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A method of constructing Krall's polynomials

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Abstract

We propose a method of constructing orthogonal polynomials $P_n(x)$ (Krall's polynomials) that are eigenfunctions of higher-order differential operators. Using this method we show that recurrence coefficients of Krall's polynomials $P_n(x)$ are rational functions of n. Let $P_n^{(a,b;M)}(x)$ be polynomials obtained from the Jacobi polynomials $P_n^{(a,b)}(x)$ by the following procedure. We add an arbitrary concentrated mass M at the endpoint of the orthogonality interval with respect to the weight function of the ordinary Jacobi polynomials. We find necessary conditions for the parameters a, b in order for the polynomials $P_n^{(a,b;M)}(x)$ to obey a higher-order differential equation. The main result of the paper is the following. Let a be a positive integer and $b \geqslant -1/2$ an arbitrary real parameter. Then the polynomials $P_n^{(a,b;M)}(x)$ are Krall's polynomials satisfying a differential equation of order $2a+4$. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Formal orthogonal polynomials OP are defined through the three-term recurrence relation [1]

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x), \quad P_0 = 1,\ P_1 = x - b_0. \tag{1.1}$$

The so-called Favard theorem [1] states that if $u_n \neq 0$, then there exists a linear functional $\mathscr{L}$ such that

$$\mathscr{L}\{P_n(x)P_m(x)\} = h_n\ \delta_{nm}, \tag{1.2}$$

where $h_n = u_1 u_2 \ldots u_n$ are normalization constants.

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The problem formulated first by Krall [9], is to find all formal OP satisfying the eigenvalue problem

$$LP_n(x) = \lambda_n P_n(x) \tag{1.3}$$

for the Nth-order differential operator L defined by

$$L = \sum_{k=0}^{N} a_k(x)\partial_x^k, \tag{1.4}$$

where $a_k(x)$ are polynomials such that $deg(a_k(x)) \leqslant k$.

It is easily seen that necessarily N is *even* and λ_n is a polynomial in n of degree $\leqslant N$.

The OP satisfying Eq. (1.3) for $N=2$ are well known as classical OP: Jacobi (and their specializations, like Gegenbauer ones), Laguerre and Hermite polynomials.

Krall found all possible OP satisfying 4th order differential equation (i.e. $N=4$). However already for $N=6$ a full list of all OP is unknown (apart from several explicit examples).

In all known explicit examples it appears that 'non-classical' OP satisfying Eq. (1.3) differ from classical ones (i.e. satisfying Eq. (1.3) with $N=2$) only by inserting one or two concentrated masses at the endpoints of the orthogonality interval.

For example, in [5,6] differential equations of *arbitrary order* were constructed for the generalized Laguerre and ultraspherical polynomials.

2. Representation coefficients of the operator L

Without loss of generality we can put

$$a_0 = 0. \tag{2.1}$$

Indeed, a_0 is a constant which can be incorporated into the definition of λ_n as is seen from Eq. (1.3). In what follows we will assume that condition (2.1) is fulfilled. Note that this means in particular that

$$\lambda_0 = 0. \tag{2.2}$$

It is useful to present the polynomials $a_n(x)$ in the form

$$a_n(x) = \sum_{s=0}^{n} \alpha_{ns} x^s, \quad n = 0, 1, \ldots, N, \tag{2.3}$$

where α_{ns} are some coefficients. In general degrees of the polynomials $a_n(x)$ can be less than n. This means that $\alpha_{nn} = 0$ for some n. However there exists at least one polynomial $a_n(x)$ such that $deg(a_n(x)) = n$. Indeed, otherwise the operator L acting to the polynomial $P_n(x)$ yields a polynomial of a lesser degree which is impossible due to Eq. (1.3).

Consider the action of the operator L on the monomials x^n. Using Eq. (1.4) we get

$$L\{x^n\} = \sum_{k=0}^{n} A_{nk} x^k, \tag{2.4}$$

where A_{nk} are some coefficients.

Proposition 2.1. *The coefficients A_{nk} have the following properties*:

(i) $A_{nk} = 0$ *for* $k = 0, 1, \dots, n - N - 1$;

(ii) A_{nk} *are polynomials in n of degrees not exceeding N* (*at least one of these polynomials has the exact degree N*);

(iii) *The polynomial* $A_{n,n-s}$ *has zeroes at* $n = 0, 1, \dots, s - 1$.

Proof. The property (i) follows from the fact that N is the maximal order of all derivative operators in L. The properties (ii) and (iii) follow from the explicit formula

$$A_{nk} = \sum_{j=0}^{N-n+k} \alpha_{j+n-k,j} n(n-1)\dots(1+k-j). \tag{2.5}$$

This allows one to rewrite $A_{n,n-s}$ as

$$A_{n,n-s} = n(n-1)(n-2)\dots(n-s+1)\pi_{N-s}(n), \tag{2.6}$$

where

$$\pi_{N-s}(n) = \alpha_{s0} + \sum_{j=1}^{N-s} \alpha_{s+j,j}(n-s)(n-s-1)\dots(n-s-j+1) \tag{2.7}$$

is a polynomial in n of degree not exceeding $N - s$.

We will call A_{nk} the representation coefficients of the operator L.

The inversion statement is important

Proposition 2.2. *Assume that* A_{nk} *are arbitrary coefficients possessing properties* (i)–(iii). *Then there exists a unique differential operator L having* A_{nk} *as representation coefficients. The order of this operator is the maximal degree of all coefficients* A_{nk} *considered as polynomials in n*.

Proof. Assume that the coefficients $A_{n,n-s}$ have the expression (2.6) with some known (arbitrary) polynomials $\pi_{N-s}(n)$. Then the coefficients α_{sm} are determined uniquely by means of the Newton interpolating formula

$$\alpha_{s+k,k} = \left. \frac{\Delta^k \pi_{N-s}(n)}{k!} \right|_{n=s}, \tag{2.8}$$

where the difference operator Δ is defined as $\Delta F(n) = F(n+1) - F(n)$. Hence the differential operator L is restored by Eqs. (1.4) and (2.3).

3. Basic relations

Let B_{nk} be the expansion coefficients of the polynomial $P_n(x)$

$$P_n(x) = \sum_{k=0}^{n} B_{nk} x^k. \tag{3.1}$$

Substituting Eqs. (3.1) and (2.4) into Eq. (1.3) and taking into account property (i) we get the relation between the coefficients

$$\sum_{s=k}^{n} B_{ns}A_{sk} = \lambda_n B_{nk}, \quad k = n-N, n-N+1, \ldots, n. \tag{3.2}$$

We thus have $N+1$ equations (3.2) for the coefficients B_{nk} in terms of the known coefficients A_{nk}.

Obviously, the coefficient B_{nn} is arbitrary. Without loss of generality, we can put $B_{nn}=1$. Then we have

Proposition 3.1. *All the coefficients $B_{n,n-1}, B_{n,n-2}, \ldots, B_{n0}$ and the spectral parameter λ_n are uniquely determined in terms of the representation coefficients A_{nk} provided $A_{nn} \neq A_{jj}$, $j = 1, 2, \ldots, n-1$. Moreover, all B_{nk} are rational functions of the argument n.*

Proof. For $k=n$ the equations (3.2) are reduced to

$$\lambda_n = A_{nn}. \tag{3.3}$$

Hence λ_n is a polynomial of an order not exceeding N.

For $k = n-1$ we get from Eq. (3.2)

$$B_{n,n-1}A_{n-1,n-1} + A_{n,n-1} = \lambda_n B_{n,n-1}. \tag{3.4}$$

Taking into account Eq. (3.3) one obtains the expression for $B_{n,n-1}$

$$B_{n,n-1} = \frac{A_{n,n-1}}{\lambda_n - \lambda_{n-1}}. \tag{3.5}$$

Hence $B_{n,n-1}$ is determined uniquely provided $\lambda_n \neq \lambda_{n-1}$ (or, equivalently, $A_{nn} \neq A_{n-1,n-1}$).

For $k = n-2$ we get

$$B_{n,n-2}(A_{nn} - A_{n-2,n-2}) = B_{n,n-1}A_{n-1,n-2} + A_{n,n-2}. \tag{3.6}$$

As $B_{n,n-1}$ is already known, $B_{n,n-2}$ is uniquely determined provided $A_{nn} \neq A_{n-2,n-2}$:

$$B_{n,n-2} = \frac{A_{n,n-1}A_{n-1,n-2} + A_{n,n-2}(A_{nn} - A_{n-1,n-1})}{(A_{nn} - A_{n-1,n-1})(A_{nn} - A_{n-2,n-2})}. \tag{3.7}$$

For $k = n-s$ we have

$$B_{n,n-s}A_{n-s,n-s} + B_{n,n-s+1}A_{n-s+1,n-s} + \cdots + A_{n,n-s} = \lambda_n B_{n,n-s}. \tag{3.8}$$

Assuming that all the coefficients $B_{n,n-1}, B_{n,n-2}, \ldots, B_{n,n-s+1}$ are already determined we determine $B_{n,n-s}$ from Eq. (3.8) provided that $A_{nn} \neq A_{n-s,n-s}$. Moreover, it is seen from Eqs. (3.5) and (3.7) that $B_{n,n-1}$ and $B_{n,n-2}$ are rational functions of n. By induction it is proved that all $B_{n,n-s}$ are rational functions of n.

Thus for the given operator L there is a unique polynomial solution $P_n(x)$ of the eigenvalue problem (1.3) provided that $\lambda_n \neq \lambda_k$ for $n \neq k$.

So far, the polynomials $P_n(x)$ need not be orthogonal. Now we consider the case of formal orthogonal polynomials satisfying three-term recurrence relation (1.1). In terms of the expansion coefficients B_{nk} recurrence relation (1.1) means

$$B_{n+1,k} + u_n B_{n-1,k} + b_n B_{nk} - B_{n,k-1} = 0, \quad k = 0, 1, \ldots, n. \tag{3.9}$$

Remark. It is assumed in Eq. (3.9) that $B_{n-1,n} = B_{n-1,n+1} = B_{n,-1} = 0$.

From Eq. (3.9) we obtain an important result:

Proposition 3.2. *If the formal orthogonal polynomials* $P_n(x)$ *are solutions of eigenvalue problem* (1.3), *then their recurrence coefficients are rational functions of* n.

Proof. From Eq. (3.9) at $k = n$ and $k = n-1$ we get the expression of the recurrence coefficients in terms of the expansion ones

$$b_n = B_{n,n-1} - B_{n+1,n}, \tag{3.10}$$

$$u_n = B_{n,n-2} - B_{n+1,n-1} - B_{n,n-1}(B_{n,n-1} - B_{n+1,n}). \tag{3.11}$$

The coefficients B_{ns} are rational functions of n, hence b_n, u_n are rational functions as well.

4. Inverse problem

In this section we consider an inverse problem: assume that the polynomials $P_n(z)$ are given. Then the coefficients B_{nk} are known explicitly. Assume also that the polynomials $P_n(z)$ satisfy the eigenvalue equation (1.3) but with some unknown operator L. We would like to reconstruct the coefficients A_{nk}. For simplicity it is convenient to change the notation:

$$B_{n,n-s} = B_n^{(s)}, \qquad A_{n,n-s} = A_n^{(s)}, \quad s = 0, 1, 2, \ldots \,. \tag{4.1}$$

We start from relation Eq. (3.8), which can be rewritten in the form

$$\sum_{i=0}^{s} B_n^{(s-i)} \, A_{n-s+i}^{(i)} = \lambda_n \, B_n^{(s)}, \quad s = 0, 1, 2, \ldots \,. \tag{4.2}$$

For $s = 0$ we get the relation $A_n^{(0)} = \lambda_n$, which is not a restriction but rather a definition of λ_n.

For $s = 1$ we have from Eq. (4.2)

$$A_n^{(1)} = (\lambda_n - \lambda_{n-1}) \, B_n^{(1)} = \Omega_n \, B_n^{(1)}, \tag{4.3}$$

where

$$\Omega_n = \lambda_n - \lambda_{n-1}. \tag{4.4}$$

Relation (4.3) imposes some restrictions for λ_n. Indeed, both λ_n and $A_n^{(1)}$ should be polynomials in n, while $B_n^{(1)}$ is a rational function:

$$B_n^{(1)} = \frac{Q^{(1)}(n)}{R^{(1)}(n)}, \tag{4.5}$$

where $Q^{(1)}(n)$ and $R^{(1)}(n)$ are some polynomials in n having no common zeroes.

Hence we get the *necessary* condition for Ω_n:

$$\Omega_n = r(n) R^{(1)}(n), \tag{4.6}$$

where $r(n)$ is a polynomial in n. Assume that Ω_n is already known:

$$\Omega_n = \sum_{k=0}^{M} \omega_k n^k, \tag{4.7}$$

where ω_k are some coefficients. Then from Eq. (4.4) we can retain explicit expression of λ_n

$$\lambda_n = \sum_{k=0}^{M} \omega_k S_k(n), \tag{4.8}$$

where

$$S_k(n) = \sum_{j=1}^{n} j^k. \tag{4.9}$$

From Eqs. (4.6) and (4.8) we get

$$deg(\lambda_n) = deg(r(n)) + deg(R^{(1)}(n)) + 1 \leqslant N, \tag{4.10}$$

where N is order of the differential operator L.

For $A_n^{(1)}$ we have from Eqs. (4.3) and (4.5)

$$A_n^{(1)} = r(n)\; Q^{(1)}(n). \tag{4.11}$$

For $s = 2$ we get

$$A_n^{(2)} = (\Omega_n + \Omega_{n-1})\; B_n^{(2)} - \Omega_{n-1} B_n^{(1)} B_{n-1}^{(1)}. \tag{4.12}$$

This process can be repeated: if $A_n^{(1)}, \dots, A_n^{(s-1)}$ are already found, then from Eq. (4.2) we find $A_n^{(s)}$ uniquely. However, in order for the polynomials $P_n(x)$ to obey eigenvalue equation (1.3) the coefficients $A_n^{(s)}$ should have expression (2.6) or, in another form

$$A_n^{(s)} = n(n-1)(n-2)\dots(n-s+1)\pi_{N-s}(n), \quad s = 0, 1, \dots, N, \tag{4.13}$$

where $\pi_k(n)$ is a polynomial of degree not exceeding k. Moreover, necessarily

$$A_n^{(s)} \equiv 0, \quad \text{for} \quad s > N. \tag{4.14}$$

Proposition 4.1. *Assume that the coefficients $A_n^{(s)}$ are constructed via algorithm* (4.11), (4.12),… *Then conditions* (4.13) *and* (4.14) *are necessary and sufficient for the polynomials $P_n(z)$ to obey eigenvalue equation* (1.3).

The proof of this proposition is a direct consequence of the Propositions 1 and 2.

Note that the coefficients α_{ik} of the differential operator L are then restored via Eq. (2.8).

Assume now that some orthogonal polynomials $P_n(z)$ are given with the coefficients $B_n^{(s)}$ being rational functions of n. We would like to recognize whether or not the polynomials $P_n(z)$ satisfy differential equation (1.3). To that end let us choose an arbitrary polynomial $r(n)$ and construct Ω_n via (4.4) (hence λ_n is also known via Eq. (4.8)). Then all the coefficients $A_n^{(s)}$, $s = 1, 2, \dots$ are constructed uniquely by Eqs. (4.11), (4.12), … . Hence the differential operator L exists if and only if there exists a polynomial $r(n)$ such that the properties (4.13) and (4.14) take place. A simple criterion for verification of the condition (4.14) provides the following.

Proposition 4.2. *Assume that* $A_n^{(s)} \neq 0$ *for* $s = 1, 2, \ldots, s_0 - 1$ *and* $A_n^{(s_0)} = 0$. *In order for* $A_n^{(s)} = 0$ *for all* $s = s_0 + 1, s_0 + 2, \ldots$ *it is necessary and sufficient that the relation*

$$\sum_{m=1}^{s_0-1} B_n^{(s-m)} \, A_{n-s+m}^{(m)} = (\lambda_n - \lambda_{n-s}) \, B_n^{(s)} \tag{4.15}$$

holds for all $s = s_0 + 1, s_0 + 2, \ldots$.

The proof of this proposition is an elementary consequence of formula (4.2). Criterion (4.15) is a convenient tool in practice if s_0 is not much greater than 1.

5. The simplest example: Jacobi polynomials

Consider how our algorithm works for the simplest example of the ordinary Jacobi polynomials. The monic Jacobi polynomials are defined by the formula

$$P_n^{(a,b)}(x) = \frac{(-1)^n (a+1)_n}{(a+b+n+1)_n} \, {}_2F_1\left(\begin{matrix} -n, n+a+b+1 \\ a+1 \end{matrix} ; x \right). \tag{5.1}$$

In what follows we will assume that $a \geqslant -1/2, \; b \geqslant -1/2$. The polynomials $P_n^{(a,b)}(x)$ are orthogonal on $(0,1)$

$$\int_0^1 P_n^{(a,b)}(x) P_m^{(a,b)}(x) w(x) \, \mathrm{d}x = h_n \, \delta_{nm} \tag{5.2}$$

with the normalized weight function

$$w(x) = \frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)} x^a (1-x)^b \tag{5.3}$$

(we define the Jacobi polynomials on the shifted orthogonality interval with respect to the standard one [7]). From Eq. (5.3) we have the explicit expression for the moments

$$c_n = \int_0^1 w(x) x^n \mathrm{d}x = \frac{(a+1)_n}{(a+b+2)_n}. \tag{5.4}$$

We need also the values

$$P_n^{(a,b)}(0) = \frac{(-1)^n (a+1)_n}{(a+b+n+1)_n}, \tag{5.5}$$

$$Q_n^{(a,b)}(0) = (-1)^{n+1} \frac{(a+b+1)(b+1)_n n!}{a(a+b+1)_{2n}}, \tag{5.6}$$

where $Q_n^{(a,b)}(z)$ are the second-kind functions corresponding to the polynomials $P_n^{(a,b)}(x)$ and defined by the formula [13]

$$Q_n^{(a,b)}(z) = \int_0^1 \frac{P_n^{(a,b)}(x)w(x)\mathrm{d}x}{z-x}. \tag{5.7}$$

From Eq. (5.1) we find the coefficients $B_n^{(s)}$:

$$B_n^{(s)} = \frac{(-n)_s(-a-n)_s}{s!(-a-b-2n)_s}. \tag{5.8}$$

The relation (4.3) is written as

$$A_n^{(1)} = -\Omega_n \frac{n(a+n)}{a+b+2n}. \tag{5.9}$$

Obviously, the simplest choice for Ω_n is the denominator of $B_n^{(1)}$, i.e.

$$\Omega_n = a+b+2n. \tag{5.10}$$

Then from Eqs. (4.4) and (2.2)

$$\lambda_n = n(n+a+b+1). \tag{5.11}$$

From Eq. (5.9) we find $A_n^{(1)}$:

$$A_n^{(1)} = -n(a+n). \tag{5.12}$$

From Eq. (4.12) we then find that $A_n^{(2)}=0$. It is easily verified that $A_n^{(s)}=0$ for $s=2,3,\dots$. Indeed, the criterion (4.15) is rewritten in our case as

$$B_n^{(s-1)}A_{n-s+1}^{(1)} = (\lambda_n - \lambda_{n-s})B_n^{(s)}. \tag{5.13}$$

Substituting Eqs. (5.11), (5.8) and (5.12) we easily find that Eq. (5.13) is fulfilled identically for all $s=2,3,\dots$. Thus the Jacobi polynomials indeed satisfy second-order differential equation (1.3). The coefficients α_{ik} of the operator L are restored via formulas (2.8), where, in our case $\pi_2(x)=x(x+a+b+1)$ and $\pi_1(x)=-x-a$. The only non-zero coefficients are: $\alpha_{22}=-\alpha_{21}=1$, $\alpha_{11}=a+b+2$, $\alpha_{10}=-a-1$. We thus have the differential operator

$$L = x(1-x)\partial_x^2 + ((a+b+2)x-a-1)\partial_x. \tag{5.14}$$

Operator (5.14) indeed has Jacobi polynomials as eigenfunctions (see, e.g. [7]).

6. Geronimus transforms

In all known examples of ‘non-classical’ Krall’s polynomials their orthogonality measure differs from that for the classical OP by inserting one or two arbitrary concentrated masses at endpoints of the orthogonality interval. Such procedure is connected with the so-called Geronimus transform [3,4].

In this section we consider basic properties of the Geronimus transform (see also [11,12,14]).

Recall that the Geronimus transform is

$$\tilde{P}_n(x) = P_n(x) - C_nP_{n-1}(x), \tag{6.1}$$

where

$$C_n = \frac{\phi_n}{\phi_{n-1}} \tag{6.2}$$

and ϕ_n is an arbitrary solution of the recurrence relation (1.1), that is

$$\phi_{n+1} + b_n\phi_n + u_n\phi_{n-1} = \mu\phi_n, \tag{6.3}$$

where μ is an arbitrary parameter (not belonging to the interior of the spectral interval).

According to general theory, ϕ_n can be presented as a linear combination of two linear independent solutions of recurrence equation (6.3). There are several possibilities to choose such solutions. For example, one can choose the OP $P_n(x)$ themselves and the associated polynomials $R_n(x)$, which satisfy the recurrence relation

$$R_{n+1} + u_{n+1}R_{n-1} + b_{n+1}R_n = xR_n(x), \quad R_0 = 1, \;\; R_1(x) = x - b_1. \tag{6.4}$$

Then

$$\phi_n = \beta_1 R_{n-1}(\mu) + \beta_2 P_n(\mu), \tag{6.5}$$

where β_1, β_2 are two arbitrary constants.

For our purposes another choice is more convenient:

$$\phi_n = Q_n(\mu) + \beta P_n(\mu), \tag{6.6}$$

where $Q_n(\mu)$ are the functions of the second kind [1,13]

$$Q_n(z) = \mathscr{L}\left\{\frac{P_n(x)}{z-x}\right\} \tag{6.7}$$

(the linear functional $\mathscr{L}$ acts on the x variable). Of course, expression (6.6) is valid only if the value $Q_n(\mu)$ exists. It is so if μ lies outside the spectral interval. For the endpoints of the spectral interval the situation is more complicated. However for the Jacobi polynomials the values $Q_n(0)$ and $Q_n(1)$ do exist (if, say $a > -1/2$, $b > -1/2$). We will use expression (6.6) in the sequel.

If $w(x)$ is the weight function of the polynomials $P_n(x)$ then

$$\tilde{w}(x) = \frac{w(x)}{x-\mu} - \beta\delta(x-\mu) \tag{6.8}$$

is the weight function of the polynomials $\tilde{P}_n(x)$ (see, e.g. [14]).

Consider 2 succeeded GT with the parameters μ_1, β_1 and μ_2, β_2. There are two ways to perform these transformations. One can first perform GT with the parameters μ_1, β_1 obtaining the polynomials $\tilde{P}_n(x)$, and then perform the second GT

$$P_n^{(2)}(x) = \tilde{P}_n(x) - \frac{\psi_n}{\psi_{n-1}}\tilde{P}_{n-1}(x), \tag{6.9}$$

where

$$\psi_n = \tilde{Q}_n(\mu_2) + \tilde{\beta}\tilde{P}_n(\mu_2). \tag{6.10}$$

For $\tilde{Q}_n(z)$ we have

$$\tilde{Q}_n(z) = \int \frac{\tilde{P}_n(x)\tilde{w}(x)\,\mathrm{d}x}{z-x} = \frac{Q_n(z) - \phi_n/\phi_{n-1}Q_{n-1}(z)}{z-\mu}, \tag{6.11}$$

where we used expression (6.8) for the weight function and (6.1) for $\tilde{P}_n(x)$. Using Eq. (6.11) we present ψ_n in the form

$$\psi_n = \frac{\Phi_n - \phi_n/\phi_{n-1}\Phi_{n-1}}{\mu_2 - \mu_1}, \tag{6.12}$$

where

$$\Phi_n = Q_n(\mu_2) + \tilde{\beta}(\mu_2 - \mu_1)P_n(\mu_2). \tag{6.13}$$

On the other hand, we can start from the functions $\phi_n(\mu_1)=Q_n(\mu_1)+\beta_1 P_n(\mu_1)$ and $\phi_n(\mu_2)=Q_n(\mu_2)+\beta_2 P_n(\mu_2)$ to get the expression

$$P_n^{(2)}(x) = \kappa_n^{-1} \begin{vmatrix} P_n(x) & P_{n-1}(x) & P_{n-2}(x) \\ \phi_n(\mu_1) & \phi_{n-1}(\mu_1) & \phi_{n-2}(\mu_1) \\ \phi_n(\mu_2) & \phi_{n-1}(\mu_2) & \phi_{n-2}(\mu_2) \end{vmatrix}, \tag{6.14}$$

where $\kappa_n = \phi_{n-1}(\mu_1)\phi_{n-2}(\mu_2) - \phi_{n-1}(\mu_2)\phi_{n-2}(\mu_1)$.

Comparing Eqs. (6.14) and (6.9) we get the relation between the parameters

$$\tilde{\beta} = \frac{\beta_2}{\mu_2 - \mu_1}. \tag{6.15}$$

Thus for the given parameters β_1, β_2 we get the expression for the transformed weight function under two GT

$$\begin{aligned} w^{(2)}(x) &= \frac{\tilde{w}(x)}{x - \mu_2} - \tilde{\beta}\delta(x - \mu_2) \\ &= \frac{w(x)}{(x-\mu_1)(x-\mu_2)} - \frac{\beta_1}{\mu_1 - \mu_2}\delta(x-\mu_1) - \frac{\beta_2}{\mu_2 - \mu_1}\delta(x - \mu_2). \end{aligned} \tag{6.16}$$

In what follows we use the notation $\tilde{P}_n(x) = \mathcal{G}(\mu;\beta)\{P_n(x)\}$ for the Geronimus transformation at the point μ with the parameter β.

7. Necessary conditions for Koornwinder's polynomials to be Krall's polynomials

In this section we derive necessary conditions for Koornwinder's generalized Jacobi polynomials $P_n^{(a,b;M_1,M_2)}(x)$ to satisfy differential equation (1.3).

Recall that the polynomials $P_n^{(a,b;M_1,M_2)}(x)$ were introduced by Koornwinder [8] and obtained from the ordinary Jacobi polynomials $P_n^{(a,b)}(x)$ by inserting of an arbitrary mass M_1 at the endpoint $x=0$ and an arbitrary mass M_2 at the endpoint $x=1$. Koornwinder raised a problem: whether or not the polynomials $P_n^{(a,b;M_1,M_2)}(x)$ satisfy a differential equation of type (1.3). Some special cases do satisfy this requirement:

(i) Legendre-type polynomials with $a=b=0$, $M_1=M_2$. These polynomials satisfy 4th order differential equation [10].

(ii) Jacobi-type polynomials with $a=0$, b arbitrary, $M_2=0$. These polynomials also satisfy 4th order differential equation [10].

(iii) The polynomials with $a=b=0$ and M_1 and M_2 arbitrary. These polynomials satisfy 6th order differential equation [10].

(iv) The polynomials with $a=b=j$ where $j=0,1,2,\dots$ and $M_1=M_2$. These polynomials are generalization of the ultraspherical polynomials and satisfy an equation of order $2j+4$ [6].

(v) The polynomials with $b=\pm 1/2$, $a=j=0,1,2\dots$ and $M_2=0$. These polynomials satisfy an equation of order $2j+4$ [6].

In the next section we show that the polynomials with *an arbitrary* parameter $b \geqslant -1/2$, $M_2=0$ and $a=j=0,1,2\dots$ satisfy the equation of the order $2j+4$.

As we know the necessary condition for polynomials to satisfy equation Eq. (1.3) is the following: the coefficients $B_n^{(s)}$ are rational functions of the argument n. So we would like to recognize whether or not the expansion coefficients $B_n^{(s)}$ for Koornwinder's polynomials are rational functions of n.

The crucial observation is:

Proposition 7.1. *Koornwinder's polynomials* $P_n^{(a,b;M_1,0)}(x)$ *coincide with the polynomials* $\mathscr{G}(0;\beta)$ $\{P_n^{(a+1,b)}(x)\}$ *which are obtained from the Jacobi polynomials* $P_n^{(a+1,b)}(x)$ *by application of the Geronimus transform at the endpoint* $x=0$ *with the parameter* $\beta=-M_1(a+b+2)/(a+1)$.

Proof. For Koornwinder's polynomials $P_n^{(a,b;M_1,0)}(x)$ the weight function is (up to a normalization constant)

$$w(x;a,b;M_1,0)=w(x;a,b)+M_1\delta(x). \tag{7.1}$$

From Eq. (6.8) we have

$$\tilde{w}(x;a+1,b)=w(x;a+1,b)/x-\beta\delta(x). \tag{7.2}$$

On the other hand, from Eq. (5.3) we see that

$$w(x;a+1,b)/x=\frac{a+b+2}{a+1}w(x;a,b). \tag{7.3}$$

Hence

$$\tilde{w}(x;a+1,b)=\left(\frac{a+b+2}{a+1}\right)\left(w(x;a,b)-\frac{\beta(a+1)}{a+b+2}\delta(x)\right). \tag{7.4}$$

Comparing Eqs. (7.1) and (7.4) we arrive at the statement of the proposition.

Quite similarly we get

Proposition 7.2. *Koornwinder's polynomials* $P_n^{(a,b;0,M_2)}(x)$ *coincide with the polynomials* $\mathscr{G}(1;\beta)$ $\{P_n^{(a,b+1)}(x)\}$ *which are obtained from the Jacobi polynomials* $P_n^{(a,b+1)}(x)$ *by application of the Geronimus transform at the endpoint* $x=1$ *with the parameter* $\beta=-M_2(a+b+2)/(b+1)$.

Combining these two statements we obtain

Proposition 7.3. *Koornwinder's polynomials* $P_n^{(a,b;M_1,M_2)}(x)$ *coincide with the polynomials* $\mathscr{G}(0;\beta_1)$ $\mathscr{G}(1;\beta_2)\{P_n^{(a+1,b+1)}(x)\}$.

Hence it is sufficient to study the Geronimus transform of the coefficients $B_n^{(s)}$ of the Jacobi polynomials.

Consider the coefficients $\tilde{B}_n^{(s)}$ for the polynomials $\tilde{P}_n(x) = \mathscr{G}(0;\beta)\{P_n^{(a,b)}(x)\}$. Using Eq. (6.1) we have

$$\tilde{B}_n^{(0)} = 1, \qquad \tilde{B}_n^{(s)} = B_n^{(s)} - \phi_n/\phi_{n-1}\, B_{n-1}^{(s-1)}, \quad s = 1,2,\ldots,n, \tag{7.5}$$

where $B_n^{(s)}$ are expansion coefficients (5.8) for the ordinary Jacobi polynomials, and

$$\phi_n = Q_n^{(a,b)}(0) + \beta P_n^{(a,b)}(0). \tag{7.6}$$

From explicit formulas (5.5) and (5.6) we have

$$\phi_n = \frac{(-1)^n (a+1)_n}{(a+b+n+1)_n}\left(\beta + \frac{a+b+1}{a} g_n\right), \tag{7.7}$$

where

$$g_n = \frac{n!(b+1)_n}{(a+1)_n(a+b+1)_n}. \tag{7.8}$$

We would like to find conditions under which the coefficients $\tilde{B}_n^{(s)}$ are rational functions of the argument n, since $B_n^{(s)}$ are rational functions of n (see Eq. (5.8)), it follows that necessarily ϕ_n/ϕ_{n-1} should be a rational function of n. In turn, as the parameter β is arbitrary, g_n should be a rational function of n. Let us assume that

$$a > -1/2, \qquad b > -1/2. \tag{7.9}$$

Then we can rewrite Eq. (7.8) in the form

$$g_n = \frac{\Gamma(a+b+1)\Gamma(a+1)}{\Gamma(b+1)} \frac{\Gamma(n+1)\Gamma(b+n+1)}{\Gamma(a+n+1)\Gamma(a+b+n+1)}. \tag{7.10}$$

If g_n is a rational function of n then the number of its zeroes and poles is finite. On the other hand, the number of poles of the function $\Gamma(n+a+1)$ is infinite: these poles are located at the points $n+a+1 = 0,-1,-2,\ldots$. Hence, some poles coming from the numerator of Eq. (7.10) should coincide with some poles coming from the denominator. It is clear from Eq. (7.10) that there are 2 possibilities of such coincidence:

(i) $a = j = 0,1,2,\ldots$, b is arbitrary;

(ii) $a+b = j_1$, $a-b = j_2$, where $j_{1,2}$ are arbitrary integers. Because of restriction (7.9) we have in the case (ii) that both a and b are simultaneously non-negative integers or half-integers. In fact, it is sufficient to consider the case of half-integer numbers: $a = m_1 + 1/2$, $b = m_2 + 1/2$, where $m_1, m_2 = 0,1,2,\ldots$, because the case of integer a,b is a special case of (i).

It is easily verified that indeed g_n is a rational function under the conditions (i) and (ii).

Taking into account Proposition 7.1 we have the following

Proposition 7.4. *Koornwinder's polynomials $P_n^{(a,b;M_1,0)}(x)$ with arbitrary M_1 and under conditions (7.9) have rational coefficients $B_n^{(s)}$ (and hence rational recurrence coefficients u_n, b_n) only in the two cases*:

(i) *a takes the values $0,1,2,\ldots$ whereas b is arbitrary*;

(ii) *both a and b take independently half-integer values $1/2$, $3/2$, $5/2,\ldots$.*

Obviously the same statement is valid for the case of the polynomials $P_n^{(a,b;0,M_2)}(x)$ with the replacement $a \leftrightarrow b$. Combining we arrive at the following:

Proposition 7.5. *Koornwinder's polynomials $P_n^{(a,b;M_1,M_2)}(x)$ with arbitrary M_1, M_2 and under the conditions $a, b > -1/2$ have rational expansion coefficients $B_n^{(s)}$ (and hence rational recurrence coefficients u_n, b_n) only in the two cases*:

(i) *both a and b take independently non-negative integer values* $0, 1, 2, \ldots$;

(ii) *both a and b take independently half-integer values* $1/2, 3/2, 5/2, \ldots$.

Proof. As M_1 and M_2 are arbitrary, we can put $M_2 = 0$. Then we see that the parameter a should be either integer (and then b is arbitrary) or half-integer (and then b is half-integer as well). Similarly, the case $M_1 = 0$ means that the parameter b should be either integer (and then a is arbitrary) or half-integer (and then a is half-integer as well). Overlapping of these two conditions yields the statement of the proposition.

Thus if the parameters a, b do not belong to the classes (i) and (ii) then corresponding Koornwinder's polynomials do not satisfy Eq. (1.3). It is interesting to note that in all known examples (i)–(v) listed above we deal only with the case (i) of Propositions 7.4 and 7.5. No examples with half-integers parameters a, b are known to satisfy Eq. (1.3) (excepting the trivial case $M_1 = M_2 = 0$, when we deal with the second-order differential equation for the ordinary Jacobi polynomials with arbitrary parameters a, b). $\square$

8. Analysis of the case of the polynomials $P_n^{(j,b;M)}(x)$

Assume that the parameter a of the Jacobi polynomials is a positive integer: $a = j = 1, 2, \ldots$ whereas the parameter $b \geqslant -1/2$ is arbitrary. In this section we show that the polynomials $\mathscr{G}(0;\beta)\{P_n^{(j,b)}(x)\}$ satisfy a differential equation of order $2j + 2$.

First of all note that

$$\phi_n = Q_n(0) + \beta P_n(0) = \frac{(-1)^n n! Y_j(n)}{j!(b+n+1)_{n+j}}, \tag{8.1}$$

where

$$Y_j(n) = \beta(n+1)_j(b+n+1)_j - (j-1)!(b+1)_{j+1} \tag{8.2}$$

is a polynomial in n of degree $2j$. Thus

$$C_n = \frac{\phi_n}{\phi_{n-1}} = -\frac{n(b+n)}{(j+b+2n)(j+b+2n-1)} \frac{Y_j(n)}{Y_j(n-1)}. \tag{8.3}$$

For the transformed polynomials one has

$$\mathscr{G}(0;\beta)\{P_n^{(j,b)}(x)\} = P_n^{(j,b)}(x) - C_n P_{n-1}^{(j,b)}(x). \tag{8.4}$$

The normalized weight function for these polynomials is

$$w(x) = (1+M_1)^{-1}\left(\frac{(b+1)_j}{(j-1)!} x^{j-1}(1-x)^b + M_1\delta(x)\right), \tag{8.5}$$

where

$$M_1 = -\beta \frac{j}{j+b+1} \tag{8.6}$$

is the value of concentrated mass added at the point $x=0$. The corresponding moments are

$$c_n = (1+M_1)^{-1}\left(\frac{(j)_n}{(j+b+1)_n} + M_1\delta_{n0}\right). \tag{8.7}$$

Using Eqs. (8.4) and (8.3) we can write down explicitly the coefficients $B_n^{(s)}$ in the expansion $\mathcal{G}(0;\beta)\{P_n^{(j,b)}(x)\} = \sum_{s=0}^{n} B_n^{(s)} x^{n-s}$. We have

$$B_n^{(s)} = \frac{(-n)_s(-n-j)_s}{s!(-j-b-2n)_s} \times \left(1 - \frac{s(b+n)}{(j+n)(j+b+2n-s)}\frac{Y_j(n)}{Y_j(n-1)}\right). \tag{8.8}$$

Now we find the coefficients $A_n^{(s)}$ of the operator L. Consider the formula

$$B_n^{(1)} = \frac{A_n^{(1)}}{\Omega_n}, \tag{8.9}$$

where

$$B_n^{(1)} = \frac{n(-\beta(n-1)(n+1)_j(b+n)_j + (j-1)!(j+n-1)(b+1)_{j+1})}{(j+b+2n-1)Y_j(n-1)}. \tag{8.10}$$

It is natural to choose Ω_n as the denominator of $B_n^{(1)}$, i.e.

$$\Omega_n = (j+b+2n-1)Y_j(n-1). \tag{8.11}$$

Then from Eq. (8.9) we find that

$$A_n^{(1)} = n(-\beta(n-1)(n+1)_j(b+n)_j + (j-1)!(j+n-1)(b+1)_{j+1}). \tag{8.12}$$

We see that $A_n^{(1)}$ satisfies condition (2.6) for $s=1$ with $N=2j+2$.

The coefficient $A_n^{(0)} = \lambda_n$ can be found from the relation

$$\Omega_n = \lambda_n - \lambda_{n-1} \tag{8.13}$$

with the additional condition $\lambda_0 = 0$. It is easily verified that

$$\lambda_n = A_n^{(0)} = n(b+n+j)\left(\frac{\beta(b+n)_j(n+1)_j}{j+1} - (j-1)!(b+1)_{j+1}\right). \tag{8.14}$$

We thus found the first coefficients $A_n^{(0)}$ and $A_n^{(1)}$. Other coefficients $A_n^{(s)}$, $s=2,3,\ldots$ can be found (step-by-step) from basic relations (3.2).

We have the following

Proposition 8.1. *The coefficients $A_n^{(s)}$ have the explicit expression*

$$A_n^{(s)} = -\beta\frac{(-j)_{s-1}(n-s)_{s+j+1}(b+n)_{j-s+1}}{s!} + \xi_n\delta_{s1} + \eta_n\delta_{s0}, \quad s=0,1,2,\ldots, \tag{8.15}$$

where

$$\xi_n = (j-1)!n(j+n-1)(b+1)_{j+1}, \qquad \eta_n = -n(b+n+j)(j-1)!(b+1)_{j+1}.$$

Proof. It is sufficient to prove the relation

$$\sum_{i=0}^{s} B_n^{(s-i)} A_{n-s+i}^{(i)} = A_n^{(0)} B_n^{(s)}, \tag{8.16}$$

where $B_n^{(s)}$ are given by Eq. (8.8) and $A_n^{(s)}$ by Eq. (8.15).

We have after simple transformations

$$\sum_{i=0}^{s} B_n^{(s-i)} A_{n-s+i}^{(i)} = \eta_{n-s} B_n^{(s)} + \xi_{n-s+1} B_n^{(s-1)} + \left(S_1 + \frac{(b+n)Y_j(n)}{(j+n)Y_j(n-1)} S_2 \right) \kappa_{ns}, \tag{8.17}$$

where

$$\kappa_{ns} = \beta \frac{(-n)_s(-j-n)_s(n-s)_{j+1}(b+n-s)_{j+1}}{s!(-j-b-2n)_s(j+1)}$$

and S_1, S_2 are the sums

$$S_1 = \sum_{i=0}^{s} \frac{(-s)_i(-1-j)_i(1+j+b+2n-s)_i}{i!(1+n-s)_i(b+n-s)_i};$$

$$S_2 = \sum_{i=0}^{s} \frac{(-s)_i(-1-j)_i(1+j+b+2n-s)_i(i-s)}{i!(1+n-s)_i(b+n-s)_i(j+b+2n-s+i)}.$$

These sums are calculated using Pfaff–Saalschütz identity [2]

$$ {}_3F_2\left(\left. \begin{matrix} -s, a, b \\ c, 1+a+b-c-s \end{matrix} \right| 1 \right) = \frac{(c-a)_s(c-b)_s}{(c)_s(c-a-b)_s}, \quad s = 0, 1, 2, \dots \, . \tag{8.18}$$

We thus have

$$S_1 = \frac{(-1-j-n)_s(-n-j-b)_s}{(-n)_s(1-n-b)_s}; \tag{8.19}$$

$$S_2 = -\frac{s}{j+b+2n-s} \frac{(-n-j)_{s-1}(-n-j-b+1)_{s-1}}{(1-n)_{s-1}(2-n-b)_{s-1}}. \tag{8.20}$$

Substituting Eqs. (8.19) and (8.20) into Eq. (8.17) we arrive at relation (8.16).

Proposition 8.2. *The polynomials* $\mathscr{G}(0;\beta)\{P_n^{(j,b)}(x)\}$ *with* $j = 1, 2, \dots$ *and* $b \geqslant -1/2$ *satisfy differential equation* (1.3) of order $N = 2j+2$.

Proof. Note that the coefficients $A_n^{(s)}$ given by Eqs. (8.14), (8.12), (8.15) satisfy the properties:

(i) $A_n^{(s)} = n(n-1)\dots(n-s+1)\pi_{N-s}(n)$, where $\pi_{N-s}(n)$ are polynomials of order $2j+2-s$;

(ii) $A_n^{(s)} = 0$, $s \geqslant j+2$.

Thus all the characteristic properties of the coefficients $A_n^{(s)}$ are fulfilled. From property (i) it follows that degree of the polynomials $A_n^{(s)}$ is $2j+2$. Hence the polynomials $\mathscr{G}(0;\beta)\{P_n^{(j,b)}(x)\}$

indeed satisfy a differential equation of order $2j+2$. The coefficients α_{ik} are calculated explicitly in the next section.

As we know, the polynomials $\mathscr{G}(0;\beta)\{P_n^{(j,b)}(x)\}$ coincide with the polynomials $P_n^{(j-1,b;M)}(x)$. Hence we have

Proposition 8.3. *The polynomials $P_n^{(j,b;M)}(x)$ with $j=0,1,2,\dots$ and $b \geqslant -1/2$ satisfy a differential equation of order $2j+4$.*

For $j=0$ we return to the so-called Jacobi-type polynomials $P_n^{(0,b;M)}(x)$ satisfying 4th order differential equation [10]. As far as we know, for $j \geqslant 1$ our result is new. Note that in [6] this result was obtained only for the special cases $b=\pm 1/2$.

9. Calculation of the coefficients α_{ik} of the differential operator L

In this section we calculate the coefficients α_{ik} of the differential operator $L=\sum_{k=0}^{N}\sum_{i=0}^{k}\alpha_{ki}x^i\partial_x^k$ using formula (2.8). From Eq. (8.15) we find

$$\begin{aligned}\pi_{N-s}(n) &= \frac{A_n^{(s)}}{n(n-1)\dots(n-s+1)} \\ &= \kappa_s(n-s)(n+1)_j(b+n)_{j-s+1} + \xi_n/n\delta_{s1} + \eta_n\delta_{s0},\end{aligned} \tag{9.1}$$

where

$$\kappa_s = -\beta\frac{(-j)_{s-1}}{s!}. \tag{9.2}$$

We employ the well known formula (being a discrete analogue of the binomial theorem)

$$\Delta^k\{f(n)g(n)\} = \sum_{i=0}^{k}\binom{k}{i}\Delta^i f(n)\Delta^{k-i}g(n+i). \tag{9.3}$$

We need also an obvious formula

$$\Delta^k(n+\gamma)_m = m(m-1)\dots(m-k+1)(n+\gamma+k)_{m-k}, \quad k \leqslant m. \tag{9.4}$$

As a consequence of Eq. (9.3) we have (for arbitrary function $g(n)$)

$$\Delta^k\{(n-s)g(n)\}\,|_{n=s} = k\Delta^{k-1}g(n+1)\big|_{n=s}. \tag{9.5}$$

Using Eqs. (9.5) and (2.8) we find for $s \geqslant 2$

$$\alpha_{s+k,k} = \kappa_s\left.\frac{\Delta^{k-1}\{(n+2)_j(b+n+1)_{j-s+1}\}}{(k-1)!}\right|_{n=s}. \tag{9.6}$$

In order to calculate (9.6) we need the following

Lemma 9.1. *For any a,b and any nonnegative integers p,m the following formula*:

$$\Delta^k\{(n+a)_p(n+b)_m\} = (-1)^k \frac{(-m-p)_k(n+a)_p(n+b)_m}{(n+b)_k} \times {}_3F_2\left(\begin{matrix} -k,-p,a-m-b \\ n+a,-p-m \end{matrix}\,\middle|\, 1\right) \tag{9.7}$$

takes place

Proof. It is sufficient to apply formulas (9.3) and (9.4) to get

$$\begin{aligned}\Delta^k\{(n+a)_p(n+b)_m\} &= \sum_{i=0}^{k}\binom{k}{i}\Delta^i(n+a)_p\Delta^{k-i}(n+b+i)_m \\ &= (-1)^k\frac{(-m)_k(n+a)_p(n+b)_m}{(n+b)_k}\sum_{i=0}^{k}\frac{(-k)_i(-p)_i(n+m+b)_i}{i!(n+a)_i(1+m-k)_i}.\end{aligned} \tag{9.8}$$

Then the sum in Eq. (9.8) is reduced to the hypergeometric function ${}_3F_2(1)$, which can be then transformed to Eq. (9.7).

Using this lemma we calculate the coefficients $\alpha_{s+k,k}$:

$$\begin{aligned}\alpha_{s+k,k} &= \beta(-1)^k\frac{(-j)_{s-1}(s-2j-1)_{k-1}(s+2)_j(s+b+1)_{j-s+1}}{(s+b+1)_{k-1}(k-1)!s!} \\ &\times {}_3F_2\left(\begin{matrix} -j,s-j-b,1-k \\ s-2j-1,s+2 \end{matrix}\,\middle|\, 1\right)(1-\delta_{k0}) \\ &+(b+1)_{j+1}((j-1)!\delta_{k1}+j!\delta_{k0})\delta_{s1} \\ &-(j-1)!(b+1)_{j+1}((b+j+1)\delta_{k1}+\delta_{k2})\delta_{s0}, \\ &\quad s=0,1,\dots,j+1, \quad k=0,1,\dots,2j+2-s.\end{aligned} \tag{9.9}$$

Thus the coefficients of the differential operator L can be expressed in terms of the hypergeometric function ${}_3F_2(1)$.

In general, the expression for the polynomials $a_i(x)$, $i=0,1,\dots,N$ is rather complicated. However for polynomials of the maximal order $i=N=2j+2$ we have a simplification. Indeed, in this case $s+k=2j+2$ and the hypergeometric function ${}_3F_2(1)$ in Eq. (9.9) is reduced to ${}_2F_1(1)$, which can be simplified using Chu-Vandermonde formula [2]. After simple calculations one obtains

$$a_N(x) = \frac{\beta}{j+1}x^{j+1}(x-1)^{j+1}. \tag{9.10}$$

Quite analogously one gets

$$a_{N-1}(x) = \beta x^j(x-1)^j((b+3j+1)x-2j). \tag{9.11}$$

10. The case of the generalized Laguerre polynomials

In this section we consider the limit $b\to\infty$ leading to Koornwinder's generalization of the Laguerre polynomials.

Let us recall the scaling property of OP. Define new OP by

$$\tilde{P}_n(x) = \gamma^{-n} P_n(\gamma x), \tag{10.1}$$

where $\gamma \neq 0$ is an arbitrary parameter. Clearly, $\tilde{P}_n(x)$ are monic OP satisfying recurrence relation (1.1) with $\tilde{b}_n = \gamma^{-1} b_n$, $\tilde{u}_n = \gamma^{-2} u_n$. If the polynomials $P_n(x)$ are orthogonal on the interval $[a, b]$

$$\int_a^b P_n(x) P_m(x)\, \mathrm{d}\mu(x) = h_n \delta_{nm}, \tag{10.2}$$

then the polynomials $\tilde{P}_n(x)$ are orthogonal on the interval $[a/\gamma, b/\gamma]$

$$\int_{a/\gamma}^{b/\gamma} \tilde{P}_n(x) \tilde{P}_m(x)\, \mathrm{d}\mu(x/\gamma) = \tilde{h}_n \delta_{nm}. \tag{10.3}$$

The expansion coefficients $\tilde{B}_{nk}$ for the polynomials $\tilde{P}_n(x)$ are

$$\tilde{B}_{nk} = \gamma^{k-n} B_{nk}, \qquad \tilde{B}_n^{(s)} = \gamma^{-s} B_n^{(s)}. \tag{10.4}$$

If the polynomials $P_n(x)$ satisfy differential equation (1.3), then the polynomials $\tilde{P}_n(x)$ satisfy the equation

$$\tilde{L}\tilde{P}_n(x) = \tilde{\lambda}_n \tilde{P}_n(x), \tag{10.5}$$

where the operator $\tilde{L}$ has the representation coefficients

$$\tilde{A}_{nk} = \gamma^{k-n} A_{nk}, \qquad \tilde{A}_n^{(s)} = \gamma^{-s} A_n^{(s)}. \tag{10.6}$$

In particular, $\tilde{\lambda}_n = \lambda_n$.

Let us put $\gamma = 1/b$ and take the limit $b \to \infty$ in expression (5.1) for the Jacobi polynomials. We then obtain the monic Laguerre polynomials:

$$L_n^{(a)}(x) = (-1)^n (a+1)_n \, {}_1F_1\left({-n \atop a+1} ; x \right) \tag{10.7}$$

having the normalized weight function

$$w(x) = \frac{\mathrm{e}^{-x} x^a}{\Gamma(a+1)}. \tag{10.8}$$

Now we put $a = j = 1, 2 \ldots$ and perform the GT at $x = 0$ for the Laguerre polynomials $L_n^{(j)}(x)$. We then get the polynomials

$$\mathcal{G}(0; \beta)\{L_n^{(j)}(x)\} = L_n^{(j)}(x) - \frac{\phi_n}{\phi_{n-1}} L_{n-1}^{(j)}(x), \tag{10.9}$$

where $\phi_n = Q_n(0) + \beta P_n(0) = (-1)^n(-n!/j + (j+1)_n)$.

The weight function of these polynomials is

$$\tilde{w}(x; j) = \frac{\mathrm{e}^{-x} x^{j-1}}{j!} - \beta\delta(x) = j^{-1}(w(x; j-1) + M\delta(x)), \tag{10.10}$$

where $M = -\beta j$.

The coefficients $A_n^{(s)}$ for these polynomials are obtained from the coefficients (8.15) by limiting procedure (10.6):

$$A_n^{(s)} = -\beta \frac{(-j)_{s-1}(n-s)_{s+j+1}}{s!} + n(n+j-1)(j-1)!\delta_{s1} - n(j-1)!\delta_{s0}, \quad s = 0, 1, 2, \dots . \tag{10.11}$$

It is clear from Eq. (10.11) that the coefficients $A_n^{(s)}$ satisfy all needed requirements:

(i) $A_n^{(s)} = n(n-1)\dots(n-s+1)\pi_s(n)$, where $\pi_s(n)$ are polynomials of the order $j+1$;

(ii) $A_n^{(s)} = 0$, $s \geqslant j+2$.

Note that the maximal degree of the polynomials $A_n^{(s)}$ is equal to $2j+2$ (for $s = j+1$).

Hence the generalized Laguerre polynomials $\mathscr{G}(0;\beta)\{L_n^{(j)}(x)\}$ satisfy a differential equation of order $2j+2$. From Eq. (10.10) we see that the polynomials $L_n^{(j,M)}(x)$ satisfy a differential equation of order $2j+4$. Here $L_n^{(j;M)}(x)$ are polynomials obtained from the Laguerre polynomials $L_n^{(j)}(x)$ by inserting the mass M at the endpoint $x = 0$ of the orthogonality interval.

This result was firstly obtained in [5] using quite a different method. In our approach this result is a consequence of the corresponding result for the generalized Jacobi polynomials $P_n^{(j,b;M)}(x)$.

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