = \phi_q(s)\tilde{\Pi}_q(s),$$
$$\phi_q(s) = \frac{q^{s/2}+q^{-s/2}}{q^{1/2}+q^{-1/2}}. \quad (1.48)$$

The Askey-Wilson polynomials correspond to the choice of the lattice

$$y(s) = \frac{1}{2}(q^s + q^{-s}) = cos\theta,\ q^s = e^{i\theta}, \quad (1.49)$$

and

$$q^{s_1} = a; q^{s_2} = b; q^{s_3} = c; q^{s_4} = d, \quad (1.50)$$

in Eq. (1.47). They were originally defined by a basic hypergeometric function (see [16]):

$$p_n(cos\theta; a, b, c, d)$$
$$= \frac{(ab, ac, ad; q)_n}{a^n}\, {}_4\phi_3\left(\begin{array}{c} q^{-n}, abcdq^{n-1}, aq^s, aq^{-s} \\ ab, ac, ad \end{array}\middle| q; q\right). \quad (1.51)$$

They admit a continuous orthogonality on $[-1, +1]$ with weight

$$\varrho(y) = (1-y^2)^{-1/2}\frac{\prod_{k=0}^{\infty}(1-2(2y^2-1)q^k+q^{2k})}{\prod_{v=a,b,c,d}\prod_{k=0}^{\infty}(1-2vyq^k+v^2q^{2k})}$$
$$max(|a|, |b|, |c|, |d|) < 1,\ -1 < q < 1. \quad (1.52)$$

The Askey-Wilson $p_n(cos\theta;a,b,c,d)$ and the q-Racah $qR_n(\bar{\mu}(x);\alpha,\beta,\delta,\gamma)$ polynomials are related as follows: Setting

$$\begin{aligned} a^2=\gamma\delta q; b^2=\alpha^2\gamma^{-1}\delta^{-1}q; c^2=\beta^2\gamma^{-1}\delta q,\\ d^2=\gamma\delta^{-1}q; e^{2i\theta}=\gamma\delta q^{2x+1}, \end{aligned} \tag{1.53}$$

in the Askey-Wilson polynomials $p_n(cos\theta;a,b,c,d)$, one finds

$$\begin{aligned} p_n(\nu(x);\gamma^{\frac{1}{2}}\delta^{\frac{1}{2}}q^{\frac{1}{2}},\alpha\gamma^{-\frac{1}{2}}\delta^{-\frac{1}{2}}q^{\frac{1}{2}};\beta\gamma^{-\frac{1}{2}}\delta^{\frac{1}{2}}q^{\frac{1}{2}},\gamma^{\frac{1}{2}}\delta^{-\frac{1}{2}}q^{\frac{1}{2}})\\ =\tfrac{(\alpha q,\beta\delta q,\gamma q;q)_n qR_n(\bar{\mu}(x);\alpha,\beta,\gamma,\delta)}{(\gamma\delta q)^{\frac{n}{2}}}, \end{aligned} \tag{1.54}$$

$$\nu(x)=\frac{1}{2}\gamma^{\frac{1}{2}}\delta^{\frac{1}{2}}q^{\frac{1}{2}+x}+\frac{1}{2}\gamma^{-\frac{1}{2}}\delta^{-\frac{1}{2}}q^{-\frac{1}{2}-x}. \tag{1.55}$$

On the other side setting

$$\alpha=abq^{-1};\beta=cdq^{-1};\gamma=adq^{-1};\delta=ad^{-1};\ q^x=a^{-1}e^{-i\theta}, \tag{1.56}$$

in the q-Racah polynomials $qR_n(\bar{\mu}(x);\alpha,\beta,\delta,\gamma)$, one finds

$$qR_n(2acos\theta;abq^{-1},cdq^{-1},adq^{-1},ad^{-1})=\frac{a^np_n(cos\theta;a,b,c,d)}{(ab,ac,ad;q)_n}. \tag{1.57}$$

Other particular cases of polynomials defined on q-nonlinear lattices $y(s)=c_1(q^s+q^{-s-\mu})+c_3$, of less interest (corresponding to two, one or zero roots for $\sigma(s)$) will appear in the enclosed at the end of the chapter global hierarchic scheme. We will note before closing this subsection that the polynomials dual to q-Racah are nothing else than the q-Racah with parameters exchanging places.

1.2.2 The case of nonlinear lattices $y(s)=\tilde{c}_2s\left(1+\frac{s}{\mu}\right)+\tilde{c}_3$.

In the general case of nonlinear lattices, $\sigma(s)$ reads

$$\sigma(s)=A\prod_{i=1}^{4}(s-s_i),\ A=const. \tag{1.58}$$

It is clear that Eq. (1.58) can be obtained from Eq. (1.47) by sending q to 1. The most general case of (1.58) leads to the Racah polynomials, while one root of σ in less leads to the dual Hahn polynomials. Basic data for the

Racah on $y(s) = \eta(s) = s(s+\gamma+\delta+1)$ (see [64, 62] and references therein) and dual Hahn on $y(s) = s(s+1)$ polynomials are displayed in Table 1.2. As for the q-Racah polynomials, the polynomials dual to Racah polynomials are nothing else than the Racah polynomials with parameters exchanging places.

The cases when $\sigma(s)$ has less than three roots are not of interest, since in such cases, as it is easily verified from Eq. (1.46), $\tau(s)$ becomes a constant.

1.2.3 The case of q-linear lattices $y(s) = c_1 q^s + c_3$.

We will remark first that, in this case, the formula (1.45) is no more valid. Hence, in contrary to the preceding situations, the knowledge of the unique "coefficients" of $\sigma(s)$ is no more sufficient for the definition of the polynomials. Besides $\sigma(s)$, the knowledge of $\hat{\sigma}(s)$ is required. In the present situation, they read

$$\sigma(s) = \breve{A}\prod_{i=1}^{2}(q^s - q^{s_i}), \tag{1.59}$$

$$\hat{\sigma}(s) = \breve{A}\prod_{i=1}^{2}(q^s - q^{\bar{s}_i}), \; \breve{A} = const. \tag{1.60}$$

It is clear that Eqs. (1.59)-(1.60) can be obtained from the corresponding equations in the case of q-nonlinear lattices,

$$\hat{\sigma}(s) = \sigma(-s-\mu) = A\prod_{i=1}^{4}\psi_q(s+s_i+\mu), \; A = const. \tag{1.61}$$

$\sigma(s)$, being given in Eq. (1.47), by sending $q^{-\mu}$ to 0. For that it suffices to set,

$$\begin{gathered} s_1(\mu) = s_1, s_2(\mu) = s_2, s_3(\mu) = -\bar{s}_1 - \mu, s_4(\mu) = -\bar{s}_2 - \mu, \\ A(\mu) = \kappa^4 q^{-\mu-(s_1+s_2+\bar{s}_1+\bar{s}_2)/2}\breve{A}. \end{gathered} \tag{1.62}$$

The most general situation of Eqs.(1.59)-(1.60) corresponds to the so-called "q-Hahn polynomials", while one root of $\sigma(s)$ and $\hat{\sigma}(s)$ in less (the degrees remaining unchanged) leads to the "q-Meixner" and "q-Kravchuk polynomials", and one root of $\sigma(s)$ in less (the degree remaining unchanged) and two roots and one degree of $\hat{\sigma}(s)$ in less leads to the "q-Charlier polynomials" (see Table 1.3) (for $q \to 1$, those q-polynomials converge to the Hahn, Meixner, Kravchuk and Charlier polynomials, respectively (see Table 1.4)). Basic data for q-Hahn, q-Meixner, q-Kravchuk and q-Charlier polynomials on $y(s) = q^s$ are displayed in Table 1.3.

Besides the evoked q-versions of the Hahn, Meixner, Kravchuk and Charlier polynomials, there exists a lot of other polynomials of less interest on q-linear lattices, corresponding to various specializations of Eqs. (1.59)-(1.60) (see the enclosed scheme at the end of the chapter). We will note also that there exists a possibility of constructing "q-versions" of the evoked polynomials (Hahn,...), on the lattice $y(s) = \frac{1}{2}(q^s - q^{-s})$.

1.2.4 The case of linear lattices $y(s) = \tilde{c}_2 s + \tilde{c}_3$.

Here also, the formula (1.45) is no more valid. Hence, together with $\sigma(s)$, we consider $\hat{\sigma}(s)$:

$$\sigma(s) = \breve{A}(s - s_1)(s - s_2); \quad \hat{\sigma}(s) = \breve{A}(s - \bar{s}_1)(s - \bar{s}_2)(\breve{A} = const). \quad (1.63)$$

It is useful to emphasize the simple form that takes the equation (1.4) in this situation of linear lattices:

$$[\sigma(s)\Delta\nabla + \tilde{c}_2\tau(s)\Delta]\, v(s) = \tilde{c}_2^2\lambda v(s). \quad (1.64)$$

It is clear that Eq. (1.63) can be obtained from Eqs. (1.59)-(1.60) by sending q to 1, $c_1(q)$ and $c_3(q)$ being adequately chosen . Less naturally, one can obtain the formulas (1.63) as a limit for $\mu \to \infty$ of the corresponding equations on nonlinear lattices

$$\sigma(s) = A\prod_{i=1}^{4}(s - s_i); \quad \hat{\sigma}(s) = A\prod_{i=1}^{4} = \sigma(-s-\mu) = A\prod_{i=1}^{4}(s + s_i + \mu) \quad (1.65)$$

if one puts

$$s_1(\mu) = s_1, s_2(\mu) = s_2, s_3(\mu) = -\bar{s}_1 - \mu, s_4(\mu) = -\bar{s}_2 - \mu, A = \frac{\breve{A}}{\mu^2}. \quad (1.66)$$

The most general polynomials on linear lattices are the Hahn polynomials $H_n^{(\alpha,\beta)}(s)$, while one degree in less for $\sigma(s)$ and $\hat{\sigma}(s)$ leads to the Meixner $M_n^{(\delta,c)}(s)$ and Kravchuk $K_n^{(p)}(s;N)$ polynomials, two degrees in less for $\hat{\sigma}(s)$ and one degree in less for $\sigma(s)$, leads to the Charlier $C_n^{(\mu)}(s)$ polynomials. Basic data for the Hahn, Meixner, Kravchuk and Charlier polynomials (on $y(s) = s$) are displayed in Table 1.4. One remarks clearly that the Kravchuk $K_n^{(p)}(s;N)$ can be obtained from the Meixner $M_n^{(\delta,c)}(s)$ by the elementary change $\delta = -N$; $c = \frac{p}{p-1}$.

Remark 1.1 Concerning the general hierarchic scheme at the end of the chapter: One will note first that the variants of Racah and q-Racah polynomials retained in that scheme differ with the ones retained in Tables 1.2 and 1.1. The variants of Racah and q-Racah polynomials retained in Tables 1.2 and 1.1 are the more commonly considered and are those from [64, 62, 16]), but we decided not to modify that original scheme. The two versions of canonical forms of general hypergeometric polynomials on q-nonlinear and nonlinear lattices are clearly linked by simple transformations. We will signal next that the lattice in the general scheme is denoted by $x(s)$. It is also perhaps worth noting that when speaking about "hypergeometric polynomials on lattices" or equally the polynomials from that general scheme, we have not in mind that those polynomials are orthogonal or not. Clearly, the conditions of orthogonality (1.24)-(1.25) are not necessary satisfied for all the given there classes of polynomials. We will note finally that the general classification scheme needs to be read following the orientation of the arrows (but not in the converse or any other sense). There is the philosophy of the matter.

Remark 1.2 The reader interested in concrete formulas for the difference and recurrence relations (not treated in this chapter) for hypergeometric polynomials on lattices can refer to the evoked references. We will note also that for the most useful of them (Charlier, Meixner, Kravchuk, Hahn, Askey-Wilson (q-Racah) and polynomials dual to them), those formulas will be obtained simpler using the factorization technique in the third chapter.

Remark 1.3 Before giving the next remark on "completeness" of orthogonal polynomials on lattices, for avoiding any confusion, let us distinguish the notion of *completeness* and that of *closure* for a given system in an inner product space, seen that their meanings sometimes permute depending on authors. A system of elements in an inner product space is said *complete* in that space if any element of that space can be approximated arbitrarily closely by a finite linear combinations of its elements. It is said *closed* in a given inner product space if there is no a non-vanishing element in the space that can be simultaneously orthogonal to all the elements of the system.

Remark 1.4 Suppose a system S of polynomials on lattices orthogonal on $[c, d]$ with a discrete (of course *positive*) weight say $\varrho(s)$ and consider the Hilbert space $\ell^2(c, d; \varrho)$ of sequences say $(\ldots v(s) \ldots)$ for which converges the series

$$\sum_{s=c}^{d} |v(s)|^2 \varrho(s). \tag{1.67}$$

Then, according to the theorem 5.3.3 in [20], the system S is complete in $\ell^2(c, d; \varrho)$. Thus, any system of hypergeometric polynomials on lattices, orthogonal with a discrete weight $\varrho(s)$ on $[c, d]$, is complete in the corresponding Hilbert space $\ell^2(c, d; \varrho)$. Concerning the system of Askey-Wilson polynomials admitting a continuous orthogonality, its completion in the corresponding Hilbert space follows from usual considerations seen that the interval of orthogonality is finite. The completion of the polynomials on linear lattices was also proved directly in [58] (theorem 4.4.1).

Table 1.1: Basic data for the q-Racah on $\bar{\mu}(s) = q^{-s} + \gamma\delta q^{s+1}$ and q-dual Hahn on $y(s) = \frac{1}{2}(q^s + q^{-s})$ polynomials

$P_n(s)$	$qR_n(\bar{\mu}(s))$	$qH_n^d(y(s))$
(a,b)	$(0,N)$	(a,b)
$\rho(s)$	$\frac{(\alpha q,\beta\delta q,\gamma q,\gamma\delta q;q)_s}{(q,\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_s(\alpha\beta q)^s}$ $\alpha q = q^{-N}$ or $\beta\delta q = q^{-N}$ or $\gamma q = q^{-N}$	$\frac{\tilde{\Gamma}_q(s+a)\tilde{\Pi}_q(s+a)\tilde{\Gamma}_q(s+c+1/2)}{\tilde{\Gamma}_q(s-a+1)\tilde{\Pi}_q(s-a+1)\tilde{\Gamma}_q(s+b) \times\tilde{\Gamma}_q(s-c+1/2)\tilde{\Gamma}_q(b-s)}$ $(a>0, \lvert c\rvert < a+1/2, b=a+N)$
$\sigma(s)$	$q(1-q^s)(1-\delta q^s)$ $\times(\beta-\gamma q^s)(\alpha-\gamma\delta q^s)$	$\psi_q(s-a)\psi_q(s+b-1)$ $\times\psi_q(s-c-1/2)\phi_q(s-a)$
$\hat{\sigma}(s)$	$(1-\alpha q^{s+1})(1-\beta\delta q^{s+1})$ $\times(1-\gamma q^{s+1})(1-\gamma\delta q^{s+1})$	$\psi_q(s+a)\psi_q(b-s-1)$ $\times\psi_q(s+c+1/2)\phi_q(s+a)$
B_n	B_n	$\frac{(-1)^n}{2^n n!} \times (q-1)^{2n}$

Table 1.2: Basic data for the Racah on $\eta(s) = s(s+\gamma+\delta+1)$ and dual Hahn on $y(s) = s(s+1)$ polynomials.

$P_n(s)$	$R_n(\eta(s))$	$H_n^d(y(s))$
(a,b)	$(0,N)$	(a,b)
$\rho(s)$	$\frac{(\alpha+1)_s(\beta+\delta+1)_s(\gamma+1)_s(\gamma+\delta+1)_s}{(-\alpha+\gamma+\delta+1)_s(-\beta+\gamma+1)_s(\delta+1)_s s!}$ $\alpha+1=-N$ or $\beta+\delta+1=-N$ or $\gamma+1=-N$	$\frac{\Gamma(s+a+1)\Gamma(s+c+1)}{\Gamma(s-a+1)\Gamma(s-c+1) \times\Gamma(s+b+1)\Gamma(b-s)}$ $-1/2 < a < b, \lvert c\rvert < 1+a,$ $b=a+N$
$\sigma(s)$	$s(s-\alpha+\gamma+\delta)$ $\times(s-\beta+\gamma)(s+\delta)$	$(s-a)(s+b)(s-c)$
$\hat{\sigma}(s)$	$(s+\alpha+1)(s+\beta+\delta+1)$ $\times(s+\gamma+1)(s+\gamma+\delta+1)$	$(s+a+1)(s+c+1)$ $\times(b-s-1)$
B_n	B_n	$\frac{(-1)^n}{n!}$

Table 1.3: $y(s) = q^s$. Basic data for q-Hahn, q-Meixner, q-Kravchuk and q-Charlier polynomials

$P_n(y)$	$qH_n^{(\alpha,\beta)}(y)$	$qM_n^{(\delta,c)}(y)$	$qK_n^p(y)$	$qC_n^{\breve{\mu}}(y))$
(a,b)	$(0,N)$	$(0,+\infty)$	$(0,N+1)$	$(0,+\infty)$
$\rho(s)$	$\frac{q^{s(\alpha+\beta)/2} \times\tilde{\Gamma}_q(s+\beta+1) \times\tilde{\Gamma}_q(N-s+\alpha)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s)}$ $\alpha > -1, \beta > -1$	$\frac{q^{s(\delta-1)/2} \times c^s\tilde{\Gamma}_q(s+\delta)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(\delta)}$ $\delta > 0,$ $0 < q < 1,$ $0 < c < 1$	$\frac{q^{-(N+1)s/2}p^s \times\tilde{\Gamma}_q(N+1) \times(1-p)^{N-s}}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s+1)}$ $0 < p < 1$	$\frac{q^{-s(s+1)/4} \times e^{-\breve{\mu}}\breve{\mu}^s}{\tilde{\Gamma}_q(s+1)}$ $0 < q < 1,$ $\breve{\mu} < \frac{1}{(1-q)q^{\frac{1}{2}}}$ or $q > 1, \breve{\mu} > 0$
$\sigma(s)$	$q^{s/2}\psi_q(s)$ $\times q^{(s-N-\alpha)/2}$ $\times\psi_q(N+\alpha-s)$	$q^{s/2}\psi_q(s)q^{s-1}$	$q^{s/2}\psi_q(s)q^{s-1}$	$q^{s/2}\psi_q(s)q^{s-1}$
$\hat{\sigma}(s)$	$q^{(s+\beta+1)/2}$ $\times\psi_q(s+\beta+1)$ $\times q^{(s-N+1)/2}$ $\times\psi_q(N-1-s)$	$cq^{(s+\delta)/2}$ $\times\psi_q(s+\delta)q^s$	$\frac{p}{1-p}q^{(s-N)/2}$ $\times\psi_q(N-s)q^s$	$\breve{\mu}q^s$
B_n	$\frac{(1-q)^n}{n!}$	$\left(\frac{q-1}{c}\right)^n$	$\frac{(1-q)^n(1-p)^n}{n!}$	$\left(\frac{q-1}{\breve{\mu}}\right)^n$

Table 1.4: $y(s) = s$. Basic data for the Hahn, Meixner, Kravchuk and Charlier polynomials

$P_n(s)$	$H_n^{(\alpha,\beta)}(s)$	$M_n^{(\delta,c)}(s)$	$K_n^{(p)}(s)$	$C_n^{(\mu)}(s)$
(a,b)	$(0,N)$	$(0,+\infty)$	$(0,N+1)$	$(0,+\infty)$
$\rho(s)$	$\frac{\Gamma(N+\alpha-s)\Gamma(\beta+1+s)}{\Gamma(s+1)\Gamma(N-s)}$ $(\alpha > -1, \beta > -1)$	$\frac{c^s\Gamma(s+\delta)}{\Gamma(s+1)\Gamma(\delta)}$ $(\delta > 0, 0 < c < 1)$	$\frac{N!p^s(1-p)^{N-s}}{\Gamma(s+1)\Gamma(N+1-s)}$ $(0 < p < 1)$	$\frac{e^{-\mu}\mu^s}{\Gamma(s+1)}$ $(\mu > 0)$
$\sigma(s)$	$s(N+\alpha-s)$	s	s	s
$\sigma(s)+\tau(s)$	$(s+\beta+1)(N-1-s)$	$c(\delta+s)$	$\frac{p}{1-p}(N-s)$	μ
B_n	$\frac{(-1)^n}{n!}$	$\frac{1}{c^n}$	$\frac{(-1)^n(1-p)^n}{n!}$	$\frac{1}{\mu^n}$

Chapter 2

Discrete factorization techniques

The first concern of this chapter consists in recalling the general practical realizations of the technique of factorization, some of its essential restrictions and potentialities (section 2.1). A special emphasis is put on three fundamental methods ("shape-invariance", "quasi-periodicity", "modification").
Secondly, our concern consists in formulating and explaining some special structures necessary for the study of the second and fourth order difference eigenelement problems (section 2.2). Most of emphasis is put to the case of second-order difference eigenelement question as the whole chapters 3 and 4 are devoted to problems related to that.
Of course, more specific explanations are to be found in each chapter in rapport with the specific techniques to be applied there.

2.1 Definitions, basic ideas and potentialities.

Let H_j be a linear difference (this part is also valid for the differential situation) operator. One says that H_j is *factorizable* iff the following product can be performed

$$\begin{aligned} H_j(x) - \mu_j &= L_j(x)R_j(x) \\ H_{j+1}(x) - \mu_j &= R_j(x)L_j(x). \end{aligned} \tag{2.1}$$

The operators L_j and R_j are called *lowering* and *raising* respectively. The difference-recurrence relations connecting the coefficients of L_j and R_j will

be called the *factorization chains* or *systems*. The term "factorization chain" can also be used to call the operatorial relations in Eq. (2.1) or the related sequence $\dots H_{j-1}, H_j, H_{j+1}, \dots$ itself. The variable j will be referred to as the *variable of factorization*. In some cases, this "variable of factorization" is an actual independent variable belonging to $\mathbf{R}$. However, in most of the cases, $j \in \mathcal{Z}$ and is considered all simply as an index.
Basic useful formulas deduced from Eq. (2.1) are:

$$\begin{aligned} H_{j+1}R_j &= R_jH_j \\ H_jL_j &= L_jH_{j+1}. \end{aligned} \tag{2.2}$$

More generally,

$$\begin{gathered} H_{j+N}R_{j+N-1}\dots R_j = R_{j+N-1}\dots R_jH_j \\ H_jL_j\dots L_{j+N-1} = L_j\dots L_{j+N-1}H_{j+N}, \;\; N \in \mathbf{N}. \end{gathered} \tag{2.3}$$

Those intertwining relations allow clearly to interconnect the spectral data of the H-operators.

2.1.1 Three fundamental methods.

The shape-invariance method.

Some eigenvalue problems can be reduced to the search of the solution of the equation

$$H(x;j)\psi(x;j) = 0, \tag{2.4}$$

where j is a variable on which depends the spectral parameter. This motivates us to consider the situation when effectively in Eq. (2.1), j acts as a full independent variable. In that case, Eq. (2.1) needs to be written as

$$\begin{aligned} H(x;j) - \mu(j) &= L(x;j)R(x;j) \\ H(x;j+1) - \mu(j) &= R(x;j)L(x;j). \end{aligned} \tag{2.5}$$

Thus $LR + \mu$ and $RL + \mu$ have the same shape and differ essentially only in parameters which appear in them. In other words there is *shape-invariance* during the permutation. The term "shape-invariance" used here is borrowed from Quantum Mechanics where two Hamiltonians say

$$H_{\pm} = -D^2 + V_{\pm}(x), \quad D^2 = \frac{d^2}{dx^2}, \tag{2.6}$$

related by supersymmetry i.e. $V_\pm(x) = W^2(x) \pm W'(x)$, are said *shape-invariant* iff

$$V_+(x; a_0) - V_-(x; a_1) = S(a_0); \; a_1 = Q(a_0) \tag{2.7}$$

where Q and S are some functions.
The equation (2.4) can then be solved using the "shape-invariant" factorization (2.5) as follows. We first remark that if $\psi(x; j_0)$ is such that

$$H(x; j_0)\psi(x; j_0) = 0, \tag{2.8}$$

then, unless it vanishes, the function

$${}^\uparrow\psi(x; j_0, n) = \prod_{j=0}^{n-1} R(x; j_0 + n - 1 - j)\psi(x; j_0), \; (n \geq 1) \tag{2.9}$$

is a nontrivial solution of the equation

$$H(x; j_0 + n)y(x; n) = 0. \tag{2.10}$$

Identically, unless it vanishes, the function

$${}^\downarrow\psi(x; j_0, n) = \prod_{j=1}^{n} L(x; j_0 - n - 1 + j)\psi(x; j_0), \; (n \geq 1) \tag{2.11}$$

is a nontrivial solution of the equation

$$H(x; j_0 - n)y(x; n) = 0. \tag{2.12}$$

It is clearly expected that the functions in (2.9) and (2.11) do not vanish simultaneously. In that case, for solving Eq. (2.4), the remaining task consists essentially in finding the "starting" function $\psi(x; j_0)$. For that, there is not a general way, but some hints are notable. Remark first that if for some j_0 one has $\mu(j_0) = 0$, then the starting function can be found solving all simply a linear first order difference equation. Indeed, if $\mu(j_0) = 0$, it follows from Eq. (2.5) that either

$$R(x; j_0)\psi(x; j_0) = 0 \tag{2.13}$$

or

$$L(x; j_0)\psi(x; j_0 + 1) = 0. \tag{2.14}$$

In the event $L(x;j)$ and $R(x;j)$ are mutually adjoint (for this hint, we are supposed to work in a convenient inner product space) and $\mu(j)$ monotonous (increasing or decreasing function of j), "solvable" situations can be reached in the following way:
If $\mu(j)$ is an increasing function, consider the following ladder

$$\begin{aligned} \psi(x;j+1) &= R(x;j)\psi(x;j) \\ -\mu(j)\psi(x;j) &= L(x;j)\psi(x;j+1). \end{aligned} \tag{2.15}$$

$L(x;j)$ and $R(x;j)$ being mutually adjoint, one easily finds that

$$\Big(\psi(x;j+1),\psi(x;j+1)\Big) = -\mu(j)\Big(\psi(x;j),\psi(x;j)\Big) \tag{2.16}$$

where $\Big(.,.\Big)$ means the scalar product. But as $\mu(j)$ is increasing, there exists necessarily some J for which $-\mu_J \leq 0$, $(-\mu(j) > 0, j < J)$, which implies that necessarily for some j_0, $\psi(x;j_0{+}1) = 0$ or identically $R(x;j_0)\psi(x;j_0) = 0$ and the starting function $\psi(x;j_0)$ for the "ladder" (2.11) is found.
If in the opposite $\mu(j)$ is decreasing, we similarly write

$$\begin{aligned} \psi(x;j) &= L(x;j)\psi(x;j+1) \\ -\mu(j)\psi(x;j+1) &= R(x;j)\psi(x;j), \end{aligned} \tag{2.17}$$

and

$$\Big(\psi(x;j),\psi(x;j)\Big) = -\mu(j)\Big(\psi(x;j+1),\psi(x;j+1)\Big) \tag{2.18}$$

such that for some j_0, $L(x;j_0)\psi(x;j_0{+}1) = 0$, leading to a starting function $\psi(x;j_0+1)$ for the "ladder" (2.9).
It is important to note that in eigenvalue problems, the task of guessing the structure of the equation (2.4), corresponding to the original eigenvalue equation, so that the operator $H(x;j)$ may be factorized as in Eq. (2.5), is very crucial for the success of the application of the method. It appeared also from the preceding reasonings that "generate" a "solvable" operator is almost equivalent to "generate" a "shape-invariantly factorizable" operator. In the following example, we "generate" a simple "shape-invariantly factorizable" operator. Its solutions appear be the Meixner $M_n^{(\gamma,\mu)}(x)$ polynomials.

Example 2.1.[87]

Consider the operator

$$X(x;n) = \mathbf{E}_x^2 + A(x;n)\mathbf{E}_x + B(x;n) \tag{2.19}$$

where $\mathbf{E}_x(h(x)) = h(x+1)$, and $A(x;n)$, $B(x;n)$ are some functions in x and n. Next, search for $y_n(x)$ such that

$$X(x;n)y_n(x) = 0. \tag{2.20}$$

For that, set the factorization

$$\begin{aligned} X(x;n) - \mu(n) &= L^+(x;n)L^-(x;n) \\ X(x;n-1) - \mu(n) &= L^-(x;n)L^+(x;n) \end{aligned} \tag{2.21}$$

where

$$L^+(x;n) := \mathbf{E}_x + q(x;n); \qquad L^-(x;n) := \mathbf{E}_x + Q(x;n). \tag{2.22}$$

We obtain the system

$$\begin{aligned} q(x;n-1) + Q(x+1;n-1) &= Q(x;n) + q(x+1;n) \\ q(x;n-1)Q(x;n-1) &= Q(x;n)q(x;n) + \mu(n) - \mu(n-1). \end{aligned} \tag{2.23}$$

Setting

$$Q(x;n) := (ax+b) + n(cx+d); \qquad q(x;n) := (ex+f) + n(vx+w), \tag{2.24}$$

one easily finds

$$\begin{aligned} L^+(x;n) := \mathbf{E}_x - wx + f + nw; \qquad L^-(x;n) := \mathbf{E}_x + dx + b + nd \\ \mu(n) := -n[dwn + df + bw]. \end{aligned} \tag{2.25}$$

From Eqs. (2.20) and (2.21), we obtain

$$\begin{aligned} c_1 y_{n-1}(x) &= L^-(x;n)y_n(x) \\ c_2 y_n(x) &= L^+(x;n)y_{n-1}(x) \end{aligned} \tag{2.26}$$

where

$$c_1c_2 = -\mu(n). \tag{2.27}$$

Considering the first equation in (2.21) and the last one in (2.25), one finds that a starting function $y_0(x)$ is given by

$$L^-(x;0)y_0(x) = 0 \tag{2.28}$$

and leads, by the second equation in (2.26), to a solution of Eq. (2.20). For $d=-1$, this function is easily seen to be

$$y_0(x) = \Gamma(x-b) \tag{2.29}$$

leading to the Meixner polynomials. More precisely for $w=-f; b=-\gamma; f=-\frac{1}{\mu}$, the operator $X(x;n)$ reads

$$X(x;n) = \mathbf{E}_x^2 + [(-1-\frac{1}{\mu})x - 1 - n + \frac{n}{\mu} - \frac{1}{\mu} - \gamma]\mathbf{E}_x + [\frac{1}{\mu}x^2 + \frac{\gamma+1}{\mu}x + \frac{\gamma}{\mu}]$$

and a solution of Eq. (2.20) is given by

$$\begin{gathered} y_0(x) = \Gamma(x+\gamma) \\ y_n(x) = \prod_{j=0}^{n-1} \frac{1}{\gamma+n-1-j}[\mathbf{E}_x + \frac{\gamma}{\mu}(-x+n-1-j)]y_0(x), \ \ (n \geq 1) \end{gathered} \tag{2.30}$$

and precisely

$$y_n(x) = \Gamma(x+\gamma)\ {}_2F_1(-n,-x;\gamma;1-\frac{1}{\mu}), \tag{2.31}$$

the functions $(\gamma)_n\ {}_2F_1(-n,-x;\gamma;1-\frac{1}{\mu})$, with $(\gamma)_n = \gamma(\gamma+1)\ldots(\gamma+n-1)$, being exactly the Meixner $M_n^{(\gamma,\mu)}(x)$ polynomials [94].

In [87], the Meixner polynomials were thus shown to be "generable" from type (2.21) factorization chain. In the same work, for generating the Hahn polynomials, another kind of factorization, not taking in particular the degree of the polynomials n as the variable of factorization, was used. Later in this thesis, we will show practically that the kind of factorization used here (shape-invariant Infeld-Hull-Miller factorization) for the Meixner polynomials can be extended not only to all the hypergeometric polynomials on linear lattices (Hahn polynomials, their various specializations, and the polynomials dual to all that), but also to all the hypergeometric polynomials on $[q]$-nonlinear lattices (Askey-Wilson polynomials, their various specializations, and the polynomials dual to them).

The modification method.

We now fix $j := p$ and consider the unique pair of operators H_p and H_{p+1} (linked by Eq. (2.1)), to study the interconnection of their spectral data. More precisely, we need to show how to generate the eigenelements of H_{p+1} from the ones for H_p. In other words, we need to "*modify*" H_p into H_{p+1}.
Let

$$(\psi_p^k, \lambda_p^k), \;\; k = 0, 1, 2, \dots \tag{2.32}$$

be a sequence of eigenelements of H_p with

$$\begin{gathered} \lambda_p^k \neq \lambda_p^l, \;\; k \neq l \\ \lambda_p^k \neq \mu_p, \;\; k = 0, 1, \dots \end{gathered} \tag{2.33}$$

From Eq. (2.1), the corresponding eigenelements for H_{p+1} are

$$(\phi_p^{k+1}, \lambda_p^k), \tag{2.34}$$

where

$$\phi_p^{k+1} = R_p \psi_p^k, \;\; k = 0, 1, \dots \tag{2.35}$$

Let us remark directly that $R_p \psi_p^k \neq 0$, $\forall k$ according to the second condition in (2.33).
On the other side, according to Eq. (2.1), we have that the function, say ϕ_p^0 given by

$$L_p \phi_p^0 = 0 \tag{2.36}$$

is an eigenfunction of H_{p+1} corresponding to the eigenvalue μ_p.
In the event L_p and R_p are mutually adjoint (in a convenient inner product space in which we are for the occasion supposed to work), so the functions ψ_p^k are mutually orthogonal as they correspond to mutually different eigenvalues for a symmetric operator $H_p = L_p R_p + \mu_p$. Adding the second condition in Eq. (2.33), we obtain that the set ϕ_p^l, $l = 0, 1, 2 \dots$ is also orthogonal as corresponding to mutually different eigenvalues for a symmetric operator $H_{p+1} = R_p L_p + \mu_p$. More precisely, we prove

$$\left(\phi_p^{k+1}, \phi_p^{l+1}\right) = \left(R_p \psi_p^k, R_p \psi_p^l\right) = (\lambda_p^l - \mu_p)\left(\psi_p^k, \psi_p^l\right) = 0 \tag{2.37}$$

and

$$\left(\phi_p^0, \phi_p^{k+1}\right) = \left(\phi_p^0, R_p\psi_p^k\right) = \left(L_p\phi_p^0, \psi_p^k\right) = 0, \ k, l = 0, 1, \ldots \tag{2.38}$$

In summary, to a set of eigenelements (ψ_p^k, λ_p^k), $k = 0, 1, \ldots$ of H_p correspond for H_{p+1} a set of eigenelements $(\phi_p^{k+1}, \lambda_p^k)$, $k = 0, 1, \ldots$ obtained by the formula (2.35) plus an extra eigenelement (ϕ_p^0, μ_p) obtained by Eq. (2.36). In the sequel, we will not only let us say that H_p is "modified" into H_{p+1} but also that the eigenelements of H_p are "modified" into the corresponding eigenelements of H_{p+1}.

Example 2.2.[104]

Consider the Hamiltonian

$$X(x) = -D^2 + x^2 \tag{2.39}$$

where $D^2h(x) = \frac{d^2}{dx^2}h(x)$. We suppose that it is known that

$$X(x)y_n(x) = (2n+1)y_n(x) \tag{2.40}$$

where $y_n(x) = e^{-\frac{x^2}{2}}\mathcal{H}_n(x)$, $\mathcal{H}_n(x)$ being the Hermite polynomials of degree $n \in \mathcal{Z}^+$, $\mathcal{H}_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$.
On the other side, replacing in Eq. (2.40), x by xi, ($i^2 = -1$), it is not difficult to verify that

$$X(x)y_n(xi) = -(2n+1)y_n(xi). \tag{2.41}$$

Hence, fixing $m \in \mathbf{Z}^+$, one easily factorizes

$$\begin{aligned} X(x) + 2m + 1 &= (D + f_m(x))(-D + f_m(x)) \\ \tilde{X}^{(m)}(x) + 2m + 1 &= (-D + f_m(x))(D + f_m(x)) \end{aligned} \tag{2.42}$$

where

$$f_m(x) = \frac{[y_m(xi)]'}{y_m(xi)} \tag{2.43}$$

and

$$\tilde{X}^{(m)}(x) = X(x) - 2f_m'(x). \tag{2.44}$$

As a consequence, the new Hamiltonian $\tilde{X}^{(m)}$ admits as eigenelements

$$(\tilde{y}_{m,0}; -2m-1) ; \qquad (\tilde{y}_{m,n+1}; 2n+1) , n = 0, 1, 2 \ldots \tag{2.45}$$

where

$$\tilde{y}_{m,n+1}(x) = [-D + f_m(x)] y_n(x), n = 0, 1, 2 \ldots \tag{2.46}$$

and

$$\tilde{y}_{m,0}(x) := [y_m(ix)]^{-1} ; \qquad [D + f_m(x)]\tilde{y}_{m,0}(x) = 0. \tag{2.47}$$

In order that the new eigenfunctions be square integrable on the whole axis $(-\infty, +\infty)$, only the cases of even m need to be considered (the cases of odd m correspond to square integrable on $[0, +\infty)$ functions).

The case $m = 0$ does not lead to new polynomials.

For $m = 2$, the new eigenfunctions read

$$\begin{aligned} \tilde{y}_{2,0}(x) &= Q_2^{-1}(x) e^{-\frac{x^2}{2}} \\ \tilde{y}_{2,n+1}(x) &= Q_2^{-1}(x) \left[Q_2(x) \mathcal{H}_{n+1}(x) + Q_2'(x) \mathcal{H}_n(x) \right] e^{-\frac{x^2}{2}} \end{aligned} \tag{2.48}$$

where

$$Q_2(x) = 1 + 2x^2. \tag{2.49}$$

The corresponding polynomials

$$\tilde{\mathcal{H}}_{2,n} = y_{2,n}(x) Q_2(x) e^{\frac{x^2}{2}}, n = 0, 1, 2 \ldots \tag{2.50}$$

are orthogonal on $(-\infty, +\infty)$ with non-classical weight $Q_2^{-2}(x)e^{-x^2}$. Those polynomials were obtained differently (solving operator relations) in [40]. In the subsection "quasi-periodicity method" (just after this one), they will serve as a welcome illustrative example (see in " example 2.3.", equations (2.86)-(2.87)), considering that it surprisingly allows to intersect the two approaches.

For $m = 4$, the new eigenfunctions read

$$\begin{aligned} \tilde{y}_{4,0}(x) &= Q_4^{-1}(x) e^{-\frac{x^2}{2}} \\ \tilde{y}_{4,n+1}(x) &= Q_4^{-1}(x) \left[Q_4(x) \mathcal{H}_{n+1}(x) + Q_4'(x) \mathcal{H}_n(x) \right] e^{-\frac{x^2}{2}} \end{aligned} \tag{2.51}$$

where

$$Q_4(x) = 4x^4 + 12x^2 + 3. \tag{2.52}$$

The corresponding polynomials

$$\tilde{\mathcal{H}}_{4,n} = y_{4,n}(x) Q_4(x) e^{\frac{x^2}{2}}, n = 0, 1, 2 \dots \tag{2.53}$$

are orthogonal on $(-\infty, +\infty)$ with non-classical weight $Q_2^{-4}(x)e^{-x^2}$. The first six examples are:

$\tilde{\mathcal{H}}_{4,0}(x) = 1$; $\tilde{\mathcal{H}}_{4,1}(x) = 2x(4x^4 + 20x^2 + 15)$; $\tilde{\mathcal{H}}_{4,2}(x) = 16x^6 + 72x^4 + 36x^2 - 6$; $\tilde{\mathcal{H}}_{4,3}(x) = 4x(8x^6 + 28x^4 - 14x^2 - 21)$; $\tilde{\mathcal{H}}_{4,4}(x) = 4(16x^8 + 32x^6 - 120x^4 - 72x^2 + 9)$; $\tilde{\mathcal{H}}_{4,5}(x) = 8x(16x^8 - 216x^4 + 81)$.

The absence in the sequence of polynomials of degree 2, 3 and 4 needs to be underlined. More details on those polynomials can be found in [104].

We will refer to this kind of transformation of classical polynomials as the "*modification*" of the polynomials. It is clear that the present "modification" of polynomials has any equivalence with the ones from [7, 8, 12, 15, 27]. On the other side, this example of modified Hermite polynomials is, in our best knowledge, the unique in literature, case of non-classical polynomials obtained using the above scheme. In chapter 4, we will follow the above scheme ("modification method") for obtaining modifications of the special Meixner $M_n^{(2,c)}(x+1)$ polynomials.

The quasi-periodicity method.

Above, it appeared that every time we aimed to obtain concrete applications of the factorization chain (2.1), we were assigned to demand it to satisfy some additional constraints: In the example 2.1, the operator $X(x;n)$ needs to be factorizable in the "shape-invariant" form. In the example 2.2, the operator $\tilde{X}(x)$ needs to be of different family than $X(x)$ from which we generate it, and so on.
Here, as in the first case, we are searching again for a certain "symmetry", reason for which we need to impose to the factorization (2.54) additional self-similarity constraints.
To explain conveniently the situation, write the factorization in Eq. (2.1) as

follows

$$\begin{aligned}\hat{H}_j &= L_j R_j \\ \hat{H}_{j+1} &= R_j L_j + \alpha_j\end{aligned} \tag{2.54}$$

where

$$\begin{gathered}\hat{H}_j = H_j - \mu_j, \\ \alpha_j = \mu_j - \mu_{j+1}.\end{gathered} \tag{2.55}$$

In this form, the intertwining relations in Eqs. (2.2), (2.3) read

$$\begin{aligned}\hat{H}_{j+1} R_j &= R_j(\hat{H}_j + \alpha_j) \\ (\hat{H}_j + \alpha_j) L_j &= L_j \hat{H}_{j+1}\end{aligned} \tag{2.56}$$

and

$$\begin{gathered}\hat{H}_{j+N} R_{j+N-1} \dots R_j = R_{j+N-1} \dots R_j(\hat{H}_j + \alpha_j + \dots + \alpha_{j+N-1}) \\ (\hat{H}_j + \alpha_j + \dots + \alpha_{j+N-1}) L_j \dots L_{j+N-1} = L_j \dots L_{j+N-1} \hat{H}_{j+N}.\end{gathered} \tag{2.57}$$

Next, impose to the factorization chain (2.54), the following so-called "quasi-periodicity" closure conditions [109]

$$\begin{aligned}\hat{H}_{j+N} &= \hat{H}_j(x - \delta) \\ \alpha_{j+N} &= \alpha_j\end{aligned} \tag{2.58}$$

and for simplicity set$j := 0$. So the relations (2.57) read

$$\begin{gathered}\hat{H}_0(x - \delta) R = R(\hat{H}_0 + \alpha) \\ (\hat{H}_0 + \alpha) L = L \hat{H}_0(x - \delta)\end{gathered} \tag{2.59}$$

where

$$\begin{gathered}\alpha = \mu_0 - \mu_N \\ R = R_{N-1} \dots R_0 \\ L = L_0 \dots L_{N-1}\end{gathered} \tag{2.60}$$

For the particular case $\delta = 0$, the relations took the form

$$\begin{aligned}\left[\hat{H}_0, R\right] &= \alpha R \\ \left[\hat{H}_0, L\right] &= -\alpha L\end{aligned} \tag{2.61}$$

where $[A, B] = AB - BA$.
For $\alpha = 0$, the equations in (2.61) reduce in commutation ones:

$$\begin{aligned} \left[\hat{H}_0, R\right] &= 0 \\ \left[\hat{H}_0, L\right] &= 0. \end{aligned} \tag{2.62}$$

The equations (2.61),(2.62) lead exceptionally to elementary functions. The solutions of Eq. (2.61) are generally expected to belong to Painlevé transcendents [106, 109], while the commutation equations (2.62) lead to hyperelliptic functions [32, 33, 70, 71, 72, 73, 91, 106, 109]. But as we are dealing with polynomials, we are clearly interested in those rare situations where the solutions are elementary.
Letting H_0 be an operator obtained from the general equation (2.59), we explain further the method by showing how to generate its eigenelements. Let $\phi_0(x)$ be a "starting" eigenfunction obtained for example by solving the equation $L_{N-1}\phi_0(x-\delta) = 0$, and let λ_0 be the corresponding eigenvalue. We have

$$H_0\phi_0(x) = \lambda_0\phi_0(x). \tag{2.63}$$

Hence, from (2.59)

$$H_0\phi_1(x) = (\lambda_0 + \mu_0 - \mu_N)\phi_1(x), \tag{2.64}$$

where

$$\phi_1(x-\delta) = R\phi_0(x). \tag{2.65}$$

More generally,

$$H_0(x)\phi_n(x) = \lambda_n\phi_n(x) \tag{2.66}$$

where

$$\begin{aligned} \phi_{n+1}(x-\delta) &= R\phi_n(x) \\ c_n\phi_n(x) &= L\phi_{n+1}(x-\delta), \end{aligned} \tag{2.67}$$

for some constant (in x) c_n, and

$$\lambda_n = (\mu_0 - \mu_N)n + \lambda_0. \tag{2.68}$$

It is not difficult to remark that the role of the "starting function" can be played by any eigenfunction of any one of the operators $\hat{H}_i, i = \overline{0, N-1}$. Thus imposing the quasi-periodicity condition (2.58) to the chain (2.54), leads to an operatorial equation (see (2.59)), from which one can generate an automatically solvable (in principle) operator. Here below, will be given various examples (inside the "example 2.3") where Eq. (2.59) is solved by a Schrödinger operator, potential of which is either a Painlevé transcendent, an elliptic function or an elementary one (leading to a sequence of polynomial eigenfunctions).

Example 2.3.[40, 106]

To illustrate our situation, consider the Schrödinger operators

$$H_i = -D^2 + v_i(x), i = 0, 1, \ldots \tag{2.69}$$

and suppose that the potentials $v_i(x)$ are such that there exists well-behaved f_i such that $v_i(x) = f_i'(x) + f_i^2(x)$. In that case, one easily verifies that the operators H_i are positive and symmetric (over the convenient space). Suppose next that one can make the intertwining relations

$$\begin{aligned} H_i &= H_i^+ H_i^- \\ H_{i+1} &= H_i^- H_i^+ + \alpha_i, \end{aligned} \tag{2.70}$$

where

$$H_i^+ = D + f_i(x); \qquad H_i^- = -D + f_i(x). \tag{2.71}$$

In terms of potentials and superpotentials, they are respectively equivalent to

$$v_{i+1}(x) = v_i(x) - 2f_i'(x) + \alpha_i \tag{2.72}$$

or

$$f_i'(x) - f_i^2(x) = -f_{i+1}'(x) - f_{i+1}^2(x) + \alpha_i, i = 0, 1, 2 \ldots. \tag{2.73}$$

Imposing the periodicity condition $H_N = H_0$ or $f_N = f_0$; $\alpha_0 = \alpha_N$, to this chain leads to (see Eq. (2.61))

$$\begin{aligned} [H_0, R] &= \alpha R \\ [H_0, L] &= -\alpha L \end{aligned} \tag{2.74}$$

where $R = H^-_{N-1} \dots H^-_0$; $L = H^+_0 \dots H^+_{N-1}$; $\alpha = \alpha_0 + \dots + \alpha_{N-1}$.

For $N = 1$, the solution of the equations is easily seen to be the harmonic oscillator

$$H_0 = -D^2 + x^2. \tag{2.75}$$

However, for illustrating our saying, it appears convenient to solve the corresponding eigenelement problem directly, following the first initiators (Schrödinger [105] and Dirac [41]) of the method in the quantum mechanics area. For this goal, we will write the harmonic oscillator operator as follows

$$H = -\frac{D^2}{2} + \frac{x^2}{2} + \frac{1}{2}. \tag{2.76}$$

It factorizes in

$$\begin{aligned} H &= H^+H^- \\ H &= H^-H^+ + 1, \end{aligned} \tag{2.77}$$

where

$$H^+ = \tfrac{D+x}{\sqrt{2}}; \qquad H^- = \tfrac{-D+x}{\sqrt{2}}. \tag{2.78}$$

Hence

$$\begin{aligned} HH^- &= H^-(H+1) \\ HH^+ &= H^+(H-1). \end{aligned} \tag{2.79}$$

There is clearly a particular solution of Eq. (2.61) for $N = 1$. Also, according to Eq. (2.77), H admits a unit eigenvalue corresponding to the eigenstate $\Psi_0 = e^{-\frac{x^2}{2}}$ obtained from the equation $H^+\Psi_0 = 0$. Considering Eq. (2.79) and the positivity of the operator $H - \frac{1}{2}$, it becomes clear that Ψ_0 is the groundstate of H. The "higher" eigenstates Ψ_n, $n = 1, 2 \dots$ are obtained applying iteratively to it the "raising" operator H^- i.e. $\Psi_n(x) = [H^-]^n e^{-\frac{x^2}{2}}$, $n = 1, 2 \dots$. By (2.68), the corresponding eigenvalues are $\lambda_n = 1 + n$, $n = 0, 1, 2 \dots$. Clearly, $\Psi_n = c_n \mathcal{H}_n(x) e^{-\frac{x^2}{2}}$, $n = 0, 1, 2 \dots$, where $\mathcal{H}_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ are the Hermite polynomials.
The preceding was probably the first example of pure application of the factorization method in quantum mechanics [105].

For $N=2$, no much hopes of obtaining interesting solutions seen that any normalized differential operator of even order can be written as a combination of an operator of lower order and an even power of the Schrödinger operator .

For $N=3$, ($\alpha \neq 0$), the chain (2.73) can be reduced to the fourth order Painlevé equation [106]:

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2-a)y + \frac{b}{y}, \qquad y = -f_0 - x. \quad (2.80)$$

The corresponding potential $v_0(x)$ can be written as

$$v_0(x) = -2z'(x) + \frac{1}{4}\alpha^2 x^2 \quad (2.81)$$

where

$$(z'')^2 - \alpha^2(z - xz')^2 + 4P(z') = 0; \qquad P(t) = t(t+\alpha_1)(t-\alpha_2). \quad (2.82)$$

For $N=3$, $\alpha=0$ (see also [106]), $v_0(x)$ is the elliptic Weierstrass function $\wp(x)$, $(\wp')^2 = 4(\wp - r_1)(\wp - r_2)(\wp - r_3)$.

Let us remark that the operators L and R in Eq. (2.74) are of order exactly equal to N.
In [40] a type

$$[H, R] = \alpha R \quad (2.83)$$

operator equation (see equation (2.10) in [40]) was considered with H a general Schrödinger $H = -D^2 + v(x)$ and R a differential operator of N^{th} order. For $N=3$, it was shown that the corresponding equations in the coefficients can be reduced into a unique equation for the potential, itself solved (not uniquely) explicitly by (see equation (4.6) in [40])

$$v(x) = \frac{1}{4}x^2 + \frac{4}{1+x^2} - \frac{8}{(1+x^2)^2} + \frac{2}{3}. \quad (2.84)$$

Remembering the theorem 5 from [106], according to which any type (2.83) operator equation can be obtained imposing the periodicity condition to a factorization chain (i.e. in the same way as we obtained the first equation in (2.74)), one can conclude that the function in Eq. (2.84) is a particular solution of the Painlevé IV in Eq. (2.80) (i.e. a particular case of $v_0(x)$ in Eq. (2.81)).

Another interesting interconnection is also in order: In [40], after obtaining the potential in Eq. (2.84), a classical method (separation of equations) was used to solve the corresponding eigenelement problem. As a result, the eigenfunctions can be written as

$$\Psi_n(x) = \frac{F_n(x)}{1+x^2} e^{-\frac{x^2}{4}} \tag{2.85}$$

where

$$F_0(x) = 1, \tag{2.86}$$

$$F_n(x) = -(1+x^2)^2 e^{\frac{x^2}{2}} \frac{d}{dx}\left[\frac{He_{n-1}(x)}{1+x^2} e^{-\frac{x^2}{2}}\right], \tag{2.87}$$

and $He_{n-1}(x)$ are the Hermite polynomials

$$He_{n-1}(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left[e^{-\frac{x^2}{2}}\right]. \tag{2.88}$$

Remarking that the present polynomials are essentially the ones in Eq. (2.50) (remark also given in [104]), the situation testifies to the possibility for the "quasi-periodicity" and the "modification" methods to lead to the same result when applied to the eigenelement problem for the Schrödinger operator.

2.1.2 Discrete spectrum, algebraic interpretation.

Discrete spectrum.

We now show how to obtain, using the techniques of factorization, a discrete point spectrum for a given Hamiltonian, under additional suppositions (positivity, symmetry).
For that consider again the intertwining relations (2.57). Those relations allow of course to interconnect the eigenelements of $\hat{H}_j$ to those of $\hat{H}_{j+N}$.
Let $\hat{\psi}_j$ be the eigenfunction of $\hat{H}_j$ corresponding to $\hat{\lambda} = 0$ and such that $L_j\hat{\psi}_j \neq 0$. It becomes clear from the second equation in (2.57) that for any N, the function

$$L_j \ldots L_{j+N-1}\hat{\psi}_{j+N} \tag{2.89}$$

is an eigenfunction of $\hat{H}_j$ corresponding to the eigenvalue $-(\alpha_j + \ldots + \alpha_{j+N-1})$. In other words, the set

$$0,\ -\alpha_j,\ -(\alpha_j + \alpha_{j+1}),\ -(\alpha_j + \alpha_{j+1} + \alpha_{j+2}),\ \ldots \tag{2.90}$$

is included in the point spectrum of $\hat{H}_j$.
Supposing that the operators $\hat{H}_j$ are all positive and symmetric (in a convenient inner product space in which we are for the occasion supposed to work) and α_j all negative ($i.e.\mu_j$ increasing), then one easily shows that this part of point spectrum is purely discrete: The function $\hat{\psi}_j$ is clearly the ground state for $\hat{H}_j$ as the latter is positive. Then, supposing that $\hat{\alpha}$ is an eigenvalue of $\hat{H}_j$ such that $0 < \hat{\alpha} < -\alpha_j$, so according to Eq. (2.56), the number $\hat{\hat{\alpha}} = \hat{\alpha} + \alpha_j < 0$ will be an eigenvalue of $\hat{H}_{j+1}$ which contradicts the fact that $\hat{H}_{j+1}$ is positive. In the same manner, one shows that there is no eigenvalue of $\hat{H}_j$ in the interval $(-(\alpha_j + \alpha_{j+1} + \ldots + \alpha_{j+i}), -(\alpha_j + \alpha_{j+1} + \ldots + \alpha_{j+i+1}))$ and so the set in Eq. (2.90) is a purely discrete part of the point spectrum of H_j. It is clear that in the event the set is unbounded (set $-\alpha_j > \epsilon > 0$ for example), then the point spectrum of $\hat{H}_j$ is purely discrete and coincides with the set in Eq. (2.90).

Similarly, letting $\bar{\psi}_j$ be the eigenfunction of $\hat{H}_j$ corresponding to $\bar{\lambda} = 0$, and such that $R_j\hat{\psi}_j \neq 0$, then the function

$$R_{j+N-1} \ldots R_j \hat{\psi}_j \tag{2.91}$$

is an eigenfunction of $\hat{H}_{j+N}$ corresponding to the eigenvalue $\alpha_j + \ldots + \alpha_{j+N-1}$, and so, similarly, for $\hat{H}_j$ positive, $\alpha_j > \bar{\epsilon} > 0$, the set

$$0,\ \alpha_{j+N-1},\ \alpha_{j+N-1} + \alpha_{j+N-2}, \ldots, \tag{2.92}$$

is a purely discrete point spectrum of $\hat{H}_N$.
In the case when the chain $\hat{H}_j$ satisfies the additional quasi-periodicity conditions (2.58), then more interesting situations appear:
For α_j negative, it follows from Eqs. (2.57) and (2.58) that the point spectrum of $\hat{H}_0$ is purely discrete and is composed by N arithmetic progressions with starting points

$$0,\ -\alpha_0,\ -(\alpha_0 + \alpha_1),\ \ldots\ , -(\alpha_0 + \alpha_1 + \ldots + \alpha_{N-2}). \tag{2.93}$$

and step $-\alpha$.
Similarly for α_j positive, the point spectrum of $\hat{H}_0$ is purely discrete and is

composed of N arithmetic progressions with starting points

$$0,\ \alpha_{N-1},\ \alpha_{N-1}+\alpha_{N-2},\ldots,\alpha_{N-1}+\ldots+\alpha_1 \tag{2.94}$$

and step α.
More discussions and applications to the Schrödinger operator $-D^2+v(x)$ can be found in [106].

Algebraic interpretation.

Remarkable is also the algebraic interpretation of the formulas (2.54)-(2.61), in the case of "periodicity" (i.e. $\delta=0$). To see this, remark first that

$$\begin{aligned} LR &= P(\hat{H}_0) \\ RL &= Q(\hat{H}_0) \end{aligned} \tag{2.95}$$

where

$$\begin{aligned} P(\hat{H}_0) &= \hat{H}_0(\hat{H}_0+\mu_0-\mu_1)(\hat{H}_0+\mu_0-\mu_2)\ldots(\hat{H}_0+\mu_0-\mu_{N-1}) \\ Q(\hat{H}_0) &= P(\hat{H}_0+\mu_N-\mu_0) \end{aligned} \tag{2.96}$$

In other words, $[L,R]$ is a polynomial of degree $N-1$ in $\hat{H}_0$ with constant coefficients. Combining this with the relations in Eq. (2.61), one obtains that the operators $\hat{H}_0$, L and R generate a so-called "polynomial algebra" (see for example [109]).
For $N=2$, we particularly obtain

$$\begin{aligned} \left[\hat{H}_0,R\right] &= (\mu_0-\mu_2)R \\ \left[\hat{H}_0,L\right] &= -(\mu_0-\mu_2)L \end{aligned} \tag{2.97}$$

and

$$[L,R] = 2a^2\hat{H}_0 - bI \tag{2.98}$$

where $a^2=\mu_0-\mu_2$, $b=(\mu_0-\mu_2)(\mu_1-\mu_2)$ and I is the unit operator. Setting $a^2=1$ and $b=0$, this algebra becomes isomorphic to $sl(2)$. On the other side, setting $a^2=1$, the four-dimensional algebra generated by $\hat{H}_0$, L, R and I is isomorphic to $\mathcal{G}(1,0)$. The equivalence between the factorization types discussed in [56, 87] and irreducible representations of $\mathcal{G}(a,b)$ and $\mathcal{F}_6$ was already explained in [86, 87].

2.2 Discrete factorization types.

The main ideas of the factorization techniques having been given and illustrated in the preceding section, as well for the difference operators as for the differential ones, the objective of the present section consists in giving a detailed discussion on the factorization technique as well for the second order difference operator as for the fourth order one. We classify the factorization techniques in types, in rapport with their intrinsic structure. We aim of course such factorization types potentially allowing to solve one of the first three problems raised in the introduction. We prefer to collect them into the two categories that follow: The *Spiridonov-Vinet-Zhedanov factorization types*, and (a special case of) the *Infeld-Hull-Miller factorization types* (see introduction).

2.2.1 Discrete factorization types for the second-order difference operator.

The second-order difference operators in which we are concerned are expected to admit complete set of polynomial eigenfunctions, among which the constant one. So, we can imagine their general form to be:

$$\mathcal{H}(s) = u(s)\mathbf{E}_s - v(s) + w(s)\mathbf{E}_s^{-1}. \tag{2.99}$$

where

$$v(s) = u(s) + w(s), \tag{2.100}$$

u and w being polynomial or rational functions in s. It is then expected that

$$\mathcal{H}(s)P_t(y(s)) = \lambda(t)P_t(y(s)), \tag{2.101}$$

for a sequence of polynomials (in $y(s)$), $P_t(y(s))$, of degrees, $t = 0, 1, 2, \dots$ We, in the first step, analyze possible factorizations of $\mathcal{H}(s)$, while in the second, we study those for $\mathbf{E}_s.[H(s) - \lambda(t)]$.

Spiridonov-Vinet-Zhedanov factorization types.

Applying first, the isospectral deformation

$$H(s) = \rho(s)\mathcal{H}(s)\rho^{-1}(s) \tag{2.102}$$

where

$$\frac{\rho^2(s+1)}{\rho^2(s)} = \frac{u(s)}{w(s+1)}, \tag{2.103}$$

we obtain a formal symmetric one

$$H(s) = \sqrt{u(s)w(s+1)}\mathbf{E}_s - v(s) + \sqrt{u(s-1)w(s)}\mathbf{E}_s^{-1}. \tag{2.104}$$

If

$$\mathcal{H}(s)\psi_n(s) = \lambda_n\psi_n(s) \tag{2.105}$$

then

$$H(s)(\rho(s)\psi_n(s)) = \lambda_n(\rho(s)\psi_n(s)) \tag{2.106}$$

The corresponding factorization chain then reads ($H_0 := H$):

$$\begin{aligned} H_j(s) - \mu_j &= (f_j(s+1)\mathbf{E}_s + g_j(s))(g_j(s) + f_j(s)\mathbf{E}_s^{-1}) \\ H_{j+1}(s) - \mu_j &= (g_j(s) + f_j(s)\mathbf{E}_s^{-1})(f_j(s+1)\mathbf{E}_s + g_j(s)) \end{aligned} \tag{2.107}$$

where

$$\begin{aligned} f_j^2(s+1) + g_j^2(s) &= -v_j(s) - \mu_j \\ f_j(s)g_j(s) &= \sqrt{u_j(s-1)w_j(s)} \end{aligned} \tag{2.108}$$

and

$$\begin{aligned} f_{j+1}^2(s+1) + g_{j+1}^2(s) &= f_j^2(s) + g_j^2(s) + \mu_j - \mu_{j+1} \\ f_{j+1}(s)g_{j+1}(s) &= f_j(s)g_j(s-1). \end{aligned} \tag{2.109}$$

Taking for example

$$\mathcal{H}(s) = A(s)\mathbf{E}_s - (A(s) + B(s)) + B(s)\mathbf{E}_s^{-1} \tag{2.110}$$

the usual canonical form of the second order-difference operators in orthogonal polynomials theory, then a transformation as in Eqs. (2.102)-(2.103) gives

$$H(s) = \sqrt{A(s)B(s+1)}\mathbf{E}_s - (A(s) + B(s)) + \sqrt{A(s-1)B(s)}\mathbf{E}_s^{-1} \tag{2.111}$$

having a direct algebraic factorization

$$H(s) = \left(\sqrt{A(s)} - \sqrt{B(s)}\mathbf{E}_s^{-1}\right)\left(\sqrt{B(s+1)}\mathbf{E}_s - \sqrt{A(s)}\right). \tag{2.112}$$

When one is dealing with the difference hypergeometric operator, the commutation of the first-order operators in Eq. (2.112) is equivalent to the application (see [107])

$$P_{n-m}^{(m)}(y(s)) \longrightarrow P_{n-m-1}^{(m+1)}(y(s)), \tag{2.113}$$

lowering the degree of the polynomials by the difference derivative.
Let now see how to avoid the square roots in Eq. (2.104). By squaring the last equations in (2.108), (2.109), one easily finds that the factorization (2.107) is equivalent to ($\hat{H} := \hat{H}_0$):

$$\begin{aligned} \hat{H}_j - \hat{\mu}_j &= (\mathbf{E}_s + \hat{g}_j(s))\left(1 + \hat{f}_j(s)\mathbf{E}^{-1}\right) \\ \hat{H}_{j+1} - \hat{\mu}_j &= \left(1 + \hat{f}_j(s)\mathbf{E}_s^{-1}\right)(\mathbf{E}_s + \hat{g}_j(s)) \end{aligned} \tag{2.114}$$

with $\hat{H}$ given by

$$\hat{H}(s) = \hat{\rho}(s)\mathcal{H}(s)\hat{\rho}^{-1}(s) = \mathbf{E}_s - v(s) + w(s)u(s-1)\mathbf{E}_s^{-1} \tag{2.115}$$

where

$$\frac{\hat{\rho}(s+1)}{\hat{\rho}(s)} = u(s). \tag{2.116}$$

Eqs. (2.108), (2.109) now read ($\hat{f}_j = f_j^2; \hat{g}_j = g_j^2$):

$$\begin{aligned} \hat{f}_j(s+1) + \hat{g}_j(s) &= -v_j(s) - \mu_j \\ \hat{f}_j(s)\hat{g}_j(s) &= w_j(s)u_j(s-1) \end{aligned} \tag{2.117}$$

and

$$\begin{aligned} \hat{f}_{j+1}(s+1) + \hat{g}_{j+1}(s) &= \hat{f}_j(s) + \hat{g}_j(s) + \hat{\mu}_j - \hat{\mu}_{j+1} \\ \hat{f}_{j+1}(s)\hat{g}_{j+1}(s) &= \hat{f}_j(s)\hat{g}_j(s-1). \end{aligned} \tag{2.118}$$

There are remarkable interconnections between the factorization chain (2.114)-(2.118) and discrete-time integrable systems. To see this (we proceed as in [111]) write first

$$\hat{H}_j(x) = \mathbf{E}_x - v_j(x) + h_j(x)\mathbf{E}_x^{-1} \tag{2.119}$$

and consider then the system (2.117)-(2.118) in term of the coefficients of $H_j(x)$.
We quickly find

$$\begin{aligned} v_{j+1}(x) &= v_j(x) + \hat{f}_j(x+1) - \hat{f}_j(x) \\ h_{j+1}(x) &= h_j(x-1)\frac{\hat{f}_j(x)}{\hat{f}_j(x-1)}. \end{aligned} \tag{2.120}$$

Eliminating $\hat{f}_j$, we obtain

$$\begin{aligned} &(h_{j+1}(x) - h_j(x))\,(h_j(x) - h_{j+1}(x+1)) = \\ &(v_{j+1}(x) - v_j(x-1))\,(v_j(x) - v_{j+1}(x))\,h_j(x); \end{aligned} \tag{2.121}$$

$$\begin{aligned} &(h_{j+1}(x+1) - h_j(x+1))\,(h_j(x) - h_{j+1}(x+1)) = \\ &(v_{j+1}(x+1) - v_j(x))\,(v_j(x) - v_{j+1}(x))\,h_{j+1}(x+1), \end{aligned} \tag{2.122}$$

a system which constitutes a discrete-time version of the Toda lattice [111].
Indeed, setting

$$\begin{aligned} h_j(x) &:= h(t + j\epsilon; x) \\ v_j(x) &:= v(t + j\epsilon; x) \end{aligned} \tag{2.123}$$

and letting $\epsilon \longrightarrow 0$, we obtain (the dot means the derivative in time t):

$$\begin{aligned} \dot{h}(x)(h(x) - h(x+1)) &= h(x)(v(x-1) - v(x))\dot{v}(x) \\ \dot{h}(x+1)(h(x) - h(x+1)) &= h(x+1)(v(x+1) - v(x))\dot{v}(x). \end{aligned} \tag{2.124}$$

Division in Eqs. (2.124) members by members leads to

$$\dot{h}(x) = k_1(t)h(x)(v(x-1) - v(x)). \tag{2.125}$$

On the other side, shifting in x the second equation in (2.124) and then dividing the resulting equations members by members, we obtain:

$$\dot{v}(x) = k_2(t)(h(x-1) - h(x)). \tag{2.126}$$

Dividing Eq. (2.125) by Eq. (2.126) and considering the second equation in (2.124) one easily finds that $k_1(t) = k_2(t)$ after which the equations (2.125),(2.126) give the continuous-time Toda lattice [117].
To go further in the same direction, let us consider the operator $\hat{H}_j$ with $v_j = 0$ and chose $\hat{H}_{j+1}$ to satisfy

$$\hat{H}_{j+1}\left(1 + \hat{A}_j\mathbf{E}_x^{-2}\right) = \left(1 + \hat{A}_j\mathbf{E}_x^{-2}\right) H_j. \tag{2.127}$$

This is a double application of the factorization technique as one can take

$$\left(1+\hat{A}_j\mathbf{E}_x^{-2}\right)=\left(1-\hat{f}_j\mathbf{E}_x^{-1}\right)\left(1+\hat{f}_j\mathbf{E}_x^{-1}\right) \tag{2.128}$$

Eq. (2.127) gives

$$\begin{aligned} h_{j+1}(x)&=h_j(x)+\hat{A}_j(x)-\hat{A}_j(x+1)\\ h_{j+1}(x)&=h_j(x-2)\frac{\hat{A}_j(x)}{\hat{A}_j(x-1)} \end{aligned} \tag{2.129}$$

Eliminating $\hat{A}_j$, we obtain

$$\begin{aligned} h_{j+1}(x+1)\left(h_j(x)-h_{j+1}(x)\right)\left(h_{j+1}(x+2)-h_j(x)\right)=\\ h_j(x)\left(h_j(x+1)-h_{j+1}(x+1)\right)\left(h_{j+1}(x+1)-h_j(x-1)\right) \end{aligned} \tag{2.130}$$

a system that constitutes a discrete-time Kac-van Moerbeke lattice [111]. In the continuous limit as above, one obtains

$$\begin{aligned} &h(x+1)\dot{h}(x)\left(h(x)-h(x+2)\right)\\ &=h(x)\dot{h}(x+1)\left(h(x-1)-h(x+1)\right) \end{aligned} \tag{2.131}$$

or equally

$$\dot{h}(x)=k(t)h(x)\left(h(x+1)-h(x-1)\right) \tag{2.132}$$

which is the continuous-time Kac-van Moerbeke system [59].
If we set

$$h_j(x)=\bar{h}_j(x)\left(\nu_j+\bar{h}_j(x-1)\right), \tag{2.133}$$

then, the discrete-time Kac-van Moerbeke system (2.130) is transformed into [111]:

$$\bar{h}_j(x)(\nu_j+\bar{h}_j(x-1))=\bar{h}_{j+1}(x)(\nu_{j+1}+\bar{h}_{j+1}(x+1)) \tag{2.134}$$

which is linked with the discrete-time Toda lattice (2.121)-(2.122) by

$$\begin{aligned} \hat{f}_j(x)&=\hat{h}_j(2x)\hat{h}_j(2x+1)\\ \hat{g}_j(x)&=\left(\nu_j+\bar{h}_j(2x+1)\right)\left(\nu_j+\bar{h}_j(2x+2)\right)\\ \hat{\mu}_j&=const-\nu_j^2. \end{aligned} \tag{2.135}$$

For this reason, the transformation in Eq. (2.135) can be considered as a discrete version of the type Miura maps connecting the continuous-time Toda and Kac-van Moerbeke lattices [51].
Let us note finally that since the system (2.118) is known to converge in the continuous x to the corresponding differential version [106] itself known namely as a delay-Painlevé I [52], it can be considered as a discrete delay-Painlevé I. The system (2.118) is also known to admit symmetries [108]

Remark finally that the operator

$$\mathcal{H}(s) = u(s)\mathbf{E}_s - v(s) + w(s)\mathbf{E}_s^{-1}, \tag{2.136}$$

itself should be reasonably factorized without any preliminary transformation (such as (2.115) (or (2.102))). Indeed, one could set

$$\begin{aligned} \mathcal{H}_j - \hat{\mu}_j &= (u(s)\mathbf{E}_s + \hat{g}_j(s))\left(1 + \hat{f}_j(s)\mathbf{E}^{-1}\right) \\ \mathcal{H}_{j+1} - \hat{\mu}_j &= \left(1 + \hat{f}_j(s)\mathbf{E}_s^{-1}\right)(u(s)\mathbf{E}_s + \hat{g}_j(s)) \end{aligned} \tag{2.137}$$

so that the factorization chain (2.118) and the relations (2.120) become respectively

$$\begin{aligned} F_{j+1}(s+1) + \hat{g}_j(s) &= F_j(s) + \hat{g}_j(s) + \hat{\mu}_j - \hat{\mu}_{j+1} \\ F_{j+1}(s)\hat{g}_{j+1}(s) &= F_j(s)\hat{g}_j(s-1). \end{aligned} \tag{2.138}$$

and

$$\begin{aligned} v_{j+1}(x) &= v_j(x) + F_j(x+1) - F_j(x) \\ w_{j+1}(x) &= w_j(x-1)\frac{F_j(x)}{F_j(x-1)}. \end{aligned} \tag{2.139}$$

where $F_j(s) = u(s-1)\hat{f}_j(s)$. It is precisely this kind of factorization that we will adopt in chapter 4 for the modification of the special $M_n^{(2,\mu)}(x+1)$ Meixner polynomials.

Let us return for a moment to the system (2.118). For such a system, the quasi-periodicity condition (2.58) (the first equation) is equivalent to

$$\begin{aligned} \hat{f}_{j+N}(x-\delta) &= \hat{f}_j(x) \\ \hat{g}_{j+N}(x-\delta) &= \hat{g}_j(x) \end{aligned} \tag{2.140}$$

In the section 3.1, we will show that the case $N = 1, \delta = 0$ leads to the Charlier polynomials while $N = 2, \delta = 1$ leads to the Meixner and Kravchuk polynomials. This will appear as a consequence of relations (2.59) and Eqs. (2.63)-(2.68).

However, already from Eq. (2.68), the limits of the "quasi-periodicity" method appear: The eigenvalues of the constructed operators are necessarily linear functions of their variable (here n). In other words, we can not by that approach obtain solution for example of types problem 1 and 2 for the Hahn polynomials or any other "higher" class of orthogonal polynomials.
In [40], as already noted in example 2.3, the authors aimed to solve the operator equation

$$\left[\hat{H}_0, R\right] = \alpha R \tag{2.141}$$

under the constraints that if for example

$$\hat{H}_0 \Psi_n(x) = \gamma_n \Psi_n(x) \tag{2.142}$$

then, up to a multiplication by a constant,

$$\Psi_{n+1}(x) = R\Psi_n(x); \qquad \Psi_{n-1}(x) = R^a \Psi_n(x) \tag{2.143}$$

where R^a is the operator adjoint to R. In the example 2.3, we have given the essential of the corresponding solution given in [40] (modified Hermite polynomials). However, as quickly remarked in [40], a consequence of Eqs. (2.141) - (2.143) is that the eigenvalue γ_n satisfies

$$\gamma_{n+1} - \gamma_n = \alpha \tag{2.144}$$

i.e. is necessarily linear (a one dimensional equidistant discrete point spectrum of step α). To get over this barrier, the authors of [40] suggested to set, instead of Eq.(2.141), the more wider operator equation

$$[[\hat{H}_0, R], R] = 2\alpha R^2. \tag{2.145}$$

Under the additional constraints in Eq. (2.143), this leads to the following equation for the corresponding eigenvalue γ_n

$$\gamma_{n+2} - 2\gamma_{n+1} + \gamma_n = 2\alpha \tag{2.146}$$

leading clearly to a quadratic in n function γ_n. More generally, they suggested to consider the operator equation

$$[[[\hat{H}_0, R], R], \ldots R] = N\alpha R^N \tag{2.147}$$

(R appears in the l.h.s. N times), leading to γ_n, as a polynomial of degree N in n.
In our case of quasi-periodicity, one can adapt the extension of the formula (2.59) in the following way. First set the functions of operators

$$\begin{aligned} F_0\left(\hat{H}_0; R\right) &= \hat{H}_0 \\ F_1\left(\hat{H}_0; R\right) &= F_0\left(\hat{H}_0(x-\delta); R\right) R - R F_1\left(\hat{H}_0; R\right) \\ \ldots &.. \ldots \\ F_N\left(\hat{H}_0; R\right) &= F_{N-1}\left(\hat{H}_0(x-\delta); R\right) R - R F_{N-1}\left(\hat{H}_0; R\right) \end{aligned} \quad (2.148)$$

and then generalize for $\delta \neq 0$ the operator, equation in (2.147) as

$$F_N = N\alpha R^N, N = 1, 2, \ldots . \quad (2.149)$$

Apparently, the preceding formula constitutes a possible generalization of the "quasi-periodicity method" to some situations of non-linear eigenvalues. However, below we adopt a more natural generalization of that method to the cases of non-linear eigenvalues. It consists in adapting the approach used in example 2.1 (the shape-invariant Infeld-Hull-Miller method) in the general situation of difference "polynomial eigenelement equation". In sum, the method works for all the hypergeometric polynomials on lattices and some special Laguerre-Hahn polynomials on lattices.

Infeld-Hull-Miller factorization types.

Consider again the eigenvalue problem

$$\left(A\mathbf{E}_s - (A+B) + B\mathbf{E}_s^{-1}\right) \Psi_t(s) = \lambda(t)\psi_t(s). \quad (2.150)$$

Instead of applying the factorization technique to the operator in the left hand side of Eq.(2.150), we will apply it to the operator

$$\mathbf{E}_s^2 - (A(s+1) + B(s+1) + \lambda(t))\mathbf{E}_s + B(s+1)A(s). \quad (2.151)$$

We will search for the factorization

$$\begin{aligned} H(s;t) - \mu(t) &= (\mathbf{E}_s + G(s;t))(\mathbf{E}_s + F(s;t)) \\ H(s;t+1) - \mu(t) &= (\mathbf{E}_s + F(s;t))(\mathbf{E}_s + G(s;t)) \end{aligned} \quad (2.152)$$

where we noted

$$H(s;t) = \mathbf{E}_s^2 - (A(s+1)+B(s+1)+\lambda(t))\mathbf{E}_s + B(s+1)A(s). \tag{2.153}$$

The factorization (2.152) leads to the chain

$$\begin{aligned} F(s+1;t+1)+G(s;t+1) &= F(s;t)+G(s+1;t) \\ F(s;t+1)G(s;t+1) &= F(s;t)G(s;t)+\mu(t)-\mu(t+1) \end{aligned} \tag{2.154}$$

with

$$\begin{aligned} F(s+1;t)+G(s;t) &= -(A(s+1)+B(s+1)+\lambda(t)) \\ F(s;t)G(s;t) &= B(s+1)A(s)-\mu(t). \end{aligned} \tag{2.155}$$

Before going further on, let us remark as in the previous case that for the operator

$$A(s)\mathbf{E}_s - (A(s)+B(s)) + B(s)\mathbf{E}_s^{-1} - \lambda(t), \tag{2.156}$$

the factorization (2.152) is equivalent to

$$\begin{aligned} \bar{H}(s;t)-\mu(t) &= (A(s)\mathbf{E}_s+G(s;t))\,(A(s)\mathbf{E}_s+F(s;t)) \\ \bar{H}(s;t+1)-\mu(t) &= (A(s)\mathbf{E}_s+F(s;t))\,(A(s)\mathbf{E}_s+G(s;t)) \end{aligned} \tag{2.157}$$

where

$$\begin{aligned} &\bar{H}(s;t) = \\ &A(s)[A(s+1)\mathbf{E}_s^2 - (A(s+1)+B(s+1) \\ &+\lambda(t))\mathbf{E}_s + B(s+1)], \end{aligned} \tag{2.158}$$

the factorization system (2.154) remaining unchanged.

Consider next in more general situations, type (2.152) factorization for a given operator

$$H(s;t) = \mathbf{E}_s^2 - V(s;t)\mathbf{E}_s + W(s). \tag{2.159}$$

Contrary to the one in Eq. (2.114), such a factorization satisfies naturally some self-similarity condition. Namely, we easily remark that

$$H(s;t+1) = H(s;t) + (V(s;t)-V(s;t+1))\mathbf{E}_s. \tag{2.160}$$

Hence presumably, the system in Eq. (2.154) can be applied in our eigenvalue problems without additional reductions .
Indeed, one can prove the following

Proposition 2.1 *Let $\mu(t)$, $F(s;t)$ and $G(s;t)$ be any solutions of the system (2.154) such that*

$$F(s;t) - G(s;t-1) \neq 0, t = 0, 1, 2, \dots \tag{2.161}$$

then the equation

$$\begin{aligned} G(s;-1)\Psi_t(s+1) - [F(s;t) + G(s-1;t)]\Psi_t(s) \\ +F(s-1;-1)\Psi_t(s-1) = 0 \end{aligned} \tag{2.162}$$

admits for $t = 0, 1, 2, \dots$ non-trivial solutions satisfying difference (in s) and three-term recurrence (in t) relations. Moreover, if

$$F(s;t) - G(s;t-1) = c_1(t)y(s) + c_2(t) \tag{2.163}$$

$c_1(t) \neq 0$, those solutions are of polynomial (in $y(s)$) type (with degrees $t = 0, 1, 2, \dots$).

Proof. We need basically to ensure the equivalence between the system (2.154) and the factorization

$$\begin{aligned} H(s;t) - \bar{\mu}(t) &= (\mathbf{E}_s + G(s;t))(\mathbf{E}_s + F(s;t)) \\ H(s;t+1) - \bar{\mu}(t) &= (\mathbf{E}_s + F(s;t))(\mathbf{E}_s + G(s;t)) \end{aligned} \tag{2.164}$$

where

$$\begin{aligned} H(s;t) &= \mathbf{E}_s^2 - [F(s+1;t) + G(s;t)]\mathbf{E}_s + F(s;-1)G(s;-1) \\ \bar{\mu}(t) &= \mu(t) - \mu(-1). \end{aligned} \tag{2.165}$$

This is easily done after "integration" (in t) of the second equation in (2.154) transforming the system in

$$\begin{aligned} F(s+1;t+1) + G(s;t+1) = F(s;t) + G(s+1;t) \\ F(s;t)G(s;t) = F(s;-1)G(s;-1) - \mu(t) + \mu(-1). \end{aligned} \tag{2.166}$$

Next, it follows from the first equation in (2.154) that the unit is a solution of the equation in (2.162) for $t = 0$. In other words, the function $\Phi_0(s)$ given by

$$\Phi_0(s+1) = -G(s;-1)\Phi_0(s) \tag{2.167}$$

is a solution of $H(s;t)$ in Eq. (2.165) for $t=0$.
Now, let $\Phi_t(s)$ be a function such that

$$H(s;t)\Phi_t(s)=0. \tag{2.168}$$

According to Eq. (2.164),

$$\begin{aligned}\Phi_{t+1}(s) &= (\mathbf{E}_s+F(s;t))\,\Phi_t(s)\\ -\bar{\mu}(t)\Phi_t(s) &= (\mathbf{E}_s+G(s;t))\,\Phi_{t+1}(s).\end{aligned} \tag{2.169}$$

Subtracting the two equations gives

$$\Phi_{t+1}(s)+\bar{\mu}(t-1)\Phi_{t-1}(s)=(F(s;t)-G(s;t-1))\Phi_t(s) \tag{2.170}$$

If Eq. (2.161) is satisfied, then we obtain as solutions of (2.168) the sequence

$$\begin{aligned}\Phi_0(s) &\neq 0\\ \Phi_1(s) &= (\mathbf{E}_s+F(s;0))\,\Phi_0(s)\\ &\overset{byEq.(2.167)}{=} \Phi_0(s)(-G(s;-1)+F(s;0)) \overset{byEq.(2.161)}{\neq} 0\\ \Phi_t(s) &\overset{byEq.(2.170)}{\neq} 0, \quad t=2,3,\ldots.\end{aligned} \tag{2.171}$$

Consequently, the corresponding non-trivial solutions of the equation in (2.162) read

$$\Psi_t(s)=\Phi_0^{-1}(s).\Phi_t(s) \tag{2.172}$$

and satisfy difference and three-term recurrence relations analog to (2.169) and (2.170) respectively. More precisely, from Eqs. (2.169), (2.172) and (2.167) we have

$$\begin{aligned}\Psi_{t+1}(s) &= (-G(s;-1)\mathbf{E}_s+F(s;t))\,\Psi_t(s)\\ -\bar{\mu}(t)\Psi_t(s) &= (-G(s;-1)\mathbf{E}_s+G(s;t))\,\Psi_{t+1}(s),\end{aligned} \tag{2.173}$$

from which naturally follows

$$\Psi_{t+1}(s)+\bar{\mu}(t-1)\Psi_{t-1}(s)=(F(s;t)-G(s;t-1))\Psi_t(s) \tag{2.174}$$

or equally (use Eq. (2.166) for $s=-1$)

$$\begin{aligned}\Psi_{t+1}(s)+[F(-1;-1)G(-1;-1)-F(-1;t)G(-1;t)]\Psi_{t-1}(s)\\ =(F(s;t)-G(s;t-1))\Psi_t(s)\end{aligned} \tag{2.175}$$

In the event where the condition (2.163) is satisfied, then the functions $\Psi_t(s)$ are, by Eq. (2.174) or (2.175), of polynomial type and one can set

$$\Psi_t = \varrho(t)P_t(y(s)), t = 0, 1, 2, \dots \tag{2.176}$$

Using this relation and (2.173) and (2.174), one easily finds that the polynomials $P_t(y(s))$ satisfy the following difference relations

$$\begin{aligned} c_1(t)a_{t+1}P_{t+1}(y(s)) &= (-G(s;-1)\mathbf{E}_s + F(s;t))\, P_t(y(s)) \\ -c_1(t+1)a_{t+1}P_t(y(s)) &= (-G(s;-1)\mathbf{E}_s + F(s;t))\, P_{t+1}(y(s)) \end{aligned} \tag{2.177}$$

and the recurrence relations

$$a_{t+1}P_{n+1}(y(s)) + a_t P_{n-1}(y(s)) = (x - b_t)P_n(y(s)) \tag{2.178}$$

where

$$a_t^2 = \frac{\bar{\mu}(t-1)}{c_1(t-1)c_1(t)}; \qquad b_t = -\frac{c_2(t)}{c_1(t)}; \qquad \frac{\varrho(t+1)}{\varrho(t)} = a_{t+1}c_1(t) \tag{2.179}$$

which can close the proof of the proposition.

Remark 2.1 Let us now return to the system (2.154) (in general) for a more close observation. The first remarkable of its properties lies on the existence for it of at least one symmetry. To find it, one needs firstly to "integrate" the second equation in (2.154) relatively to t , secondly "differentiate" it relatively to s. We obtain

$$\begin{aligned} \tilde{F}(s+1;t+1) + \tilde{G}(s+1;t) &= \tilde{F}(s;t) + \tilde{G}(s;t+1) \\ \tilde{F}(s+1;t)\tilde{G}(s+1;t) &= \tilde{F}(s;t)\tilde{G}(s;t) \\ &\quad + \tilde{\mu}(s) - \tilde{\mu}(s+1) \end{aligned} \tag{2.180}$$

where

$$\begin{aligned} \tilde{F}(s;t) &:= F(s;t), \\ \tilde{G}(s;t) &:= -G(s;t), \\ \tilde{\mu}(s) &:= \tilde{\mu}(-1) - F(-1;-1)G(-1;-1) + F(s;-1)G(s;-1) \end{aligned} \tag{2.181}$$

So, while the system (2.154) is an Infeld-Hull-Miller factorization system with t as the "variable of factorization", the one in Eq. (2.180) is an Infeld-Hull-Miller factorization system with now s as the "variable of factorization".

Next, from Eqs. (2.154), (2.180) and (2.181) we see that the correspondence

$$\begin{aligned} \breve{F}(s;t) &:= F(t;s), \\ \breve{G}(s;t) &:= -G(t;s), \\ \breve{\mu}(t) &:= \breve{\mu}(-1) - F(-1;-1)G(-1;-1) + F(t;-1)G(t;-1) \end{aligned} \quad (2.182)$$

(remark that $\hat{\mu}(t) := \hat{\mu}(-1) + F(-1;-1)G(-1;-1) - F(-1;t)G(-1;t)$) is a *symmetry* of the Infeld-Hull-Miller factorization system (2.154) i.e. the functions in left hand side of Eq. (2.182) are solutions of Eq. (2.154) as well. This symmetry is probably the main characteristic of the system under consideration. It becomes then of much necessity to understand which is its intrinsic meaning.

As proved in the preceding proposition, the functions $\Psi_t(s), t = 0, 1, 2, \ldots$ of the independent variable $s \in Z^+$ "generated" by the Infeld-Hull-Miller factorization system (2.154), satisfy the second-order difference equation

$$\begin{aligned} G(s;-1)\Psi_t(s+1) - (F(s;t) + G(s-1;t))\Psi_t(s) \\ +F(s-1;-1)\Psi_t(s-1) = 0 \end{aligned} \quad (2.183)$$

and the three-term recurrence relation

$$\begin{aligned} \Psi_{t+1}(s) - (F(s;t) - G(s;t-1))\Psi_t(s) + (F(-1;-1)G(-1;-1) \\ -F(-1;t-1)G(-1;t-1))\Psi_{t-1}(s) = 0. \end{aligned} \quad (2.184)$$

On the other side, the functions $\tilde{\Psi}_s(t)$, $s = 0, 1, 2, \ldots$ of the independent variable $t \in Z^+$ "generated" by the Infeld-Hull-Miller factorization system (2.180), satisfy the second-order difference equation

$$\begin{aligned} G(-1;t)\tilde{\Psi}_s(t+1) + (F(s;t) - G(s;t-1))\tilde{\Psi}_s(t) \\ -F(-1;t-1)\tilde{\Psi}_s(t-1) = 0 \end{aligned} \quad (2.185)$$

and the three-term recurrence relation

$$\begin{aligned} \tilde{\Psi}_{s+1}(t) - (F(s;t) + G(s-1;t))\tilde{\Psi}_s(t) + (F(s-1;-1)G(s-1;-1) \\ -F(-1;-1)G(-1;-1))\tilde{\Psi}_{s-1}(t) = 0. \end{aligned} \quad (2.186)$$

Comparing Eqs. (2.183) and (2.186) on the one side, Eqs. (2.184) and (2.185) on the other side, we remark that the sequences of functions $\widehat{\Psi}_t(s) := \frac{\Psi_t(s)}{\varrho(t)}$

and $\hat{\tilde{\Psi}}_t(s) := \frac{\tilde{\Psi}_t(s)}{\tilde{\varrho}(t)}$ where $\frac{\varrho(t+1)}{\varrho(t)} = -G(-1;t)$ and $\frac{\tilde{\varrho}(t+1)}{\tilde{\varrho}(t)} = G(t;-1))$ are mutually dual, i. e. $\hat{\tilde{\Psi}}_s(t) \equiv \hat{\Psi}_t(s)$, iff

$$F(-1;-1)G(-1;-1) = 0. \tag{2.187}$$

Remark that in the case $F(s;t) + G(s-1;t)$ and $F(s;t) - G(s;t-1)$ are linear functions respectively of say, $\lambda(t)$ (with coefficients not depending on t) and $y(s)$ (with coefficients not depending on s), $\hat{\Psi}_t(s)$ and $\hat{\tilde{\Psi}}_s(t)$ are (bispectral) orthogonal polynomials of the variables $y(s)$ and $\lambda(t)$ respectively. For such polynomials, the symmetry (2.182), constrained by Eq. (2.187), becomes clearly the usual duality relation between (bispectral) orthogonal polynomials (see for example [75]).
But as we will see in chapter 3, all the classical (up to the Askey-Wilson polynomials) polynomials on lattices can be generated from a type (2.154) Infeld-Hull-Miller factorization system and the corresponding "superpotentials" satisfy the constraint (2.187). The latter is easily seen verifying for those polynomials the equality $\sigma(0) = 0$ (see section 1.2), $\sigma(s)$ in section 1.2 being comparable with $F(s-1;-1)$ here (for the Askey-Wilson polynomials, it is necessary first to transform them into equivalent to them q-Racah polynomials (see such transformation also in section 1.2), in order that Eq. (2.187) be satisfied, the dual Hahn and their q-versions from the section 1.2 need also additional simple transformations). As a consequence of this and the preceding observations, we obtain that each class of polynomials dual to one of the classes of classical (up to q-Racah polynomials) polynomials on lattices, can also be generated from a type (2.154) Infeld-Hull-Miller factorization system (now written as Eq. (2.180)). This observation is clearly very useful and we will refer frequently to it later. It is worth noting that the cases of self-duality (Charlier, Meixner and Kravchuk cases) correspond to one of the following identities (to be verified in section 3.2)

$$F(t;s) = F(s;t); \qquad G(t;s) = -G(s;t) \tag{2.188}$$

or

$$F(t;s) = -F(s;t); \qquad G(t;s) = G(s;t). \tag{2.189}$$

Remark 2.2 Before closing this subsection, another observation is in order. Let us combine the Infeld-Hull-Miller factorization systems (2.154) and (2.163). We obtain,

$$F(s;t) \quad = \quad G(s;t-1) + c_1(t)y(s) + c_2(t)$$

$$\begin{aligned} G(s;t) &= F(s;t-1) - (c_1(t)y(s+1) + c_2(t)) \\ F(s;t)G(s;t) &= F(s;t-1)G(s;t-1) + \hat{\mu}(t-1) - \hat{\mu}(t). \end{aligned} \quad (2.190)$$

It is interesting to note that such systems are already known in orthogonal polynomials theory. Indeed, a large subset of so-called Laguerre-Hahn orthogonal polynomials (including particularly the Askey-Wilson polynomials) can be defined by such a system. This fact attesting of the interconnection between the factorization technique and the Laguerre-Hahn approach to orthogonal polynomials [79, 80, 82] will be discussed in more details later in section 5.2.

2.2.2 Discrete factorization types for the fourth-order difference operator.

The fourth-order difference operators in which we are concerned are expected to admit complete set of polynomial eigenfunctions, among which the constant one. So, we can imagine their general form to be:

$$\breve{L} = \breve{A}(s)\mathbf{E}_s^2 + \breve{B}(s)\mathbf{E}_s + \breve{V}(s) + \breve{C}(s)\mathbf{E}_s^{-1} + \breve{D}(s)\mathbf{E}_s^{-2} \quad (2.191)$$

where

$$\breve{V} = -(\breve{A} + \breve{B} + \breve{C} + \breve{D}), \quad (2.192)$$

$\breve{A}, \breve{B}, \breve{C}$ and $\breve{D}$, being polynomial or rational functions in s. It is then expected that

$$\breve{L}P_t(y(s)) = \lambda(t)P_t(y(s)), \quad (2.193)$$

for a sequence of polynomials (in $y(s)$), $P_t(y(s))$, of degrees, $t = 0, 1, 2, \ldots$ As for the case of second-order difference operators, we collect possible types of factorizations in two categories: Spiridonov-Vinet-Zhedanov types and Infeld-Hull-Miller types.

Spiridonov-Vinet-Zhedanov factorization types.

As in the case of second-order difference operators, we firstly try to apply to $\breve{L}$, a similarity reduction leading to a formal symmetric operator

$$\breve{\breve{L}} = \breve{\rho}(s)\breve{L}\breve{\rho}^{-1}(s), \quad (2.194)$$

for some function $\breve{\rho}(s)$. It is easily seen that the requirement that $\breve{\breve{L}}$ be formal symmetric is equivalent to that $\breve{\rho}(s)$ satisfies simultaneously

$$\begin{aligned} \frac{\breve{\rho}^2(s+2)}{\breve{\rho}^2(s)} &= \frac{\breve{A}(s)}{\breve{D}(s+2)}, \\ \frac{\breve{\rho}^2(s+1)}{\breve{\rho}^2(s)} &= \frac{\breve{B}(s)}{\breve{C}(s+1)}. \end{aligned} \tag{2.195}$$

So, contrary to the second-order case, the problem admits solutions iff

$$\frac{A(s)}{D(s+2)} = \frac{B(s+1)B(s)}{C(s+2)C(s+1)}. \tag{2.196}$$

Supposing that this is actually the case, $\breve{\breve{L}}$ is formal symmetric and consequently can be factorized as follows ($\breve{\breve{L}}_0 = \breve{\breve{L}}$):

$$\begin{gathered} \breve{\breve{L}}_j - \breve{\mu}_j = \\ \left(\breve{f}_j(s+1)\mathbf{E}_s + \breve{g}_j(s) + \breve{q}_j(s-1)\mathbf{E}_s^{-1}\right)\left(\breve{q}_j(s)\mathbf{E}_s + \breve{g}_j(s) + \breve{f}_j(s)\mathbf{E}_s^{-1}\right) \\ \breve{\breve{L}}_{j+1} - \breve{\mu}_j = \\ \left(\breve{q}_j(s)\mathbf{E}_s + \breve{g}_j(s) + \breve{f}_j(s)\mathbf{E}_s^{-1}\right)\left(\breve{f}_j(s+1)\mathbf{E}_s + \breve{g}_j(s) + \breve{q}_j(s-1)\mathbf{E}_s^{-1}\right). \end{gathered}$$

The corresponding factorization system then reads

$$\begin{aligned} \breve{f}_{j+1}(s+1)\breve{g}_{j+1}(s+1) &= \breve{q}_j(s)\breve{f}_j(s+2) \\ \breve{f}_{j+1}(s+1)\breve{g}_{j+1}(s+1) + \breve{g}_{j+1}(s)\breve{q}_{j+1}(s) &= \breve{q}_j(s)\breve{g}_j(s+1) + \breve{g}_j(s)\breve{f}_j(s+1) \\ \breve{f}_{j+1}^2(s+1) + \breve{g}_{j+1}^2(s) &= \breve{q}_j^2(s) + \breve{g}_j^2(s) + \breve{\mu}_j - \breve{\mu}_{j+1} \end{aligned} \tag{2.197}$$

In its general form (i. e. without the constraint (2.196)), the operator $\breve{L}$ admits also three other "compatible" (i.e. for which the corresponding factorization system is compatible) types of factorization. The first one reads

$$\begin{gathered} \hat{\breve{L}}_j - \breve{\mu}_j = \\ \left(\breve{h}(s)\mathbf{E}_s + \breve{q}_j(s) + \breve{g}_j(s)\mathbf{E}_s^{-1}\right)\left(\breve{h}(s)\mathbf{E}_s + \breve{p}_j(s) + \breve{f}_j(s)\mathbf{E}_s^{-1}\right) \\ \hat{\breve{L}}_{j+1} - \breve{\mu}_j = \\ \left(\breve{h}(s)\mathbf{E}_s + \breve{p}_j(s) + \breve{f}_j(s)\mathbf{E}_s^{-1}\right)\left(\breve{h}(s)\mathbf{E}_s + \breve{q}_j(s) + \breve{g}_j(s)\mathbf{E}_s^{-1}\right), \end{gathered} \tag{2.198}$$

where

$$\hat{\breve{L}}_j = \breve{h}(s)\breve{L}_j; \qquad \breve{L}_0 = \breve{L}; \qquad \breve{h}(s+1)\breve{h}(s) = \breve{A}(s). \tag{2.199}$$

The corresponding factorization system reads

$$\begin{aligned}\breve{p}_{j+1}(s+1) + \breve{q}_{j+1}(s) &= \breve{q}_j(s+1) + \breve{p}_j(s)\\ \breve{G}_{j+1}(s) + \breve{F}_{j+1}(s+1) + \breve{q}_{j+1}(s)\breve{p}_{j+1}(s) &= \breve{G}_j(s+1) + \breve{F}_j(s) + \breve{p}_j(s)\breve{q}_j(s)\\ \breve{q}_{j+1}(s)\breve{F}_{j+1}(s) + \breve{p}_{j+1}(s-1)\breve{G}_{j+1}(s) &= \breve{F}_j(s)\breve{q}_j(s-1) + \breve{p}_j(s)\breve{G}_j(s)\\ \breve{G}_{j+1}(s)\breve{F}_{j+1}(s-1) &= \breve{F}_j(s)\breve{G}_j(s-1),\end{aligned} \tag{2.200}$$

where

$$\breve{F}_j(s) = \breve{h}(s)\breve{f}_j(s); \qquad \breve{G}_j(s) = \breve{h}(s)\breve{g}_j(s). \tag{2.201}$$

In the two remaining types, we avoid the additional equation $\breve{h}(s+1)\breve{h}(s) = \breve{A}(s)$. We, on the one side, set ($\breve{L}_0 = \breve{L}$):

$$\begin{aligned}\breve{L}_j - \breve{\mu}_j &= \left(\breve{A}(s)\mathbf{E}_s^2 + \breve{q}_j(s)\mathbf{E}_s + \breve{g}_j(s)\right)\left(1 + \breve{p}_j(s)\mathbf{E}_s^{-1} + \breve{f}_j(s)\mathbf{E}_s^{-2}\right)\\ \breve{L}_{j+1} - \breve{\mu}_j &= \left(1 + \breve{p}_j(s)\mathbf{E}_s^{-1} + \breve{f}_j(s)\mathbf{E}_s^{-2}\right)\left(\breve{A}(s)\mathbf{E}_s^2 + \breve{q}_j(s)\mathbf{E}_s + \breve{g}_j(s)\right)\end{aligned}$$

leading to the system

$$\begin{aligned}\breve{A}(s)\breve{p}_{j+1}(s+2) + \breve{q}_{j+1}(s) &= \breve{q}_j(s) + \breve{A}(s-1)\breve{p}_j(s)\\ \breve{A}(s)\breve{f}_{j+1}(s+2) + \breve{q}_{j+1}(s)\breve{p}_{j+1}(s+1) + \breve{g}_{j+1}(s) &=\\ \breve{g}_j(s) + \breve{p}_j(s)\breve{q}_j(s-1) &+ \breve{A}(s-2)\breve{f}_j(s)\\ \breve{q}_{j+1}(s)\breve{f}_{j+1}(s+1) + \breve{g}_{j+1}(s)\breve{p}_{j+1}(s) &= \breve{p}_j(s)\breve{g}_j(s-1) + \breve{f}_j(s)\breve{q}_j(s-2)\\ \breve{g}_{j+1}(s)\breve{f}_{j+1}(s) &= \breve{f}_j(s)\breve{g}_j(s-2).\end{aligned} \tag{2.202}$$

On the other side, we set ($\breve{L}_0 = \breve{L}$):

$$\begin{aligned}\breve{L}_j - \breve{\mu}_j &= \left(\breve{A}(s)\mathbf{E}_s^2 + \breve{q}_j(s)\mathbf{E}_s + \breve{p}_j(s) + \breve{g}_j(s)\mathbf{E}_s^{-1}\right)\left(1 + \breve{f}_j(s)\mathbf{E}_s^{-1}\right)\\ \breve{L}_{j+1} - \breve{\mu}_j &= \left(1 + \breve{f}_j(s)\mathbf{E}_s^{-1}\right)\left(\breve{A}(s)\mathbf{E}_s^2 + \breve{q}_j(s)\mathbf{E}_s + \breve{p}_j(s) + \breve{g}_j(s)\mathbf{E}_s^{-1}\right)\end{aligned}$$

leading to the system

$$\begin{aligned}\breve{A}(s)\breve{f}_{j+1}(s+2) + \breve{q}_{j+1}(s) &= \breve{q}_j(s) + \breve{A}(s-1)\breve{f}_j(s)\\ \breve{q}_{j+1}(s)\breve{f}_{j+1}(s+1) + \breve{p}_{j+1}(s) &= \breve{p}_j(s) + \breve{f}_j(s)\breve{q}_j(s-1)\\ \breve{p}_{j+1}(s)\breve{f}_{j+1}(s) + \breve{g}_{j+1}(s) &= \breve{g}_j(s) + \breve{f}_j(s)\breve{p}_j(s-1)\\ \breve{g}_j(s)\breve{f}_j(s-1) &= \breve{f}_j(s)\breve{g}_j(s-1).\end{aligned} \tag{2.203}$$

Remark that this factorization leads to the intertwining relation

$$\breve{L}_{j+1}\left(1+\breve{f}_j(s)\mathbf{E}_s^{-1}\right)=\left(1+\breve{f}_j(s)\mathbf{E}_s^{-1}\right)\breve{L}_j, \tag{2.204}$$

so that the factorization chain (2.203) is equivalent to the following relations between the coefficients of $\breve{L}_j$ and $\breve{L}_{j+1}$:

$$\begin{aligned}
\breve{A}_{j+1}(s) &= \breve{A}_j(s)\\
\breve{B}_{j+1}(s) &= \breve{B}_j(s)+\breve{A}_j(s-1)\breve{f}_j(s)-\breve{A}_j(s)\breve{f}_j(s+2)\\
\breve{V}_{j+1}(s) &= \breve{V}_j(s)+\breve{B}_j(s-1)\breve{f}_j(s)-\breve{B}_{j+1}(s)\breve{f}_j(s+1)\\
\breve{C}_{j+1}(s) &= \breve{C}_j(s)+\breve{V}_j(s-1)\breve{f}_j(s)-\breve{V}_{j+1}(s)\breve{f}_j(s)\\
\breve{D}_{j+1}(s) &= \breve{f}_j(s)\frac{\breve{D}_j(s-1)}{\breve{f}_j(s-2)},
\end{aligned} \tag{2.205}$$

or equally

$$\begin{aligned}
\breve{A}_{j+1}(s) &= \breve{A}_j(s)\\
\breve{B}_{j+1}(s) &= \breve{B}_j(s)+\breve{A}_j(s-1)\breve{f}_j(s)-\breve{A}_j(s)\breve{f}_j(s+2)\\
\breve{V}_{j+1}(s) &= \breve{V}_j(s)+\breve{B}_j(s-1)\breve{f}_j(s)-\breve{B}_j(s)\breve{f}_j(s+1)\\
&\quad -\breve{A}_j(s-1)\breve{f}_j(s)\breve{f}_j(s+1)+\breve{f}_j(s+1)\breve{f}_j(s+2)\breve{A}_j(s)\\
\breve{C}_{j+1}(s) &= \breve{C}_j(s)+\breve{f}_j(s)[\breve{V}_j(s-1)-\breve{V}_j(s)-\breve{f}_j(s)\breve{B}_j(s-1)\\
&\quad +\breve{f}_j(s+1)\breve{B}_j(s)+\breve{A}_j(s-1)\breve{f}_j(s)\breve{f}_j(s+1)\\
&\quad -\breve{A}_j(s)\breve{f}_j(s+1)\breve{f}_j(s+2)]\\
\breve{D}_{j+1}(s) &= \breve{f}_j(s)\frac{\breve{D}_j(s-1)}{\breve{f}_j(s-2)},
\end{aligned} \tag{2.206}$$

knowing that

$$\breve{L}_j\Phi_j(s)=\breve{\mu}_j\Phi_j(s);\qquad \breve{f}_j(s)=-\frac{\Phi_j(s)}{\Phi_j(s-1)}. \tag{2.207}$$

The formulas (2.206) extend in some sense, the ones in Eq. (2.139), to the fourth-order case. They should serve for "modification" of some polynomials satisfying a fourth-order difference eigenvalue equation, a question which will not be treated in this thesis.

Infeld-Hull-Miller factorization types.

As in the case of second-order, we consider now the operator $H(s;t) = \mathbf{E}_s^2.[\breve{L} - \lambda(t)]$ (for simplicity, we consider here the case of linear lattices). The factorization then reads

$$\begin{gathered}\mathcal{H}(s;t) - \breve{\mu}_j = \\ \left(\breve{A}(s)\mathbf{E}_s^2 + \mathcal{Q}(s;t)\mathbf{E}_s + \mathcal{G}(s;t)\right)\left(\breve{A}(s)\mathbf{E}_s^2 + \mathcal{P}(s;t)\mathbf{E}_s + \mathcal{F}(s;t)\right) \\ \mathcal{H}(s;t+1) - \breve{\mu}_j = \\ \left(\breve{A}(s)\mathbf{E}_s^2 + \mathcal{P}(s;t)\mathbf{E}_s + \mathcal{F}(s;t)\right)\left(\breve{A}(s)\mathbf{E}_s^2 + \mathcal{Q}(s;t)\mathbf{E}_s + \mathcal{G}(s;t)\right),\end{gathered} \tag{2.208}$$

where

$$\mathcal{H}(s;t) = \breve{A}(s)H(s;t) \tag{2.209}$$

that is

$$\begin{aligned}\breve{A}(s)\breve{B}(s+2) &= \breve{A}(s)\mathcal{P}(s+2;t) + \breve{A}(s+1)\mathcal{Q}(s;t) \\ \breve{A}(s)\breve{V}(s+2) &= \breve{A}(s)\mathcal{F}(s+2;t) + \mathcal{P}(s+1;t)\mathcal{Q}(s;t) + \mathcal{G}(s;t)\breve{A}(s) \\ \breve{A}(s)\breve{C}(s+2) &= \mathcal{Q}(s;t)\mathcal{F}(s+1;t) + \mathcal{G}(s;t)\mathcal{P}(s;t) \\ \breve{A}(s)\breve{D}(s+2) &= \mathcal{G}(s;t)\mathcal{F}(s;t) - \breve{\mu}(t).\end{aligned} \tag{2.210}$$

The corresponding factorization chain reads

$$\begin{gathered}\breve{A}(s)\mathcal{P}(s+2;t+1) + \mathcal{Q}(s;t+1)\breve{A}(s+1) = \breve{A}(s)\mathcal{Q}(s+2;t) \\ +\mathcal{P}(s;t)\breve{A}(s+1); \\ \breve{A}(s)\mathcal{F}(s+2;t+1) + \mathcal{Q}(s;t+1)\mathcal{P}(s+1;t+1) \\ +\mathcal{G}(s;t+1)\breve{A}(s) = \\ \breve{A}(s)\mathcal{G}(s+2;t) + \mathcal{Q}(s+1;t)\mathcal{P}(s;t) + \mathcal{F}(s;t)\breve{A}(s); \\ \mathcal{Q}(s;t+1)\mathcal{F}(s+1;t+1) + \mathcal{G}(s;t+1)\mathcal{P}(s;t+1) = \\ \mathcal{P}(s;t)\mathcal{G}(s+1;t) + \mathcal{F}(s;t)\mathcal{Q}(s;t); \\ \mathcal{F}(s;t+1)\mathcal{G}(s;t+1) = \mathcal{F}(s;t)\mathcal{G}(s;t) + \breve{\mu}(t) - \breve{\mu}(t+1).\end{gathered} \tag{2.211}$$

Consider next the operator

$$\hat{\mathcal{H}}(s;t) = H^+(s;t)H^-(s;t) + (\breve{\mu}(t) - \breve{\mu}(-1)) \tag{2.212}$$

where

$$\begin{aligned}H^+(s;t) &= \breve{A}(s)\mathbf{E}_s^2 + \mathcal{Q}(s;t)\mathbf{E}_s + \mathcal{G}(s;t) \\ H^-(s;t) &= \breve{A}(s)\mathbf{E}_s^2 + \mathcal{P}(s;t)\mathbf{E}_s + \mathcal{F}(s;t) \\ \breve{A}(s) &= -(\mathcal{Q}(s;-1) + \mathcal{G}(s;-1)).\end{aligned} \tag{2.213}$$

Analogue to the proposition 2.1 is the following

Proposition 2.2 *Letting $\mathcal{F}(s;t)$, $\mathcal{G}(s;t)$, $\mathcal{P}(s;t)$, $\mathcal{Q}(s;t)$, and $\breve{\mu}(t)$ be solutions of the factorization system (2.211), such that*

$$\mathcal{P}(s;t) - \mathcal{Q}(s;t-1) \neq 0; \qquad \mathcal{F}(s;t) - \mathcal{G}(s;t-1) \neq 0, t = 0, 1, \ldots \tag{2.214}$$

then, the equation

$$\hat{\mathcal{H}}(s;t)\Phi_t(s) = 0 \tag{2.215}$$

admits for $t = 0, 1, 2, \ldots$ non-trivial solutions

$$\begin{aligned} \Phi_0(s) &:= 1, \\ \Phi_{t+1}(s) &:= H^-(s;t)\Phi_t(s), t = 0, 1, 2, \ldots \end{aligned} \tag{2.216}$$

Moreover, if

$$\begin{aligned} \mathcal{P}(s;t) = \mathcal{Q}(s;t-1) + c_0(t)s + c_1(t) \\ \mathcal{F}(s;t) = \mathcal{G}(s;t-1) + c_2(t)s + c_3(t) \end{aligned} \tag{2.217}$$

then the obtained solutions are of polynomial (in s) type.

Proof. It is clear that for $\mathcal{F}, \mathcal{G}, \mathcal{P}, \mathcal{Q}$ and $\breve{\mu}$ given by Eq. (2.211), the factorization

$$\begin{aligned} \hat{\mathcal{H}}(s;t) - (\breve{\mu}(t) - \breve{\mu}(-1)) &= H^+(s;t)H^-(s;t) \\ \hat{\mathcal{H}}(s;t+1) - (\breve{\mu}(t) - \breve{\mu}(-1)) &= H^-(s;t)H^+(s;t) \end{aligned} \tag{2.218}$$

takes place. Hence we can write

$$\begin{aligned} \hat{\mathcal{H}}(s;t+1)H^-(s;t) &= H^-(s;t)\hat{\mathcal{H}}(s;t) \\ \hat{\mathcal{H}}(s;t)H^+(s;t) &= H^+(s;t)\hat{\mathcal{H}}(s;t+1). \end{aligned} \tag{2.219}$$

That justifies the difference relations

$$\begin{aligned} \Phi_{t+1}(s) &= H^-(s;t)\Phi_t(s) \\ -(\breve{\mu}(t) - \breve{\mu}(-1))\Phi_t(s) &= H^+(s;t)\Phi_{t+1}(s) \end{aligned} \tag{2.220}$$

and consequently the recurrence relation

$$\begin{gathered} \Phi_{t+1}(s) + (\breve{\mu}(t-1) - \breve{\mu}(-1))\Phi_{t-1}(s) \\ = \\ (\mathcal{P}(s;t) - \mathcal{Q}(s;t-1))\Phi_t(s+1) + (\mathcal{F}(s;t) - \mathcal{G}(s;t-1))\Phi_t(s). \end{gathered} \tag{2.221}$$

The solution $\Phi_0(s)$ in Eq. (2.216) is obtained by loading $t = -1$ in the second equation in (2.218) while the remaining solutions $\Phi_t(s)$, $t = 1, 2, \ldots$ in Eq. (2.216) are obtained using the first equation in (2.220) and (2.221), considering Eq. (2.214). Moreover, using Eq. (2.221), it becomes clear that under the constraints (2.217), the full sequence is of polynomial (in s) type, which proves the proposition.

Remark that in the case of orthogonal polynomials, one obtains, using the corresponding three-term recurrence relation and the recurrence relation in Eq. (2.221), that the obtained polynomials satisfy an usual three-term difference relation. As a consequence (see Theorem 1 in [80]), the obtained polynomials will be of semi-classical (see Chapter 5) type .

Let us now combine the system (2.211) and the conditions (2.217). The result is

$$\begin{gathered}
\mathcal{F}(s;t) = \mathcal{G}(s;t-1) + M(s;t); \\
\mathcal{P}(s;t) = \mathcal{Q}(s;t-1) + N(s;t); \\
\mathcal{Q}(s;t) = \mathcal{P}(s;t-1) - \tfrac{\breve{A}(s)}{\breve{A}(s+1)} N(s+2;t); \\
\mathcal{G}(s;t) = \mathcal{F}(s;t-1) + \tfrac{N(s+2;t)}{\breve{A}(s+1)} (\mathcal{Q}(s+1;t-1) \\
+N(s+1;t)) - \tfrac{N(s+1;t)}{\breve{A}(s)} \mathcal{P}(s;t-1) - M(s+2;t); \\
\mathcal{Q}(s;t+1)\mathcal{F}(s+1;t+1) + \mathcal{G}(s;t+1)\mathcal{P}(s;t+1) = \\
\mathcal{P}(s;t)\mathcal{G}(s+1;t) + \mathcal{F}(s;t)\mathcal{Q}(s;t); \\
\mathcal{F}(s;t+1)\mathcal{G}(s;t+1) = \mathcal{F}(s;t)\mathcal{G}(s;t) + \breve{\mu}(t) - \breve{\mu}(t+1).
\end{gathered} \tag{2.222}$$

where

$$\begin{aligned}
N(s;t) &= c_0(t)s + c_1(t) \\
M(s;t) &= c_2(t)s + c_3(t).
\end{aligned} \tag{2.223}$$

Further simplifications give

$$\begin{gathered}
\mathcal{Q}(s;t) = \mathcal{Q}(s;t-2) + N(s;t-1) - \tfrac{\breve{A}(s)}{\breve{A}(s+1)} N(s+2;t); \\
\mathcal{P}(s;t) = \mathcal{P}(s;t-2) - \tfrac{\breve{A}(s)}{\breve{A}(s+1)} N(s+2;t-1) + N(s;t); \\
\mathcal{G}(s;t) = \mathcal{G}(s;t-2) + M(s;t-1) + \tfrac{N(s+2;t)}{\breve{A}(s+1)} [\mathcal{Q}(s+1;t-3)
\end{gathered}$$

$$
\begin{aligned}
&+N(s+1;t-2)-\tfrac{\breve{A}(s+1)}{\breve{A}(s+2)}N(s+3;t-1)+N(s+1;t)]\\
&-\tfrac{N(s+1;t)}{\breve{A}(s)}\left(\mathcal{Q}(s;t-2)+N(s;t-1)\right)-M(s+2;t);\\
&\mathcal{F}(s;t)=\mathcal{F}(s;t-2)+\tfrac{N(s+2;t-1)}{\breve{A}(s+1)}(\mathcal{Q}(s+1;t-2)\\
&+N(s+1;t-1))-\tfrac{N(s+1;t-1)}{\breve{A}(s)}\mathcal{P}(s;t-2)+M(s;t)\\
&-M(s+2;t-1);\\
&M(s+1;t+1)(\mathcal{P}(s;t)-\tfrac{\breve{A}(s)}{\breve{A}(s+1)}N(s+2;t+1))\\
&-\tfrac{\breve{A}(s)}{\breve{A}(s+1)}N(s+2;t+1)\mathcal{G}(s+1;t)+\mathcal{F}(s;t)N(s;t+1)\\
&+(\mathcal{Q}(s;t)+N(s;t+1))[\mathcal{F}(s;t)+\tfrac{N(s+2;t+1)}{\breve{A}(s+1)}(\mathcal{Q}(s+1;t)\\
&+N(s+1;t+1))-\tfrac{N(s+1;t+1)}{\breve{A}(s)}\mathcal{P}(s;t)-M(s+2;t+1)]=0;\\
&\mathcal{F}(s;t)M(s;t+1)+(\mathcal{G}(s;t)+M(s;t+1))[\tfrac{N(s+2;t+1)}{\breve{A}(s+1)}(\mathcal{Q}(s+1;t)\\
&+N(s+1;t+1))-\tfrac{N(s+1;t+1)}{\breve{A}(s)}\mathcal{P}(s;t)-M(s+2;t+1)]-\breve{\mu}(t)\\
&+\breve{\mu}(t+1)=0.
\end{aligned}
\tag{2.224}
$$

Remark 2.3 The system (2.222) (or (2.224)) is in our sense a certain extension of the one in Eq. (2.190), from the second-order situation to the fourth-order one. As the system (2.190) can serve as a starting point for the definition of the Laguerre-Hahn polynomials, it is legitimate to expect that the system (2.222) (or (2.224)) can serve as a starting point for the definition of a class of polynomials extending the usual Laguerre-Hahn polynomials. This question will not however be treated in this thesis.

Chapter 3

Discrete factorization techniques for the hypergeometric polynomials on lattices

The major concern of this chapter consists in revisiting the "eigenelement problem" (1.4), and its particular case (1.64) using the factorization technique approach. More precisely, we need to show practically that the operators in Eqs. (1.4) and (1.64) can on the one side be generated and on the other be solved using the factorization technique (see problem 1 and 2, in the introduction).

The cases of Charlier, Meixner and Kravchuk polynomials are firstly treated separately, to illustrate the performance of the quasi-periodicity method (section 3.1). Even if we performed those results independently, a q-version of the quasi-periodicity method was already used in [109] to generate the q-versions of the Charlier, Meixner and Kravchuk polynomials. Shape-invariant Infeld-Hull-Miller factorization technique is next used for solving the problems (1 and 2) relatively to the operator in Eqs. (1.4) and (1.64) (see section 3.2 and 3.3 respectively). Details are also given for each special class of polynomials. Also, difference hypergeometric functions generalizing the Askey-Wilson polynomials are given (see section 3.3).

3.1 Discrete factorization techniques for the Charlier, the Meixner and Kravchuk polynomials.

Let us write the system (2.118) under the constraints (2.140) and set for simplicity $j := 0$. We have

$$\begin{aligned} f_1(x+1)+g_1(x) &= f_0(x)+g_0(x)+\mu_0-\mu_1 \\ f_1(x)g_1(x) &= f_0(x)g_0(x-1) \\ \dots \quad &\dots \quad \dots \\ f_0(x+1-\delta)+g_0(x-\delta) &= f_{N-1}(x)+g_{N-1}(x)+\mu_{N-1}-\mu_N \\ f_0(x-\delta)g_0(x-\delta) &= f_{N-1}(x)g_{N-1}(x-1). \end{aligned} \tag{3.1}$$

For a given solution of that system, set

$$H_0 = (\mathbf{E}_x + g_0)\left(1 + f_0\mathbf{E}_x^{-1}\right) + \mu_0 \tag{3.2}$$

and write

$$R = R_{N-1}\dots R_0; \qquad L = L_0 \dots L_{N-1} \tag{3.3}$$

as given in Eq. (2.60). According to Eqs. (2.59) and (2.67), it is clear that, a solution of Eq. (3.1) having been found, for establishing polynomial eigenfunctions for H_0, it remains basically to find the starting function for the iterations. As we are searching polynomial eigenfunctions, we in preference suppose that the ladder is "bounded below" and search the starting function $\Phi_0(x)$ as a solution of

$$\begin{aligned} H_0\Phi_0(x) &= 0 \\ R\Phi_0(x) &\neq 0 \\ L\Phi_0(x-1) &= 0. \end{aligned} \tag{3.4}$$

3.1.1 The Charlier case.

For this case, we set $N = 1, \delta = 0$ and the system (3.1) becomes

$$\begin{aligned} f_0(x+1) &= f_0(x)+\mu_0-\mu_1 \\ g_0(x) &= g_0(x-1). \end{aligned} \tag{3.5}$$

A standard solving of it gives

$$\begin{aligned} f_0 &= (\mu_0-\mu_1)x+c_1 \\ g_0 &= c_0. \end{aligned} \tag{3.6}$$

On the other side, Eq. (3.4) reads

$$\begin{gathered}\left(\mathbf{E}_x + (f_0(x+1) + g_0(x) + \mu_0) + f_0(x)g_0(x)\mathbf{E}_x^{-1}\right)\Phi_0(x) = 0\\ \left(1 + f_0(x)\mathbf{E}_x^{-1}\right)\Phi_0(x) \neq 0\\ \left(\mathbf{E}_x + g_0(x)\right)\Phi_0(x-1) = 0,\end{gathered}\tag{3.7}$$

giving

$$\begin{gathered}\frac{\Phi_0(x+1)}{\Phi_0(x)} = -g_0(x+1)\\ g_0(x) \neq f_0(x)\\ (f_0(x+1) - g_0(x+1)) - (f_0(x) - g_0(x)) = -\mu_0.\end{gathered}\tag{3.8}$$

Loading the formulas (3.6) in (3.8) allows to fix the coefficients of f_0 and g_0:

$$\mu_1 = 2\mu_0; \qquad \mu_0 \neq 0 \tag{3.9}$$

and we will take for simplicity $\mu_0 = 1$, $\mu_1 = 2$.
Thus according to Eq. (2.59), we are led to

$$\begin{gathered}H_0\Phi_n(x) = \lambda_n\Phi_n(x)\\ \lambda_n = -n\\ \Phi_{n+1}(x) = \left(1 - (x - c_1)\mathbf{E}_x^{-1}\right)\Phi_n(x)\\ k_n\Phi_{n-1}(x) = \left(\mathbf{E}_x + c_0\right)\Phi_n(x), \qquad n \in Z^+.\end{gathered}\tag{3.10}$$

Then, using the equations in (3.10), we arrive to the three-term recurrence relations for the functions Φ_n (and so for the relating polynomials):

$$c_0\Phi_{n+1}(x) + (n - c_0)\Phi_n(x) + k_n\Phi_{n-1}(x) = (x - c_1)\Phi_n(x). \tag{3.11}$$

From the third equation in (3.10), we have that $\Phi_n(c_1) = constant$ (in x and n). As a consequence, we find from Eq. (3.11), $k_n = -n$. Up to a linear change of variable $X = x - c_1$, the obtained polynomials $\mathcal{C}_n(x) = \Phi_0^{-1}(x)\Phi_n(x)$ and second-order difference operator $\bar{H}_0 = \Phi_0^{-1}H_0\Phi_0$ are the Charlier ones (see for example the first chapter with $\breve{\mu} = -c_0$). *This solves simultaneously the "problem 1" and the "problem 2", for the Charlier polynomials.*

3.1.2 The Meixner and Kravchuk cases.

Here, we set $N = 2, \delta = 1$ and the system (3.1) becomes

$$\begin{aligned} f_1(x+1) + g_1(x) &= f_0(x) + g_0(x) + \mu_0 - \mu_1 \\ f_1(x) g_1(x) &= f_0(x) g_0(x-1) \\ f_0(x) + g_0(x-1) &= f_1(x) + g_1(x) + \mu_1 - \mu_2 \\ f_0(x-1) g_0(x-1) &= f_1(x) g_1(x-1). \end{aligned} \tag{3.12}$$

Solving it gives

$$\begin{aligned} f_0(x) &= c_0 \frac{\mu_2 - \mu_0}{1 - c_0} x + c_0 c_1 + c_0 \frac{\mu_1 - \mu_0}{c_0 - 1} \\ f_1(x) &= c_0 \frac{\mu_2 - \mu_0}{1 - c_0} x + c_1 c_0 - c_0 \frac{\mu_2 - \mu_0}{1 - c_0} \\ g_0(x) &= \frac{\mu_2 - \mu_0}{1 - c_0} x + c_1 \\ g_1(x) &= \frac{\mu_2 - \mu_0}{1 - c_0} x + c_1 + \frac{\mu_1 - \mu_0}{c_0 - 1}. \end{aligned} \tag{3.13}$$

The equations (3.4) now read

$$\begin{aligned} \left(\mathbf{E}_x + (f_0(x+1) + g_0(x) + \mu_0) + f_0(x) g_0(x) \mathbf{E}_x^{-1}\right) \Phi_0(x) = 0 \\ \left(1 + f_1(x) \mathbf{E}_x^{-1}\right) \left(1 + f_0(x) \mathbf{E}_x^{-1}\right) \Phi_0(x) \neq 0 \\ \left(\mathbf{E}_x + g_0(x)\right) \left(\mathbf{E}_x + g_1(x)\right) \Phi_0(x-1) = 0. \end{aligned} \tag{3.14}$$

Or equivalently

$$\begin{aligned} \tfrac{\Phi_0(x+1)}{\Phi_0(x)} + (f_0(x+1) + g_0(x) + \mu_0) + f_0(x) g_0(x) \tfrac{\Phi_0(x-1)}{\Phi_0(x)} = 0 \\ \tfrac{\Phi_0(x+1)}{\Phi_0(x)} + (f_0(x+1) + f_1(x+1)) + f_0(x) f_1(x+1) \tfrac{\Phi_0(x-1)}{\Phi_0(x)} \neq 0 \\ \tfrac{\Phi_0(x)}{\Phi_0(x-1)} = \tfrac{g_0(x) g_1(x) - g_0(x) f_0(x)}{f_0(x+1) + g_0(x) + \mu_0 - g_1(x+1) - g_0(x)}. \end{aligned} \tag{3.15}$$

Loading the solutions from Eq. (3.13) into the basic conditions (3.15) gives

$$\mu_2 = 2\mu_0; \qquad \mu_0 \neq 0; \qquad c_0 \neq 1 \tag{3.16}$$

(the relations (3.16) of course suffice as well for (3.13) to satisfy Eq. (3.15)) and we chose for convenience $\mu_0 = 1 - 1/c_0$, having in mind that c_0 does never equal to the unit. Let $\tilde{f}_0, \tilde{f}_1, \tilde{g}_0$ and $\tilde{g}_1$ be the corresponding solutions from Eq. (3.13). So, again, according to Eqs. (2.59) and (2.67), we are led to the following relations

$$
\begin{aligned}
H_0\Phi_n(x) &= \lambda_n\Phi_n(x)\\
\lambda_n &= -n(1-\tfrac{1}{c_0}), \qquad n\in Z^+\\
\Phi_{n+1}(x) &= \left(\mathbf{E}_x + (\tilde f_1(x+1)+\tilde f_0(x+1)) + \tilde f_1(x+1)\tilde f_0(x))\mathbf{E}_x^{-1}\right)\Phi_n(x)\\
d_n\Phi_{n-1}(x) &= \left(\mathbf{E}_x + (\tilde g_1(x+1)+\tilde g_0(x)) + \tilde g_1(x)\tilde g_0(x))\mathbf{E}_x^{-1}\right)\Phi_n(x)
\end{aligned} \tag{3.17}
$$

Next, using the equations in (3.17), we obtain the following three-term recurrence relations for the eigenfunctions $\Phi_n(x)$ (and for the corresponding polynomials):

$$
\alpha(n)\Phi_{n+1}(x) + \beta(n)\Phi_n(x) + \gamma(n)\Phi_{n-1}(x) = -x(1-\frac{1}{c_0})\Phi_n(x) \tag{3.18}
$$

where

$$
\begin{aligned}
\alpha(n) = \tfrac{1}{1-c_0}; \qquad \gamma(n) = d_n\tfrac{c_0}{1-c_0};\\
\beta(n) = \tfrac{c_0(c_0^2c_1+c_0\mu_1-c_0-2c_0c_1+2+c_1)-1+(c_0^2-1)n}{c_0(1-c_0)}.
\end{aligned} \tag{3.19}
$$

(There is of course self-duality between Eq. (3.18) and the first equation in (3.17) (up to similarity reductions)). For determining d_n, we first fix $\mu_1 = (1-1/c_0)(1-c_0c_1)$ (for this choice, the coefficients of $\mathbf{E}_x^{-1}$ in all the equations in (3.17) become "multiple" of x) and then set $x=0$ into the four equations in (3.17) and (3.18) (after which the evoked coefficients of course vanish). As a result, we can evaluate d_n in standard ways and find $d_n = (1-1/c_0)^2 n(n-c_0c_1)$. The resulting polynomials $\mathcal{M}_n(x) = \Phi_0^{-1}\Phi_n(x)$ and second-order difference operator $\bar H_0 = \Phi_0^{-1}H_0\Phi_0$ are the Meixner ones (see for example the second chapter with $\mu = 1/c_0, \gamma = 1-c_0c_1$), up to a multiplication by $(\mu-1)^n$. The Kravchuk polynomials are obtained formally from the Meixner ones by setting $\gamma=-N; \mu=\frac{p}{p-1}$. *This of course solves simultaneously the "problem 1" and the "problem 2", for the Meixner and Kravchuk polynomials.*

3.2 Discrete factorization techniques for the hypergeometric polynomials on linear lattices.

3.2.1 The general case.

Consider the hypergeometric operator on linear lattice $y(s) = s$ (see Eq. (1.64))

$$L = \sigma(x)\Delta\nabla + \tau(x)\Delta \tag{3.20}$$

where

$$\sigma(x) = \sigma_0 x^2 + \sigma_1 x + \sigma_2; \tau(x) = \tau_0 x + \tau_1. \tag{3.21}$$

We wish to use the factorization technique for studying the eigenelement problem for L. We particularly show how to solve types 1 and 2 problems (see introduction) for that operator.
Let

$$\begin{aligned} H(x;t) &= \mathbf{E}_x.[\varrho(x)(L-\lambda(t))\varrho^{-1}(x)] \\ &= \mathbf{E}_x^2 - (\sigma(x+1)+\tau(x+1)+\lambda(t))\mathbf{E}_x + (\sigma(x)+\tau(x))\sigma(x+1) \end{aligned} \tag{3.22}$$

where

$$\varrho(x+1) = (\sigma(x)+\tau(x))\varrho(x), \tag{3.23}$$

$\lambda(t)$ a function to be determined later.
We can show the following

Lemma 3.1 *There exists functions $\lambda(t)$, $\mu(t)$, $f(x;t)$ and $g(x;t)$ such that the operator $H(x;t)$ factorizes into*

$$\begin{aligned} H(x;t) - \mu(t) &= (\mathbf{E}_x + g(x;t))(\mathbf{E}_x + f(x;t)) \\ H(x;t+1) - \mu(t) &= (\mathbf{E}_x + f(x;t))(\mathbf{E}_x + g(x;t)). \end{aligned} \tag{3.24}$$

Proof. The factorization (3.24) can be obtained by searching the unknowns $\lambda(t)$, $\mu(t)$, $f(x;t)$ and $g(x;t)$ in the equations

$$\begin{aligned} f(x+1;t) + g(x;t) &= -(\sigma(x+1)+\tau(x+1)+\lambda(t)) \\ f(x;t)g(x;t) &= (\sigma(x)+\tau(x))\sigma(x+1) - \mu(t) \\ \Delta(f-g) &= \lambda(t+1) - \lambda(t). \end{aligned} \tag{3.25}$$

The two first equations in (3.25) suggest to search f and g under the forms

$$\begin{aligned} f(x;t) &= -\sigma(x) - \tau(x) - \frac{1}{2}\lambda(t) + \varphi(x;t) \\ g(x;t) &= -\sigma(x+1) - \frac{1}{2}\lambda(t) + \varphi(x+1;t) \end{aligned} \tag{3.26}$$

where $\varphi(x;t) = \phi(t)x + \psi(t)$. The corresponding system in the variables $\phi(t)$, $\psi(t)$, $\lambda(t)$ and $\mu(t)$ resulting in the comparison of the coefficients in x is compatible and we find

$$\begin{aligned}
\phi(t) &= \frac{1}{2}\tau_0 - \sigma_0 - \frac{1}{2}\lambda(t) + \frac{1}{2}\lambda(t+1) \\
\psi(t) &= \frac{\phi(t)(\tau_1 + \tau_0 - \sigma_0 - \phi(t)) + (\sigma_0 + \sigma_1 + \frac{1}{2}\tau_0)\lambda(t)}{2\phi(t) + 2\sigma_0 - \tau_0} \\
\mu(t) &= \psi(t)(\psi(t) + \phi(t) + \sigma_1 + \sigma_0 - \tau_1) - \frac{1}{2}\lambda(t)(\sigma_0 + \sigma_1 \\
&\quad +2\sigma_2 + \tau_1) - \phi(t)(\sigma_2 + \tau_1 + \frac{1}{2}\lambda(t)) - \frac{1}{4}\lambda^2(t),
\end{aligned} \tag{3.27}$$

while $\lambda(t)$ satisfies

$$\lambda^2(t+1) - 2(\sigma_0 + \lambda(t))\lambda(t+1) + \lambda^2(t) - 2\sigma_0\lambda(t) + 2\sigma_0\tau_0 - \tau_0^2 = 0 \tag{3.28}$$

which proves the lemma.
The preceding lemma allows to prove the following

Proposition 3.1 *The operator L in Eq. (3.20) admits a sequence of non-trivial eigenfunctions satisfying difference and three-term recurrence relations. In particular, that operator admits necessarily a sequence of polynomial eigenfunctions $P_t(x)$ of degree $t = 0, 1, 2, \ldots$.*

Proof. Supposing that $\Phi_t(x)$ is an eigenfunction of L corresponding to $\lambda(t)$ from Eq. (3.28), so the functions $\Phi_{t+1}(x)$ obtained using Eqs. (3.24) and (3.23) as satisfying the relations

$$\begin{aligned}
\Phi_{t+1}(x) &= [(\sigma(x) + \tau(x))\mathbf{E}_x + f(x;t)]\,(\Phi_t(x)) \\
-\mu(t)\Phi_t(x) &= [(\sigma(x) + \tau(x))\mathbf{E}_x + g(x;t)]\,(\Phi_{t+1}(x))
\end{aligned} \tag{3.29}$$

are eigenfunctions (if not zeros) of L corresponding to $\lambda(t+1)$. A combination of the equations in (3.29) gives

$$\Phi_{t+1}(x) + \mu(t-1)\Phi_{t-1}(x) = (f(x;t) - g(x;t-1))\Phi_t(x). \tag{3.30}$$

A non-trivial (i.e. non-vanishing) starting function Φ_0 having been chosen, the condition $\mu(0) \neq 0$ (this condition is satisfied for a general solution of Eq. (3.28) as it is satisfied for example for $\lambda'(t)$(see Eq. (3.31) below)) guarantees the fact that Φ_1 is also non-trivial, after which Eq. (3.30) leads to the expected sequence.

For searching the polynomial eigenfunctions, let us return to the equation for $\lambda(t)$ in (3.28). Polynomials that we are looking for are given on linear lattice. We can for this reason suppose that the polynomials dual to them are given "at most" on non-linear lattice $y(s) = \tilde{c}_0 s^2 + \tilde{c}_1 s + \tilde{c}_2$ (this is of course in reality correct). As a consequence, the eigenvalue $\lambda(t)$ needs to be searched under the form $\lambda(t) = c_0 t^2 + c_1 t + c_2$. Loading this in Eq. (3.28), we obtain

$$\lambda^\iota(t) = \sigma_0 t^2 + [(\tau_0 - \sigma_0)^2 + 4\sigma_0 \iota]^{\frac{1}{2}} t + \iota. \tag{3.31}$$

where ι is a free parameter. The constraint of existence of constant eigenfunction of L (the polynomial of degree zero) leads to ($\iota = 0$):

$$\lambda^0(t) = \tau_0 t + \sigma_0 t(t-1) \tag{3.32}$$

(one can of course find this classically using L).
Let $f^0(x;t)$, $g^0(x;t)$, $\phi^0(t)$, $\psi^0(t)$ and $\mu^0(t)$ be the corresponding functions in Eqs. (3.26) and (3.27). We have according to Eq. (3.26)

$$f^0(x;t) - g^0(x;t-1) = c_3(t)x + c_4(t) \tag{3.33}$$

where

$$\begin{aligned} c_3(t) &= \phi^0(t) + \phi^0(t-1) - \tau_0 + 2\sigma_0 \\ c_4(t) &= \psi^0(t) + \psi^0(t-1) + \phi^0(t-1) + \sigma_1 - \tau_1 + \sigma_0 \\ &\quad + \tfrac{1}{2}(\lambda^0(t-1) - \lambda^0(t)). \end{aligned} \tag{3.34}$$

Consequently, according to Eqs. (3.29), (3.30), the polynomials that we are searching for satisfy the difference relations (see Eq. (2.177))

$$\begin{aligned} c_3(t)a_{t+1}P_{t+1}(x) &= [(\sigma(x) + \tau(x))\mathbf{E}_x + f(x;t)]\,(P_t(x)) \\ -c_3(t+1)a_{t+1}P_t(x) &= [(\sigma(x) + \tau(x))\mathbf{E}_x + g(x;t)]\,(P_{t+1}(x)) \end{aligned} \tag{3.35}$$

and the three-term recurrence relations

$$a_{t+1}P_{t+1} + b_t P_t + a_t P_{t-1} = xP_t \tag{3.36}$$

where

$$b_t = -\frac{c_4(t)}{c_3(t)}; \qquad a_t^2 = \frac{\mu^0(t-1)}{c_3(t)c_3(t-1)}. \tag{3.37}$$

Thus we can take $P_0 = constant$, $P_1(x)$ following the first equation in (3.35) and $P_t(x), t = 2, 3, \ldots$ following Eq. (3.36). However, we must keep in mind some additional constraint: the polynomials $P_t(x), t = 0, 1, 2, 3, \ldots$ must be of degree t exactly (or equally, the eigenvalues $\lambda(t)$ must be mutually-different i.e. $\lambda(t_1) \neq \lambda(t_2)$ for $t_1 \neq t_2$). It can be seen that that condition is satisfied iff

$$\tau_0 + m\sigma_0 \neq 0, m = -2, -1, 0, 1, 2, 3 \ldots \tag{3.38}$$

This completes the proof of the proposition. At the same time, *the "problem 1" for the hypergeometric polynomials on linear lattices is solved.*
For solving the problem 2, we need to make a somewhat converse way. Recall that the problem in itself consists in generating from solutions of the factorization chain (2.118) a second-order difference operator having a complete set of polynomial eigenfunctions. Let us remark first that as the functions $f^0(x;t)$ and $g^0(x;t)$ from the preceding proposition are solutions of the equations in (3.25), so they are also solutions of a type (2.118) factorization chain. As a consequence, according to the proposition 2.1, the operator

$$\tilde{L} = g^0(x;-1)\mathbf{E}_x - (f^0(x;t) + g^0(x-1;t)) + f^0(x-1;-1)\mathbf{E}_x^{-1} \tag{3.39}$$

admits a sequence of polynomial eigenfunctions. On the other side, one easily verifies that

$$g^0(x;-1) = -(\sigma(x) + \tau(x)); \qquad f^0(x-1;-1) = -\sigma(x) \tag{3.40}$$

Taking into account next of the first equation in (3.25), we obtain that

$$\tilde{L} = -(L - \lambda(t)). \tag{3.41}$$

As we showed in the preceding that the operator L admits a complete set of polynomial eigenfunctions, *this solves the "problem 2" for the hypergeometric polynomials on linear lattices (and the polynomials dual to them (such as dual Hahn), according to the remark 2.1).*

Remark 3.1 This solution of the problem 2 is found by converting the way followed for solving the problem 1. In section 6.2, we will show that a solution to the problem 2 can be given more originally i.e. starting more purely from a type (2.118) factorization chain.

Remark 3.2 If we consider the functions $f^\iota(x;t)$ and $g^\iota(x;t)$ in Eq. (3.26) corresponding to $\lambda^\iota(t)$ in Eq. (3.31), then, according to the proposition 2.1, the operator

$$g^\iota(x;-1)\mathbf{E}_x - (f^\iota(x;t) + g^\iota(x-1;t)) + f^\iota(x-1;-1)\mathbf{E}_x^{-1} \tag{3.42}$$

admits a sequence of polynomial solutions as well but now depending to an additional parameter ι. A simple verification shows however that this parameter (ι) is inessential and the concerned polynomials are essentially the already evoked ones. Indeed, we note first that the expression in center of Eq. (3.42) can be written as

$$\begin{aligned} f^\iota(x;t) + g^\iota(x-1;t) &= f^\iota(x;0) + g^\iota(x-1;0) - (\lambda^\iota(t) - \lambda^\iota(0)) \\ &= g^\iota(x;-1) + f^\iota(x-1;-1) - (\lambda^\iota(t) - \lambda^\iota(0)). \end{aligned} \tag{3.43}$$

As a consequence, the operator in Eq. (3.42) is also a type $-(L - \lambda^0(t))$ operator with

$$\begin{aligned} \widehat{\sigma}(x) := -f^\iota(x-1;-1); \widehat{\tau}(x) := -(g^\iota(x;-1) - f^\iota(x-1;-1)) \\ \widehat{\lambda^0}(t) = \lambda^\iota(t) - \lambda^\iota(0). \end{aligned} \tag{3.44}$$

However, let us remark that if the eigenfunction of L corresponding to $\lambda^\iota(0)$ is known, one should generate a set of eigenfunctions (not of polynomial type) of L generalizing (in ι) the hypergeometric polynomials. Similar question will be treated in the following section for the hypergeometric difference operator on q-nonlinear lattices.

3.2.2 The concrete cases.

The polynomial eigenfunctions of L studied here can be given by a "Rodrigues type formula"

$$P_t(x) = \frac{c(t)}{\varrho(x)} \prod_{i=0}^{t-1} (E + f(x;t-1-i))\varrho(x), \tag{3.45}$$

$c(t)$ being some constant (in x), easily determined from Eq. (3.35). They satisfy type (3.35) difference relations and three-term recurrence relations in Eq. (3.36). They constitute the hypergeometric polynomials or equally the classical polynomials on linear lattices. Data necessary for the factorization of the hypergeometric operator on linear lattice, for the concrete cases of Charlier, Meixner (the Kravchuk case is found by replacing $\gamma = -N, \mu = \frac{p}{p-1}$) and Hahn polynomials are displayed in Table 3.1, Table 3.2 and Table 3.3 respectively. One will note for example the evoked in Eqs. (2.188)-(2.189) "self-duality" of $f(x;t)$ and $g(x;t)$ for the Charlier and Meixner polynomials.

Table 3.1: Data for the Charlier case

$H(x;t)$	$E^2-(x+\mu+\lambda(t)+1)E+\mu(x+1)$
$\varrho(x)$	μ^x
$f(x;t)$	$-x+t$
$g(x;t)$	$-\mu$
$\mu(t)$	$\mu t+\mu$
$\lambda(t)$	t

Table 3.2: Data for the Meixner case

$H(x;t)$	$E^2-[(\mu+1)x+\mu(\gamma+1)+1+\lambda(t)]E+\mu x^2+\mu(\gamma+1)x+\gamma\mu$
$\varrho(x)$	$\mu^x\Gamma(x+\gamma)$
$f(x;t)$	$-x+t$
$g(x;t)$	$-\mu(x+\gamma+t+1)$
$\mu(t)$	$\mu(t\gamma+t^2+t+\gamma)$
$\lambda(t)$	$t(1-\mu)$

3.3 Discrete factorization techniques for the hypergeometric polynomials on q-nonlinear lattices.

3.3.1 Discrete factorization techniques for the Askey-Wilson polynomials.

Consider now the hypergeometric operator on q-nonlinear lattice $y(s)=\frac{q^s+q^{-s}}{2}$ (recall that for μ finite, this is "equivalent" to the general case in Eq. (9)) (see Eq. (1.4)):

$$\pounds=\sigma(s)\frac{\Delta}{\Delta y(s-\frac{1}{2})}.\frac{\nabla}{\nabla y(s)}+\tau(s)\frac{\Delta}{\Delta y(s)} \tag{3.46}$$

The preceding study of the eigenelements problem for the operator L is now going to be extended to that of $\pounds$.
Let us rewrite the operator $\pounds$ under the form

$$\frac{1}{\Delta y(s-\frac{1}{2})}.\left[\frac{\hat{\sigma}(s)}{\Delta y(s)}\mathbf{E}_s-\left(\frac{\sigma_A(s)}{\Delta y(s)}+\frac{\sigma(s)}{\nabla y(s)}\right)+\frac{\sigma(s)}{\nabla y(s)}\mathbf{E}_s^{-1}\right] \tag{3.47}$$

Table 3.3: Data for the Hahn case

$H(x;t)$	$E^2+[2x^2+(6+\beta-\alpha-2N)x+(5+2\beta-\alpha-3N-\beta N$ $-\lambda(t))]E+[x^4+(4+\beta-\alpha-2N)x^3+(6+3\beta-3\alpha-6N$ $+N^2-2N\beta+\alpha N-\alpha\beta)x^2+(4+3\beta-3\alpha-6N+2N^2$ $-4N\beta+2N\alpha-2\alpha\beta+N^2\beta+N\alpha\beta)x+1+\beta-\alpha-2N$ $+N^2-2N\beta+\alpha N-\alpha\beta+N^2\beta+N\alpha\beta$
$\varrho(x)$	$\frac{\Gamma(x+\beta+1)}{\Gamma(-x+N)}$
$f(x;t)$	$x^2-(N+\alpha+t-1)x-(\beta+1)(N-1)-\frac{1}{2}\lambda(t)+\psi(t)$
$g(x;t)$	$x^2+(3+t+\beta-N)x+2+\beta+t-N-\frac{1}{2}\lambda(t)-\psi(t)$
$\mu(t)$	$\psi(t)(\psi(t)-1-\beta N-t)-\frac{1}{2}\lambda(t)(\beta+1)(N-1)$ $-\frac{1}{2}\lambda(t)(N+\alpha-1)+(t+\alpha+\beta+1)(\beta+1)(N-1)$ $+\frac{1}{4}\lambda(t)(t+2)(t+\alpha+\beta+1)$
$\lambda(t)$	$t(\alpha+\beta+t+1)$
$\psi(t)$	$\frac{(t+\alpha+\beta+1)(\beta+1)(N-1)-\lambda(t)(N+\alpha)+\frac{1}{2}\lambda(t)(\alpha+\beta+2)}{2+2t+\alpha+\beta}$

where

$$\hat{\sigma}(s)=\sigma(-s). \tag{3.48}$$

Evaluating

$$\begin{aligned}\tilde{\sigma}(y(s)) &= c_1q^{2s}+c_2q^s+c_3+c_2q^{-s}+c_1q^{-2s}\\ \frac{1}{2}\tilde{\tau}(y(s))\Delta y(s-\frac{1}{2}) &= c_4q^{2s}+c_5q^s-c_5q^{-s}-c_4q^{-2s}\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}\Delta y(s-\frac{1}{2}) &= c_6(q^s-q^{-s})\\ \Delta y(s) &= c_7(qq^s-q^{-s})\\ \nabla y(s) &= c_7(q^s-qq^{-s}).\end{aligned} \tag{3.50}$$

where c_i, $i=1,\ldots,7$ are some constants (in s) and letting s_1, s_2, s_3 and s_4 be the (set, mutually-different) roots of $\sigma(s)$, one verifies that, up to a multiplication by a constant, the operator in Eq. (3.47) can be written as

$$\tilde{\mathcal{L}}=\frac{1}{z-z^{-1}}\left(\mathcal{A}(z)\mathbf{E}_q-[\mathcal{A}(z)+\mathcal{B}(z)]+\mathcal{B}(z)\mathbf{E}_q^{-1}\right) \tag{3.51}$$

where

$$\begin{gathered}
\mathcal{A}(z) = \frac{A_{-2}z^{-2}+A_{-1}z^{-1}+A_0+A_1z+A_2z^2}{qz-z^{-1}} \\
\mathcal{B}(z) = \frac{A_2z^{-2}+A_1z^{-1}+A_0+A_{-1}z+A_{-2}z^2}{z-qz^{-1}} \\
A_{-2}=1;\, A_{-1}=-(a+b+c+d);\, A_0 = ab+ac+ad+bc+bd+cd \\
A_1 = -(abc+abd+bcd+acd);\, A_2 = abcd \\
z = q^s \\
a=q^{s_1};\, b=q^{s_2};\, c=q^{s_3};\, d=q^{s_4} \\
\mathbf{E}_q^i(h(\chi(z))) = h(\chi(q^iz)),\, i\in Z \\
\chi(z) = \frac{z+z^{-1}}{2}.
\end{gathered} \tag{3.52}$$

The operator $\tilde{\mathcal{L}}$ can also be written as [62]:

$$\begin{gathered}
\tilde{\mathcal{L}}(z) = v(z)\mathbf{E}_q - (v(z)+v(z^{-1})) + v(z^{-1})\mathbf{E}_q^{-1}, \\
v(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.
\end{gathered} \tag{3.53}$$

Letting $\mathcal{D}_q$,

$$\begin{gathered}
\mathcal{D}_q h(\chi(z)) = \frac{h(\chi(q^{\frac{1}{2}}z)) - h(\chi(q^{-\frac{1}{2}}z))}{\chi(q^{\frac{1}{2}}z)-\chi(q^{-\frac{1}{2}}z)}, \\
\chi(z) = \frac{z+z^{-1}}{2},
\end{gathered} \tag{3.54}$$

be the Askey-Wilson first-order divided difference operator [16], one can also write $\tilde{\mathcal{L}}$ as:

$$\tilde{\mathcal{L}} = \left[\frac{(q-1)^2}{4q}\frac{z^2-1}{z\omega(z)}\mathcal{D}_q\frac{z^2-1}{z}v(q^{-\frac{1}{2}}z)\omega(q^{-\frac{1}{2}}z)\right]\mathcal{D}_q, \tag{3.55}$$

where

$$\frac{\omega(qz)}{\omega(z)} = \frac{v(z)}{v((qz)^{-1})}. \tag{3.56}$$

The Askey-Wilson polynomials $\mathcal{P}_n(\chi(z))$ (see chapter 1), satisfy the second order q-difference equation:

$$\tilde{\mathcal{L}}\mathcal{P}_n(\chi(z)) = \lambda(n)\mathcal{P}_n(\chi(z)) \tag{3.57}$$

where

$$\lambda(n) = -(1-q^{-n})(1-abcdq^{n-1}). \tag{3.58}$$

The operator $\tilde{\mathcal{L}}$ was shown in [60] to be symmetric (more precisely, the operator in square brackets, in the r.h.s. of Eq. (3.55), is adjoint to the first-order Askey-Wilson divided difference operator) in the space $S_{a,b,c,d}$ of real polynomials in $\chi(z)$ with the inner product

$$(h_1, h_2) := \frac{1}{2\pi i} \oint_C h_1(\chi(z)) h_2(\chi(z)) \omega(z) \frac{dz}{z}, \tag{3.59}$$

where $\omega(z)$ is given by Eq. (3.56) and C is a deformation of the unit circle. Considering the representation in Eq. (3.55) and the equation in (3.57), we will refer to $\tilde{\mathcal{L}}$, everywhere in this thesis, as the *Askey-Wilson second order q-difference operator.*

It is clear that the $\lambda(t)$-eigenfunctions of $\tilde{\mathcal{L}}$ are exactly the zero-functions of the operator

$$\tilde{\tilde{\mathcal{L}}} = \left(z - z^{-1}\right)\left(\tilde{\mathcal{L}} - \lambda(t)\right). \tag{3.60}$$

Let

$$\begin{aligned} \mathcal{H}(z;t) &= \mathbf{E}_q . \hat{\varrho}(z) \tilde{\tilde{\mathcal{L}}} \hat{\varrho}^{-1}(z) \\ &= \mathbf{E}_q^2 - [\mathcal{A}(qz) + \mathcal{B}(qz) + \mathcal{K}(qz)\lambda(t)]\mathbf{E}_q + \mathcal{B}(qz)\mathcal{A}(z) \end{aligned} \tag{3.61}$$

where

$$\begin{aligned} \hat{\varrho}(qz) &= \mathcal{A}(z)\hat{\varrho}(z) \\ \mathcal{K}(z) &= z - z^{-1}. \end{aligned} \tag{3.62}$$

We can prove the following

Lemma 3.2 *There exists functions $\lambda(t)$, $\mu(t)$, $\mathcal{F}(z;t)$ and $\mathcal{G}(z;t)$ such that the operator $\mathcal{H}(z;t)$ factorizes into*

$$\begin{aligned} \mathcal{H}(x;t) - \mu(t) &= (\mathbf{E}_q + \mathcal{G}(z;t))(\mathbf{E}_q + \mathcal{F}(z;t)) \\ \mathcal{H}(x;t+1) - \mu(t) &= (\mathbf{E}_q + \mathcal{F}(z;t))(\mathbf{E}_q + \mathcal{G}(z;t)). \end{aligned} \tag{3.63}$$

Proof. The operatorial relations (3.63) are equivalent to the system:

$$\begin{aligned} \mathcal{F}(qz;t) + \mathcal{G}(z;t) &= -(\mathcal{A}(qz) + \mathcal{B}(qz) + \mathcal{K}(qz)\lambda(t)) \\ \mathcal{F}(z;t)\mathcal{G}(z;t) &= \mathcal{A}(z)\mathcal{B}(qz) - \mu(t) \\ \Delta_q(\mathcal{F}(z;t) - \mathcal{G}(z;t)) &= (\lambda(t+1) - \lambda(t))\mathcal{K}(qz) \end{aligned} \tag{3.64}$$

where $\Delta_q h(z) = h(qz) - h(z)$.
Using the q-integration, we first transform the system (3.64) in :

$$\begin{aligned}
\mathcal{F}(z;t) + \mathcal{F}(qz;t) = &\quad -(\mathcal{A}(qz) + \mathcal{B}(qz)) - (\beta_{-1} - \tfrac{\lambda(t)}{q})z^{-1} - \beta_0 \\
& \quad -(\beta_1 + \lambda(t)q)z \\
\mathcal{F}(z;t)\mathcal{G}(z;t) = &\quad \mathcal{A}(z)\mathcal{B}(qz) - \mu(t) \\
\mathcal{G}(z;t) - \mathcal{F}(z;t) = &\quad \textstyle\sum_{i=-1}^{i=1} \beta_i z^i
\end{aligned} \tag{3.65}$$

where $\beta_{-1} = \frac{\lambda(t+1)-\lambda(t)}{1-q}$; $\beta_1 = q\beta_{-1}$; β_0 remaining arbitrary for the moment. Observing the first equation in (3.65) and then using the last one, it becomes sensible to search $\mathcal{F}(z;t)$ and then $\mathcal{G}(z;t)$ under the forms:

$$\begin{aligned}
\mathcal{F}(z;t) &:= \frac{F_{-2}z^{-2}+F_{-1}z^{-1}+F_0+F_1z+F_2z^2}{qz-z^{-1}} \\
\mathcal{G}(z;t) &:= \frac{(F_{-2}-\beta_{-1})z^{-2}+(F_{-1}-\beta_0)z^{-1}+F_0+(F_1+\beta_0 q)z+(F_2+\beta_1 q)z^2}{qz-z^{-1}}
\end{aligned} \tag{3.66}$$

Taking λ, μ, β_0, F_{-2}, F_{-1}, F_0, F_1, F_2 as unknowns, the system (3.64) will then be transformed in an algebraic system of 16 equations for 8 unknowns. To solve it (*by hand*), up to λ excluded, one first determines the last 7 unknowns from 7 equations as functions of λ and then very delicately ensures himself that they satisfy other 8 equations, while λ satisfies its own equation. The result is:

$$\begin{aligned}
F_{-2}(t) &: \quad \frac{\lambda(t)-q\lambda(t+1)}{q^2-1} - \frac{q+A_2}{q^2+q} \qquad (3.67)\\
F_2(t) &: \quad \frac{\lambda(t)q-\lambda(t+1)}{1-q^2}q^2 - \frac{q^2+qA_2}{q+1} \\
\beta_0(t) &: \quad \frac{1-q}{(\lambda(t)-\lambda(t+1))q^3}\{(2\frac{\lambda(t)q-\lambda(t+1)}{1-q^2}q^2 + \frac{\lambda(t+1)-\lambda(t)}{1-q}q^2 \\
&\quad -2\frac{q^2+qA_2}{1+q})(A_1+qA_{-1}) + (2A_1q^2 + 2A_2A_{-1}q)\} \\
F_{-1}(t) &: \quad \frac{\beta_0(t)}{2} - \frac{A_1+qA_{-1}}{2q} \\
F_1(t) &: \quad -\frac{q\beta_0(t)}{2} - \frac{A_1+qA_{-1}}{2} \\
F_0(t) &: \quad \frac{1}{q+q^2}\{q^2 - q^3 - A_0(q+q^2) + A_2(q-1) + q^2(\lambda(t)+\lambda(t+1))\} \\
\mu(t) &: \quad A_0 + A_1A_{-1}q^{-1} + A_0A_2q^{-2} + F_0(t)\beta_{-1}(t) + F_{-1}(t)\beta_0(t) \\
&\quad -2F_{-2}(t)F_0(t) - F_{-1}^2(t).
\end{aligned}$$

and

$$\begin{aligned}
& q^2(\lambda(t)-q\lambda(t+1))^2 - q^2(q+1)(\lambda(t)-q\lambda(t+1))(\lambda(t) \\
& -\lambda(t+1)) - 2q(q-1)(q+A_2)(\lambda(t)-q\lambda(t+1)) + q(q^2-1) \\
& (q+A_2)(\lambda(t)-\lambda(t+1)) + (1-A_2)(q-1)^2(q^2-A_2) = 0,
\end{aligned} \tag{3.68}$$

which proves the lemma.
Using this lemma, we prove the following

Proposition 3.2 *The operator $\tilde{\mathcal{L}}$ in Eq. (3.51) (or equally $\mathcal{L}$ in Eq. (3.46) admits a sequence of non-trivial eigenfunctions satisfying difference and three-term recurrence relations. In particular, that operator admits necessarily a sequence of polynomial eigenfunctions $\mathcal{P}_t(\chi(z))$ of degree $t = 0, 1, 2, \ldots$.*

Proof. The proof of this proposition is similar to that given for the proposition 3.1, reason for which we give it only under a sketched form. The eigenfunctions Ψ_{t+1} of $\tilde{\mathcal{L}}$ corresponding to $\lambda(t)$ given by Eq. (3.68) satisfy the difference relations

$$\begin{aligned} \Psi_{t+1}(z) &= [\mathcal{A}(z)\mathbf{E}_q + \mathcal{F}(z;t)]\,(\Psi_t(z)) \\ -\mu(t)\Psi_t(z) &= [\mathcal{A}(z)\mathbf{E}_q + \mathcal{G}(z;t)]\,(\Psi_{t+1}(z)) \end{aligned} \tag{3.69}$$

from which we deduce

$$\Psi_t(z) + \mu(t-1)\Psi_{t-1}(z) = (\mathcal{F}(z;t) - \mathcal{G}(z;t-1))\Psi_t(z). \tag{3.70}$$

A non-trivial (i.e. non-vanishing) starting function Ψ_0 having been chosen, the condition $\mu(0) \neq 0$ (this condition is satisfied for a general solution of Eq. (3.68) as it is satisfied for example for $\lambda^{\varepsilon}(t)$(see Eq. (3.71) below)) guarantees the fact that Ψ_1 is also non-trivial, after which Eq. (3.70) leads to the expected sequence.
The eigenvalue $\lambda(t)$ corresponding to polynomial eigenfunctions is now searched under the form $\lambda(t) = c_0 q^t + c_1 q^{-t} + c_2$ (since the dual polynomials are expected to be given on q-nonlinear lattice $y(s) = \tilde{c}_0 q^s + \tilde{c}_1 q^{-s} + \tilde{c}_2$). Loading this in Eq. (3.68), we obtain

$$\lambda^{\varepsilon}(t) = -(1 - \varepsilon q^{-t})(1 - A_2 \varepsilon^{-1} q^{t-1}) \tag{3.71}$$

where ε is a free parameter. The constraint of existence of constant eigenfunction (the polynomial of degree zero) gives ($\varepsilon = 1$):

$$\lambda^1(t) = -(1 - q^{-t})(1 - A_2 q^{t-1}). \tag{3.72}$$

Letting $\mathcal{F}^1(z;t)$, $\mathcal{G}^1(z;t)$ and $\mu^1(t)$ be the corresponding solutions of Eq. (3.64), we obtain

$$\mathcal{F}^1(z;t) - \mathcal{G}^1(z;t-1) = c_8(t)\chi(z) + c_9(t) \tag{3.73}$$

where

$$\begin{aligned} c_8(t) &= 2A_2 q^{-1} q^t - 2q^{-t} \\ c_9(t) &= -\frac{\beta_0(t)+\beta_0(t-1)}{2}, \quad (\forall t \in Z). \end{aligned} \tag{3.74}$$

In particular,

$$\begin{aligned} const.\mathcal{P}_1(\chi(z)) &= (\mathcal{A}(z)\mathbf{E}_q + \mathcal{F}^1(z;0))(1) \\ &= [2A_2 q^{-1} - 2]\chi(z) - \frac{\beta_0(0)}{2} + \frac{A_1 - qA_{-1}}{2q}. \end{aligned} \tag{3.75}$$

As a consequence, the polynomials that we are searching for satisfy the difference relations (see Eq. (2.177))

$$\begin{aligned} c_8(t)a_{t+1}\mathcal{P}_{t+1}(\chi(z)) &= \left[\mathcal{A}(z)\mathbf{E}_q + \mathcal{F}^1(z;t)\right](\mathcal{P}_t(\chi(z))) \\ -c_8(t+1)a_{t+1}\mathcal{P}_t(\chi(z)) &= \left[\mathcal{A}(z)\mathbf{E}_q + \mathcal{G}^1(z;t)\right](\mathcal{P}_{t+1}(\chi(z))) \end{aligned} \tag{3.76}$$

and the three-term recurrence relations

$$a_{t+1}\mathcal{P}_{t+1} + b_t\mathcal{P}_t + a_t\mathcal{P}_{t-1} = \chi(z)\mathcal{P}_t \tag{3.77}$$

where

$$b_t = -\frac{c_9(t)}{c_8(t)}; \qquad a_t^2 = \frac{\mu^1(t-1)}{c_8(t)c_8(t-1)}. \tag{3.78}$$

We thus (using Eqs. (3.75) and (3.77)) reach the expected set of polynomial eigenfunctions $\mathcal{P}_t(\chi(z))$, $t = 0, 1, 2, \ldots$ of $\tilde{\mathcal{L}}$. To guarantee that $\mathcal{P}_t$ be of degree equal exactly to t or equally $\lambda(t_1) \neq \lambda(t_2)$, for $t_1 \neq t_2$, we need to assure

$$c_8(m) \neq 0, m \neq -1, 0, 1, 2 \ldots. \tag{3.79}$$

This completes the proof of the proposition and at the same time, *the "problem 1", for the hypergeometric polynomials on q-nonlinear lattices is solved.* As for the linear case, the problem 2 is solved converting the process of solving the problem 1. Now the second-order operator generated by solving the factorization chain reads:

$$\widehat{\mathcal{L}} = \mathcal{G}^1(z;-1)\mathbf{E}_q - (\mathcal{F}^1(z;t) + \mathcal{G}^1(z-1;t)) + \mathcal{F}^1(z-1;-1)\mathbf{E}_q^{-1} \tag{3.80}$$

with, as one can verify,

$$\mathcal{G}^1(z;-1) = -\mathcal{A}(z); \qquad \mathcal{F}^1(z-1;-1) = -\mathcal{B}(z). \tag{3.81}$$

Consequently, taking into account the first equation in (3.64), we obtain that

$$\widehat{\mathcal{L}} = -\tilde{\tilde{\mathcal{L}}} \tag{3.82}$$

As we have already shown that the operator $\tilde{\mathcal{L}}$ admits a sequence of polynomial eigenfunctions and taking into account of Eq. (3.60), *this solves the "problem 2", for the hypergeometric polynomials on q-nonlinear lattices (and the polynomials dual to them, according to the remark 2.1).*

Remark 3.3 As the Askey-Wilson polynomials are the unique polynomial eigenfunctions of $\tilde{\mathcal{L}}$ ($\widehat{P}_t, t = 0, 1, 2, \ldots$, of degree exactly t) corresponding to the eigenvalue in Eq. (3.72) (see theorem 3.4 in [31]), the produced hypergeometric polynomials are necessarily the Askey-Wilson ones. More precisely, letting $\mathcal{P}_t, t = 0, 1, 2\ldots$ be the set of polynomials produced by the first relation in Eq. (3.69) starting at $\mathcal{P}_0 = 1$ (they are all polynomials in $\chi(z)$ according to Eqs. (3.75) and (3.77)):

$$\tilde{c}(t)\mathcal{P}_t(\chi(z)) = \prod_{i=0}^{t-1}[\mathcal{A}(z)\mathbf{E}_q + \mathcal{F}(z; t-1-i)](1), t = 1, 2, 3, \ldots \tag{3.83}$$

$\tilde{c}(t)$ being some non-null constant (in z), or equally

$$\tilde{c}(t)\mathcal{P}_t(\chi(z)) = \frac{1}{\hat{\varrho}(z)}\prod_{i=0}^{t-1}[\mathbf{E}_q + \mathcal{F}(z; t-1-i)](\hat{\varrho}(z)), t = 1, 2, 3, \ldots \tag{3.84}$$

where

$$\frac{\hat{\varrho}(qz)}{\hat{\varrho}(z)} = \mathcal{A}(z),$$

and letting $p_t(z; a, b, c, d), t = 0, 1, 2, \ldots$ be the Askey-Wilson polynomials, so we have necessarily

$$\mathcal{P}_t = \tilde{\tilde{c}}(t).p_t \tag{3.85}$$

for some constant (in z) $\tilde{\tilde{c}}(t)$.

Remark 3.4 Remarks similar to remark 3.1 and remark 3.2 are adaptable to the case of the q-nonlinear lattices.
We pass now to the discussion concerning the generalization of the Askey-Wilson polynomials.

3.3.2 The generalization of the Askey-Wilson polynomials.

We envisage to show how to produce a sequence of functions $\Phi_n(z,\varepsilon)$ generalizing the Askey-Wilson polynomials in that sense that they are eigenfunctions of the Askey-Wilson operator $\tilde{\pounds}$ and $\Phi_n(z,\varepsilon) \longrightarrow \mathcal{P}_t(\chi(z))$ while $\varepsilon \longrightarrow 1$.
Let us consider the equation:

$$\tilde{\pounds} y(z,\varepsilon) = \lambda^{\varepsilon}(0) y(z,\varepsilon), \tag{3.86}$$

This equation is not new: A little wider equation (with $\pounds$ instead of $\tilde{\pounds}$) has been explicitly solved in [19] in terms of difference hypergeometric functions. In our situation, according to [19], one first solves the non-homogeneous equations:

$$\left(\tilde{\pounds} - \lambda^{\varepsilon}(0)\right) u(z,\varepsilon,\alpha) = G(z,\varepsilon,\alpha) \tag{3.87}$$

where

$$G(z,\varepsilon,\alpha) = \tfrac{(\alpha\varepsilon, abcd\alpha\varepsilon^{-1}q^{-1};q)_\infty}{\alpha(ab\alpha, ac\alpha, ad\alpha, \alpha q;q)_\infty}(az, az^{-1};q)_\infty \tag{3.88}$$

and $(\sigma;q)_\infty := \lim_{i\to\infty}(\sigma;q)_i$, for the following values of α:

$$\alpha := 1 \qquad \alpha := \frac{q}{ab} \qquad \alpha := \frac{q}{ac} \qquad \alpha := \frac{q}{ad}.$$

The corresponding solutions are the functions [19]:

$$u(z,\varepsilon,\alpha)$$
$$= \tfrac{(az,az^{-1};q)_\infty}{(\alpha az, \alpha az^{-1};q)_\infty} \textstyle\sum_{i=0}^{\infty} \tfrac{(\alpha\varepsilon, abcd\alpha\varepsilon^{-1}q^{-1}, \alpha az, \alpha az^{-1};q)_i}{(ab\alpha, ac\alpha, ad\alpha, \alpha q;q)_i} q^i. \tag{3.89}$$

Let us remark that the Askey-Wilson polynomials appear here as the functions $u(z,q^{-t},1)$, $t = 0,1,2,\ldots$
The solutions of the homogeneous equation (3.86) are then obtained by operating adequate (considering Eq. (3.88)) linear combinations of any two of the functions (3.89).

Among the solutions of the homogeneous equation (3.86), we are interested in those having a constant limit when ε converges to the unit. Let us take the "adequate" linear combination of $u(z,\varepsilon,1)$ and $u(z,\varepsilon,\alpha)$, $\alpha \neq 1$:

$$y(z,\varepsilon) = u(z,\varepsilon,1) - \alpha \frac{(\varepsilon, abcd\varepsilon^{-1}q^{-1}, ab\alpha, ac\alpha, ad\alpha, \alpha q; q)_\infty}{(ab, ac, ad, q, \alpha\varepsilon, abcd\alpha\varepsilon^{-1}q^{-1}; q)_\infty} u(z,\varepsilon,\alpha). \tag{3.90}$$

It is easily seen that $y(z,1) = 1$. However, if $\tilde{y}(z,\varepsilon)$ is another "adequate" linear combination of $u(z,\varepsilon,\alpha_1)$ and $u(z,\varepsilon,\alpha_2)$ where α_1 and α_2 differ both from the unit and each other, then $\tilde{y}(z,1) \neq$ *constant*. In other words $\tilde{y}(z,1)$ is the nonconstant solution of the Askey-Wilson equation for $t = 0$.
Our point here resides in the following

Proposition 3.3 *If $y(z,\varepsilon)$ is the solution in Eq. (3.90) of Eq. (3.86) , then the functions*

$$\begin{aligned} \Phi_0(z,\varepsilon) &:= y(z,\varepsilon), \\ \Phi_t(z,\varepsilon) &:= \prod_{i=0}^{t-1} [\mathcal{A}(z)\mathbf{E}_q + \mathcal{F}^\varepsilon(z; t-1-i)] y(z,\varepsilon), \\ & \quad\; t = 1,2,3,\ldots, \end{aligned} \tag{3.91}$$

generalize the Askey-Wilson polynomials in that sense that they are eigenfunctions of the Askey-Wilson operator $\tilde{\mathcal{L}}$ (corresponding to $\lambda^\varepsilon(t)$) and they converge to them when ε converges to the unit.

Proof. The fact that the functions in Eq. (3.91) are eigenfunctions of the Askey-Wilson operator $\tilde{\mathcal{L}}$ corresponding to $\lambda^\varepsilon(t)$, is a consequence of the fact that $\Phi_0(z;\varepsilon)$ is an eigenvector of $\tilde{\mathcal{L}}$ corresponding to $\lambda^\varepsilon(0)$. To be assured that the functions in Eq. (3.91) converge to the Askey-Wilson polynomials (of course, up to a multiplication by a non-null constant), we need essentially to remember that $\lambda^\varepsilon(t)$ converges to $\lambda^1(t)$ when ε converges to the unit and then compare the r.h.s. of Eqs. (3.91) and (3.83), which proves the proposition.

Remark 3.5 By a direct verification, one finds that

$$(\mathcal{A}(z)\mathbf{E}_q + \mathcal{G}^1(z;-1))(1) = 0. \tag{3.92}$$

For this reason, the functions extending the Askey-Wilson polynomials , using lowering operators $\mathcal{A}(z)\mathbf{E}_q + \mathcal{G}^1(z;t)$ relatively to t from Z^+ to Z^-, are all vanishing. They are on their side generalized by the functions obtained by applying the lowering operators $\mathcal{A}(z)\mathbf{E}_q + \mathcal{G}^{\varepsilon}(z; -(i+1)), i = 0, 1, 2\ldots$ to $y(z;\varepsilon)$. Moreover if ${}^{\uparrow}\tilde{\Psi}_t(z)$ (or ${}^{\downarrow}\tilde{\Psi}_t(z)$) are the eigenfunctions obtained by applying t-times the "raising" (or "lowering") operator to $\tilde{y}(z, 1)$, so their generalizations are obtained by performing similar operations starting now at $\tilde{y}(z, \varepsilon)$. Let us note finally that the present generalizations are also difference hypergeometric functions (in the sense of [19]) reason for which they are closely related to the Askey-Wilson polynomials.

Chapter 4

Discrete factorization techniques for the modification of the hypergeometric polynomials on linear lattices

In the preceding chapter, Sturm-Liouville difference operators (more precisely, hypergeometric difference operators of Nikiforov-Suslov-Uvarov) were generated and solved from "quasi-periodicity" or simple shape-invariance (symmetry) non-linear difference equations. In the present chapter, we aim to "modify" known solvable Sturm-Liouville difference operators into new ones, avoiding in the same time any significant self-similarity such as shape-invariance or "quasi-periodicity" so that the new difference operators do not belong to the same family as the old one. A special case of the Meixner polynomials $M_n^{(2,c)}(x+1)$ will be "modified" efficiently, into a new complete sequence of non-classical orthogonal polynomials.

4.1 Generalities.

Clarifying the leading idea, let us first consider a general situation. The studied second-order difference operator H,

$$H(x) = u(x)\mathbf{E}_x + v(x) + w(x)\mathbf{E}_x^{-1}, \tag{4.1}$$

is in the first place, supposed to act in a linear space of functions, say $\mathcal{L}$, and to admit a non-empty set of eigenfunctions. One can fix $(\Psi_{\hat{\alpha}}, \lambda_{\hat{\alpha}})$ as one of its pairs of eigenfunctions and corresponding eigenvalues. In that case, one can make the following factorization

$$\begin{aligned} H - \lambda_{\hat{\alpha}} &= L_{\hat{\alpha}} R_{\hat{\alpha}} \\ \tilde{H} - \lambda_{\hat{\alpha}} &= R_{\hat{\alpha}} L_{\hat{\alpha}} \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} R_{\hat{\alpha}} = 1 + f_{\hat{\alpha}}(x)\mathbf{E}_x^{-1}; \qquad L_{\hat{\alpha}} &= u(x)\mathbf{E}_x + g_{\hat{\alpha}}(x); \\ f_{\hat{\alpha}}(x) &= -\tfrac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)}. \end{aligned} \tag{4.3}$$

As it can be easily seen, the similarity reductions

$$\rho H \rho^{-1}; \qquad \tilde{\rho}\tilde{H}\tilde{\rho}^{-1} \tag{4.4}$$

where

$$\tfrac{\rho^2(x+1)}{\rho^2(x)} = \tfrac{u(x)}{f_{\hat{\alpha}}(x+1)g_{\hat{\alpha}}(x+1)}; \qquad \tfrac{\tilde{\rho}^2(x+1)}{\tilde{\rho}^2(x)} = \tfrac{u(x)}{f_{\hat{\alpha}}(x+1)g_{\hat{\alpha}}(x)}, \tag{4.5}$$

allow to transform H and $\tilde{H}$ into their *formal* symmetric form (i.e. like $A(x+1)\mathbf{E}_x + B(x) + A(x)\mathbf{E}_x^{-1}$). Denote by $\ell^2(a, b; \rho^2)$ the linear space of vectorial functions $(\psi(a), \psi(a+1), \ldots, \psi(b))$, in which is defined a discrete-weighted inner product

$$(\psi, \phi)_\rho = \sum_a^b \psi(x)\phi(x)\rho^2(x), -\infty < a < b \leq +\infty. \tag{4.6}$$

Such defined $\ell^2(a, b; \rho^2)$ is well known to be a separable Hilbert space (see for example [2], for a general theory).
The similar space for $\tilde{\rho}^2$ will be denoted by $\ell^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$.

Letting $\{\Psi_\alpha, \lambda_\alpha\}$, $\alpha \in N_\alpha$ (a denumerable set of index , not containing $\hat{\alpha}$), be a set of eigenfunctions and corresponding mutually different eigenvalues of H, one easily finds that the set $(\tilde{\Psi}_{\hat{\alpha}}, \lambda_{\hat{\alpha}})$, $\{\tilde{\Psi}_\alpha, \lambda_\alpha\}$, $\alpha \in N_\alpha$ where

$$\begin{aligned} \tilde{\Psi}_\alpha &= R_{\hat{\alpha}}\Psi_\alpha \\ L_{\hat{\alpha}}\tilde{\Psi}_{\hat{\alpha}} &= 0, \end{aligned} \tag{4.7}$$

is a resulting set of eigenfunctions and corresponding mutually different eigenvalues of $\tilde{H}$. Moreover, the completeness of $\tilde{\Psi}_{\hat{\alpha}}$, $\tilde{\Psi}_\alpha$, $\alpha \in N_\alpha$ in $\ell^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$ can be deduced from that of Ψ_α, $\alpha \in N_\alpha$ in $\ell^2(a, b; \rho^2)$ in the sense of the following proposition.

Proposition 4.1 *Supposing that*

(c1) $L_{\hat\alpha}\tilde\Theta_{\hat\alpha} \in \ell^2(a,+\infty;\rho^2)$*, and* $\{R_{\hat\alpha}\Theta_\alpha\} \in \ell^2(a+1,+\infty;\tilde\rho^2)$*, for* $\{\Theta_\alpha\}$ *and* $\tilde\Theta_{\hat\alpha}$ *belonging to the first and second space respectively.*

(c2) $\Theta_\alpha(x) = \tilde\Theta_{\hat\alpha}(x) = 0,\ x < a+1.$

(c3) $\{\Theta_\alpha\}$ *is complete in* $\ell^2(a+1,+\infty;\rho^2)$.

Then

(r1) The operators $L_{\hat\alpha}$ *and* $R_{\hat\alpha}$ *are* $(\rho^2,\tilde\rho^2)$*-mutually adjoint, in that sense that*

$$(\tilde\Theta_{\hat\alpha}, R_{\hat\alpha}\Theta_\alpha)_{\tilde\rho^2} = (L_{\hat\alpha}\tilde\Theta_{\hat\alpha}, \Theta_\alpha)_{\rho^2} \tag{4.8}$$

(r2) $\tilde\Theta^0_{\hat\alpha}$ *and* $\{R_{\hat\alpha}\Theta_\alpha\}$ *are complete in* $\ell^2(a+1,+\infty;\tilde\rho^2)$*, where*

$$\tilde\Theta^0_{\hat\alpha}(x) = \begin{cases} \tilde Y_{\hat\alpha}(x), & x \geq a+1 \\ 0, & x < a+1; \end{cases}$$

$\tilde Y_{\hat\alpha}(x)$ *being defined by*

$$L_{\hat\alpha}\tilde Y_{\hat\alpha}(x) = 0. \tag{4.9}$$

Before proving this proposition, some remarks are in order. Let us first note that $\{.\}$ symbolizes the *set* of elements as the one in the brackets for varying index α in N_α. Next, in the choice of $+\infty$ as the right boundary of the interval of orthogonality, one needs to see only the search of completeness avoiding at the same time nonsignificant particularities.

Proof of the proposition: Simple summation by parts gives

$$\begin{aligned}(L_{\hat\alpha}\tilde\Theta_{\hat\alpha}, \Theta_\alpha)_{\rho^2} &= \textstyle\sum_a^{+\infty} u(x)\tilde\Theta_{\hat\alpha}(x+1)\Theta_\alpha(x)\rho^2(x) \\ &\quad + \textstyle\sum_a^{+\infty} g_{\hat\alpha}(x)\tilde\Theta_{\hat\alpha}(x)\Theta_\alpha(x)\rho^2(x) \\ &= \textstyle\sum_{a+1}^{+\infty} u(x-1)\tilde\Theta_{\hat\alpha}(x)\Theta_\alpha(x-1)\rho^2(x-1) \\ &\quad + \textstyle\sum_a^{+\infty} g_{\hat\alpha}(x)\tilde\Theta_{\hat\alpha}(x)\Theta_\alpha(x)\rho^2(x) \\ &= \textstyle\sum_{a+1}^{+\infty} u(x-1)\tilde\Theta_{\hat\alpha}(x)\Theta_\alpha(x-1)\rho^2(x-1) \\ &\quad + \textstyle\sum_{a+1}^{+\infty} g_{\hat\alpha}(x)\tilde\Theta_{\hat\alpha}(x)\Theta_\alpha(x)\rho^2(x),\end{aligned} \tag{4.10}$$

where we used $(c2)$ for the last equality. On the other side

$$(\tilde{\Theta}_{\hat{\alpha}}, R_{\hat{\alpha}}\Theta_\alpha)_{\tilde{\rho}^2} = \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x)[\Theta_\alpha(x) + f_{\hat{\alpha}}(x)\Theta_\alpha(x-1)]\tilde{\rho}^2(x)$$
$$= \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x)\Theta_\alpha(x)\tilde{\rho}^2(x) + \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x)f_{\hat{\alpha}}(x)\Theta_\alpha(x-1)\tilde{\rho}^2(x). \quad (4.11)$$

Remarking that, the equality of (4.10) and (4.11) requires the conditions set by the definitions in Eq. (4.5), this proves $(r1)$.
Remark next that for proving the completeness of the system in $(r2)$ in $\ell^2(a+1,+\infty;\tilde{\rho}^2)$, it suffices to prove its closure in that space seen that the latter is separable and Hilbertian , while the equivalence between the completeness and the closure of systems in such spaces is a well-known result (see for example [63], theorem 4, parag.4, chap.3).
Let $\tilde{\Theta}_{\hat{\alpha}}$ be an element satisfying $(c1)$. One can suppose without annoying any hypothesis that it satisfies $(c2)$ as well. Suppose next that, that element is orthogonal to the full set $\{R_{\hat{\alpha}}\Theta_\alpha\}$. So by the already proved $(r1)$ and using $(c3)$ and $(c2)$, one obtains that the unique possibly non-vanishing coordinate of $L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}}$ is the one in the a^{th} place. Clearly, this coordinate reads $u(a)\tilde{\Theta}_{\hat{\alpha}}(a+1)$. But as it is easily seen, up to a multiplication by a constant, this is exactly the structure of the semi-infinite vector $L_{\hat{\alpha}}\tilde{\Theta}^0_{\hat{\alpha}}$. In other words, $\tilde{\Theta}^0_{\hat{\alpha}}$ is essentially the unique element orthogonal to the whole set $\{R_{\hat{\alpha}}\Theta_\alpha\}$ which proves $(r2)$ and the proposition is completely proved.

Type (4.8) "mutually adjointness" was intensively used in [60, 89] to give simplest proves of the orthogonality relations for most of classical polynomials (including the Askey-Wilson class). In those works, the role of $L_{\hat{\alpha}}$ and $R_{\hat{\alpha}}$ were played by the usual difference relations (considered there as the starting point) lowering and raising the degree of the polynomials and "perturbing" the parameters. Thus, ρ^2 and $\tilde{\rho}^2$ differed only by a "perturbation" of parameters.
Here, ρ^2 and $\tilde{\rho}^2$ will differ more than in the unique shape: We expect $\tilde{\rho}^2$ to be "non-classical" for given "classical" ρ^2. This will be done efficiently on the special case of Meixner polynomials $M_n^{(2,c)}(x+1)$.

For treating the question of orthogonality for the new functions $\tilde{\Psi}_{\hat{\alpha}}$, $\tilde{\Psi}_\alpha$, $\alpha \in N_\alpha$, in $\ell^2(\tilde{a},\tilde{b};\tilde{\rho}^2)$, one can adopt the following way. Firstly, certify, using the reasoning of the preceding proposition, the orthogonality of $\tilde{\Psi}_{\hat{\alpha}}$ and every $\tilde{\Psi}_\alpha$, $\alpha \in N_\alpha$ (provided that $\tilde{a} = a+1; b = \tilde{b} = +\infty$). Secondly, deduce the orthogonality of $\tilde{\Psi}_\alpha$ and $\tilde{\Psi}_\beta$ for $\alpha \neq \beta$, $\alpha, \beta \in N_\alpha$, as that of eigenfunctions corresponding to mutually different eigenvalues for a symmetric operator $\tilde{H}$, provided that (obtained using simple summation by parts)

$$u(x)\tilde{\rho}^2(x)\left(\tilde{\Psi}_\alpha \Delta \tilde{\Psi}_\beta - \tilde{\Psi}_\beta \Delta \tilde{\Psi}_\alpha\right)|_{\tilde{a}-1}^{\tilde{b}} = 0. \qquad (4.12)$$

It is this way that will be followed later to verify the orthogonality relations for the functions (polynomials) modifying the already evoked special case of Meixner polynomials $M_n^{(2,c)}(x+1)$.

Thus, from a Sturm-Liouville operator for which the eigenfunction expansion is known, we can generate a new (in that sense that the operators need to belong to different families) Sturm-Liouville operator and the corresponding eigenfunction expansion.
But as we are "transforming" polynomials, the situation is delicate seen that, even found, the "*transformation function*" $\Psi_{\hat{\alpha}}$ can't be generally taken as a polynomial, while the "*transformed functions*", $\tilde{\Psi}_{\hat{\alpha}}$, $\tilde{\Psi}_\alpha, \alpha \in N_\alpha$ need to be polynomials or rational functions satisfying additional constraints. So, for a given Sturm-Liouville difference operator, a copious reservoir of "good" transformation functions, can't be expected.
Remark that, in what precedes, the transformation function was deliberately chosen outside of the "*transformable system*" $\Psi_\alpha, \alpha \in N_\alpha$. Supposing that the transformable functions $\Psi_\alpha(x)$ are, not only explicit functions of x, but also of α, so it has also a sense to chose as the transformation function, a function Ψ_γ for which $\gamma \in N_\alpha$. In that case, the new functions need to be searched as

$$\tilde{\Psi}_\alpha = \frac{R_\gamma \Psi_\alpha}{\lambda_\alpha - \lambda_\gamma}, \qquad (4.13)$$

so that the transformed $\tilde{\Psi}_\gamma$ must be understood as a limit (Hospital rule) of the right hand side of Eq. (4.13) for $\alpha \longrightarrow \gamma$. In orthogonal polynomials theory, transformations as in Eq. (4.13) (we mean here, when the acting functions are and are considered as explicit functions of the independent variable α) are referred to as the *Christoffel transformations* having as inverse, the so-called *Geronimus transformations*. Much interesting studies and applications of those transformations can be found in [113, 44] and references therein.

Let us recall that in the preceding chapter, we have shown (practically) that the operator H is "generable" and "solvable" from simple shape-invariance (symmetry) factorization chains.

In this chapter below, we "modify" a special case of the Meixner polynomials, namely the $M_n^{(2,c)}(x+1)$, sending them into a complete system of non-classical orthogonal rational (polynomial) functions.
Consider the operator

$$H = \sigma\Delta\nabla + \tau\Delta = (\sigma+\tau)\mathbf{E}_x - (2\sigma+\tau) + \sigma\mathbf{E}_x^{-1} \tag{4.14}$$

and let $\lambda_n = n\tau' + \frac{1}{2}\sigma'' n(n-1)$; $P_n(x)$, $n = 0, 1, 2, \ldots$, be its eigenvalues and the above evoked corresponding polynomial eigenfunctions. According to the above discussions, we can transform the polynomials as

$$\tilde{P}_n(x) = P_n(x) + f_{\hat{\alpha}}(x) P_n(x-1) \tag{4.15}$$

where

$$f_{\hat{\alpha}}(x) = -\tfrac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)}, \tag{4.16}$$

obviously provided that the function $\Psi_{\hat{\alpha}}(x)$ does not belong to the sequence $P_n(x)$, $n = 0, 1, 2, \ldots$. Moreover, since the hypergeometric polynomials on linear lattices are (if taken in their canonical forms), not only polynomials in x but also in n (called dual polynomials), we can also use the Christoffel transformation (we here shifted x comparatively to Eq. (4.13), in order that the obtained polynomials (in n) be of degree exactly equal to x):

$$\tilde{P}_x^{(\gamma)}(n) = \frac{P_n(x+1) - \frac{P_\gamma(x+1)}{P_\gamma(x)} P_n(x)}{\lambda_n - \lambda_\gamma}. \tag{4.17}$$

Thus by Eq. (4.17), one can transform the polynomials dual to the hypergeometric polynomials (recall that for the Charlier, Meixner and Kravchuk polynomials, there exists self-duality) into non-classical polynomials with rational coefficients in the three-term recurrence relation. In [44], one can find detailed discussions concerning the applications of the Christoffel transformation to any family of orthogonal polynomials admitting a dual sequence of orthogonal polynomials (i.e. q-Racah polynomials and specializations).
It is worth here noting that the functions dual to the polynomials obtained by Eq. (4.17) are not necessarily of polynomial type. It is indeed easily seen that the functions

$$\lim_{n\to\gamma} \tilde{P}_x^{(\gamma)}(n) \tag{4.18}$$

are not necessarily polynomials in x. Taking for example the Hahn case, choosing for simplicity $\gamma = 0$, one obtains

$$\tilde{P}_x^{(0)}(0) = -{}_3F_2\left(\begin{array}{c} 1, -x, \alpha+\beta+2 \\ \alpha+1, -N \end{array} \middle| 1\right). \tag{4.19}$$

Here, the interesting us transformation is the one given by Eqs. (4.15)-(4.16). In this case, for already explained reasons, the transformation function $\Psi_{\hat{\alpha}}(x)$ would lie outside the set of transformable polynomials $P_n(x)$. Moreover as it can be easily seen, if we expect that the transformed polynomials be of different degree (and so be independent), we must avoid the choice of $\Psi_{\hat{\alpha}}(x)$ as a polynomial. However, for evident reasons, the rapport $\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)}$ must be in preference a rational function, say $\frac{N_{\hat{\alpha}}(x)}{D_{\hat{\alpha}}(x)}$. Thus, if for example the new ground state (provided it belongs to $\ell^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$) appears in the form $\tilde{\Psi}_{\hat{\alpha}(x)} = \frac{\hat{N}_0(x)}{D_{\hat{\alpha}}(x)}$, the new polynomials should be checked inside the set of $\hat{N}_0(x)$ and the different numerators of the fractions

$$\tilde{P}_n(x) = P_n(x) - \frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)}P_n(x-1), \; n = 0, 1, \ldots \tag{4.20}$$

where

$$\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)} = \frac{N_{\hat{\alpha}}(x)}{D_{\hat{\alpha}}(x)}, \tag{4.21}$$

while the actual new weight should read as $D_{\hat{\alpha}}^{-2}(x)\tilde{\rho}^2(x)$. The example 2.2 (see the first chapter) of Samsonov-Ovcharov (see [104]), on the "modification" of Hermite polynomials, is a typical example (unique in the literature, to our best knowledge) of a realization of this scheme. For the polynomials on lattice, this scheme will be realized below only for a special case of the Meixner polynomials, namely the $M_n^{(2,c)}(x+1)$, the question remaining open for higher polynomials.

Remark that once the transformation (4.20)-(4.21) is realized, the obtained functions are not only rational functions in x, but also polynomials in n (of degree x). For the Charlier case, such polynomials can be found in [44]. It is however clear that for any choice the transformation function $\Psi_{\hat{\alpha}}(x)$ (not necessarily satisfying Eq. (4.21)) outside the transformable sequence, the transformation in Eq. (4.20) will lead necessarily to polynomials in n, under the unique condition that the transformable polynomials in x, be also polynomials in n (i.e. q-Racah polynomials and specializations). In this approach (i.e. n as a variable and x as an index), the interest of the condition in Eq. (4.21) resides only in the fact that the functions (in x) dual to the so-obtained polynomials (in n) will be of rational type.

The last remark, before opening calculations, concerns the degrees of the polynomials (in x) to be obtained by Eqs. (4.20)-(4.21). Writing the operator $\tilde{H}.\frac{1}{D_{\hat{\alpha}}(x)}$ under the form

$$\tilde{H}.\frac{1}{D_{\hat{\alpha}}(x)} = \tilde{\sigma}\Delta\nabla + \tilde{\tau}\Delta + \tilde{\nu}, \tag{4.22}$$

one easily notes that the functions $\tilde{\sigma}$, $\tilde{\tau}$ and $\tilde{\nu}$ are not necessarily of degree 2, 1 and 0 respectively. In other words, $\tilde{H}.\frac{1}{D_{\hat{\alpha}}(x)}$ is not necessarily of hypergeometric type. Consequently, as in the case of "modified" by Samsonov-Ovcharov Hermite polynomials, we can not expect that the corresponding polynomials (numerators in Eq. (4.20)) be of degrees exactly equal $n = 0, 1, 2, 3, \ldots$. There is clearly, no matter to be worried by this seen that the polynomials conserve most of other properties common to usual orthogonal polynomials (completeness, orthogonality, difference and recurrence relations, difference equations, duality,...). The question of global study of orthogonal polynomials of such a category have been already raised in [44].

4.2 The Meixner case.

The special case of Meixner polynomials $M_n^{(2,c)}(x+1)$ can be transformed as follows. From the equation,

$$\left[(\sigma+\tau)\mathbf{E}_x - (2\sigma+\tau) + \sigma\mathbf{E}_x^{-1}\right] P_n(x) = \lambda_n P_n(x), \tag{4.23}$$

one can, specializing the coefficient of σ and τ, deduce the following

$$\begin{aligned} K(c,d;x)\hat{P}_n(x) &= (c-1)n\hat{P}_n(x) \\ K(c,d;x)c^{-x}\hat{\hat{P}}_n(x) &= -(c-1)nc^{-x}\hat{\hat{P}}_n(x) \end{aligned} \tag{4.24}$$

where

$$K(c,d;x) = c(x+d)\mathbf{E}_x - (c+1)(x+d) + (x+d)\mathbf{E}_x^{-1} \tag{4.25}$$

and the polynomials $\hat{\hat{P}}_n(x)$ are obtained from $\hat{P}_n(x)$ replacing x by $-x$ and d by $-d$.

As it can be easily seen, the polynomials $\hat{P}_n(x)$ are not orthogonal (the interval of orthogonality does not exist). However, they can be written as

$$\hat{P}_{n+1}(x) = (x+d)\bar{P}_n(x), n = 0, 1, 2, \ldots \tag{4.26}$$

so that the polynomials $\bar{P}_n(x)$ satisfy

$$\begin{aligned} \bar{K}(c,d;x)\bar{P}_n(x) &= (c-1)n\bar{P}_n(x) \\ \bar{K}(c,d;x)c^{-x}\bar{\bar{P}}_n(x) &= -(c-1)(n+2)c^{-x}\bar{\bar{P}}_n(x) \end{aligned} \tag{4.27}$$

where

$$\bar{K}(c,d;x) = c(x+d+1)\mathbf{E}_x - [(c+1)(x+d)+c-1] + (x+d-1)\mathbf{E}_x^{-1}$$

and the polynomials $\bar{\bar{P}}_n(x)$ are obtained from $\bar{P}_n(x)$ replacing x by $-x$ and d by $-d$.
As is easily seen, the polynomials $\bar{P}_n(x)$, $n = 0, 1, 2, \ldots$, are orthogonal on $[-d+1, +\infty)$ with the weight $\rho^2(x) = c^x(x+d)$, $(0 < c < 1)$. The corresponding coefficients in the three-term recurrence relations are

$$b_n = -\frac{(n+1)(c+1)+dc-d}{c-1}; \qquad a_n^2 = \frac{n(n+1)c}{(c-1)^2}. \tag{4.28}$$

It is clear that the parameter d is essentially a translating one. Namely, setting $d = 1$, the polynomials $\bar{P}_n(x)$ are exactly the special Meixner $M_n^{(2,c)}(x)$. However, for future convenience, we will fix $d = 2$, so that the polynomials candidate to the transformation are the translated Meixner $M_n^{(2,c)}(x+1)$. Thus, we have shown

$$\begin{aligned} &\mathcal{M}(c;x)M_n^{(2,c)}(x+1) = (c-1)nM_n^{(2,c)}(x+1), \\ &\mathcal{M}(c;x)c^{-x}\hat{M}_n(x) = -(c-1)(n+2)c^{-x}\hat{M}_n(x) \end{aligned} \tag{4.29}$$

where

$$\mathcal{M}(c;x) = c(x+3)\mathbf{E}_x - [(c+1)x+3c+1] + (x+1)\mathbf{E}_x^{-1} \tag{4.30}$$

and the polynomials $\hat{M}_n(x)$ are obtained from $\bar{\bar{P}}_n(x)$ replacing d by 2.
Let us remark that the candidate transformation functions $\Psi_\gamma(x) = c^{-x}\hat{M}_\gamma(x)$ need not to be "quadratically integrable". As already noted, the essential is that the ratio $\frac{\Psi_\gamma(x)}{\Psi_\gamma(x-1)}$ simplifies in a rational function say $\frac{N_\gamma(x)}{D_\gamma(x)}$, so that the transformed function $\tilde{\Psi}_n(x) = P_n(x) - \frac{N_\gamma(x)}{D_\gamma(x-1)}P_n(x-1)$ would have as

numerators, polynomial functions.

Recapitulation.

The polynomials to "modify" are the Meixner $M_n^{(2,c)}(x+1)$. They satisfy

$$\left[u(x)\mathbf{E}_x + v(x) + w(x)\mathbf{E}_x^{-1}\right] M_n^{(2,c)}(x+1) = (c-1)nM_n^{(2,c)}(x+1) \quad (4.31)$$

where

$$u(x) = c(x+3); \qquad v(x) = -[(c+1)x + 3c + 1]; \qquad w(x) = x+1, (4.32)$$

as well as the usual recurrence relations with

$$b_n = -\frac{(c+1)n + 3c - 1}{c-1}; \qquad a_n^2 = \frac{n(n+1)c}{(c-1)^2} \quad (4.33)$$

and the difference relations

$$\begin{aligned} (c-1)a_{n+1}M_{n+1}^{(2,c)}(x+1) &= [u(x)\mathbf{E}_x + k(x;n)]M_n^{(2,c)}(x+1), \\ -(c-1)a_n M_{n-1}^{(2,c)}(x+1) &= [u(x)\mathbf{E}_x + l(x;n)]M_n^{(2,c)}(x+1), \quad (4.34) \end{aligned}$$

where

$$\begin{aligned} k(x;n) &= -x + n - 1, \\ l(x;n) &= -c(x+3+n). \quad (4.35) \end{aligned}$$

They are orthogonal on $[-1, +\infty)$ with weight $\rho^2(x) = c^x(x+2)$.

The transformation functions are the functions $c^{-x}\hat{M}_n(x)$, eigenfunctions of the same operator as the Meixner $M_n^{(2,c)}(x+1)$, corresponding to the eigenvalues $-(c-1)(n+2)$. Precisely, $\hat{M}_n(x) = M_n^{(2,c)}(-x-3)$.

The new rational type functions are $\tilde{\Psi}_{\gamma,0}(x)$, $\tilde{\Psi}_{\gamma,n+1}(x)$, $n, \gamma = 0, 1, 2, \ldots$

$$\begin{aligned} [c(x+3)\mathbf{E}_x + g_\gamma(x)]\,\tilde{\Psi}_{\gamma,0}(x) = 0, \\ \tilde{\Psi}_{\gamma,n+1}(x) = \left[1 + f_\gamma(x)\mathbf{E}_x^{-1}\right] M_n^{(2,c)}(x+1) \quad (4.36) \end{aligned}$$

where

$$f_\gamma(x) = -\frac{\hat{M}_\gamma(x)}{c\hat{M}_\gamma(x-1)}; \; g_\gamma(x) = v(x) - u(x)f_\gamma(x+1) + (c-1)(\gamma+2).$$

They satisfy the second-order eigenvalue difference equation

$$\begin{gathered}\left[u(x)\mathbf{E}_x + \tilde{v}_\gamma(x) + \tilde{w}_\gamma(x)\mathbf{E}_x^{-1}\right]\tilde{\Psi}_{\gamma,0}(x) = -(c-1)(\gamma+2)\tilde{\Psi}_{\gamma,0}(x),\\ \left[u(x)\mathbf{E}_x + \tilde{v}_\gamma(x) + \tilde{w}_\gamma(x)\mathbf{E}_x^{-1}\right]\tilde{\Psi}_{\gamma,n+1}(x) = (c-1)n\tilde{\Psi}_{\gamma,n+1}(x),\\ n,\gamma = 0,1,2,\ldots \end{gathered} \tag{4.37}$$

where

$$\begin{aligned}\tilde{v}_\gamma(x) &= v(x) + f_\gamma(x)u(x-1) - u(x)f_\gamma(x+1),\\ \tilde{w}_\gamma(x) &= f_\gamma(x)\frac{w(x-1)}{f_\gamma(x-1)}.\end{aligned} \tag{4.38}$$

The case $\gamma = 0$ does not lead to non-classical polynomials. The cases $\gamma \geq 1$ lead to rational functions orthogonal with non-classical weights and from which one can extract non-classical polynomials. Below, the computations results and various discussions are given for the cases of $\gamma = 1$, $\gamma = 2$, and $\gamma = 3$.

Second-order difference eigenvalue equations, eigenfunctions.

In this subsection, in each case, are written explicitly: The coefficients in the factorization product (4.2), the coefficients in the second-order difference eigenvalue equations as well as the first six eigenfunctions.

The case $\gamma = 1$.

The coefficients in the factorization product (4.2) are

$f_1(x) = -\frac{(c-1)x+c-3}{c((c-1)x-2)}$;

$g_1(x) = -\frac{c(x+1)((c-1)x-2)}{(c-1)x+c-3}$;

The second order difference eigenvalue equation reads as in Eq. (4.37) with

$\tilde{v}_1(x) = \{-((c+1)(c-1)^2x^3 + (c-1)(4c^2-7c-5)x^2 + (6-20c^2+19c+3c^3)x + 16c - 4c^2)\}$
$/\{((c-1)x-2)((c-1)x+c-3)\}$;

$\tilde{w}_1(x) = \frac{((c-1)x+c-3)x((c-1)x-1-c)}{((c-1)x-2)^2}$;

The new eigenfunctions are

$$\tilde{\Psi}_{1,0}(x) = \frac{1}{(x+2)(x+1)((c-1)x-2)};$$

$$\tilde{\Psi}_{1,n}(x) = \frac{\mathcal{P}_{1,n}(x)}{Q_1(x)}, \; n = 1, 2, 3, 4, 5; \ldots$$

where

$$Q_1(x) = (c-1)x - 2;$$

$$\mathcal{P}_{1,1}(x) = (c-1)^2 x - 3c + 3;$$

$$\mathcal{P}_{1,2}(x) = (c-1)((c-1)^2 x^2 + 3(c-1)^2 x - 8c);$$

$$\mathcal{P}_{1,3}(x) = (c-1)((c-1)^3 x^3 + (7c-2)(c-1)^2 x^2 + (c-1)(12c^2 - 19c - 3)x - 30c^2);$$

$$\mathcal{P}_{1,4}(x) = (c-1)((c-1)^4 x^4 + 12c(c-1)^3 x^3 + (47c^2 - 22c - 7)(c-1)^2 x^2 + 6(c-1)(10c^3 - 20c^2 - 5c - 1)x - 144c^3);$$

$$\mathcal{P}_{1,5}(x) = (c-1)((c-1)^5 x^5 + 3(6c+1)(c-1)^4 x^4 + 7(17c^2-1)(c-1)^3 x^3 + 9(38c^2+15c+3)(c-1)^3 x^2 + 2(c-1)(180c^4 - 419c^3 - 131c^2 - 41c - 9)x - 840c^4);$$

$$\ldots$$

The case $\gamma = 2$.

The coefficients in the factorization product (4.2) are

$$f_2(x) = -\frac{(c-1)^2 x^2 + (c-1)(c-7)x - 6c + 12}{c((c-1)^2 x^2 - (c+5)(c-1)x + 6)};$$

$$g_2(x) = -\frac{c(x+1)((c-1)^2 x^2 - (c+5)(c-1)x + 6)}{(c-1)^2 x^2 + (c-1)(c-7)x - 6c + 12};$$

The second order difference eigenvalue equation reads as in Eq. (4.37) with

$$\tilde{v}_2(x) = \{-((c+1)(c-1)^4 x^5 + 3(c^2 - 5c - 4)(c-1)^3 x^4 - (c^3 + 41c^2 - 85c - 53)(c-1)^2 x^3 - 3(c-1)(c^4 - c^3 - 63c^2 + 77c + 34)x^2 + (72 - 348c^2 + 312c + 24c^4 + 12c^3)x + 180c - 72c^2)\}$$

$/\{((c-1)^2x^2-(c+5)(c-1)x+6)((c-1)^2x^2+(c-1)(c-7)x-6c+12)\};$

$\tilde{w}_2(x) = \{((c-1)^2x^2+(c-1)(c-7)x-6c+12)x((c-1)^2x^2-3(c-1)(c+1)x+2c+2c^2+2)\}$
$/\{((c-1)^2x^2-(c+5)(c-1)x+6)^2\};$

The new eigenfunctions are

$\tilde{\Psi}_{2,0}(x) = \{1\}/\{(x+2)(x+1)(2(c-1)^2x-c^2+((c^2+10c+1)(c-1)^2)^{1/2}-4c+5)(2(c-1)^2x-c^2-((c^2+10c+1)(c-1)^2)^{1/2}-4c+5)\};$

$\tilde{\Psi}_{2,n}(x) = \frac{\mathcal{P}_{2,n}(x)}{Q_2(x)}, n = 1,2,3,4,5;\ldots$

where

$Q_2(x) = (c-1)^2x^2-(c+5)(c-1)x+6;$

$\mathcal{P}_{2,1}(x) = (c-1)((c-1)^2x^2-(c+7)(c-1)x+12);$

$\mathcal{P}_{2,2}(x) = (c-1)((c-1)^3x^3+(2c-7)(c-1)^2x^2-(c-1)(3c^2+19c-12)x+30c);$

$\mathcal{P}_{2,3}(x) = (c-1)((c-1)^4x^4+6(c-1)^4x^3+(5c^2-46c+5)(c-1)^2x^2-12(c^2+7c+1)(c-1)^2x+108c^2);$

$\mathcal{P}_{2,4}(x) = (c-1)((c-1)^5x^5+(11c-4)(c-1)^4x^4+7(5c^2-10c-1)(c-1)^3x^3+(13c^3-298c^2+53c+22)(c-1)^2x^2-6(c-1)(10c^4+58c^3-73c^2-19c-4)x+504c^3);$

$\mathcal{P}_{2,5}(x) = (c-1)((c-1)^6x^6+(17c-1)(c-1)^5x^5+(101c^2-74c-19)(c-1)^4x^4+(223c^3-685c^2-115c+1)(c-1)^3x^3+2(9c^4-1060c^3+242c^2+164c+45)(c-1)^2x^2-24(c-1)(15c^5+85c^4-124c^3-40c^2-13c-3)x+2880c^4);$

$\ldots$

The case $\gamma = 3$.

In order to avoid a fourth order algebraic equation, we have fixed the parameter c taking $c = 1/2$.

The coefficients in the factorization product (4.2) now read

$f_3(x) = -\frac{2(x+7)(x^2+17x+48)}{(x+6)(x^2+15x+32)};$

$g_3(x) = -\frac{(x+1)(x+6)(x^2+15x+32)}{2(x+7)(x^2+17x+48)};$

The second order difference eigenvalue equation reads as in Eq. (4.37) with

$\tilde{v}_3(x) = \{-3(x+3)(37536x + 709x^4 + 5621x^3 + 21946x^2 + 43x^5 + x^6 + 18432)\}$
$/\{2(x+6)(x^2+15x+32)(x+7)(x^2+17x+48)\};$

$\tilde{w}_3(x) = \frac{(x+7)(x^2+17x+48)x(x+5)(x^2+13x+18)}{(x+6)^2(x^2+15x+32)^2};$

The new eigenfunctions are

$\tilde{\Psi}_{3,0}(x) = \frac{1}{(x+1)(x+2)(x+6)(2x+15-97^{1/2})(2x+15+97^{1/2})};$

$\tilde{\Psi}_{3,n}(x) = \frac{\mathcal{P}_{3,n}(x)}{Q_3(x)}, n = 1, 2, 3, 4, 5; \ldots$

where

$Q_3(x) = (x+6)(x^2+15x+32);$

$\mathcal{P}_{3,1}(x) = -(x+15)(x+4)(x+8);$

$\mathcal{P}_{3,2}(x) = \frac{1}{2}(x^4 + 24x^3 + 137x^2 - 66x - 1152);$

$\mathcal{P}_{3,3}(x) = -\frac{1}{4}(x^5 + 18x^4 - 7x^3 - 960x^2 - 1740x + 4032);$

$\mathcal{P}_{3,4}(x) = \frac{1}{8}(19704x - 18432 - 179x^4 - 1209x^3 + 4714x^2 + 9x^5 + x^6);$

$\mathcal{P}_{3,5}(x) = -\frac{1}{16}(-199296x + 103680 + 435x^4 + 18958x^3 - 9864x^2 - 311x^5 - 3x^6 + x^7);$

$\ldots$

Orthogonality, weight functions.

We must refer literally to the results of the discussions in the first section, particularly those related to the proposition 4.1. Here, $a = -1$, and the interval of orthogonality is waited to be $[0, +\infty)$. On the other side, various solutions of the equation in (4.5) for $\tilde{\rho}_\gamma$ are

$$\tilde{\rho}_1^2(x) = \frac{\tilde{\rho}_1^2(0)(3-c)(cx-x-2)(x+2)(x+1)c^x}{4(cx+c-x-3)};$$

$$\tilde{\rho}_2^2(x) = \{c^x(x+1)(x+2)(2xc^2-4cx+2x+5-4c-c^2+((c^2+10c+1)(c-1)^2)^{1/2})(2xc^2-4cx+2x+5-4c-c^2-((c^2+10c+1)(c-1)^2)^{1/2})(c^2-8c+7-((c^2+10c+1)(c-1)^2)^{1/2})(c^2-8c+7+((c^2+10c+1)(c-1)^2)^{1/2})\tilde{\rho}_2^2(0)\} /\{2(-5+4c+c^2-((c^2+10c+1)(c-1)^2)^{1/2})(-5+4c+c^2+((c^2+10c+1)(c-1)^2)^{1/2})(2xc^2-4cx+2x+c^2+7-8c+((c^2+10c+1)(c-1)^2)^{1/2})(2xc^2-4cx+2x-((c^2+10c+1)(c-1)^2)^{1/2}+c^2+7-8c)\};$$

$$\tilde{\rho}_3^2(x) = \{2^{-x}7(x+1)(x+2)(x+6)(2x+15-97^{1/2})(2x+15+97^{1/2})(-17+97^{1/2})(17+97^{1/2})\tilde{\rho}_3^2(0)\} /\{12(x+7)(2x-97^{1/2}+17)(2x+17+97^{1/2})(15+97^{1/2})(-15+97^{1/2})\};$$

Recalling that the bottom function $\tilde{\Psi}_{\gamma,0}(x)$ is the solution of the equation (see Eq. (4.9))

$$L_{\hat{\alpha}}\tilde{Y}(x) = 0, \tag{4.39}$$

one directly deduces from the proposition 4.1, the orthogonality, on the interval $[0, +\infty)$, of the bottom function $\tilde{\Psi}_{\gamma,0}(x)$ with each of the elements from the higher ladder $\tilde{\Psi}_{\gamma,n}(x)$, $n = 1, 2, \ldots$, relatively to the weights $\tilde{\rho}_\gamma^2(x)$. On the other side, as one can easily verify, the equation (4.12) is verified for $\alpha, \beta = 1, 2, \ldots$, $\alpha \neq \beta$. Consequently, the transformed operator $\tilde{H}_\gamma$ (in l.h.s. of Eq. (4.37)) is symmetric in the subspace of $\ell^2(0, +\infty; \tilde{\rho}_\gamma^2(x))$ generated by the higher ladder $\tilde{\Psi}_{\gamma,n}(x)$, $n = 1, 2, \ldots$. Hence, the functions from this ladder are there mutually orthogonal. Thus, we have obtained that all the new functions $\tilde{\Psi}_{\gamma,n}(x)$, $n = 0, 1, 2, \ldots$ are mutually orthogonal on the interval $[0, +\infty)$ relatively to the weights $\tilde{\rho}_\gamma^2(x)$ given above.

A direct consequence of this, is the orthogonality relations between polynomials related to the $\tilde{\Psi}$-functions. Namely:

The polynomials

$$\bar{\mathcal{P}}_{1,0}(x) = 1;$$

$$\bar{\mathcal{P}}_{1,n}(x) = (x+2)(x+1)\mathcal{P}_{1,n}(x), n = 1, 2, 3, 4, 5; \ldots$$

are orthogonal on the same interval as $\tilde{\Psi}_{1,n}(x)$ but now with the weight

$$\tilde{\varrho}_1^2(x) = [(x+1)(x+2)Q_1(x)]^{-2}\,\tilde{\rho}_1^2(x).$$

Identically, the polynomials

$$\bar{\mathcal{P}}_{2,0}(x) = Q_2(x);$$

$$\bar{\mathcal{P}}_{2,n}(x) = [\tilde{\Psi}_{2,0}(x)]^{-1}\mathcal{P}_{2,n}(x), n = 1, 2, 3, 4, 5; \ldots$$

are orthogonal on the same interval as $\tilde{\Psi}_{2,n}(x)$ but now with the weight

$$\tilde{\varrho}_2^2(x) = \left[[\tilde{\Psi}_{2,0}(x)]^{-1}Q_2(x)\right]^{-2}\tilde{\rho}_2^2(x).$$

Finally, the polynomials

$$\bar{\mathcal{P}}_{3,0}(x) = (x+6)^{-1}Q_3(x);$$

$$\bar{\mathcal{P}}_{3,n}(x) = [(x+6)\tilde{\Psi}_{3,0}(x)]^{-1}\mathcal{P}_{3,n}(x), n = 1, 2, 3, 4, 5; \ldots$$

are orthogonal on the same interval as $\tilde{\Psi}_{3,n}(x)$ but now with the weight

$$\tilde{\varrho}_3^2(x) = \left[[(x+6)\tilde{\Psi}_{3,0}(x)]^{-1}Q_3(x)\right]^{-2}\tilde{\rho}_3^2(x).$$

Completion.

The completion of the functions $\tilde{\Psi}_{\gamma,n}(x)$, $n = 0, 1, 2, \ldots$, in $\ell^2(0, +\infty; \tilde{\rho}_\gamma^2(x))$ follows from the one of the Meixner $M_n^{(2,c)}(x+1)$ in $\ell^2(-1, +\infty; \rho^2(x))$. This is a direct consequence of the proposition 4.1. That proposition was in reality formulated so that this completion must hold, once the proposition was proved. In particular here, $a = -1$, so that $\tilde{\Theta}_{\hat{\alpha}}$ and Θ_α considered in the proposition 4.1 are obtained respectively from $\tilde{\Psi}_{\gamma,0}(x)$ and $M_\alpha^{(2,c)}(x+1)$ by

sending coordinates corresponding to negative x, to 0.
On the other side , the completion of the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x)$, $n = 0, 1, 2 \dots$ in $\ell^2(0, +\infty; \tilde{\varrho}^2_\gamma(x))$ follows from that of $\tilde{\Psi}_{\gamma,n}(x)$, $n = 0, 1, 2 \dots$ in $\ell^2(0, +\infty; \tilde{\rho}^2_\gamma(x))$. This completion holds well in spite of the fact that, in each of the constructed sequences of polynomials, there exists at least one number $n \in Z^+$ such that, no polynomial from that sequence has degree exactly equal n.

Difference and recurrence relations.

As other deductible properties of the new functions, one easily finds from the formula (4.36) and the recurrence relations for the Meixner $M_n^{(2,c)}(x+1)$ (see Eq. (4.33)) polynomials, the following five-term recurrence relations:

$$\tilde{\Psi}_{\gamma,n+2}(x) + [b_{n+1} + b_n - 2x + 1]\tilde{\Psi}_{\gamma,n+1}(x) + [a^2_{n+1} + a^2_n + (b_n - x + 1)(b_n - x)]\tilde{\Psi}_{\gamma,n}(x) + [a^2_n(b_n + b_{n-1} - 2x + 1)]\tilde{\Psi}_{\gamma,n-1}(x) + a^2_n a^2_{n-1}\tilde{\Psi}_{\gamma,n-2}(x) = 0, n = 3, 4, \dots.$$

But as it is easily seen, those recurrence relations are satisfied by any linear combination (with coefficients depending of x and not of n) of Meixner $M_n^{(2,c)}(x+1)$, and $M_n^{(2,c)}(x)$. In other words, the relations do not depend of $f_\gamma(x)$.
More characteristic recurrence relations for those functions are the following three-term recurrence relations that one can establish from the formula (4.36), the difference eigenvalue equation (4.31) and the difference relations (4.34)-(4.35) satisfied by the Meixner $M_n^{(2,c)}(x+1)$ polynomials:

$$[(c-1)a_{n+1}(f_\gamma(x)l(x-1;n) - w(x) - f_\gamma(x)l(x;n) + f_\gamma(x)(v(x) - \lambda_n) - f^2_\gamma(x)u(x-1))]\tilde{\Psi}_{\gamma,n+1}(x) + [f_\gamma(x)k(x-1;n)l(x;n) - f_\gamma(x)k(x-1;n)(v(x) - \lambda_n) + f^2_\gamma(x)k(x-1;n)u(x-1) - w(x)l(x;n) - f_\gamma(x)k(x;n)l(x-1;n) + k(x;n)w(x) + f_\gamma(x)l(x-1;n)(v(x) - \lambda_n) - f^2_\gamma(x)u(x-1)l(x-1;n)]\tilde{\Psi}_{\gamma,n}(x) + [(c-1)a_n(f_\gamma(x)(v(x) - \lambda_n) - f_\gamma(x)k(x;n) - f^2_\gamma(x)u(x-1) + f_\gamma(x)k(x-1;n) - w(x))]\tilde{\Psi}_{\gamma,n-1}(x) = 0, n = 2, 3, \dots$$

The obtained functions satisfy also the following third-order difference relations, which follow clearly from the difference relations (4.34)-(4.35) satisfied by the Meixner $M_n^{(2,c)}(x+1)$ polynomials, together with the formula (4.36) and its inverse:

$-\lambda_\gamma(c-1)a_{n+1}\tilde{\Psi}_{\gamma,n+1}(x) = \{[1+f_\gamma(x)\mathbf{E}_x^{-1}][u(x)\mathbf{E}_x + k(x;n)][u(x)\mathbf{E}_x + g_\gamma(x)]\}\tilde{\Psi}_{\gamma,n}(x)$

$\lambda_\gamma(c-1)a_n\tilde{\Psi}_{\gamma,n-1}(x)$
$= \{[1+f_\gamma(x)\mathbf{E}_x^{-1}][u(x)\mathbf{E}_x + l(x;n)][u(x)\mathbf{E}_x + g_\gamma(x)]\}\tilde{\Psi}_{\gamma,n}(x) n = 2,3,\ldots.$

Using the second-order difference eigenvalue equation (4.37) satisfied by the new functions and the preceding relations, one can reach if necessary, the following first-order difference relations

$-\lambda_\gamma(c-1)a_{n+1}\tilde{\Psi}_{\gamma,n+1}(x) = \{[-u(x)(v(x+1)-\lambda_n) + u(x)g_\gamma(x+1) + k(x;n)u(x) + f_\gamma(x)u(x-1)u(x) - u(x)f_\gamma(x)k(x-1;n)g_\gamma(x-1)/w(x)]\mathbf{E}_x + [-u(x)w(x+1) + k(x;n)g_\gamma(x) + f_\gamma(x)g_\gamma(x)u(x-1) + f_\gamma(x)k(x-1;n)u(x-1) - f_\gamma(x)g_\gamma(x-1)k(x-1;n)(v(x)-\lambda_n)/w(x)]\}\tilde{\Psi}_{\gamma,n}(x)$

$\lambda_\gamma(c-1)a_n\tilde{\Psi}_{\gamma,n-1}(x) = \{[-u(x)(v(x+1)-\lambda_n)+u(x)g_\gamma(x+1)+l(x;n)u(x)+ f_\gamma(x)u(x-1)u(x) - u(x)f_\gamma(x)l(x-1;n)g_\gamma(x-1)/w(x)]\mathbf{E}_x + [-u(x)w(x+1) + l(x;n)g_\gamma(x) + f_\gamma(x)g_\gamma(x)u(x-1) + f_\gamma(x)l(x-1;n)u(x-1) - f_\gamma(x)g_\gamma(x-1)l(x-1;n)(v(x)-\lambda_n)/w(x)]\}\tilde{\Psi}_{\gamma,n}(x) n = 2,3,\ldots.$

It is clear that the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x)$ satisfy the same recurrence relations as the functions $\tilde{\Psi}_{\gamma,n}(x)$. Their difference relations are also obviously deduced from those of $\tilde{\Psi}_{\gamma,n}(x)$.

Duality.

In a parallel to the above results for the functions $\tilde{\Psi}_{\gamma,n}(x)$ and the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x)$, one can of course deduce the corresponding results for the polynomials dual to the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x)$ (more precisely, dual to the functions $\tilde{\Psi}_{\gamma,n}(x)$), $n = 1,2,\ldots$, thanks to the existence of dual (self-duality) polynomials for the Meixner polynomials. We will denote $\mathcal{D}_{\gamma,x}(n) = \breve{\rho}(x)\tilde{\Psi}_{\gamma,n+1}(x)$, $n,\ x = 0,1,2,\ldots$, $\breve{\rho}(x) = c^x\Gamma(x+3)$.
$\mathcal{D}_{\gamma,x}(n)$ are polynomials in n, of degree exactly equal $x+1$ and satisfy the usual three-term recurrence relations (in x) with (see Eq. (4.37)):

$$b_{\gamma,x} = \tilde{v}_\gamma(x); \qquad a^2_{\gamma,x} = \tilde{w}_\gamma(x)u(x-1). \tag{4.40}$$

The polynomials in n

$$\check{\mathcal{D}}_{\gamma,0} = 1; \qquad \check{\mathcal{D}}_{\gamma,1} = (c-1)n - b_{\gamma,0}; \tag{4.41}$$

$$\check{\mathcal{D}}_{\gamma,x+1} = ((c-1)n - b_{\gamma,x})\check{\mathcal{D}}_{\gamma,x} - a^2_{\gamma,x}\check{\mathcal{D}}_{\gamma,x-1}, \qquad x = 1,2,\ldots \tag{4.42}$$

are for $x = 0,1,2\ldots$ of degree exactly equal x and are clearly orthogonal, the condition required in the well known Favard theorem ($b_{\gamma,x}$ reel and $a^2_{\gamma,x} > 0$, $x \geq 1$), being satisfied. On the other side, $\mathcal{D}_{\gamma,x}(n) = \check{\mathcal{D}}_{\gamma,x}(n)\check{\rho}(0)\tilde{\Psi}_{\gamma,n+1}(0)$, where from Eq. (4.36), $\tilde{\Psi}_{\gamma,n+1}(0) = (\gamma)_n(\frac{c}{c-1})^n[f_\gamma(0) + 1 + \frac{1}{2}n(1-\frac{1}{c})]$, $(\gamma)_0 = 1$, $(\gamma)_n = \gamma(\gamma+1)\ldots(\gamma+n-1)$, $n = 1,2\ldots$. Consequently,

$$\tilde{\Psi}_{\gamma,n+1}(x) = \frac{(\gamma)_n(\frac{c}{c-1})^n}{c^x\Gamma(x+3)}.[2f_\gamma(0) + 2 + n(1-\tfrac{1}{c})]\check{\mathcal{D}}_{\gamma,x}(n), \quad x,n = 0,1,\ldots. \tag{4.43}$$

This ends the solution of the "problem 3" for the special Meixner $M_n^{(2,c)}(x+1)$.

Concluding this chapter, let us remark that, on their turn, the obtained polynomials and functions can be transformed in a similar manner. Here, the reservoir of transformation functions is composed of

$$\Phi_{\hat{\alpha},n}(x) = [1 + f_{\hat{\alpha}}\mathbf{E}_x^{-1}]c^{-x}\hat{M}_n(x), \; n,\hat{\alpha} \in \mathbf{N}, \; n \neq \hat{\alpha}, \tag{4.44}$$

and clearly, the process can, in principle, be repeated infinitely.

The existence of other efficiently "modifiable" particular cases of the hypergeometric difference operator H in Eq. (4.14), "higher" to the special Meixner $M_n^{(2,c)}(x+1)$ treated here, is not of course precluded. As noted in [36], it is neither precluded that such generated solvable Hamiltonians should be generated by type quasi-periodicity or shape-invariance (symmetry) techniques. The non-classical polynomials for example in ([40], equation (4.13)) related to the Hermite polynomials generated by "quasi-periodicity" methods had been rediscovered in [104], using the "modification method" on the Hermite polynomials (see the case $m = 2$ in [104]). Those related to the Jacobi polynomials in [77] were generated by shape-invariance (symmetry) techniques. On the side, we know that in "Laguerre-Hahn" theory [79], a whole subset of non-semi-classical Laguerre-Hahn polynomials can be generated by simple shape-invariance (symmetry) of factorization chains (see explicit examples in the second-section of the following chapter).

It is worth remarking that no efficient "modification" of a fourth-order Sturm-Liouville difference (or differential) operator (using for example one

of the formulas from section 2.2.2) is known in literature.
Remark next that if the constraint of being polynomials is rejected as well for the "transformable" functions as for the "transformed" ones (retaining only the constraints of integrability and completeness), the situation becomes much easier. Thus, nowadays, the used here method is one of the main tools for generating new exactly solvable Hamiltonians in quantum mechanics (see for example [90, 98, 99, 100, 120]).
In [118] were studied transformations of orthogonal polynomials such that the weights of the new polynomials are obtained by multiplying a *rational* function to the weights of the original ones. Let us note that although the weights $\tilde{\varrho}_\gamma^2(x)$, $\gamma = \overline{1,3}$ are obtained by multiplying rational functions to $\rho_\gamma^2(x)$, $\gamma = \overline{1,3}$, respectively, the situation here is generally that the new weights are obtained from the old ones by multiplying to them not by rational functions but a ratio of products of Gamma and may be exponential functions. So that the polynomials studied here are not in principle of the same kind than those from [118]. Another remarkable difference resides in that the polynomials studied in [118] are "ordinary" orthogonal polynomials satisfying a three-term recurrence relation and having degrees exactly equal $0, 1, 2, \ldots$, while the orthogonal polynomials studied here do not satisfy the usual three-term recurrence relation and are not of degrees exactly equal $0, 1, 2, \ldots$ (see above).
As already noted the polynomials studied here conserve most of the properties of "ordinary" orthogonal polynomials (completeness, orthogonality, difference and recurrence relations, difference (eigenvalue) equations, duality,...). Other explicit examples of such orthogonal polynomials are in [104, 40, 44].

Chapter 5

The Laguerre-Hahn orthogonal polynomials on special non-uniform lattices

The Laguerre researches in orthogonal polynomials theory were essentially based on the theory of continued fractions satisfying first order differential equations. The corresponding orthogonal polynomials appeared then as the denominators in the n^{th} Padé approximation. The crucial question solved by Laguerre him self [74], consisted then of showing how to construct the differential equations satisfied by those polynomials.
The Hahn works went generally in the converse sense [53, 54]: Characterize orthogonal polynomials satisfying linear differential equations. It appeared that those are nothing else than the polynomials studied by Laguerre from continued fractions point of view.
In such a situation, a question naturally arises: What is the essential characteristic of the polynomials appearing in the Laguerre and Hahn researches? In other words, what can be seen as the "Laguerre-Hahn" polynomials? The answer is in [79] (see [78] for the continuous case): The *Laguerre-Hahn polynomials* can be considered as those for which the corresponding Stieltjes function (the continued fraction) satisfies a certain Riccati equation. It appears that most of known orthogonal polynomials belong to this class. This chapter is devoted to them considering that we surprisingly have noted the existence of a significant interconnection between the so-called "Laguerre-Hahn" approach to orthogonal polynomials and the discussed above shape-invariant Infeld-Hull-Miller factorization technique.

The plan of the chapter reads as follows: We firstly recall notions and results on Laguerre-Hahn polynomials on special nonuniform lattices [79, 80, 82], that will be useful in the following section and in the chapter 6 (section 5.1) (this matter and other results can be found in details in the cited Magnus works). Next, we attempt to show the interconnection between the factorization technique and the Laguerre-Hahn approach to orthogonal polynomials on special nonuniform lattices (section 5.2).

5.1 Definitions and properties.

Searching for two functions $\eta_2(x)$ and $\eta_1(x)$ such that the difference operator

$$(\mathcal{D}f)(x) = \frac{f(\eta_2(x)) - f(\eta_1(x))}{\eta_2(x) - \eta_1(x)} \tag{5.1}$$

leaves a polynomial of degree $n-1$ when applied to a polynomial of degree n, one finds that $\eta_2(x)$ and $\eta_1(x)$ must be two roots y of some quadratic equation [79, 80]:

$$F(x,y) := c_0 y^2 + 2c_1 xy + c_2 x^2 + 2c_3 y + 2c_4 x + c_5 = 0.$$

Searching next for a parameterization $x(s)$, $y(s)$ such that $\eta_2(x(s)) = y(s+1)$, $\eta_1(x(s)) = y(s)$, one is led to [79, 80]:

$$\begin{aligned} x(s) &= \tilde{c}_1 q^s + \tilde{c}_2 q^{-s} + \tilde{c}_3, \\ y(s) &= \tilde{c}_4 x(s - \frac{1}{2}) + \tilde{c}_5, \end{aligned} \tag{5.2}$$

the so-called special non-uniform lattice (snul).
Let $P_n(y(s))$ be a sequence of orthogonal on snul lattice polynomials with the orthogonality measure $d\varepsilon$,

$$S(y) := \int_{Supp.\varepsilon} \frac{d\varepsilon(\tau)}{y - \tau},$$

the corresponding Stieltjes function. The polynomials $P_n(y(s))$ are called (class κ) Laguerre-Hahn orthogonal on snul polynomials (LHP) iff the Stieltjes function $S(y(s))$ satisfies the Riccati equation [79]:

$$\begin{aligned} \mathcal{A}(x(s))\tfrac{S(y(s+1))-S(y(s))}{y(s+1)-y(s)} &= \mathcal{B}(x(s))S(y(s+1))S(y(s)) \\ &+ \mathcal{C}(x(s))\tfrac{S(y(s+1))+S(y(s))}{2} + \mathcal{D}(x(s)) \end{aligned} \tag{5.3}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are polynomials of degrees $\leq \kappa+2$, $\kappa+2$, $\kappa+1$ and κ, respectively. Most of nowadays known orthogonal polynomials belong to this class. The subclass of semi-classical orthogonal polynomials [79, 80], corresponds to the case $\mathcal{B}=0$. The classical polynomials appear then as the semi-classical of class $\kappa=0$.

Next, let $P_{n-m}^{(m)}(y(s))$, $m \in Z^+$ be the m-associated polynomials of $P_n(y(s))$, i.e.

$$\begin{aligned} a_{n+1}P_{n+1-m}^{(m)} + (b_n - y(s))P_{n-m}^{(m)} + a_n P_{n-1-m}^{(m)} = 0, \\ a_{n+1}P_{n+1} + (b_n - y(s))P_n + a_n P_{n-1} = 0 \end{aligned} \tag{5.4}$$

$(n,\, m = 0, 1, 2, \ldots)$ and let $S_m(y(s))$ be the corresponding Stieltjes functions. It can be proved that (see [79]) $S_m(y(s))$ also satisfies a Riccati equation similar to Eq. (5.3):

$$\begin{aligned} \mathcal{A}_m(x(s))\tfrac{S_m(y(s+1))-S_m(y(s))}{y(s+1)-y(s)} = \mathcal{B}_m(x(s))S_m(y(s+1))S_m(y(s)) \\ +\mathcal{C}_m(x(s))\tfrac{S_m(y(s+1))+S_m(y(s))}{2} + \mathcal{D}_m(x(s)), m = 0, 1, 2, \ldots \end{aligned} \tag{5.5}$$

where $\mathcal{A}_m$, $\mathcal{B}_m$, $\mathcal{C}_m$ and $\mathcal{D}_m$ ($\mathcal{A}_0 = \mathcal{A}$, $\mathcal{B}_0 = \mathcal{B}$, $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{D}_0 = \mathcal{D}$) are as well polynomials of degree $\leq \kappa+2$, $\kappa+2$, $\kappa+1$ and κ, respectively. In other words, the class of Laguerre-Hahn polynomials is invariant under the operation of passage to associated polynomials (see another approach in [122]).
Let us write Eq. (5.5) in the homographic form:

$$S_m(y(s+1)) = \frac{\left(\frac{\mathcal{A}_m(x(s))}{y(s+1)-y(s)} + \frac{\mathcal{C}_m(x(s))}{2}\right)S_m(y(s)) + \mathcal{D}_m(x(s))}{\frac{\mathcal{A}_m(x(s))}{y(s+1)-y(s)} - \frac{\mathcal{C}_m(x(s))}{2} - \mathcal{B}_m(x(s))S_m(y(s))}. \tag{5.6}$$

The coefficients of this transformation iterate as follows [79]:

$$\begin{aligned} &\frac{\mathcal{A}_{m+1}(x(s))}{y(s+1)-y(s)} + \frac{\mathcal{C}_{m+1}(x(s))}{2} \\ &= \frac{\mathcal{A}_m(x(s))}{y(s+1)-y(s)} - \frac{\mathcal{C}_m(x(s))}{2} - (y(s+1) - b_m)\mathcal{D}_m(x(s)); \\ &\frac{\mathcal{A}_{m+1}(x(s))}{y(s+1)-y(s)} - \frac{\mathcal{C}_{m+1}(x(s))}{2} \\ &= \frac{\mathcal{A}_m(x(s))}{y(s+1)-y(s)} + \frac{\mathcal{C}_m(x(s))}{2} + (y(s) - b_m)\mathcal{D}_m(x(s)); \\ &\left(\frac{\mathcal{A}^2_{m+1}(x(s))}{(y(s+1)-y(s))^2} - \frac{\mathcal{C}^2_{m+1}(x(s))}{4}\right) - \left(\frac{\mathcal{A}^2_m(x(s))}{(y(s+1)-y(s))^2} - \frac{\mathcal{C}^2_m(x(s))}{4}\right) \\ &= a_m^2\mathcal{D}_m\mathcal{D}_{m-1} - a_{m+1}^2\mathcal{D}_{m+1}\mathcal{D}_m \end{aligned} \tag{5.7}$$

with

$$\mathcal{B}_m(x(s)) := a_m^2 \mathcal{D}_{m-1}(x(s)), \qquad m := 0, 1, 2, \ldots. \tag{5.8}$$

Consider next the following notations (letting from now on $m \equiv n$):

$$\alpha_n(x(s)) = -\left(\frac{\mathcal{A}_n(x(s))}{y(s+1)-y(s)} - \frac{\mathcal{C}_n(x(s))}{2}\right); \beta_n(x) = \frac{\mathcal{A}_n(x(s))}{y(s+1)-y(s)} + \frac{\mathcal{C}_n(x(s))}{2}$$
$$\gamma_n(x(s)) = \frac{\mathcal{B}_n(x(s))}{\mu_0}; \delta_n(x(s)) = \ddot{\mu}_0 \mathcal{D}_n(x(s)), \tag{5.9}$$

$\ddot{\mu}_n$ being the moments of $d\varepsilon$.
Using the relations (5.7),(5.8) and (5.4), it has been established in [79] that the Laguerre-Hahn polynomials satisfy the following difference-recurrence relations:

$$\begin{aligned}
&\beta_n(x(s))P_n(y(s)) + a_n\mathcal{D}_n(x(s))P_{n-1}(y(s)) \\
&= \beta_0(x(s))P_n(y(s+1)) + \gamma_0(x(s))P_{n-1}^{(1)}(y(s+1)), && (5.10)\\
&\alpha_n(x(s))P_n(y(s+1)) + a_n\mathcal{D}_n(x(s))P_{n-1}(y(s+1)) \\
&= \alpha_0(x(s))P_n(y(s)) + \gamma_0(x(s))P_{n-1}^{(1)}(y(s)), && (5.11)\\
&\beta_{n+1}(x(s))P_{n-1}^{(1)}(y(s+1)) + a_{n+1}\mathcal{D}_n(x(s))P_n^{(1)}(y(s+1)) \\
&= \beta_0(x(s))P_{n-1}^{(1)}(y(s)) + \delta_0(x(s))P_n(y(s)), && (5.12)\\
&\alpha_{n+1}(x(s))P_{n-1}^{(1)}(y(s)) + a_{n+1}\mathcal{D}_n(x(s))P_n^{(1)}(y(s)) \\
&= \alpha_0(x(s))P_{n-1}^{(1)}(y(s+1)) + \delta_0(x(s))P_n(y(s+1)). && (5.13)
\end{aligned}$$

Remark 5.1 Considering the relations in Eqs. (5.7) and (5.8) for $m := r, r+1, \ldots,$ one finds that the difference-recurrence relations for the polynomials $P_{n-r}^{(r)}$, r-associated to P_n, are found from the preceding ones by shifting the initial value for n, from n:=0 to n:=r. One then obtains in place of Eqs. (5.10)-(5.13):

$$\begin{aligned}
&\beta_n(x(s))P_{n-r}^{(r)}(y(s)) + a_n\mathcal{D}_n(x(s))P_{n-1-r}^{(r)}(y(s)) \\
&= \beta_r(x(s))P_{n-r}^{(r)}(y(s+1)) + \gamma_r(x(s))P_{n-1-r}^{(1+r)}(y(s+1)), && (5.14)\\
&\alpha_n(x(s))P_{n-r}^{(r)}(y(s+1)) + a_n\mathcal{D}_n(x(s))P_{n-1-r}^{(r)}(y(s+1)) \\
&= \alpha_r(x(s))P_{n-r}^{(r)}(y(s)) + \gamma_r(x(s))P_{n-1-r}^{(1+r)}(y(s)), && (5.15)\\
&\beta_{n+1}(x(s))P_{n-1-r}^{(1+r)}(y(s+1)) + a_{n+1}\mathcal{D}_n(x(s))P_{n-r}^{(1+r)}(y(s+1))
\end{aligned}$$

$$= \beta_r(x(s))P^{(1+r)}_{n-1-r}(y(s)) + \delta_r(x(s))P^{(r)}_{n-r}(y(s)), \tag{5.16}$$

$$\alpha_{n+1}(x(s))P^{(1+r)}_{n-1-r}(y(s)) + a_{n+1}\mathcal{D}_n(x(s))P^{(1+r)}_{n-r}(y(s))$$
$$= \alpha_r(x(s))P^{(1+r)}_{n-1-r}(y(s+1)) + \delta_r(x(s))P^{(r)}_{n-r}(y(s+1)). \tag{5.17}$$

In the semi-classical situation, $\gamma_0(x) = 0$ and it was proven in [80] that the class of semi-classical on snul lattices polynomials is fully characterized by the equations (5.10) and (5.11). Similar question remains open for the Laguerre-Hahn polynomials. Moreover, in the semi-classical case, the second order difference equation is obtained directly by a combination of the equations (5.10) and (5.11). The result reads [79]:

$$\begin{aligned}&\mathcal{D}_n(x(s-1))\beta_0(x(s))P_n(y(s+1))\\ &-[\mathcal{D}_n(x(s-1))\beta_n(x(s)) - \mathcal{D}_n(x(s))\alpha_n(x(s-1))]P_n(y(s))\\ &-\mathcal{D}_n(x(s))\alpha_0(x(s-1))P_n(y(s-1)) = 0.\end{aligned} \tag{5.18}$$

In the classical case, $\mathcal{D}_n(x(s))$ is a constant (in x) and Eq. (5.18) becomes:

$$\begin{aligned}&\beta_0(x(s))P_n(y(s+1)) - [\beta_n(x(s)) - \alpha_n(x(s-1))]P_n(y(s))\\ &-\alpha_0(x(s-1))P_n(y(s-1)) = 0.\end{aligned} \tag{5.19}$$

In the Laguerre-Hahn case, $\gamma_0(x)$ is no more vanishing so that we are led to work with the presence of r-associated notion when searching for such equations. In [79] (see also [82]), the Laguerre-Hahn polynomials were shown to be expressible in a combination of products of functions each satisfying second order difference and three-term recurrence equations. Namely, consider the following equation

$$\mathcal{H}(s;n)\varphi_n(y(s)) = 0 \tag{5.20}$$

where

$$\begin{aligned}\mathcal{H}(s;n) &= \mathcal{D}_n(x(s-1))\mathbf{E}_s + [\mathcal{D}_n(x(s))\alpha_n(x(s-1))\\ &-\mathcal{D}_n(x(s-1))\beta_n(x(s))] + \mathcal{D}_n(x(s))[a_0^2\mathcal{D}_0(x(s-1))\mathcal{D}_{-1}(x(s-1))\\ &-\alpha_0(x(s-1))\beta_0(x(s-1))]\mathbf{E}_s^{-1}.\end{aligned} \tag{5.21}$$

Consider on the other side the recurrence equation

$$a_{n+1}\varphi_n(y) = (y - b_n)\varphi_n(y) - a_n\varphi_{n-1}(y). \tag{5.22}$$

It was proven in [79] (see also [82]) the following

Proposition 5.1 *There exists sequences of functions $\{\varphi_n^1\}$ and $\{\varphi_n^2\}$, $n \in Z$, satisfying simultaneously the difference equation (5.20) and the recurrence relation (5.22) such that*

$$\begin{aligned} P_n(y) &= \frac{1}{\sqrt{\mu_0}} \frac{\varphi_n^1(y)\varphi_{-1}^2(y) - \varphi_n^2(y)\varphi_{-1}^1(y)}{\varphi_0^1(y)\varphi_{-1}^2(y) - \varphi_0^2(y)\varphi_{-1}^1(y)}, \\ P_{n-r}^r(y) &= K_r \frac{1}{\sqrt{\mu_0}} \frac{\varphi_n^1(y)\varphi_{r-1}^2(y) - \varphi_n^2(y)\varphi_{r-1}^1(y)}{\varphi_0^1(y)\varphi_{-1}^2(y) - \varphi_0^2(y)\varphi_{-1}^1(y)}. \end{aligned} \tag{5.23}$$

Moreover, an algorithm for establishing fourth order difference equation satisfied by those polynomials (LHP) was given in [79]. We will not go further in this matter here: In chapter 6, we will return to this question establishing the fourth order difference equation (using an algorithm different of that from [79]) for the concerned polynomials and giving it explicitly for some particular cases of them.
For the moment, the recalled notions allow to start the following section.

5.2 Interconnection between the DFT and the Laguerre-Hahn approach to orthogonal polynomials.

We are going to show very simply but very surprisingly that the system (5.7) can be considered as an extension of the factorization chain (2.154) submitted to the condition (2.163).

Rewrite for convenience as follows the equations in (5.7) (taking into account of Eq. (5.9))

$$\begin{aligned} \beta_{n+1}(x(s)) &= -\alpha_n(x(s)) - (y(s+1) - b_n)\mathcal{D}_n(x(s)) \\ \alpha_{n+1}(x(s)) &= -\beta_n(x(s)) - (y(s) - b_n)\mathcal{D}_n(x(s)) \\ \alpha_{n+1}\beta_{n+1} &= \alpha_n\beta_n - a_n^2\mathcal{D}_n\mathcal{D}_{n-1} + a_{n+1}^2\mathcal{D}_{n+1}\mathcal{D}_n, \end{aligned} \tag{5.24}$$

the factorization chain (2.154) and the condition in Eq. (2.163)

$$\begin{aligned} f(s+1;n+1) + g(s;n+1) &= f(s;n) + g(s+1;n) \\ f(s;n+1)g(s;n+1) &= f(s;n)g(s;n) + \mu(n) - \mu(n+1) \end{aligned} \tag{5.25}$$

$$f(s;n) - g(s;n-1) = c_1(n)y(s) + c_2(n). \tag{5.26}$$

A combination of the first equation in (5.25) and the equation (5.26) transforms Eqs. (5.25)-(5.26) in

$$\begin{aligned} g(s;n) &= g(s;n-1) - (c_1(n)y(s+1) + c_2(n)) \\ f(s;n) &= g(s;n-1) + c_1(n)y(s) + c_2(n) \\ f(s;n+1)g(s;n+1) &= f(s;n)g(s;n) + \mu(n) - \mu(n+1). \end{aligned} \tag{5.27}$$

On the other side, a combination of the first and second equations in (5.24) transforms it in

$$\begin{aligned} &\tfrac{\beta_{n+1}(x(s))}{\mathcal{D}_n(x(s))} - \tfrac{\alpha_{n+1}(x(s+1))}{\mathcal{D}_n(x(s+1))} = \tfrac{\beta_{n+1}(x(s+1))}{\mathcal{D}_n(x(s+1))} - \tfrac{\alpha_{n+1}(x(s))}{\mathcal{D}_n(x(s))} \\ &\alpha_{n+1}\beta_{n+1} = \alpha_n\beta_n - a_n^2\mathcal{D}_n\mathcal{D}_{n-1} + a_{n+1}^2\mathcal{D}_{n+1}\mathcal{D}_n \end{aligned} \tag{5.28}$$

$$\alpha_{n+1}(x(s)) = -\beta_n(x(s)) - (y(s) - b_n)\mathcal{D}_n(x(s)). \tag{5.29}$$

Our point here consists in saying that the system of equations (5.24) can be considered as an extension of Eq. (5.27) or identically, Eqs. (5.28)-(5.29) relatively to Eqs. (5.25)-(5.26).

This observation is founded at least in appearance. A more founded argument consists in remarking that if for the polynomial $\mathcal{D}_n(x)$, only one term, say the leading, is different from zero i.e. $\mathcal{D}_n(x) = \hat{d}(n)x^\kappa$ (one may consider the general situation when the variables x and n in $\mathcal{D}_n(x)$ are separable i.e. $\mathcal{D}_n(x) = c(n)p(x)$, $c(n)$ and $p(x)$ constant and polynomial in x respectively), then the equations in (5.24) are made equivalent to the ones in (5.27) (or identically, Eqs. (5.28)-(5.29) relatively to Eqs. (5.25)-(5.26)) by setting

$$\begin{aligned} f(s;n) &:= -\frac{\alpha_{n+1}(x(s))}{x^\kappa}, \\ g(s;n) &:= \frac{\beta_{n+1}(x(s))}{x^\kappa}. \end{aligned} \tag{5.30}$$

In other words, under the condition

$$\mathcal{D}_n = \hat{d}_n x^\kappa, \tag{5.31}$$

the systems of equations (5.24) and (5.27) (or identically, Eqs. (5.28)-(5.29) and (5.25)-(5.26)) are first both factorization chains available in orthogonal polynomials theory, second mutually-equivalent and third allow (each on its side) in principle to generate a special (but without restriction in rapport with the class κ) part of Laguerre-Hahn orthogonal polynomials. But as

the system of equations (5.24) (or equally Eqs. (5.28)-(5.29)) fully characterizes the Laguerre-Hahn polynomials, so the system of equations (5.24) can be considered in orthogonal polynomials theory as a generalization of the Infeld-Hull-Miller factorization chain (5.27) or identically, (5.28)-(5.29) relatively to Eqs. (5.25)-(5.26).
To express this in operatorial language, we first transform Eq. (5.28) "integrating" (in n) the second equation so that to obtain

$$\begin{aligned} \tfrac{\beta_{n+1}(x(s))}{\mathcal{D}_n(x(s))} - \tfrac{\alpha_{n+1}(x(s+1))}{\mathcal{D}_n(x(s+1))} &= \tfrac{\beta_n(x(s+1))}{\mathcal{D}_n(x(s+1))} - \tfrac{\alpha_n(x(s))}{\mathcal{D}_n(x(s))} \\ \tfrac{\alpha_{n+1}(x(s))\beta_{n+1}(x(s))}{\mathcal{D}_{n+1}(x(s))\mathcal{D}_n(x(s))} &= \tfrac{\alpha_0(x(s))\beta_0(x(s))-a_0^2\mathcal{D}_0(x(s))\mathcal{D}_{-1}(x(s))}{\mathcal{D}_{n+1}(x(s))\mathcal{D}_n(x(s))} + a_{n+1}^2, \end{aligned} \tag{5.32}$$

from which we next easily verify the following product

$$\begin{gathered} \tfrac{1}{\mathcal{D}_{n+1}(x(s))} H(s;n) - a_{n+1}^2 = \\ \left[\tfrac{1}{\mathcal{D}_{n+1}(x(s))}\left(\mathbf{E}_s - \beta_{n+1}(x(s))\right)\right]\left[\tfrac{1}{\mathcal{D}_n(x(s))}\left(\mathbf{E}_s + \alpha_{n+1}(x(s))\right)\right] \\ \tfrac{1}{\mathcal{D}_n(x(s))} H(s;n+1) - a_{n+1}^2 = \\ \left[\tfrac{1}{\mathcal{D}_n(x(s))}\left(\mathbf{E}_s + \alpha_{n+1}(x(s))\right)\right]\left[\tfrac{1}{\mathcal{D}_{n+1}(x(s))}\left(\mathbf{E}_s - \beta_{n+1}(x(s))\right)\right] \end{gathered} \tag{5.33}$$

where

$$\begin{aligned} H(s;n) = {} & \tfrac{1}{\mathcal{D}_n(x(s+1))}\mathbf{E}_s^2 - [\tfrac{\beta_{n+1}(x(s))}{\mathcal{D}_n(x(s))} \\ & - \tfrac{\alpha_{n+1}(x(s+1))}{\mathcal{D}_n(x(s+1))}]\mathbf{E}_s + \tfrac{a_0^2\mathcal{D}_0(x(s))\mathcal{D}_{-1}(x(s))-\alpha_0(x(s))\beta_0(x(s))}{\mathcal{D}_n(x(s))}. \end{aligned} \tag{5.34}$$

The rapport between this second order operator and the one in Eq. (5.21) is clear (taking into account of the first equation in (5.32) as well):

$$H(s;n) = \frac{1}{\mathcal{D}_n(x(s))\mathcal{D}_n(x(s+1))}.\mathbf{E}_s(\mathcal{H}(s;n)), \tag{5.35}$$

so that we are doing effectively with the Laguerre-Hahn polynomials.
It is very important to remark that the raising and lowering operators in Eq. (5.33) were originally found in [79, 82] as those linking the zeros of $H(s;n)$ to that of $H(s;n+1)$ and vice-versa.
In the event where Eq. (5.31) is verified, it is easily seen that the product (5.33) is nothing else than the Infeld-Hull-Miller factorization product (see Eq. (2.154)), which justifies again the evoked generalization.
Let us recall that in Eq. (5.24), the semi-classical situation corresponds to the case

$$a_0^2\mathcal{D}_0(x(s))\mathcal{D}_{-1}(x(s)) = 0. \tag{5.36}$$

In the "classical" situation (Eq. (5.36) together with $\kappa = 0$), the general case consists of Askey-Wilson polynomials (or Associated Askey-Wilson polynomials for $\kappa = 0$, $a_0^2\mathcal{D}_0(x(s))\mathcal{D}_{-1}(x(s)) \neq 0$) [79] (see also section 6.2 below). On its side, the product (5.33) coincides essentially (the lattices (5.2) need first to be transformed in their canonical forms as in section 6.2 below) with the factorization (3.63) of the Askey-Wilson operator $\tilde{\mathcal{L}}$ (see Eq. (3.51)). Indeed, in section 6.2, it will appear occasionally from practical computations that the operator in left hand side of Eq. (5.19) coincides essentially with $(z - z^{-1}).[\tilde{\mathcal{L}} - \lambda^1(n)]$ (see Eq. (3.72) for $\lambda^1(n)$).

Let us remark that the present is in fact one of the possible generalizations of the technique of factorization (2.152), (2.154), the already encountered extension to the fourth-order situation (section 2.2.2, in particular the remark 2.3) being probably the most natural.

What can we in fact do explicitly? In the "classical" situation ($\kappa = 0$), the general solution will be given occasionally in section 6.2. In non-classical situations ($\kappa \geq 1$), the general solution can no more be given in term of elementary functions. Indeed, the experience from the continuous (limit of discrete case) case [81] shows that the coefficients in the three-term recurrence relation satisfy type Painlevé equations. Reason for which probably, the passage from the situations in which $\kappa = 0$ to those of $\kappa \geq 1$ quickly increases difficulties. Let us study here one of the simplest purely Laguerre-Hahn situations, namely (recall the notation $\mathcal{D}_n = \hat{\hat{d}}(n) + \hat{d}(n)x(s)$) :

$$\kappa = 1; \qquad x(s) = s = y(s); \qquad a_0^2\hat{d}_0\hat{d}_{-1} \neq 0. \tag{5.37}$$

In that case, the system of equations (5.24) can be reduced to the following system of non-linear difference equations satisfied by the coefficients in the three-term recurrence relations (with two degrees of freedom in parameters in less: the general will be given in section 6.3):

$$\begin{aligned}&-\tfrac{1}{2}\alpha_3 - \tfrac{1}{2}\alpha_2 + \alpha_3 a_{n+2}^2 b_{n+2} + \alpha_3 b_{n+1} a_{n+2}^2 \\ &+(\tfrac{1}{2}\alpha_3 n - \alpha_1 + \alpha_3)b_{n+1} + (\alpha_1 - \tfrac{1}{2}\alpha_3 n)b_n \\ &-2\alpha_3 a_{n+1}^2 b_{n+1} - 2\alpha_3 b_n a_{n+1}^2 + \alpha_3 a_n^2 b_{n-1} + \alpha_3 b_n a_n^2 + (2\alpha_2 \\ &+2\alpha_3)a_{n+1}^2 - (\alpha_3 + \alpha_2)a_n^2 - (\alpha_3 + \alpha_2)a_{n+2}^2 = 0,\end{aligned} \tag{5.38}$$

$$\begin{aligned}&-2\alpha_1 + \alpha_3 n + \alpha_3 - (\alpha_3 + \alpha_2)b_{n+1} + (\alpha_2 + \alpha_3)b_n \\ &-\alpha_3 b_n^2 + \alpha_3 b_{n+1}^2 - \alpha_3 a_n^2 + \alpha_3 a_{n+2}^2 = 0,\end{aligned} \tag{5.39}$$

while the expressions for α_n, β_n and $\mathcal{D}_n$ read respectively

$$\begin{aligned}\alpha_{n+1} := -\{[\alpha_1 - \alpha_3 - \tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\alpha_3 n + \alpha_3 a_{n+2}^2 b_{n+2} + \alpha_3 b_{n+1} a_{n+2}^2 \\ +(\tfrac{3}{2}\alpha_3 + \tfrac{1}{2}\alpha_3 n - \alpha_1 + \tfrac{1}{2}\alpha_2) b_{n+1} - \alpha_3 b_n a_{n+1}^2 \\ -\alpha_3 a_{n+1}^2 b_{n+1} - \tfrac{1}{2}\alpha_3 b_{n+1}^2 + (\alpha_2 + \tfrac{3}{2}\alpha_3) a_{n+1}^2 - (\alpha_2 \\ +\tfrac{3}{2}\alpha_3) a_{n+2}^2] + [\alpha_1 - \tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\alpha_3 - \tfrac{1}{2}\alpha_3 n] s + [\tfrac{1}{2}\alpha_3] s^2\}; \qquad (5.40)\end{aligned}$$

$$\begin{aligned}\beta_{n+1} := [\alpha_3 a_{n+2}^2 b_{n+2} + \alpha_3 b_{n+1} a_{n+2}^2 + (\tfrac{1}{2}\alpha_3 n + \tfrac{1}{2}\alpha_3 - \alpha_1 - \tfrac{1}{2}\alpha_2) b_{n+1} \\ -\alpha_3 a_{n+1}^2 b_{n+1} - \alpha_3 b_n a_{n+1}^2 + \tfrac{1}{2}\alpha_3 b_{n+1}^2 + (\alpha_2 + \tfrac{1}{2}\alpha_3) a_{n+1}^2 - (\alpha_2 \\ +\tfrac{1}{2}\alpha_3) a_{n+2}^2] + [\alpha_1 + \tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\alpha_3 - \tfrac{1}{2}\alpha_3 n] s + [-\tfrac{1}{2}\alpha_3] s^2; \qquad (5.41)\end{aligned}$$

$$\mathcal{D}_n := [-\tfrac{1}{2}\alpha_3 - \alpha_2 + \alpha_3 b_n] + [\alpha_3] s. \qquad (5.42)$$

One needs to note that although the system (5.38)-(5.39) is related to Painlevé transcendent (see [81] for the continuous case), some of its particular cases may have elementary particular solutions. For example, setting

$$\begin{aligned}(\alpha_1, \alpha_2, \alpha_3) &= (\alpha_1, 0, 1); \\ (\alpha_1, \alpha_2, \alpha_3) &= \left(\frac{1}{4}\alpha_3 + \alpha_3(a_1^2 - a_0^2), -\alpha_3, \alpha_3\right), \qquad (5.43)\end{aligned}$$

and

$$(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{1}{4}\alpha_3 + 2\alpha_3(a_1^2 - a_0^2), -\alpha_3, \alpha_3\right), \qquad (5.44)$$

we obtain as particular solutions of Eqs. (5.38)-(5.39), the pairs

$$\begin{aligned}b_n &= \frac{1}{2} \\ a_n^2 &= -\frac{1}{4}n^2 + \alpha_1 n + \frac{1}{2}(a_0^2 - a_1^2 + \alpha_1 - \frac{1}{4})(-1)^n \\ &\quad + \frac{1}{2}(a_0^2 + a_1^2 - \alpha_1 + \frac{1}{4}); \qquad (5.45)\end{aligned}$$

$$\begin{aligned}b_n &= d \\ a_n^2 &= -\frac{1}{4}n^2 + (\frac{1}{4} + a_1^2 - a_0^2)n + a_0^2; \qquad (5.46)\end{aligned}$$

and

$$
\begin{aligned}
b_n &= \frac{1}{2}(n + 4(a_0^2 - a_1^2))i \\
a_n^2 &= -\frac{1}{8}n^2 + (\frac{1}{8} + a_1^2 - a_0^2)n + a_0^2
\end{aligned}
\tag{5.47}
$$

respectively, leading to Laguerre-Hahn polynomials orthogonal with respect to quasi-definite (but not positive-definite) moment functionals. It is clear that the first example (i.e. Eq. (5.45)) satisfies the condition (5.31) and so, considering Eq. (5.29), is "generable" from Infeld-Hull-Miller factorization chain (2.154)).

Chapter 6

The fourth order difference equation for the Laguerre-Hahn polynomials on special non-uniform lattices

The Magnus result in proposition 5.1 is in our sense very interesting. It permits, when working with the Laguerre-Hahn polynomials, to limit oneself to relatively simple difference operator i.e. the operator in Eq. (5.21). This is an operator of only second-degree and we less or more handle it: We can for example factorize it as in Eq. (5.33). The unique uncomfortable thing lies in our sense in the fact that the functions φ_n^1 and φ_n^2, from which the Laguerre-Hahn polynomials are expressed, are not easily palpable.
In any case it would be useful to establish a linear fourth-order difference equation, whose coefficients are "difference" (in s and n) polynomials of $\alpha_n(x(s))$, $\beta_n(x(s))$, $\mathcal{D}_n(x(s))$ and a_n^2, satisfied by the Laguerre-Hahn polynomials themselves, in analogy with the equation (5.18) for the semi-classical polynomials. That is probably the best characterization result that we can expect to obtain in this direction. We can not expect for example to establish an eigenelement equation of a fixed order say Λ, which solutions include all the Laguerre-Hahn polynomials, in analogy with the second-order difference eigenelement equation for the "classical" polynomials. Indeed, we know on the one side that the class of Laguerre-Hahn polynomials is invariant under

a finite number of Christoffel and Geronimus transformations [122, 113]. On the other side, a combination of a finite number of such transformations can lead for example to the so-called modified-Laguerre polynomials $L_n^{\alpha,M}(x)$ [61] when applied to the Laguerre polynomials $L_n^0(x)$ [114]. But it is known from [61] that for $\alpha \in Z^+$, the polynomials $L_n^{\alpha,M}(x)$ satisfy a differential eigenvalue equation of order $2\alpha+4$. In other words, for any non-negative integer α, the polynomials $L_n^{\alpha,M}(x)$ are examples of (differential) Laguerre-Hahn polynomials satisfying a differential eigenvalue equation of order $2\alpha+4$. Which implies that the evoked number Λ does not exist and our saying is demonstrated. The demonstration should be easier if one considers the case of infinite order difference eigenvalue equation satisfied by the generalized Meixner polynomials [27], which are clearly semi-classical (and consequently of Laguerre-Hahn type).

An algorithm for establishing the fourth order-difference equation for the Laguerre-Hahn polynomials has been already given in [79].
In the first section of the present chapter, we establish (using an algorithm different of that from [79]) practically the equation in question in general case. In the second section, we give it explicitly and separately for the cases of polynomials r-associated to all "classical" polynomials, while in the last, we give it "semi-explicitly" (up to a system of non-linear difference equations satisfied by the coefficients in the three-term recurrence relations) for the class one ($\kappa = 1$) Laguerre-Hahn polynomials on linear lattice.

6.1 The general case.

Without any surprise, the equation needs to be extracted from the set of difference-recurrence relations (5.10)-(5.13). The presence of terms with associated polynomials (contrary to the semi-classical situation) does not of course facilitates the things.
Let us combine together the equations (5.10) and (5.11) on the one side and the equations (5.12) and (5.13), on the other side. As a result, we obtain :

$$\sigma_n(x(s))P_n(y(s+2)) + \zeta_n(x(s))P_n(y(s+1)) + \nu_n(x(s))P_n(y(s)) + t_n(x(s))P_{n-1}^{(1)}(y(s+2)) + z_n(x(s))P_{n-1}^{(1)}(y(s)) = 0, \tag{6.1}$$

$$f_n(x(s))P_n(y(s+2)) + g_n(x(s))P_n(y(s)) + h_n(x(s))P_{n-1}^{(1)}(y(s+2)) + v_n(x(s))P_{n-1}^{(1)}(y(s+1)) + w_n(x(s))P_{n-1}^{(1)}(y(s)) = 0. \tag{6.2}$$

where:

$$\begin{aligned}
\sigma_n(x(s)) &:= \tfrac{\beta_0(x(s+1)}{\mathcal{D}_n(x(s+1))}; \nu_n(x(s)) := -\tfrac{\alpha_0(x(s))}{\mathcal{D}_n(x(s))}; t_n(x(s)) := \tfrac{\gamma_0(x(s+1))}{\mathcal{D}_n(x(s+1))} \\
z_n(x(s)) &:= -\tfrac{\gamma_0(x(s))}{\mathcal{D}_n(x(s))}; f_n(x(s)) := -\tfrac{\delta_0(x(s+1))}{\mathcal{D}_n(x(s+1))}; g_n(x(s)) := \tfrac{\delta_0(x(s))}{\mathcal{D}_n(x(s))} \\
h_n(x(s)) &:= -\tfrac{\alpha_0(x(s+1))}{\mathcal{D}_n(x(s+1))}; w_n(x(s)) := \tfrac{\beta_0(x(s))}{\mathcal{D}_n(x(s)} \\
\zeta_n(x(s)) &:= \tfrac{\mathcal{D}_n(x(s+1))\alpha_n(x(s))-\mathcal{D}_n(x(s))\beta_n(x(s+1))}{\mathcal{D}_n(x(s+1))\mathcal{D}_n(x(s))} \\
v_n(x(s)) &:= \tfrac{\mathcal{D}_n(x(s))\alpha_{n+1}(x(s+1))-\mathcal{D}_n(x(s+1))\beta_{n+1}(x(s))}{\mathcal{D}_n(x(s+1))\mathcal{D}_n(x(s))}.
\end{aligned} \tag{6.3}$$

Solving the equations (6.1) and (6.2) relatively to $P_{n-1}^{(1)}(y(s+2))$, $P_{n-1}^{(1)}(y(s+1))$ and $P_{n-1}^{(1)}(y(s))$, as linear combinations of $P_n(y(s+3))$, $P_n(y(s+2))$, $P_n(y(s+1))$ and $P_n(y(s)))$, with coefficients depending on $x(s)$ and n and taking into account the fact that for example $P_{n-1}^{(1)}(y(s+1))$ is a shift of $P_{n-1}^{(1)}(y(s))$, we obtain the expected fourth order difference equation:

$$\begin{aligned}
&[\tfrac{A_n(x(s+1))N_n(x(s+1))}{G_n(x(s+1))N_n(x(s+1))-F_n(x(s+1))R_n(x(s+1))}] \quad P_n(y(s+2)) \\
&+[\tfrac{A_n(x(s))R_n(x(s))}{G_n(x(s))N_n(x(s))-F_n(x(s))R_n(x(s))} \\
&+\tfrac{B_n(x(s+1))N_n(x(s+1))-F_n(x(s+1))K_n(x(s+1))}{G_n(x(s+1))N_n(x(s+1))-F_n(x(s+1))R_n(x(s+1))}] \quad P_n(y(s+1)) \\
&+[\tfrac{R_n(x(s))B_n(x(s))-K_n(x(s))G_n(x(s))}{G_n(x(s))N_n(x(s))-F_n(x(s))R_n(x(s))} \\
&+\tfrac{C_n(x(s+1))N_n(x(s+1))-F_n(x(s+1))L_n(x(s+1))}{G_n(x(s+1))N_n(x(s+1))-F_n(x(s+1))R_n(x(s+1))}] \quad P_n(y(s)) \\
&+[\tfrac{C_n(x(s))R_n(x(s))-L_n(x(s))G_n(x(s))}{G_n(x(s))N_n(x(s))-F_n(x(s))R_n(x(s))} \\
&+\tfrac{E_n(x(s+1))N_n(x(s+1))-F_n(x(s+1))M_n(x(s+1))}{G_n(x(s+1))N_n(x(s+1))-F_n(x(s+1))R_n(x(s+1))}] \quad P_n(y(s-1)) \\
&+[\tfrac{R_n(x(s))E_n(x(s))-M_n(x(s))G_n(x(s))}{G_n(x(s))N_n(x(s))-F_n(x(s))R_n(x(s))}] \quad P_n(y(x(s-2)) \\
&= 0
\end{aligned} \tag{6.4}$$

where (having in mind Eqs. (6.3) and (5.9)):

$$\begin{aligned}
A_n(x(s)) &:= \tfrac{t_n(x(s-2))(h_n(x(s-1))\sigma_n(x(s-1))-f_n(x(s-1))t_n(x(s-1)))}{v_n(x(s-1))t_n(x(s-1))}; \\
B_n(x(s)) &:= \sigma_n(x(s-2)) + \tfrac{t_n(x(s-2))h_n(x(s-1))\zeta_n(x(s-1))}{v_n(x(s-1))t_n(x(s-1))}; \\
C_n(x(s)) &:= \zeta_n(x(s-2)) \\
&+\tfrac{t_n(x(s-2))(h_n(x(s-1))\nu_n(x(s-1))-t_n(x(s-1))g_n(x(s-1)))}{v_n(x(s-1))t_n(x(s-1))}; \\
E_n(x(s)) &:= \nu_n(x(s-2)); G_n(x(s)) := z_n(x(s-2));
\end{aligned}$$

$$\begin{aligned}
K_n(x(s)) &:= \tfrac{f_n(x(s-2))t_n(x(s-2))-h_n(x(s-2))\sigma_n(x(s-2))}{t_n(x(s-2))};\\
L_n(x(s)) &:= -\tfrac{h_n(x(s-2))\zeta_n(x(s-2))}{t_n(x(s-2))};\ N_n(x(s)) := v_n(x(s-2));\\
M_n(x(s)) &:= \tfrac{t_n(x(s-2))g_n(x(s-2))-h_n(x(s-2))\nu_n(x(s-2))}{t_n(x(s-2))};\\
R_n(x(s)) &:= \tfrac{t_n(x(s-2))w_n(x(s-2))-z_n(x(s-2))h_n(x(s-2))}{t_n(x(s-2))};\\
F_n(x(s)) &:= \tfrac{t_n(x(s-2))(h_n(x(s-1))z_n(x(s-1))-w_n(x(s-1))t_n(x(s-1)))}{v_n(x(s-1))t_n(x(s-1))}. \qquad (6.5)
\end{aligned}$$

This solves the "problem 5" for the Laguerre-Hahn polynomials on special nonuniform lattices.

Remark 6.1 Considering the remark 5.1, we clearly note that the fourth order difference equation for $P_{n-r}^{(r)}$ is obtained from the preceding one satisfied by P_n, by shifting the initial value for n, from $n := 0$ to $n := r$. More precisely, one needs to replace in Eq. (6.3) α_0, β_0, γ_0 and δ_0 respectively by α_r, β_r, γ_r and δ_r.

Remark 6.2 We have already in this section used the solutions of the system of equations (6.1) and (6.2), solved relatively to the $P_{n-1}^{(1)}$, as functions of the P_n. Adding a solution of that system now relatively to the P_n as functions of the $P_{n-1}^{(1)}$, we obtain the following inverse difference relations:

$$P_{n-1}^{(1)}(y(s)) := \Phi_n(x(s))[P_n(y(s))]; \qquad (6.6)$$

$$P_n(y(s)) := \Psi_n(x(s))[P_{n-1}^{(1)}(y(s))]; \qquad (6.7)$$

where

$$\begin{aligned}
&\Phi_n(x(s))[\theta_n(y(s))]\\
&:= [\tfrac{A_n(x(s+2))N_n(x(s+2))}{F_n(x(s+2))R_n(x(s+2))-G_n(x(s+2))N_n(x(s+2))}]\theta_n(y(s+3))\\
&+[\tfrac{B_n(x(s+2))N_n(x(s+2))-K_n(x(s+2))F_n(x(s+2))}{F_n(x(s+2))R_n(x(s+2))-G_n(x(s+2))N_n(x(s+2))}]\theta_n(y(s+2))\\
&+[\tfrac{C_n(x(s+2))N_n(x(s+2))-F_n(x(s+2))L_n(x(s+2))}{F_n(x(s+2))R_n(x(s+2))-G_n(x(s+2))N_n(x(s+2))}]\theta_n(y(s+1))\\
&+[\tfrac{E_n(x(s+2))N_n(x(s+2))-F_n(x(s+2))M_n(x(s+2))}{F_n(x(s+2))R_n(x(s+2))-G_n(x(s+2))N_n(x(s+2))}]\theta_n(y(s)), \qquad (6.8)
\end{aligned}$$

and

$$\begin{aligned}
&\Psi_n(x(s))[\theta_n(y(s))]\\
&:= [\tfrac{\hat{A}_n(x(s+2))\hat{N}_n(x(s+2))}{\hat{F}_n(x(s+2))\hat{R}_n(x(s+2))-\hat{G}_n(x(s+2))\hat{N}_n(x(s+2))}]\theta_n(y(s+3))
\end{aligned}$$

$$+[\frac{\hat{B}_n(x(s+2))\hat{N}_n(x(s+2))-\hat{K}_n(x(s+2))\hat{F}_n(x(s+2))}{\hat{F}_n(x(s+2))\hat{R}_n(x(s+2))-\hat{G}_n(x(s+2))\hat{N}_n(x(s+2))}]\theta_n(y(s+2))$$
$$+[\frac{\hat{C}_n(x(s+2))\hat{N}_n(x(s+2))-\hat{F}_n(x(s+2))\hat{L}_n(x(s+2))}{\hat{F}_n(x(s+2))\hat{R}_n(x(s+2))-\hat{G}_n(x(s+2))\hat{N}_n(x(s+2))}]\theta_n(y(s+1))$$
$$+[\frac{\hat{E}_n(x(s+2))\hat{N}_n(x(s+2))-\hat{F}_n(x(s+2))\hat{M}_n(x(s+2))}{\hat{F}_n(x(s+2))\hat{R}_n(x(s+2))-\hat{G}_n(x(s+2))\hat{N}_n(x(s+2))}]\theta_n(y(s)), \tag{6.9}$$

where the hat has the meaning that $\hat{X}$ is obtained from X by replacing $\sigma_n, \zeta_n, \nu_n, t_n, z_n, f_n, g_n, h_n, v_n, w_n$ by $h_n, v_n, w_n, f_n, g_n, t_n, z_n, \sigma_n, \zeta_n, \nu_n$ *respectively* (thanks to the symmetric form of the system (6.1)-(6.2)).
Shifting in Φ_n and Ψ_n the initial value for n from $n := 0$ to $n := r$, it becomes clear that the obtained operators play the role of "raising" and "lowering" operators within the sequence of (normalized) polynomials satisfying the following three-term recurrence relation:

$$P_n^{(r)}(y(s)) = (y(s) - b_r)P_{n-1}^{(r+1)}(y(s)) - a_{r+1}^2 P_{n-2}^{(r+2)}(y(s)). \tag{6.10}$$

6.2 The fourth order difference equation for the polynomials r-associated to the classical polynomials.

We now go further and calculate explicitly the coefficients in the fourth order difference equation satisfied by the polynomials $P_{n-r}^{(r)}$ r-associated to the classical orthogonal polynomials (up to the Askey-Wilson polynomials). Let us remark first that the problem consists essentially in solving the system (5.7) (in the classical case i.e. $\kappa = 0$, $\mathcal{B}_0 = 0$). The coefficients in the fourth order difference equation are then obtained directly using Eqs. (6.3) and (6.5) and considering the remark 6.1.
In the following, the cases of linear, q-linear and Askey-Wilson lattices are treated in details separately. For simplicity, canonical forms of lattices are chosen.

6.2.1 The linear case: $y(s) := s$, $x(s) = s$.

Considering Eq. (5.9), one notes that $\alpha_n(x(s))$ and $\beta_n(x(s))$ need to be searched under the forms (recall that we are searching classical solutions, so $\kappa = 0$ in Eq. (5.3)):

$$\alpha_n(x(s)) \ := \ -(\alpha_n^0 + \alpha_n^1 s + \alpha_n^2 s^2),$$

$$\beta_n(x(s)) \quad := \quad \beta_n^0 + \beta_n^1 s + \alpha_n^2 s^2. \tag{6.11}$$

Those expressions satisfy the system (5.7) iff (here and in all that follows, x_0, x_1, x_2, y_0, y_1, are arbitrary constants):

$$\begin{aligned}\alpha_n(x(s)) := \{&[2x_2^2 n + x_2 y_1 - x_2 x_1]s^2 + [-2x_2^2 n^2 + (-x_2 y_1 + 3x_2 x_1)n \\ &+ y_1 x_1 - x_1^2]s + [x_2^2 n^3 + (x_2 y_1 - 2x_2 x_1)n^2 + (x_2 x_0 - y_1 x_1 + x_1^2 \\ &+ x_2 y_0)n - x_0 x_1 + x_0 y_1]\}/ \\ &\{-2x_2 n - y_1 + x_1\};\end{aligned} \tag{6.12}$$

$$\begin{aligned}\beta_n(x(s)) := \{&[2x_2^2 n + x_2 y_1 - x_2 x_1]s^2 + [2x_2^2 n^2 + (-x_2 x_1 + 3x_2 y_1)n \\ &+ y_1^2 - y_1 x_1]s + [x_2^2 n^3 + (-x_2 x_1 + 2x_2 y_1)n^2 + (x_2 x_0 + x_2 y_0 - y_1 x_1 \\ &+ y_1^2)n + y_0 y_1 - y_0 x_1]\}/ \\ &\{2x_2 n + y_1 - x_1\};\end{aligned} \tag{6.13}$$

$$\mathcal{D}_n(x(s)) := -2x_2 n - x_2 - y_1 + x_1; \tag{6.14}$$

$$\begin{aligned}b_n := \{&(2x_2^2 - x_2 x_1 - x_2 y_1)n^2 + (2x_2^2 + x_2 y_1 - 3x_2 x_1 + x_1^2 - y_1^2)n \\ &+ (x_2 y_1 - x_2 x_1 + x_0 y_1 - x_0 x_1 + x_1^2 - y_0 y_1 + y_0 x_1 \\ &- y_1 x_1)\}/ \\ &\{4x_2^2 n^2 + (4x_2^2 + 4x_2 y_1 - 4x_2 x_1)n + 2x_2 y_1 - 2x_2 x_1 \\ &+ y_1^2 - 2y_1 x_1 + x_1^2\};\end{aligned} \tag{6.15}$$

$$\begin{aligned}a_n^2 := (&-x_2^4 n^6 + (3x_2^3 x_1 - 3x_2^3 y_1)n^5 + (-3x_2^2 y_1^2 - 2x_2^3 x_0 - 2x_2^3 y_0 \\ &+ 7x_2^2 y_1 x_1 - 3x_2^2 x_1^2)n^4 + (-5x_2 x_1^2 y_1 + 5x_2 y_1^2 x_1 + 4x_2^2 y_0 x_1 - x_2 y_1^3 \\ &- 4x_2^2 x_0 y_1 + 4x_2^2 x_0 x_1 + x_2 x_1^3 - 4x_2^2 y_0 y_1)n^3 + (-x_2^2 y_0^2 - x_2^2 x_0^2 \\ &+ x_1^3 y_1 + 2x_2^2 y_0 x_0 - 3x_2 x_0 y_1^2 + y_1^3 x_1 - 2x_2 y_0 y_1^2 + 5x_2 y_0 y_1 x_1 \\ &+ 5x_2 x_0 y_1 x_1 - 2x_2 x_0 x_1^2 - 3x_2 y_0 x_1^2 - 2x_1^2 y_1^2)n^2 + (-2x_2 y_0 x_0 x_1 \\ &- 2x_1^2 y_1 y_0 + 2y_1^2 x_1 x_0 + y_1^2 x_1 y_0 - x_1^2 y_1 x_0 + x_2 x_0^2 x_1 - x_2 y_0^2 y_1 \\ &+ x_2 y_0^2 x_1 - x_2 x_0^2 y_1 + x_1^3 y_0 + 2x_2 y_0 x_0 y_1 - y_1^3 x_0)n\}/ \\ &\{16x_2^4 n^4 + (32x_2^3 y_1 \\ &- 32x_2^3 x_1)n^3 + (-48x_2^2 y_1 x_1 + 24x_2^2 x_1^2 - 4x_2^4 + 24x_2^2 y_1^2)n^2 + (-24x_2 y_1^2 x_1 \\ &+ 24x_2 x_1^2 y_1 - 8x_2 x_1^3 - 4x_2^3 y_1 + 4x_2^3 x_1 + 8x_2 y_1^3)n + (-x_2^2 y_1^2 + y_1^4 - 4y_1^3 x_1 \\ &+ 6x_1^2 y_1^2 + 2x_2^2 y_1 x_1 - 4x_1^3 y_1 - x_2^2 x_1^2 + x_1^4)\};\end{aligned} \tag{6.16}$$

6.2.2 The q-linear case: $y(s) := q^s$, $x(s) := q^s$.

Now, $\alpha_n(x(s))$ and $\beta_n(x(s))$ need to be searched under the forms:

$$\begin{aligned}\alpha_n(x(s)) &:= -(\alpha_n^0 q^{-s} + \alpha_n^1 + \alpha_n^2 q^s),\\ \beta_n(x(s)) &:= \alpha_n^0 q^{-s} + \beta_n^1 + \beta_n^2 q^s.\end{aligned} \tag{6.17}$$

Those expressions satisfy the system (5.7) iff :

$$\begin{aligned}\alpha_n(x(s)) := \{&[y_2 q x_2 q^{2n} - x_2^2]q^s + [(y_1 x_2 q + q y_2 x_1)q^{2n} + (-y_1 x_2 q\\ &-x_2 x_1)q^n] + [x_0 q q^{3n} y_2 - x_0 x_2 q^n]q^{-s}\}/\\ &\{q x_2 q^n - q^2 y_2 q^{3n}\};\end{aligned} \tag{6.18}$$

$$\begin{aligned}\beta_n(x(s)) := \{&[y_2^2 q^2 q^{3n} - y_2 q x_2 q^n]q^s + [(y_1 y_2 q^2 + q y_2 x_1)q^{2n} + (-y_1 x_2 q\\ &-q y_2 x_1)q^n] + [x_0 y_2 q q^{2n} - x_0 x_2]q^{-s}\}/\\ &\{q^2 y_2 q^{2n} - q x_2\};\end{aligned} \tag{6.19}$$

$$\mathcal{D}_n(x(s)) := x_2 q^{-2} q^{-n} - y_2 q^n; \tag{6.20}$$

$$\begin{aligned}b_n := \{&-(q^3 y_2 x_1 + y_2 q^4 y_1)q^{3n} + (q^3 y_2 x_1 + q^2 y_2 x_1 + q^3 y_1 x_2\\ &+y_1 x_2 q^2)q^{2n} - (y_1 x_2 q^2 + x_2 x_1 q)q^n\}/\\ &\{q^4 y_2^2 q^{4n} + (-y_2 q x_2\\ &-x_2 y_2 q^3)q^{2n} + x_2^2\};\end{aligned} \tag{6.21}$$

$$\begin{aligned}a_n^2 := \{&-y_2^3 q^4 x_0 q^{7n} + (y_2^3 q^4 x_0 + y_2^2 q^4 y_1 x_1 + q^3 y_2^2 x_0 x_2)q^{6n} - (y_2^2 q^4 y_1 x_1\\ &+x_2 q^3 y_1 y_2 x_1 + q^3 y_2^2 x_1^2 - q^3 y_2^2 x_0 x_2 + q^4 y_1^2 y_2 x_2)q^{5n} - (-q^4 y_1^2 y_2 x_2\\ &-2x_2 q^3 y_1 y_2 x_1 - q^3 y_2^2 x_1^2 - x_2 q^2 y_2 x_1^2 - x_2^2 q^3 y_1^2 + 2q^3 y_2^2 x_0 x_2\\ &+2y_2 x_0 x_2^2 q^2)q^{4n} - (-y_2 x_0 x_2^2 q^2 + x_2 q^3 y_1 y_2 x_1 + x_2^2 q^3 y_1^2\\ &+x_2 q^2 y_2 x_1^2 + y_1 x_1 x_2^2 q^2)q^{3n} + (q x_0 x_2^3 + y_2 x_0 x_2^2 q^2 + y_1 x_1 x_2^2 q^2)q^{2n}\\ &-q x_0 x_2^3 q^n\}/\\ &\{q^4 y_2^4 q^{8n} - (y_2^3 x_2 q^2 + 2y_2^3 x_2 q^3 + y_2^3 x_2 q^4)q^{6n} + (2x_2^2 y_2^2 q^2\\ &+2x_2^2 y_2^2 q^3 + 2x_2^2 y_2^2 q)q^{4n} - (x_2^3 y_2 q^2 + x_2^3 y_2 + 2x_2^3 y_2 q)q^{2n} + x_2^4\};\end{aligned} \tag{6.22}$$

6.2.3 The Askey-Wilson case: $y(s) := \frac{q^s+q^{-s}}{2}$, $x(s) := qq^s + q^{-s}$.

In that case, one can also verify from Eq. (5.9) that $\alpha_n(x(s))$ and $\beta_n(x(s))$ need to be searched under the forms:

$$\begin{aligned}
\alpha_n(x(s)) &:= \frac{\alpha_n^0 q^{-2s} + \alpha_n^1 q^{-s} + \beta_n^2 + q\beta_n^1 q^s + q^2\beta_n^0 q^{2s}}{q^{-s} - qq^s}, \\
\beta_n(x(s)) &:= \frac{\beta_n^0 q^{-2s} + \beta_n^1 q^{-s} + \beta_n^2 + q\alpha_n^1 q^s + q^2\alpha_n^0 q^{2s}}{qq^s - q^{-s}}. \qquad (6.23)
\end{aligned}$$

Those expressions satisfy the system (5.7) iff (here as above, we exclude, of course, trivial solutions):

$$\begin{gathered}
\alpha_n(x(s)) := \{[q^2 y_0 x_0 q^{2n} - q^2 y_0^2]q^{2s} + [(y_0 x_1 q + y_1 x_0 q)q^{2n} + (-y_0 q y_1 \\
-y_0 x_1 q)q^n]q^s + [-q x_0^2 q^{4n} + (q x_0^2 + y_2 x_0 + q x_0 y_0)q^{3n} + (-q y_0^2 - q x_0 y_0 \\
-y_0 y_2)q^n + y_0^2 q] + [(y_1 x_0 + x_1 x_0)q^{3n} + (-y_0 x_1 - y_1 x_0)q^{2n}]q^{-s} + [q^{4n} x_0^2 \\
-q^{2n} x_0 y_0]q^{-2s}\}/ \\
\{[-q x_0 q^{3n} + q y_0 q^n]q^s + [x_0 q^{3n} - y_0 q^n]q^{-s}\}; \qquad (6.24)
\end{gathered}$$

$$\begin{gathered}
\beta_n(x(s)) := \{[q^2 q^{4n} x_0^2 - q^2 y_0 x_0 q^{2n}]q^{2s} + [(y_1 x_0 q + x_1 q x_0)q^{3n} + (-y_0 x_1 q \\
-y_1 x_0 q)q^{2n}]q^s + [-q x_0^2 q^{4n} + (q x_0^2 + y_2 x_0 + q x_0 y_0)q^{3n} + (-q y_0^2 - q x_0 y_0 \\
-y_0 y_2)q^n + y_0^2 q] + [(y_0 x_1 + y_1 x_0)q^{2n} + (-y_1 y_0 - y_0 x_1)q^n]q^{-s} + [q^{2n} x_0 y_0 \\
-y_0^2]q^{-2s}\}/ \\
\{[q x_0 q^{3n} - q y_0 q^n]q^s + [-x_0 q^{3n} + y_0 q^n]q^{-s}\}; \qquad (6.25)
\end{gathered}$$

$$\mathcal{D}_n(x(s)) := 2y_0 q^{-n} - 2x_0 q q^n; \qquad (6.26)$$

$$\begin{gathered}
b_n := \{(-y_1 x_0 q - x_1 q x_0)q^{3n} + (y_1 x_0 + y_0 x_1 + y_1 x_0 q + y_0 x_1 q)q^{2n} \\
+(-y_1 y_0 - y_0 x_1)q^n\}/ \\
\{2q^2 x_0^2 q^{4n} + 2(-y_0 x_0 - q^2 x_0 y_0)q^{2n} + 2y_0^2\}; \qquad (6.27)
\end{gathered}$$

$$\begin{gathered}
a_n^2 := \{q x_0^4 q^{8n} + (-x_0^3 y_0 q - x_0^3 y_2 - q x_0^4)q^{7n} + (y_2 x_0^2 y_0 + x_0^3 y_2 + x_0^2 y_1 x_1)q^{6n} \\
+(-y_0 x_1 y_1 x_0 - y_1^2 x_0^2 - y_0 x_1^2 x_0 + q x_0^2 y_0^2 - x_0^2 y_1 x_1 + y_2 x_0^2 y_0 + x_0^3 y_0 q)q^{5n} \\
+(-2q x_0^2 y_0^2 + g_0^2 x_1^2 - 2y_2 x_0^2 y_0 + y_1^2 y_0 x_0 + y_1^2 x_0^2 + y_0 x_1^2 x_0 - 2y_0^2 y_2 x_0 \\
+2y_0 x_1 y_1 x_0)q^{4n} + (-y_0 x_1 y_1 x_0 - y_0^2 x_1^2 - g_0^2 x_1 y_1 + q x_0 y_0^3 + y_0^2 y_2 x_0 + q x_0^2 y_0^2
\end{gathered}$$

$$-y_1^2y_0x_0)q^{3n} + (y_2y_0^3 + y_0^2y_2x_0 + y_0^2x_1y_1)q^{2n} + (-y_2y_0^3 - y_0^4q - qx_0y_0^3)q^n$$
$$+[qy_0^4]\}/$$
$$\{qx_0^4q^{8n} + (-x_0^3y_0 - 2x_0^3y_0q - x_0^3y_0q^2)q^{6n} + (2qx_0^2y_0^2 + 2x_0^2y_0^2q^2$$
$$+2x_0^2y_0^2)q^{4n} + (-q^2x_0y_0^3 - x_0y_0^3 - 2qx_0y_0^3)q^{2n} + [qy_0^4]\}. \quad (6.28)$$

Remark 6.3 The expressions for α_n, β_n calculated here (from Eq. (5.7)) in subsections 6.2.1, 6.2.2 and 6.2.3 allow to establish explicitly, through the second-order difference operator in the left hand side of Eq. (5.19), the second-order difference operator for "classical" polynomials on linear, q-linear and q-nonlinear lattices respectively. Let us remark that that operator admits a sequence of polynomial "solutions" thanks to the factorization in Eq. (5.33) for $\kappa = 0$, $a_0^2\mathcal{D}_0\mathcal{D}_{-1} = 0$. Moreover, the fact that the generated here "classical" polynomials on lattices are the usual ones can be easily assured by verifying directly the relation

$$\breve{H}(z,n) = (z - z^{-1}).[\tilde{\mathcal{L}} - \lambda^1(n)] \quad (6.29)$$

where $\tilde{\mathcal{L}}$ is the Askey-Wilson q-difference operator in Eq. (3.51), $\lambda^1(n)$ is from Eq. (3.72) and $\breve{H}(z,n)$ is the evoked operator in the left hand side of Eq. (5.19) with q^s replaced by z and α_n and β_n of course as given in subsection 6.2.3. For that, the correspondence between the parameters used in $\tilde{\mathcal{L}}$ and $\lambda^1(n)$ and those used in subsection 6.2.3 is as follows:

$$y_0 := -1; y_1 := a+b+c+d; y_2 := -(ab+ac+ad+bc+bd+cd);$$
$$x_1 := (abc+abd+bcd+acd)q^{-1}; x_0 := -abcdq^{-2} \quad (6.30)$$

As (as we saw above) in the "classical" situation the system (5.24) is a (pure) factorization chain, this means that we occasionally reached the expected in remarks 3.1 and 3.4 result: generate originally (exclusively) from factorization chains all "classical" polynomials on lattices i.e. *give a direct (not indirect as in sections 3.2 and 3.3) solution of the "problem 2", for those polynomials.*

Remark 6.4 One of the main advantage of the "factorization" approach (or equally, as we saw above, the Laguerre-Hahn approach) to orthogonal polynomials consists of the fact that the knowledge of α_n, β_n, $\mathcal{D}_n$ and a_n^2 (from type (5.24) "factorization chain") leads not only to the knowledge of the difference equation satisfied by the corresponding polynomials but also to that satisfied by the polynomials r-associated to them, all that thanks to the remark 6.1. In other words, the expressions calculated for α_n, β_n, $\mathcal{D}_n$ and a_n^2 in subsections 6.2.1, 6.2.2 and 6.2.3 lead directly (by the remark 6.1) to the expected fourth-order difference equation for all polynomials r-associated to the "classical" ones, namely, the polynomials r-associated to the classical polynomials on linear lattices, on q-linear lattices and on q-nonlinear (Askey-Wilson) lattices. *This solves the "problem 6", for all the cited classes of orthogonal polynomials.*

6.3 The fourth order difference equation for the class one Laguerre-Hahn polynomials on linear lattices.

We saw above that in the classical situation ($\kappa = 0$, $\mathcal{B}_0 = 0$), one success to solve the system (5.7) in term of elementary functions. We can not expect to do this in non-classical situations, as we know that in the continuous case, the coefficients in the three-term recurrence relations are related to Painlevé transcendent (see [81] in semi-classical differential situation). Below we give solutions of Eq. (5.7) for the case of class $\kappa = 1$ ($\mathcal{B}_0$ not necessary zero) and for simplicity in the case of linear lattice $y(s) = x(s) = s$, up to explicit non-linear difference equations satisfied by the coefficients in the three-term recurrence relations. This furnishes naturally "semi-explicit" fourth order difference equations for the corresponding polynomials by the formula (6.4) and naturally for the polynomials r-associated to them according to the remark 6.1.
We have ($\kappa = 1$, $y(s) = x(s) = s$):
(c_1, c_2, c_3, c_4, arbitrary parameters)

$$\begin{aligned} \alpha_n(x(s)) &:= -(\alpha_n^0 + \alpha_n^1 s + \alpha_n^2 s^2 + c_3 s^3), \\ \beta_n(x(s)) &:= \beta_n^0 + \beta_n^1 s + \beta_n^2 s^2 + c_3 s^3, \end{aligned} \tag{6.31}$$

with

$$\begin{aligned} \alpha_n^0 := [&4X_nY_nc_3^2 - 4c_2c_3c_4 + 8c_2c_1c_3 - 4Y_{n+1}X_{n+1}c_3^2 + 4c_3^2c_2n + 8c_2^2c_3n \\ &-2c_3^2c_4n - 3Y_{n+1}Y_n^2 - Y_{n+1}^3 - (4c_3n + 8c_2 + 2c_3 - 2c_4)Y_nY_{n+1} - (-2c_3n \\ &-4c_2 + c_4)Y_n^2 - (-2c_3c_4n + 4c_2c_3 + c_3^2 + 2c_3^2n^2 + 2c_3^2n - 2c_3c_4 - 4c_4c_2 \\ &+8c_3c_2n + 4c_2^2 + 4c_1c_3)Y_{n+1} - (2c_3c_4n + 4c_4c_2 - 8c_2c_3n - 2c_3^2n^2 - 4c_2^2 \\ &+c_3^2 + 2c_3c_4 - 4c_1c_3 - 2c_3^2n)Y_n - (-8c_3^3 + 8c_3^3n - 32c_3^2c_2 - 16c_2c_3c_4 \end{aligned}$$

$$-4c_3^2c_4 + 32c_2c_3^2n)X_n - (-2c_3 - 4c_2 + c_4 - 2c_3n)Y_{n+1}^2 - (-32c_2c_3^2n$$
$$-32c_2c_3^2 - 8c_3^3n + 16c_2c_3c_4 + 4c_3^2c_4 - 8c_3^3)X_{n+1} - (4c_3c_4 - 8c_3^2$$
$$-8c_3^2n)Y_nX_{n+1} - (8c_3^2n + 12c_3^2 - 4c_3c_4)Y_{n+2}X_{n+1} + 4c_2c_3^2n^2 + Y_n^3$$
$$+3Y_nY_{n+1}^2 - 4c_2c_3c_4n - c_3^2c_4 + 2c_3^3n^2 + 4c_1c_3^2 - 4c_2^2c_4 - (4c_3c_4$$
$$-8c_3^2n + 8c_3^2)X_nY_{n+1} - (8c_3^2n - 12c_3^2 - 4c_3c_4)X_nY_{n-1}]/(8c_3^2);$$

$$\alpha_n^1 := c_1 - \tfrac{1}{2}(c_3 + c_4)n + \tfrac{1}{2}c_3n^2 - \tfrac{1}{2}Y_n;\ \alpha_n^2 := c_2 + \tfrac{1}{2}c_3 + \tfrac{1}{2}c_4 - c_3n;$$

$$\beta_n^0 := [4c_3^2X_nY_n - 4c_2c_3c_4 + 8c_1c_2c_3 - 4c_3^2X_{n+1}Y_{n+1} + 4c_2c_3^2n - 8c_2^2c_3n$$
$$+(-4c_3c_4 + 8c_3^2n - 8c_3^2)Y_{n+1}X_n + (-8c_3^2n + 4c_3c_4 - 12c_3^2)Y_{n+2}X_{n+1}$$
$$+(-4c_3c_4 + 8c_3^2 + 8c_3^2n)Y_nX_{n+1} + (-8c_3^2n + 12c_3^2 + 4c_3c_4)Y_{n-1}X_n + 2c_3^2c_4n$$
$$-3Y_{n+1}Y_n^2 - Y_{n+1}^3 + 4c_2c_3^2n^2 + Y_n^3 + 3Y_nY_{n+1}^2 - 4c_2c_3c_4n + (4c_2 + c_4$$
$$-2c_3n)Y_n^2 + (32c_3^2c_2 + 32c_3^2c_2n - 8c_3^3n - 16c_2c_3c_4 - 8c_3^3 + 4c_3^2c_4)X_{n+1}$$
$$+(-4c_3^2c_4 - 8c_3^3 - 32c_3^2c_2n + 16c_2c_3c_4 + 32c_3^2c_2 + 8c_3^3n)X_n + (-2c_3n - 2c_3$$
$$+c_4 + 4c_2)Y_{n+1}^2 + (4c_4c_2 + 4c_2^2 - 2c_3c_4 - c_3^2 - 8c_3c_2n - 2c_3c_4n + 4c_3c_1$$
$$+2c_3^2n^2 + 2c_3^2n)Y_n + (8c_2c_3n - 2c_3^2n^2 - 4c_2^2 + 2c_3c_4n + 2c_3c_4$$
$$+4c_2c_3 - c_3^2 - 4c_4c_2 - 4c_1c_3 - 2c_3^2n)Y_{n+1} + (-2c_4 + 2c_3 - 8c_2$$
$$+4c_3n)Y_nY_{n+1} + c_3^2c_4 - 2c_3^3n^2 - 4c_1c_3^2 + 4c_2^2c_4]/(8c_3^2);$$

$$\beta_n^1 := \tfrac{1}{2}Y_n + c_1 - \tfrac{1}{2}(c_3 + c_4)n + \tfrac{1}{2}c_3n^2;\ \beta_n^2 := c_2 - \tfrac{1}{2}c_3 - \tfrac{1}{2}c_4 + c_3n;$$

$$\mathcal{D}_n(x(s)) := (c_4 - 2c_3n)s + [(2c_3n - c_4 - c_3)Y_n + (c_4 - c_3 - 2c_3n)Y_{n+1}$$
$$+4c_3c_2n - 2c_4c_2]/(2c_3);$$

where X_n and Y_n given by

$$\begin{aligned} a_n^2 &:= X_n \\ b_n &:= (Y_{n+1} - Y_n - 2c_2 + c_3)/(2c_3); \end{aligned} \tag{6.32}$$

are required to satisfy the following non-linear difference system

$$-8c_2Y_{n+1}Y_{n+2} - 4c_3^2Y_{n+2}X_{n+2} - Y_{n+2}^3 + 8c_3^2c_2 + 4c_2Y_{n+2}^2 - 3Y_{n+2}Y_{n+1}^2$$
$$+3Y_{n+1}Y_{n+2}^2 - Y_n^3 + 2Y_{n+1}^3 + 8c_3^2Y_{n+1}X_{n+1} - 8c_2c_3c_4 - 4c_3^2Y_nX_n$$
$$+16c_3^2c_2n + 8c_2Y_nY_{n+1} + 3Y_{n+1}Y_n^2 - 3Y_nY_{n+1}^2 - 4c_2Y_n^2 + (32c_2c_3c_4$$

$$
\begin{aligned}
&-32c_3^2c_2 - 64c_3^2c_2n)X_{n+1} + (-8c_3^2n - 20c_3^2 + 4c_3c_4)X_{n+2}Y_{n+3}\\
&+(8c_3c_4 - 4c_3^2 - 16c_3^2n)X_{n+1}Y_n + (4c_3c_4 - 8c_3^2n + 8c_3^2)X_nY_{n+1}\\
&+(8c_3^2n - 12c_3^2 - 4c_3c_4)X_nY_{n-1} + (-4c_3c_4 + 16c_3^2 + 8c_3^2n)X_{n+2}Y_{n+1}\\
&+(-8c_3c_4 + 16c_3^2n + 12c_3^2)X_{n+1}Y_{n+2} + (-2c_3^2n^2 - c_3^2 - 4c_2^2 + 2c_3c_4n\\
&-4c_1c_3 + 2c_3^2n)Y_n + (-2c_3^2n^2 - 4c_2^2 + 2c_3c_4n - 4c_1c_3 + 4c_3c_4 - 6c_3^2n\\
&-5c_3^2)Y_{n+2} + (64c_3^2c_2 + 32c_3^2c_2n - 16c_2c_3c_4)X_{n+2} + (4c_3^2n^2 + 2c_3^2 - 4c_3c_4\\
&+8c_2^2 - 4c_3c_4n + 8c_1c_3 + 4c_3^2n)Y_{n+1} + (-16c_2c_3c_4 - 32c_3^2c_2\\
&+32c_3^2c_2n)X_n = 0,
\end{aligned}
\tag{6.33}
$$

$$
\begin{aligned}
&-2c_3^3 - 8c_1c_3^2 - 8c_2^2c_3 - 4c_3^3n + 4c_3^2c_4n + 4c_3^2c_4 - 4c_3^3n^2 - 2c_3Y_{n+1}^2\\
&-8c_3^3X_{n+1} + (c_4 - 2c_3n - 4c_3)Y_{n+2}^2 + (8c_3c_2n - 4c_2c_3 - 4c_4c_2)Y_n + (12c_2c_3\\
&-4c_4c_2 + 8c_3c_2n)Y_{n+2} + (-16c_3^3 - 8c_3^3n + 4c_3^2c_4)X_{n+2} + (8c_4c_2 - 8c_2c_3\\
&-16c_3c_2n)Y_{n+1} + (-2c_3 + 2c_3n - c_4)Y_n^2 + (-8c_3^3 + 8c_3^3n - 4c_3^2c_4)X_n\\
&+(6c_3 - 2c_4 + 4c_3n)Y_{n+1}y_{n+2} + (-4c_3n + 2c_4 + 2c_3)Y_nY_{n+1} = 0.
\end{aligned}
\tag{6.34}
$$

This solves partially (up to Eqs. (6.33)-(6.34)) the "problem 6", for the class one Laguerre-Hahn polynomials on linear lattices (and for the polynomials r-associated to them).

The equations (6.33) and (6.34) are the most general (i.e. without loss of degree of freedom in parameters) connecting the coefficients in the three-term recurrence relations for the class one LHP orthogonal on linear lattice. They contain for example the ones obtained in [45]. Let us remark that the equations (6.33) and (6.34) can be written in explicit form in rapport with the highest differences (i.e. X_{n+4} and Y_{n+4}). It can be seen also that loosing one degree of freedom (in parameters c_i) allows to write Eqs. (6.33)-(6.34) explicitly not only in function of a_n^2 but also in function of b_n. Let us remark finally that in principle, in spite of the fact that the present system is related to Painlevé transcendent (see [81] for the continuous semi-classical case), some of its particular cases have rational particular solutions as predicted in [44] (see also [43]).

Summary and outlooks

The first major concern of this thesis (the first four chapters (chapter 1 is essentially for recalling related concepts)) deal with the applications of the discrete factorization techniques in orthogonal polynomials on lattices theory. There is mainly questions of "generating", "solving" or "modifying" difference operators admitting complete set of polynomial eigenfunctions. In sum, we have resorted to three kinds of factorization techniques.
The first one consists in imposing "quasi-periodicity" behaviour to the factorization chains. This method has been successfully used to generate the Charlier, the Meixner and Kravchuk polynomials (section 3.1).
The second technique consists in imposing to the factorization chains a special "shape-invariant" behaviour. The method has been successfully used to generate and to solve the Nikiforov-Suslov-Uvarov hypergeometric difference operators on linear (Charlier, Meixner, Kravchuk, Hahn), nonlinear (dual Hahn (thanks to remark 2.1)) (section 3.2) and q-nonlinear lattices (Askey-Wilson (q-Racah)) (section 3.3).
In passing, functions generalizing the Askey-Wilson polynomials were given (subsection 3.3.2).
The third technique goes in the opposite way relatively to the two preceding ones. It consists in imposing to the factorization that successive links in the chains belong to totally different families. We must avoid any kind of self-similarity, such as "quasi-periodicity" or "shape-invariance". Thus, while the two first approaches aim either to solve known difference operators or generate them from elementary factorization systems, this third method consists in "modification" of difference operators: starting from a known exactly solved Hamiltonian and then generate a new exactly solvable one. Also, besides all the usual physical requirements (integrability, completeness, ...), as we are dealing with polynomials, an additional sufficiently hard requirement demands that the new eigenelements family be constituted of polynomials. Here, we succeeded to "modify" a special case of Meixner polynomials, into

new complete sequences of non-classical orthogonal polynomials on lattices (section 4.2).

The chapter five (section 5.1 is essentially for recalling related concepts) deals with what one may call a "transition" concern: We have shown the interconnection between the factorization technique of Infeld-Hull-Miller type and the Laguerre-Hahn approach to orthogonal polynomials on lattices. In passing, explicit examples of pure (i.e. non-semiclassical) Laguerre-Hahn polynomials were given. Among them, an example of pure Laguerre-Hahn polynomials "generable" from Infeld-Hull-Miller factorization chains (section 5.2).

The Chapter six deals with the second major concern of this thesis: That of establishing difference equations (not necessarily of eigenelements type) satisfied by orthogonal polynomials. The fourth-order difference equation satisfied by the Laguerre-Hahn polynomials has been established in general situation (not explicitly) (section 6.1). Explicitly, it has been established for all the polynomials r-associated to the classical (up to Askey-Wilson) polynomials on lattices (section 6.2) and for the class one Laguerre-Hahn polynomials on linear lattices (up to a system of non-linear difference equations satisfied by the coefficients in the three-term recurrence relations)(section 6.3).

It is important to note that most of the questions treated here open naturally to remarkable outlooks:
The formulas in [40] (see Eqs. (2.145), (2.147) here), extending the "quasi-periodicity" method to the cases of eigenfunctions admitting eigenvalues as non-linear polynomials of their variables, is nowadays almost non-explored not only in the discrete situations but also in the continuous ones.
The extension of the factorization techniques, applied here to the second-order difference eigenelements problems, to the fourth-order situation (see schemes in section 2.2.2) should offer good perspectives. In particular, as noted in remark 2.3, it is sensible that the extension of the Infeld-Hull-Miller factorization technique to the fourth-order situation (see scheme in section 2.2.2, second part) should lead to a class of polynomials extending the Laguerre-Hahn polynomials.
We succeeded to "modify" the special case of Meixner polynomials. A natural question consists in finding other "modifiable" cases "higher" than the Meixner class (i.e. Hahn, Askey-Wilson, ...). Also, the possibility of "modifi-

cation" of some fourth-order situations (Krall type polynomials) is expected. Now, to extend problems treated in the 6th chapter, a natural question consists of deepening the study of the non-linear system of difference equations in (6.33)-(6.34). Here, explicit examples of pure (i.e. non-semiclassical) Laguerre-Hahn polynomials have been extracted from it. A more systematic study should open to interesting situations. A complete characterization from difference equations point of view is expected not only for the semi-classical but also for the Laguerre-Hahn polynomials on lattices.
The final question comes mostly from simple intuition: The authors of the so-called "quasi-periodicity method" have somewhere recalled that the term "quasi-periodicity" used here has nothing to see with the well known quasi-periodicity (also called almost periodicity) behaviour of Bohr type [30]. A question appears: What may be the nature of eigenelements (if any) generated from factorization chains constrained by Bohr almost periodicity closure conditions? It is worth noting for the interested reader that our "annex thesis" is devoted effectively to Bohr almost periodic functions.

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