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Raising and lowering operators, factorization and differential/difference operators of hypergeometric type

Miguel Lorente

Departamento de Física, Universidad de Oviedo, 33007 Oviedo, Spain

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Abstract

Starting from the Rodrigues formula we present a general construction of raising and lowering operators for orthogonal polynomials of continuous and discrete variables on a uniform lattice. In order to have these operators mutually adjoint we introduce orthonormal functions with respect to the scalar product of unit weight. Using the Infeld–Hull factorization method, we generate from the raising and lowering operators the second-order self-adjoint differential/difference operator of hypergeometric type.

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1. Introduction

The factorization method has become a very powerful tool to solve second-order differential equations and in application to physical models with orthonormal basis, generated by creation and annihilation operators. A classical paper by Infeld and Hull [1] defined the method and applied it to a large class of second-order Hamiltonians that generalizes the well known description of the non-relativistic oscillator by means of creation and annihilation operators. Miller [2] enlarged this method to difference equations and made a connection to the orthogonal polynomial of a discrete variable. An analysis of the factorization types led Miller to the idea that this method is a particular case of the representation theory of Lie algebras.

The two volumes of Nikiforov and collaborators [3, 4] about classical orthogonal polynomials of continuous and discrete variables opened the way to a more rigorous and systematic approach to the factorization method. In fact Atakishiev and colaborators [5–8] explored the application of Kravchuk, Meixner and Charlier polynomials to the eigenvalue problem of some dynamical systems where the energy eigenvalues are equally spaced. This particular situation makes it possible to determine the generators of the dynamical symmetry group. Also Smirnov [9] has used the properties of difference equations of hypergeometric type given in [3] to construct raising and lowering operators that generate orthonormal functions corresponding to Hamiltonians of different levels.

Bangerezako and Magnus have developed the method of the factorization of difference operators of hypergeometric type [10–12]. They proposed two different approaches for this factorization. (1) For a given operator find raising and lowering operators that generate a complete set of polynomial eigenfunctions. (2) Generate from a factorization chain an operator having a complete set of polynomial eigenfunctions.

We have presented two papers [13, 14] related to the construction of the creation and annihilation operators for the orthogonal polynomials (or functions) of continuous and discrete variables. The motivation for these papers was the construction of a mathematical model for quantum systems on discrete spacetime (such as the harmonic oscillator, the hydrogen atom, the Dirac equation) [15] and to make connection with standard quantum mechanics by the continuous limit.

In this paper we follow the second approach to the factorization method of Bangerezako and Magnus explained before. Starting from the raising and lowering operators we generate the second-order differential/difference equation corresponding to the hypergeometric functions of continuous and discrete variables. Our procedure is completely general and valid for all functions of this type.

In section 2 we use the results of Nikiforov *et al* [3, 4] connecting first-order derivatives and orthogonal polynomials (as a consequence of Rodrigues' formula) to construct raising and lowering operators (the last one with the help of recurrence relations). In general these operators are not mutually adjoint with respect to the standard scalar product. For this reason, we introduce in section 3 the orthonormalized functions of hypergeometric type and then the corresponding raising and lowering operators are always mutually adjoint.

In sections 4 and 5 we repeat the same systematic procedure, derived from Rodrigues' formula, to calculate the raising and lowering operators for orthogonal polynomials and functions of discrete variable. It can be proved that these operators are also mutually adjoint.

In section 6 we introduce the factorization method to generate the second-order differential operator of the Sturm–Liouville type having a complete set of polynomial eigenfunctions. The factorization of the raising and lowering operators fulfils (up to a factor) the defining equations of the Infeld–Hull method [1].

In section 7 we apply the same technique to the hypergeometric functions of discrete variables. As usual all these functions transform in the limit into the corresponding hypergeometric functions of continuous variables.

It is important to make clear that the raising and lowering operators, introduced in sections 2–5, are defined with respect to one index only, namely, the degree of the orthogonal polynomials or the degree of the corresponding orthonormal functions. The same definition has been used by Atakishiev and colaborators [6–8], by Bangerezako and Magnus [10–12] and by Infeld *et al* [1, 2]. Physically this situation corresponds in the case of quantum oscillators to the creation and annihilation operators with respect to the index that distinguishes different eigenvectors of the energy operators.

With respect to the factorization techniques in the case of difference equations of section 7 two types of factorization can be considered [12]. Writing a linear difference equation of second order in the form

$$H(x)y(x) = \sum_{i=-d}^{d} A_i(x) E_x^i y(x)$$

where $E_x^i[f(x)] = f(x+i), d \in Z^+, i \in Z$ and $A_i(x)$ are some scalar functions in x, the first type of factorization consists in factorizing exactly the operators $H(x) + C$, with C some constant, and the raising and lowering operators satisfying a quasi-periodicity condition (Spiridinov–Vinet–Zhedanov type) [19]. The second factorization technique consists in factorizing the operator $E_x^d \circ [H(x) + C]$ with some raising and lowering operators that are shape invariant (Infeld–Hull–Miller type) [1, 2]. In section 7 we have used the first type of factorization, but in section 6 obviously we have used the Infeld–Hull–Miller technique for differential equations of hypergeometric type.

2. Raising and lowering operators for orthogonal polynomials of continuous variable

A polynomial of hypergeometric type $y_n(s)$ of continuous variable s satisfies two fundamental equations, from which one derives the raising and lowering operator.

(i) *Differential equation.*

$$\sigma(s)y_n'' + \tau(s)y_n'(s) + \lambda_n y_n(s) = 0 \tag{C1}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of, at most, second and first degree respectively, and λ_n is a constant, related to the above functions

$$\lambda_n = -n\left(\tau' + \frac{n-1}{2}\sigma''\right).$$

The differential equation can be written in the form of an eigenvalue equation of Sturm–Liouville type:

$$(\sigma(s)\rho(s)y_n'(s))' + \lambda_n \rho(s)y_n(s) = 0$$

where $\rho(s)$ is the weight function, satisfying $(\sigma(s)\rho(s))' = \tau(s)\rho(s)$.

The solutions of the differential equation are polynomials that satisfy an orthogonality relation with respect to the scalar product

$$\int_a^b y_n(s)y_m(s)\rho(s)\,\mathrm{d}s = d_n^2\delta_{nm}$$

where d_n is a normalization constant.

The differential equation (C1) defines an operator that is self-adjoint with respect to this scalar product.

(ii) *Three-term recurrence relations.*

$$sy_n(s) = \alpha_n y_{n+1}(s) + \beta_n y_n(s) + \gamma_n y_{n-1}(s) \tag{C2}$$

where α_n, β_n, γ_n are constants.

(iii) *Raising operator.* From the Rodrigues formula (which is a consequence of the differential equation (C1)) one derives a relation for the first derivative of polynomials $y_n(s)$ in terms of the polynomials themselves

$$\sigma y_n'(s) = \frac{\lambda_n}{n\tau_n'}\left[\tau_n(s)y_n(s) - \frac{B_n}{B_{n+1}}y_{n+1}(s)\right]$$

where

$$\tau_n(s) = \tau(s) + n\sigma'(s)$$

$$\tau_n'(s) = \tau' + n\sigma'' = -\frac{\lambda_{2n+1}}{2n+1}.$$

We can modify the last equation in a more suitable form. From

$$a_n = B_n \prod_{k=0}^{n-1}\left(\tau' + \tfrac{1}{2}(n+k-1)\sigma''\right) \qquad a_0 = B_0$$

we can prove the following identity:

$$\alpha_n = \frac{a_n}{a_{n+1}} = \frac{B_n}{B_{n+1}}\frac{\tau' + \frac{n-1}{2}\sigma''}{(\tau' + \frac{2n-1}{2}\sigma'')(\tau' + n\sigma'')} = \frac{B_n}{B_{n+1}}\frac{\lambda_n}{n}\frac{2n}{\lambda_{2n}}\frac{2n+1}{\lambda_{2n+1}}$$

from which we finally obtain

$$+\frac{\lambda_n}{n}\frac{\tau_n(s)}{\tau_n'}y_n(s)-\sigma(s)y_n'(s)=\frac{\lambda_{2n}}{2n}\alpha_n y_{n+1}(s). \tag{C3}$$

The left-hand side of this equation can be considered the differential operator which, when applied to $y_n(s)$, gives a polynomial of higher degree.

(iv) *Lowering operator.* Introducing (C2) in (C3) we obtain a differential operator which, when applied to an orthogonal polynomial of some degree, gives another polynomial of lower degree:

$$-\frac{\lambda_n}{n}\frac{\tau_n(s)}{\tau_n'}+\frac{\lambda_{2n}}{2n}(s-\beta_n)+\sigma(s)y_n'(s)=\frac{\lambda_{2n}}{2n}\gamma_n y_{n-1}(s). \tag{C4}$$

Formulae (C3) and (C4) can be used to calculate solutions of the diferential equation (C1). In fact, if we put $n=0$ in (C4) we obtain $y_0(s)$. Inserting this value in (C3) we obtain by iteration all the solutions of the differential operator (C1).

The explicit expressions for orthogonal polynomials of a continuous variable are given in appendix A. The values of $\rho(s)$, $\sigma(s)$, $\tau(s)$, λ_n, α_n, β_n, γ_n, d_n are taken from [4].

3. Raising and lowering operators for orthonormal functions of a continuous variable

From the orthogonal polynomials that satisfy a scalar product with respect to the weight $\rho(s)$ we can construct a new function:

$$\psi_n(s)\equiv d_n^{-1}\sqrt{\rho(s)}\,y_n(s)$$

and obtain orthogonal functions of unit norm. Solving the last expression for $y_n(s)$ and substituting in (C1)–(C4) and using the properties of $\sigma(s)$ and $\tau(s)$ we obtain the following expressions for the normalized orthogonal functions.

(i) *Differential equation.*

$$\sigma(s)\psi_n''(s)+\sigma'(s)\psi_n'(s)-\left[\frac{1}{4}\frac{(\tau(s)-\sigma'(s))^2}{\sigma(s)}+\frac{1}{2}(\tau'-\sigma'')\right]\psi_n(s)+\lambda_n\psi_n(s)=0 \tag{NC1}$$

which corresponds to a self-adjoint operator of Sturm–Liouville type.

(ii) *Recurrence relation.*

$$\frac{\lambda_{2n}}{2n}\frac{d_{n+1}}{d_n}\alpha_n\psi_{n+1}(s)+\frac{\lambda_{2n}}{2n}\frac{d_{n-1}}{d_n}\gamma_n\psi_n(s)+\frac{\lambda_{2n}}{2n}(\beta_n-s)\psi_n(s)=0. \tag{NC2}$$

(iii) *Raising and lowering operators.*

$$\begin{aligned}L^+(s,n)\psi_n(s)&=\left[\frac{\lambda_n}{n}\frac{\tau_n(s)}{\tau_n'}+\frac{1}{2}(\tau(s)-\sigma'(s))\right]\psi_n(s)-\sigma(s)\psi_n'(s)\\&=\frac{\lambda_{2n}}{2n}\alpha_n\frac{d_{n+1}}{d_n}\psi_{n+1}(s)\end{aligned} \tag{NC3}$$

$$\begin{aligned}L^-(s,n)\psi_n(s)&=\left[-\frac{\lambda_n}{n}\frac{\tau_n(s)}{\tau_n'}+\frac{\lambda_{2n}}{2n}(s-\beta_n)-\frac{1}{2}(\tau(s)-\sigma'(s))\right]\psi_n(s)\\&\quad+\sigma(s)\psi_n'(s)=\frac{\lambda_{2n}}{2n}\gamma_n\frac{d_{n-1}}{d_n}\psi_{n-1}(s).\end{aligned} \tag{NC4}$$

Putting $n = 0$ in (NC4) we obtain $\psi_0(s)$, and inserting this value in (NC3) we obtain by iteration all the orthonormal functions of hypergeometric type.

The explicit expressions for these functions are given in appendix A.

We want to make two observations. First, the operator corresponding to (NC1) is a self-adjoint operator of Sturm–Liouville type as can be easily checked. Secondly, the raising and lowering operators (NC3) and (NC4) are mutually adjoint in the case of Laguerre and Hermite functions. For the Jacobi and Legendre functions we have to multiply both operators by $2n/\lambda_{2n}$. In fact, we have

$$\int_a^b \psi_{n+1}(s)\left[\frac{2n}{\lambda_{2n}}L^+(s,n)\psi_n(s)\right]\mathrm{d}s = \alpha_n\frac{d_{n+1}}{d_n}$$

$$\int_a^b \left[\frac{2n+2}{\lambda_{2n+2}}L^-(s,n+1)\psi_{n+1}(s)\right]\psi_n(s)\,\mathrm{d}s = \gamma_{n+1}\frac{d_n}{d_{n+1}}.$$

Both integrals are equal because $\gamma_{n+1} = \alpha_n\frac{d_{n+1}^2}{d_n^2}$.

(In the case of Hermite and Laguerre functions λ_m/m is independent of m, for any m.)

4. Raising and lowering operators for orthogonal polynomials of discrete variable

A polynomial of hypergeometric type $P_n(x)$ of discrete variable x satisfies two fundamental relations from which one derives raising and lowering operators.

(i) *Difference equation.*

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0 \tag{D1}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of, at most, second and first degree, respectively. The forward (backward) difference operators are

$$\Delta f(x) = f(x+1) - f(x) \qquad \nabla f(x) = f(x) - f(x-1).$$

This difference equation can be written in the form of an eigenvalue equation of Sturm–Liouville type

$$\Delta[\sigma(x)\rho(x)\nabla P_n(x)] + \lambda_n\rho(x)P_n(x) = 0$$

where $\rho(x)$ is a weight function satisfying

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x)$$

and λ_n is the eigenvalue corresponding to the eigenfunction $P(x)$:

$$\lambda_n = -n\Delta\tau(x) - \frac{n(n-1)}{2}\Delta^2\sigma(x) = -n\left(\tau' + \frac{n-1}{2}\sigma''\right).$$

The solutions of the difference equation are polynomials that satisfy an orthogonality relation with respect to the scalar product

$$\sum_{x=a}^{b-1} P_n(x)P_m(x)\rho(x) = d_n^2\delta_{nm}$$

where δ_{mn} is the Kronecker symbol and d_n some normalization constant. The difference equation (D1) defines an operator that is self-adjoint with respect to this scalar product.

(ii) *Three-term recurrence relations.*

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x) \tag{D2}$$

where α_n, β_n, γ_n are some constants.

(iii) *Raising operator.*
From the Rodrigues formula, one derives a relation for the first difference operator of polynomials $P_n(x)$ in terms of the polynomials themselves.

$$\sigma(x)\nabla P_n(x) = \frac{\lambda_n}{n\tau_n'}\left[\tau_n(x)P_n(x) - \frac{B_n}{B_{n+1}}P_{n+1}(x)\right]$$

where

$$\tau_n(x) = \tau(x+n) + \sigma(x+n) - \sigma(x)$$
$$\Delta\tau_n(x) = \Delta\tau(x) + n\Delta^2\sigma(x)$$
$$\text{or} \qquad \tau_n' = \tau' + n\sigma''(x) = -\frac{\lambda_{2n+1}}{2n+1}$$

because $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree respectively. We can modify the last equation to a more suitable form, as we did in the continuous case. From the definition

$$a_n = B_n \prod_{k=0}^{n-1}(\tau' + \tfrac{1}{2}(n+k-1)\sigma'') \qquad a_0 = B_0$$

we have the following identity:

$$\alpha_n = \frac{a_n}{a_{n+1}} = -\frac{2n}{\lambda_{2n}}\frac{(2n+1)}{\lambda_{2n+1}}\frac{n}{\lambda_n}\frac{B_n}{B_{n+1}}$$

from which we obtain a more simplified version

$$\sigma(x)\nabla P_n(x) = \frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'}P_n(x) - \frac{\lambda_{2n}}{2n}P_{n+1}(x). \tag{D3}$$

This equation defines the raising operator in terms of the backward difference.

(iv) *Lowering operator.*
From the expression for the raising operator we can derive another lowering operator in terms of the forward operator. We substitute the difference operator ∇ in (D3) for its equivalent $\nabla = \Delta - \nabla\Delta$, and then the difference equation (D1) and the three-term recurrence relations (D2), with the result

$$(\sigma(x) + \tau(x))\,\Delta P_n(x) = \left[-\frac{\lambda_n}{n}\frac{2n+1}{x_{2n+1}}\tau(x) - \lambda_n - \frac{\lambda_{2n}}{2n}(x-\beta_n)\right]P_n(x)$$
$$+\frac{\lambda_{2n}}{2n}\gamma_n P_{n-1}(x). \tag{D4}$$

As in the continuous case from (D4) putting $n = 0$ we obtain $P_0(x)$ and inserting this value in (D3) we obtain by iteration all the polynomials $P_n(x)$ satisfying (D1).

The explicit expressions for the orthogonal polynomials $P_n(x)$ are given in appendix B. The values of $\rho(x)$, $\sigma(x)$, $\tau(x)$, λ_n, α_n, β_n, γ_n, d_n are taken from [4].

5. Raising and lowering operators for orthonormal functions of discrete variable

In the last section we have a set of polynomials that are orthogonal with respect to the weight function $\rho(x)$. From these polynomials we construct some functions that are orthogonal with respect to the unit weight, $\rho(x) = 1$, and at the same time are normalized:

$$\phi_n(x) = d_n^{-1}\sqrt{\rho(x)}P_n(x).$$

Introducing this expression in (D1)–(D4) and using the properties of function $\sigma(x)$, $\tau(x)$ and $\rho(x)$, we obtain the following.

(i) *Difference equation.*

$$\sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}\phi_n(x+1)+\sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}\phi_n(x-1)$$
$$-(2\sigma(x)+\tau(x))\phi_n(x)+\lambda_n\phi_n(x)=0. \qquad \text{(ND1)}$$

(ii) *Three-term recursion relation.*

$$\frac{\lambda_{2n}}{2n}\alpha_n\frac{d_{n+1}}{d_n}\phi_{n+1}(x)+\frac{\lambda_{2n}}{2n}\gamma_n\frac{d_{n-1}}{d_n}\phi_{n-1}(x)+\frac{\lambda_{2n}}{2n}(\beta_n-x)\phi_n(x)=0. \qquad \text{(ND2)}$$

(iii) *Raising operator.*

$$L^+(x,n)\equiv\left[\frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'}-\sigma(x)\right]\phi_n(x)+\sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}\phi_n(x-1)$$
$$=\frac{\lambda_{2n}}{2n}\alpha_n\frac{d_{n+1}}{d_n}\phi_{n+1}(x). \qquad \text{(ND3)}$$

(iv) *Lowering operator.*

$$L^-(x,n)\equiv\left[-\frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'}+\lambda_n+\frac{\lambda_{2n}}{2n}(x-\beta_n)-\sigma(x)-\tau(x)\right]\phi_n(x)$$
$$+\sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}\phi_n(x+1)=\frac{\lambda_{2n}}{2n}\gamma_n\frac{d_{n-1}}{d_n}\phi_{n-1}(x). \qquad \text{(ND4)}$$

From the last two expressions we obtain all the solutions of the difference equation (ND1). Putting $n=0$ in (ND4) we obtain $\phi_0(x)$, and inserting this value in (DC3) we obtain, by iteration, all the normalized functions $\phi_n(x)$.

The explicit calculations for all the orthonormal functions of hypergeometric type are given in appendix B.

As in section 3, we make two observations. Firstly, the raising and lowering operators (ND3) and (ND4) are mutually adjoint in the case of Krauvchuk, Meixner and Charlier functions. For the Hahn and Chebyshev functions we have to divide both by $\lambda_{2n}/2n$; therefore, they become mutually adjoint, namely,

$$\sum_{x=a}^{b-1}\phi_{n+1}(x)\left[\frac{2n}{\lambda_{2n}}L^+(x,n)\phi_n(x)\right]=\sum_{x=a}^{b-1}\left[\frac{2n+2}{\lambda_{2n+2}}L^-(x,n+1)\phi_{n+1}(x)\right]\phi_n(x)=\alpha_n\frac{d_{n+1}}{d_n}.$$

Secondly, the operator corresponding to the eigenvalue λ_n in (ND1) is self-adjoint. In order to prove this, it is enough to show

$$\sum_{x=a}^{b-1}\phi_l(x)\left\{\sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}\phi_n(x+1)+\sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}\phi_n(x-1)\right\}$$
$$=\sum_{x=a}^{b-1}\phi_n(x)\left\{\sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}\phi_l(x-1)\right.$$
$$\left.+\sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}\phi_l(x+1)\right\}.$$

From the orthogonality conditions $\sigma(a)=\sigma(b)=0$, we can write

$$\sum_{x=a}^{b-1}\phi_n(x)\sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}\phi_l(x-1)$$
$$=\sum_{x'=a-1}^{b-2}\phi_n(x'+1)\sqrt{(\sigma(x')+\tau(x'))\sigma(x'+1)}\phi_l(x')$$

$$= \sum_{x=a}^{b-1} \phi_n(x+1)\sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}\phi_l(x).$$

Similarly

$$\sum_{x=a}^{b-1} \phi_n(x)\sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}\phi_l(x+1)$$
$$= \sum_{x=a}^{b-1} \phi_n(x-1)\sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}\phi_l(x).$$

6. Factorization for differential equation of hypergeometric type

The raising and lowering operators of sections 2 and 4 will help us to factorize the second-order differential equation of hypergeometric type into the product of two first-order operators in agreement with the general method of Infeld and Hull [1].

From (NC1) we define the operator

$$H(s,n) \equiv \sigma(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \sigma'(s)\frac{\mathrm{d}}{\mathrm{d}s} - \frac{1}{4}\frac{(\sigma(s)-\sigma'(s))^2}{\sigma(s)} - \frac{1}{2}(\tau' - \sigma'') + \lambda_n$$

that satisfies $H(s,n)\psi_n(s) = 0$.

We write the raising and lowering operators, (NC3) and (NC4) respectively, in the following way:

$$L^+(s,n) \equiv f(s,n) - \sigma(s)\frac{\mathrm{d}}{\mathrm{d}s}$$
$$L^-(s,n) \equiv g(s,n) + \sigma(s)\frac{\mathrm{d}}{\mathrm{d}s}$$

where

$$f(s,n) = \frac{\lambda_n}{n}\frac{\tau_n(s)}{\tau_n'} + \frac{1}{2}(\tau(s) - \sigma'(s))$$
$$g(s,n) = -\frac{\lambda_n}{n}\frac{\tau_n(s)}{\tau_n'} + \frac{\lambda_{2n}}{2n}(s - \beta_n) - \frac{1}{2}(\tau(s) - \sigma'(s))$$

satisfying

$$f(s,n-1) = g(s,n) \qquad \text{or} \qquad f(s,n) = g(s,n+1)$$

which can be proved by Taylor expansion.

Now we calculate

$$L^-(s,n+1)L^+(s,n) = g(s,n+1)f(s,n) + \sigma(s)\{f(s,n) - g(s,n+1)\}\frac{\mathrm{d}}{\mathrm{d}s}$$
$$+\sigma(s)\left\{f'(s,n) - \sigma'(s)\frac{\mathrm{d}}{\mathrm{d}s} - \sigma(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right\}.$$

The second term of the right-hand side becomes zero. Substituting the values for $f(s,n)$, $g(s,n)$ and $H(s,n)$ we obtain

$$L^-(s,n+1)L^+(s,n) = \left[\left(\frac{\lambda_n}{n}\right)^2\left(\frac{\tau_n(s)}{\tau_n'}\right)^2 + \frac{\lambda_n}{n}\frac{\tau_n(s)}{\tau_n'}(\tau(s) - \sigma'(s)) + (n+1)\frac{\lambda_n}{n}\sigma(s)\right]$$
$$-\sigma(s)H(s,n).$$

It can be proved that the expression in squared brackets is independent of s, say $\mu(n)$. Applying the last equality to the orthonormal functions $\psi_n(s)$ and taking into account (NC3) and (NC4) we obtain

$$\mu(n) = \frac{\lambda_{2n}}{2n}\frac{\lambda_{2n+2}}{2n+2}\alpha_n\gamma_{n+1}.$$

With the same technique we calculate

$$L^+(s,n-1)L^-(s,n) = f(s,n-1)g(s,n) + \sigma(s)\{f(s,n-1) - g(s,n)\}\frac{\mathrm{d}}{\mathrm{d}s} \\ -\sigma(s)\left\{g'(s,n) + \sigma'(s)\frac{\mathrm{d}}{\mathrm{d}s} + \sigma(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right\}.$$

From the properties between $f(s,n)$ and $g(s,n)$, the second term in the right-hand side becomes zero. Substituting the values of these functions and $H(s,n)$ we finally obtain

$$L^+(s,n-1)L^-(s,n) = \left[\left(\frac{\lambda_{n-1}}{n-1}\right)^2\left(\frac{\tau_{n-1}(s)}{\tau'_{n-1}}\right)^2 + \frac{\lambda_{n-1}}{n-1}\frac{\tau_{n-1}(s)}{\tau'_{n-1}}(\tau(s) - \sigma'(s)) \\ + n\frac{\lambda_{n-1}}{n-1}\sigma(s)\right] - \sigma(s)H(s,n).$$

It can be proved that the expression in squared brackets is independent of s, say $\nu(n)$. Applying the last equality to the orthonormal functions $\psi_n(s)$ and taking into account (NC3) and (NC4) we obtain

$$\nu(n) = \frac{\lambda_{2n-2}}{2n-2}\frac{\lambda_{2n}}{2n}\alpha_{n-1}\gamma_n.$$

Obviously, $\nu(n+1) = \mu(n)$. These constants are given explicitly in appendix A.

Finally we have the desired relation equivalent to the Infeld–Hull–Miller factorization method:

$$L^-(s,n+1)L^+(s,n) = \mu(n) - \sigma(s)H(s,n) \tag{NC5}$$

$$L^+(s,n)L^-(s,n+1) = \mu(n) - \sigma(s)H(s,n+1). \tag{NC6}$$

If we require $L^+(s,n)$ and $L^-(s,n)$ to be mutually adjoint we have to divide both sides of (NC5) and (NC6) by

$$\frac{\lambda_{2n+2}}{2n+2}\frac{\lambda_{2n}}{2n}.$$

7. Factorization of difference equation of hypergeometric type

For the case of orthonormal hypergeometric functions of a discrete variable, we define from (ND1) the operator

$$H(x,n) \equiv \sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}E^+ + \sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}E^- \\ -(2\sigma(x)+\tau(x)) + \lambda_n$$

where $E^+f(x) = f(x+1)$, $E^-f(x) = f(x-1)$ and the orthonormal functions satisfy

$$H(x,n)\phi_n(x) = 0.$$

As before we write the raising and lowering operators in the following way:

$$L^+(x,n) = u(x,n) + \sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}E^-$$

$$L^-(x,n) = v(x,n) + \sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}E^+$$

where

$$u(x,n) = \frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'} - \sigma(x)$$
$$v(x,n) = -\frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'} + \lambda_n + \frac{\lambda_{2n}}{2n}(x-\beta_n) - \sigma(x) - \tau(x).$$

Both expressions satisfy

$$u(x+1,n) = v(x,n+1) \qquad \text{or}$$
$$u(x+1,n-1) = v(x,n)$$

that can be proved by Taylor expansion.

Now we calculate

$$L^-(x,n+1)L^+(x,n) = v(x,n+1)u(x,n) + (\sigma(x)+\tau(x))\sigma(x+1) + u(x+1,n) \times \left\{\sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}E^+ + \sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}E^-\right\}.$$

Substituting the values for $u(x,n)$, $v(x,n)$ and $H(x,n)$ we obtain

$$L^-(x,n+1)L^+(x,n) = \left[\left(\frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'} - \lambda_n\right)\left(\frac{\lambda_n}{n}\frac{\tau_n(x+1)}{\tau_{n+1}'} - \sigma(x+1)\right) + \frac{\lambda_n}{n}\frac{\tau_n(x+1)}{\tau_n'}(\sigma(x)+\tau(x))\right] + u(x+1,n)H(x,n).$$

It can be proved that the expression in squared brackets is independent of x, say $\mu(n)$. Applying the last equality to the orthonormal function $\phi_n(x)$ and taking into account (ND1), (ND3) and (ND4) we obtain

$$\mu(n) = \frac{\lambda_{2n}}{2n}\frac{\lambda_{2n+2}}{2n+2}\alpha_n\gamma_{n+1}.$$

With the same technique we calculate

$$L^+(x,n-1)L^-(x,n) = \left[\left(-\frac{\lambda_n}{n}\frac{\tau_n(x-1)}{\tau_n'} + \frac{\lambda_{2n}}{2n}(x-1-\beta_n) + \lambda_n\right) \times\left(-\frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'} + \frac{\lambda_{2n}}{2n}(x-\beta_n) + \sigma(x)\right) - (\sigma(x-1)+\tau(x-1)) \times\left(-\frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'} + \frac{\lambda_{2n}}{2n}(x-\beta_n)\right)\right] + u(x,n-1)H(x,n).$$

As before the expression in squared brackets is independent of x, say $\nu(n)$. Applying both sides of the last equality to the functions $\phi_n(x)$, and taking into account (ND1), (ND3) and (ND4) we obtain

$$\nu(n) = \frac{\lambda_{2n-2}}{2n-2}\frac{\lambda_{2n}}{2n}\alpha_{n-1}\gamma_n.$$

Obviously $\nu(n+1) = \mu(n)$.

These constants are given explicitly in appendix B.

Finally the desired relations corresponding to the Spiridonov–Vinet–Zhedanov factorization method are

$$L^-(x,n+1)L^+(x,n) = \mu(n) + u(x+1,n)H(x,n) \tag{ND5}$$
$$L^+(x,n)L^-(x,n+1) = \mu(n) + u(x,n-1)H(x,n+1). \tag{ND6}$$

Again, if we want $L^{+}(x, n)$ and $L^{-}(x, n)$ to be mutually adjoint, we have to divide both expressions (ND5) and (ND6) by

$$\frac{\lambda_{2n}}{2n} \frac{\lambda_{2n+2}}{2n+2}$$

only in the case of Hahn and Chebyshev functions.

8. Some comments

The classical orthogonal polynomials we have presented in the preceding sections are solutions of the second-order differential equation

$$\sigma(s) y_n''(s) + \tau(s) y_n'(s) + \lambda_n y_n(s) = 0$$

in the continuous case, or second-order difference equation

$$\sigma(x) \Delta\nabla y_n(x) + \tau(x) \Delta y_n(x) + \lambda_n y_n(x) = 0$$

in the discrete case for uniform lattices where $\sigma(x)$ and $\tau(x)$ are polynomials of at most the second and first degree respectively.

Atakishiev and collaborators have generalized the classical orthogonal polynomials using a characterization based on the difference equation of hypergeometric type that covers all the cases defined by Andrews and Askey [16]. This characterization covers the q-analogue of classical orthogonal polynomials on non-uniform lattices.

Our paper should be implemented with the construction of raising and lowering operators for the orthogonal polynomials on non-uniform lattices, in particular the q-analogue of the classical orthogonal polynomials. For this purpose we have at our disposal the analogue of difference equations, Rodrigues formula and recurrence relations for the orthogonal polynomials on non-uniform lattice, given explicitly by Nikiforov *et al* [4].

Another approach to the same problem is given by Smirnov, via the factorization method suggested by Schrödinger for the solution of a second-order differential equation of hypergeometric type. Smirnov has applied this method to the finite-difference equation on uniform lattices [9] and on non-uniform lattices [17, 18]. In his approach the raising and lowering operators are defined with respect to two indices: the first one, the degree of the orthogonal polynomials, the second one the order of the finite derivative with respect to the discrete variable. For this reason his raising and lowering operators are not equal to ours.

As a final comment to appendices A and B, we note that in an unpublished report of Koekoek and Swarttouw [20] presents tables for orthogonal polynomials of the Askey scheme and its q-analogue, including the raising and lowering operators of classical orthogonal polynomials of hypergeometric type. There are two points by which our appendices are different from theirs. First, we have calculated the raising and lowering operators from the Rodrigues formula (see (1.2.13) and (2.2.10) of [4]), but their raising and lowering operators are connected with some recurrence relations (see (1.4.5) and (2.4.13)–(2.4.17) of [4]) which are defined with respect to two indices. Besides this, their appendices do not cover the differential/difference equations, recurrence relations and raising/lowering operators with respect to the orthonormal functions of hypergeometric type as given in our appendices.

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Appendix A. Orthogonal polynomials of continuous variable

A.1. Hermite polynomials

$$H_n''(s) - 2sH_n'(s) + 2nH_n(s) = 0 \tag{He1}$$

$$sH_n(s) = \tfrac{1}{2}H_{n+1}(s) + nH_{n-1}(s) \tag{He2}$$

$$H_{n+1}(s) = 2sH_n(s) - H_n'(s) \tag{He3}$$

$$H_{n-1}(s) = \frac{1}{2n}H_n'(s). \tag{He4}$$

A.2. Laguerre polynomials

$$sL_n^{\alpha''}(s) + (1+\alpha-s)L_n^{\alpha'}(s) + nL_n^{\alpha}(s) = 0 \tag{La1}$$

$$(n+1)L_{n+1}^{\alpha}(s) + (n+\alpha)L_{n-1}^{\alpha}(s) + (s-2n-\alpha-1)L_n^{\alpha}(s) = 0 \tag{La2}$$

$$(n+1)L_{n+1}^{\alpha}(s) = (s-n-\alpha-1)L_n^{\alpha}(s) + sL_n^{\alpha'}(s) \tag{La3}$$

$$(n+\alpha)L_{n-1}^{\alpha}(s) = nL_n^{\alpha}(s) - sL_n^{\alpha'}(s). \tag{La4}$$

A.3. Legendre polynomials

$$(1-s^2)P_n''(s) - 2sP_n'(s) + n(n+1)P_n(s) = 0 \tag{Le1}$$

$$\frac{n+1}{2n+1}P_{n+1}(s) + \frac{n}{2n+1}P_{n-1}(s) - sP_n(s) = 0 \tag{Le2}$$

$$(n+1)P_{n+1}(s) = (n+1)sP_n(s) - (1-s^2)P_n'(s) \tag{Le3}$$

$$nP_{n-1}(s) = nsP_n(s) + (1-s^2)P_n'(s). \tag{Le4}$$

A.4. Jacobi polynomials

$$(1-s^2)P_n^{(\alpha,\beta)''}(s) + [\beta-\alpha-(\alpha+\beta+2)s]P_n^{(\alpha,\beta)'}(s) + n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(s) = 0 \tag{J1}$$

$$\begin{aligned}\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(s) + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(s)\\ +\left[\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} - s\right]P_n^{(\alpha,\beta)}(s) = 0\end{aligned} \tag{J2}$$

$$\begin{aligned}&\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(s)\\ &\quad= \left[\frac{(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)}(\alpha-\beta) + (n+\alpha+\beta+1)s\right]P_n^{(\alpha,\beta)}(s) - (1-s^2)P_n^{(\alpha,\beta)'}(s)\end{aligned} \tag{J3}$$

$$\begin{aligned}\frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)}P_{n-1}^{(\alpha,\beta)}(s) = \Bigg[&\frac{(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)}(\beta-\alpha) - (n+\alpha+\beta+1)s\\ &+(2n+\alpha+\beta+1)\left(s - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)\Bigg]P_n^{(\alpha,\beta)'}(s)\\ &+(1-s^2)P_n^{(\alpha,\beta)}(s).\end{aligned} \tag{J4}$$

A.5. Normalized Hermite functions

$$\psi_n(s) = (2^n n!\sqrt{\pi})^{-1/2}\mathrm{e}^{-s^2/2}H_n(s)$$

$$\psi_n''(s) + (1-s^2)\psi_n(s) + 2n\psi_n(s) = 0 \tag{NHe1}$$

$$\sqrt{2(n+1)}\psi_{n+1}(s) + \sqrt{2n}\psi_{n-1}(s) - 2s\psi_n(s) = 0 \tag{NHe2}$$

$$L^+(s,n)\psi_n(s) = s\psi_n(s) - \psi_n'(s) = \sqrt{2(n+1)}\psi_{n+1}(s) \tag{NHe3}$$

$$L^-(s,n)\psi_n(s) = s\psi_n(s) + \psi_n'(s) = \sqrt{2n}\psi_{n-1}(s) \tag{NHe4}$$

$$\psi_0(s) = \pi^{-1/4}\mathrm{e}^{-s^2/2}$$

$$\psi_n(s) = \frac{1}{\sqrt{2^n n!}}\left(s - \frac{\mathrm{d}}{\mathrm{d}s}\right)^n \psi_0(s)$$

$$L^+(s,n)L^-(s,n) = 2n\psi_n(s)$$

$$L^-(s,n)L^+(s,n)\psi_n(s) = 2(n+1)\psi_n(s).$$

A.6. Normalized Laguerre functions

$$\psi_n(s) = \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}\mathrm{e}^{-s/2}s^{\alpha/2}L_n^\alpha(s)$$

$$s\psi_n''(s) + \psi_n'(s) - \frac{1}{4}\left(s + \frac{\alpha^2}{s} - 2\alpha - 2\right)\psi_n(s) + n\psi_n(s) = 0 \tag{NLa1}$$

$$\sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}(s) + \sqrt{n(n+\alpha)}\psi_{n-1}(s) - (2n+\alpha+1-s)\psi_n(s) = 0 \tag{NLa2}$$

$$L^+(s,n)\psi_n(s) = -\tfrac{1}{2}(2n+\alpha+2-s)\psi_n(s) - s\psi_n'(s) = -\sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}(s) \tag{NLa3}$$

$$L^-(s,n)\psi_n(s) = -\tfrac{1}{2}(2n+\alpha-s)\psi_n(s) + s\psi_n'(s) = -\sqrt{n(n+\alpha)}\psi_{n-1}(s) \tag{NLa4}$$

$$\psi_0(s) = \sqrt{\frac{1}{\Gamma(\alpha+1)}}\mathrm{e}^{-s/2}s^{\alpha/2}$$

$$\psi_n(s) = \frac{1}{\sqrt{n!(\alpha+1)_n}}\prod_{k=0}^{n-1} L^+(s, n-1-k)\psi_0(s)$$

$$L^+(s,n-1)L^-(s,n)\psi_n(s) = n(n+\alpha)\psi_n(s)$$

$$L^-(s,n+1)L^+(s,n)\psi_n(s) = (n+1)(n+\alpha+1)\psi_n(s).$$

A.7. Normalized Legendre functions

$$\psi_n(s) = \sqrt{\frac{2n+1}{2}}P_n(s)$$

$$(1-s^2)\psi_n''(s) - 2s\psi_n'(s) + n(n+1)\psi_n(s) = 0 \tag{NLe1}$$

$$(n+1)\sqrt{\frac{2n+1}{2n+3}}\psi_{n+1}(s) + n\sqrt{\frac{2n+1}{2n-1}}\psi_{n-1}(s) - (2n+1)s\psi_n(s) = 0 \tag{NLe2}$$

$$L^+(s,n)\psi_n(s) = (n+1)s\psi_n(s) - (1-s^2)\psi_n'(s) = (n+1)\sqrt{\frac{2n+1}{2n+3}}\psi_{n+1}(s) \tag{NLe3}$$

$$L^-(s,n)\psi_n(s) = ns\psi_n(s) + (1-s^2)\psi_n'(s) = n\sqrt{\frac{2n+1}{2n-1}}\psi_{n-1}(s) \tag{NLe4}$$

$$\psi_0(s) = \frac{1}{\sqrt{2}}$$
$$\psi_n(s) = \frac{1}{n!}\sqrt{2n+1}\prod_{k=0}^{n-1} L^+(s, n-1-k)\psi_0(s)$$
$$L^+(s, n-1)L^-(s, n) = n^2$$
$$L^-(s, n+1)L^+(s, n) = (n+1)^2.$$

A.8. Normalized Jacobi functions

$$\psi_n(s) = \sqrt{\frac{n!(2n+\alpha+\beta+1)(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}(1-s)^{\frac{\alpha}{2}}(1+s)^{\frac{\beta}{2}} P_n^{(\alpha,\beta)}(s)$$

$$(1-s^2)\psi_n''(s) - 2s\psi_n'(s) - \frac{1}{4}\left\{\frac{(\beta-\alpha-(\alpha+\beta)s)^2}{1-s^2} - 2(\alpha+\beta)s\right\}\psi_n(s)$$
$$+n(n+\alpha+\beta+1)\psi_n(s) = 0 \tag{NJ1}$$

$$\frac{2\sqrt{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)(2n+\alpha+\beta+1)}}{(2n+\alpha+\beta+2)\sqrt{2n+\alpha+\beta+3}}\psi_{n+1}(s)$$
$$+\frac{2\sqrt{n(n+\alpha)(n+\beta)(n+\alpha+\beta)(2n+\alpha+\beta+1)}}{(2n+\alpha+\beta+2)\sqrt{2n+\alpha+\beta-1}}\psi_{n-1}(s)$$
$$+(2n+\alpha+\beta+1)\left\{\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} - s\right\}\psi_n(s) = 0 \tag{NJ2}$$

$$L^+(s, n)\psi_n(s) = \left\{(n+\alpha+\beta+1)s - \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2}(\beta-\alpha-n^2)\right.$$
$$\left.+\tfrac{1}{2}(\beta-\alpha-(\alpha+\beta)s)\right\}\psi_n(s) - (1-s^2)\psi_n'(s)$$
$$= \frac{2\sqrt{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)(2n+\alpha+\beta+1)}}{(2n+\alpha+\beta+2)\sqrt{2n+\alpha+\beta+3}}\psi_{n+1}(s) \tag{NJ3}$$

$$L^-(s, n)\psi_n(s) = \left\{-(n+\alpha+\beta+1)s + \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2}(\beta-\alpha-n^2)\right.$$
$$+(2n+\alpha+\beta+1)\left(s - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)$$
$$\left.-\tfrac{1}{2}(\beta-\alpha-(\alpha+\beta)s)\right\}\psi_n(s) + (1-s^2)\psi_n'(s)$$
$$= \frac{2\sqrt{n(n+\alpha)(n+\beta)(n+\alpha+\beta)(2n+\alpha+\beta+1)}}{(2n+\alpha+\beta)\sqrt{2n+\alpha+\beta-1}}\psi_{n-1}(s) \tag{NJ4}$$

$$\psi_0(s) = \frac{\alpha+\beta+1}{\sqrt{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}}(1-s)^{\frac{\alpha}{2}}(1+s)^{\frac{\beta}{2}}$$
$$\psi_n(s) = \prod_{k=0}^{n-1}\left\{\frac{(2k+\alpha+\beta+2)}{2\sqrt{(k+1)(k+\alpha+1)(k+\beta+1)}}\right.$$
$$\left.\times\frac{\sqrt{2k+\alpha+\beta+3}}{\sqrt{(k+\alpha+\beta+1)(2k+\alpha+\beta+1)}}L^+(s, n-1-k)\right\}\psi_0(s)$$

$$L^{+}(s,n-1)L^{-}(s,n)\psi_n(s)=\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2}\psi_n(s)$$

$$L^{-}(s,n+1)L^{+}(s,n)\psi_n(s)=\frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)^2}\psi_n(s).$$

Appendix B. Orthogonal polynomials of a discrete variable

B.1. Kravchuk polynomials

$$\frac{p(N-x)}{q}k_n(x+1)+xk_n(x-1)+\frac{x(p-q)-Np}{q}k_n(x)+\frac{n}{q}k_n(x)=0 \tag{K1}$$

$$\frac{n+1}{q}k_{n+1}(x)+p(N-n+1)k_{n-1}(x)+[n+p(N-2n)-x]k_n(x)=0 \tag{K2}$$

$$\frac{n+1}{q}k_{n+1}(x)=\frac{p}{q}(x+n-N)k_n(x)+xk_n(x-1) \tag{K3}$$

$$p(N-n+1)=\frac{p}{q}(x+n-N)k_n(x)+\frac{p}{q}(N-x)k_n(x+1). \tag{K4}$$

B.2. Meixner polynomials

$$\mu(x+\gamma)m_n(x+1)+xm_n(x-1)-[\mu(x+\gamma)+x]m_n(x)+n(1-\mu)m_n(x)=0 \tag{M1}$$

$$\mu m_{n+1}(x)-n(n+\gamma-1)m_{n+1}(x)+[\mu(x+n+\gamma)+n-x]m_n(x)=0 \tag{M2}$$

$$-\mu m_{n+1}(x)=-\mu(x+n+\gamma)m_n(x)+xm_n(x-1) \tag{M3}$$

$$-n(n+\gamma-1)m_{n-1}(x)=-\mu(x+n+\gamma)m_n(x)+\mu(x+\gamma)m_n(x+1). \tag{M4}$$

B.3. Charlier polynomials

$$\mu c_n(x+1)+xc_n(x-1)-(x+\mu)c_n(x)+nc_n(x)=0 \tag{C1}$$

$$-\mu c_{n+1}(x)-nc_{n-1}(x)+(n+\mu-x)c_n(x)=0 \tag{C2}$$

$$-\mu c_{n+1}(x)=-\mu c_n(x)+xc_n(x-1) \tag{C3}$$

$$-nc_{n-1}(x)=-\mu c_n(x)+\mu c_n(x+1). \tag{C4}$$

B.4. Chebyshev polynomials

$$\begin{aligned}&(x+1)(N-x-1)t_n(x+1)+x(N-x)t_n(x-1)\\&\qquad-[(N-x-1)(x+1)+x(N-x)]t_n(x)+n(n+1)t_n(x)=0\end{aligned} \tag{T1}$$

$$\tfrac{1}{2}(n+1)t_{n+1}(x)+\tfrac{1}{2}n(N^2-n^2)t_{n-1}(x)+\tfrac{1}{2}(2n+1)(N-1-2x)t_n(x)=0 \tag{T2}$$

$$\tfrac{1}{2}(n+1)t_{n+1}(x)=-[\tfrac{1}{2}(n+1)(N-2x-n-1)+x(N-x)]t_n(x)+x(N-x)t_n(x-1) \tag{T3}$$

$$\begin{aligned}\frac{1}{2}n(N^2-n^2)t_{n-1}(x)=&\left[\frac{1}{2}(n+1)(N-2x-n-1)+n(n+1)+(2n+1)\left(x-\frac{N-1}{2}\right)\right.\\&\left.-(x+1)(N-x-1)\right]t_n(x)+(x+1)(N-x-1)t_n(x+1).\end{aligned} \tag{T4}$$

B.5. Hahn polynomials

$$\begin{aligned}&[x(N-x-\beta-2)+(\beta+1)(N-1)]h_n^{\alpha,\beta}(x+1)+x(N+\alpha-x)h_n^{\alpha,\beta}(x-1)\\&\qquad -[x(2N-2x+\alpha-\beta-2)+(\beta+1)(N-1)]h_n^{\alpha,\beta}(x)\\&\qquad +n(n+\alpha+\beta+1)h_n^{\alpha,\beta}(x)=0\end{aligned} \tag{Ha1}$$

$$\begin{aligned}&\frac{(n+1)(n+\alpha+\beta+1)}{2n+\alpha+\beta+2}h_{n+1}^{\alpha,\beta}(x)\\&\qquad +\frac{(n+\alpha)(n+\beta)(N+n+\alpha+\beta)(N-n)}{2n+\alpha+\beta}h_{n-1}^{\alpha,\beta}+(2n+\alpha+\beta+1)\\&\qquad \times\left[\frac{\alpha-\beta+2N-2}{4}+\frac{\left(\beta^2-\alpha^2\right)(2N+\alpha+\beta)}{4(2n+\alpha+\beta)(2n+\alpha+\beta+2)}-x\right]h_n^{\alpha,\beta}(x)=0\end{aligned} \tag{Ha2}$$

$$\begin{aligned}&\frac{(n+1)(n+\alpha+\beta+1)}{2n+\alpha+\beta+2}h_{n+1}^{\alpha,\beta}(x)=x(N+\alpha-x)h_n^{\alpha,\beta}(x-1)\\&\qquad -\left\{\frac{(n+\alpha+\beta+1)}{2n+\alpha+\beta+2}\left[(\beta+1)(N-1)-(\alpha+\beta+2+2n)x+(N-n-\beta-2)n\right]\right.\\&\qquad \left.+x(N+\alpha-x)\right\}h_n^{\alpha,\beta}(x)\end{aligned} \tag{Ha3}$$

$$\begin{aligned}&\frac{(n+\alpha)(n+\beta)(N+n+\alpha+\beta)(N-n)}{2n+\alpha+\beta}h_{n-1}^{\alpha,\beta}(x)\\&\quad =[x(N-x-\beta-2)+(\beta+1)(N-1)]h_n^{\alpha,\beta}(x+1)\\&\qquad +\left[\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2}((\beta+1)(N-1)-(\alpha+\beta+2+2n)x\right.\\&\qquad +(N-n-\beta-2)n)+n(n+\alpha+\beta+1)+(2n+\alpha+\beta+1)\\&\qquad \times\left(x-\frac{\alpha-\beta+2N-2}{4}-\frac{(\beta^2-\alpha^2)(2N+\alpha+\beta)}{4(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)\\&\qquad \left.-x(N-x-\beta-2)+(\beta+1)(N-1)\right]h_n^{\alpha,\beta}(x).\end{aligned} \tag{Ha4}$$

B.6. Normalized Kravchuk functions

$$\psi_n(x)=\sqrt{\frac{n!(N-n)!}{(pq)^n}}\sqrt{\frac{p^xq^{N-x}}{x!(N-x)!}}k_n(x)$$

$$\begin{aligned}&\sqrt{\frac{p}{q}(N-x)(x+1)}\psi_n(x+1)+\sqrt{\frac{p}{q}(N-x+1)x}\psi_n(x-1)\\&\qquad +\frac{x(p-q)-Np}{q}\psi_n(x)+\frac{n}{q}\psi_n(x)=0\end{aligned} \tag{NK1}$$

$$\begin{aligned}&\sqrt{\frac{p}{q}(N-n)(n+1)}\psi_{n+1}(x)+\sqrt{\frac{p}{q}(N-n+1)n}\psi_{n-1}(x)\\&\qquad +\frac{1}{q}[n+p(N-2n)-x]\psi_n(x)=0\end{aligned} \tag{NK2}$$

$$L^+(x,n)\psi_n(x) = \frac{p}{q}(x+n-N)\psi_n(x) + \sqrt{\frac{p}{q}(N-x+1)x}\,\psi_n(x-1)$$
$$= \sqrt{\frac{p}{q}(N-n)(n+1)}\,\psi_{n+1}(x) \tag{NK3}$$
$$L^-(x,n)\psi_n(x) = \frac{p}{q}(x+n-N)\psi_n(x) + \sqrt{\frac{p}{q}(N-x)(x+1)}\,\psi_n(x+1)$$
$$= \sqrt{\frac{p}{q}(N-n+1)n}\,\psi_{n-1}(x) \tag{NK4}$$

$$\psi_0(x) = \sqrt{\frac{N!p^x q^{N-x}}{x!(N-x)!}}$$
$$\psi_n(x) = \sqrt{\frac{q^n(N-n)!}{p^n N!n!}}\prod_{k=0}^{n-1} L^+(x,n-1-k)\psi_0(x)$$
$$L^+(x,n-1)L^-(x,n)\psi_n(x) = \frac{p}{q}(N-n+1)n\psi_n(x)$$
$$L^-(x,n+1)L^+(x,n)\psi_n(x) = \frac{p}{q}(N-n)(n+1)\psi_n(x).$$

B.7. Normalized Meixner functions

$$\psi_n(x) = \sqrt{\frac{\mu^n(1-\mu)^\gamma}{n!(\gamma)_n}}\sqrt{\frac{\mu^x\Gamma(x+\gamma)}{\Gamma(x+1)\Gamma(\gamma)}}\,m_n^\gamma(x)$$

$$\sqrt{\mu(x+\gamma)(x+1)}\,\psi_n(x+1) + \sqrt{\mu x(x+\gamma-1)}\,\psi_n(x-1)$$
$$-[\mu(x+\gamma)+x]\psi_n(x) + n(1-\mu)\psi_n(x) = 0 \tag{NM1}$$
$$-\sqrt{\mu(n+\gamma)(n+1)}\,\psi_{n+1}(x) - \sqrt{\mu n(n+\gamma-1)}\,\psi_{n-1}(x)$$
$$+[\mu(x+n+\gamma)+n-x]\psi_n(x) = 0 \tag{NM2}$$
$$L^+(x,n)\psi_n(x) = -[\mu(x+n+\gamma)]\psi_n(x) + \sqrt{\mu x(x+\gamma-1)}\,\psi_n(x-1)$$
$$= -\sqrt{\mu(n+\gamma)(n+1)}\,\psi_{n+1}(x) \tag{NM3}$$
$$L^-(x,n)\psi_n(x) = -[\mu(x+n+\gamma)]\psi_n(x) + \sqrt{\mu(x+1)(x+\gamma)}\,\psi_n(x+1)$$
$$= -\sqrt{\mu(n+\gamma-1)n}\,\psi_{n-1}(x) \tag{NM4}$$

$$\psi_0(x) = \sqrt{(1-\mu)^\gamma}\sqrt{\frac{\mu^x\Gamma(x+\gamma)}{\Gamma(x+1)\Gamma(\gamma)}}$$
$$\psi_n(x) = \frac{(-1)^n}{\sqrt{\mu^n(\gamma)_n n!}}\prod_{k=0}^{n-1} L^+(x,n-1-k)\psi_0(x)$$
$$L^+(x,n-1)L^-(x,n)\psi_n(x) = \mu(n+\gamma-1)n\psi_n(x)$$
$$L^-(x,n+1)L^+(x,n)\psi_n(x) = \mu(n+\gamma)(n+1)\psi_n(x).$$

B.8. Normalized Charlier functions

$$\psi_n(x) = \sqrt{\frac{\mu^n}{n!}}\sqrt{\frac{\mathrm{e}^{-\mu}\mu^x}{x!}}c_n^{(\mu)}(x)$$

$$\sqrt{\mu(x+1)}\psi_n(x+1) + \sqrt{\mu x}\psi_n(x-1) - (x+\mu)\psi_n(x) + n\psi_n(x) = 0 \tag{NC1}$$

$$-\sqrt{\mu(n+1)}\psi_{n+1}(x) - \sqrt{\mu n}\psi_{n-1}(x) + (n+\mu-x)\psi_n(x) = 0 \tag{NC2}$$

$$L^{+}(x,n)\psi_n(x) = -\mu\psi_n(x) + \sqrt{\mu x}\psi_n(x-1) = -\sqrt{\mu(n+1)}\psi_{n+1}(x) \tag{NC3}$$

$$L^{-}(x,n)\psi_n(x) = -\mu\psi_n(x) + \sqrt{\mu(x+1)}\psi_n(x+1) = -\sqrt{\mu n}\psi_{n-1}(x) \tag{NC4}$$

$$\psi_0(x) = \sqrt{\frac{\mathrm{e}^{-\mu}\mu^x}{x!}}$$

$$\psi_n(x) = \frac{(-1)^n}{\sqrt{\mu^n n!}}\prod_{k=0}^{n-1} L^{+}(x, n-1-k)\psi_0(x)$$

$$L^{+}(x,n-1)L^{-}(x,n)\psi_n(x) = \mu n\psi_n(x)$$

$$L^{-}(x,n+1)L^{+}(x,n)\psi_n(x) = \mu(n+1)\psi_n(x).$$

B.9. Normalized Chebyshev functions

$$\psi_n(x) = \sqrt{\frac{(2n+1)(N-n-1)}{(N+n)!}}t_n(x)$$

$$(x+1)(N-x-1) + \psi_n(x+1) + x(N-x)\psi_n(x-1)$$
$$-[(x+1)(N-x-1) + x(N-x)]\psi_n(x) + n(n+1)\psi_n(x) = 0 \tag{NT1}$$

$$\frac{n+1}{2}\sqrt{\frac{(2n+1)(N^2-n^2-2n-1)}{2n+3}}\psi_{n+1}(x) + \frac{n}{2}\sqrt{\frac{(2n+1)(N^2-n^2)}{2n-1}}\psi_{n-1}(x)$$
$$+(2n+1)\left(\frac{N-1}{2} - x\right)\psi_n(x) = 0 \tag{NT2}$$

$$L^{+}(x,n)\psi_n(x) = -[\tfrac{1}{2}(n+1)(N-2x-n-1) + x(N-x)]\psi_n(x)$$
$$+x(N-x)\psi_n(x-1) = \frac{n+1}{2}\sqrt{\frac{(2n+1)(N^2-n^2-2n-1)}{2n+3}}\psi_{n+1}(x) \tag{NT3}$$

$$L^{-}(x,n)\psi_n(x) = \left[\tfrac{1}{2}(n+1)(N-2x-n-1) + n(n+1)\right.$$
$$\left. +(2n+1)\left(x - \frac{N-1}{2}\right) - (x+1)(N-x-1)\right]\psi_n(x)$$
$$+(x+1)(N-x-1)\psi_n(x+1) = \frac{n}{2}\sqrt{\frac{(2n+1)(N^2-n^2)}{2n-1}}\psi_{n-1}(x) \tag{NT4}$$

$$\psi_0(x) = \frac{1}{\sqrt{N}}$$

$$\psi_n(x) = \prod_{k=0}^{n-1}\left\{\frac{2}{k+1}\sqrt{\frac{2k+3}{(2k+1)(N^2-k^2-2k-1)}}L^{+}(x,n-1-k)\right\}\psi_0(x)$$

$$L^{+}(x,n-1)L^{-}(x,n)\psi_n(x) = \frac{n^2}{4}(N+n)(N-n)\psi_n(x)$$

$$L^{-}(x,n+1)L^{+}(x,n)\psi_n(x)=\frac{(n+1)^2}{4}(N+n+1)(N-n-1)\psi_n(x).$$

B.10. Normalized Hahn functions

$$\psi_n(x)=\sqrt{\frac{(2n+\alpha+\beta+1)n!(N-n-1)!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(N+n+\alpha+\beta+1)}}\times\sqrt{\frac{\Gamma(N+\alpha-x)\Gamma(x+\beta+1)}{\Gamma(N-x)\Gamma(x+1)}}h_n^{(\alpha,\beta)}(x)$$

$$\sqrt{(N-x-1)(x+\beta+1)(N+\alpha-x-1)(x+1)}\psi_n(x+1)+\sqrt{(N-x)(x+\beta)(N+\alpha-x)x}\psi_n(x-1)-\{N-x-1)(x+\beta+1)+x(N+\alpha-x)\}\psi_n(x)+n(n+\alpha+\beta+1)\psi_n(x)=0 \tag{NHa1}$$

$$\frac{\sqrt{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)(2n+\alpha+\beta+1)}}{(2n+\alpha+\beta+2)}\times\sqrt{\frac{(N+n+\alpha+\beta+1)(N-n-1)}{2n+\alpha+\beta+3}}\cdot\psi_{n+1}(x)+\frac{\sqrt{n(n+\alpha)(n+\beta)(n+\alpha+\beta)(2n+\alpha+\beta+1)(N+n+\alpha+\beta)(N-n)}}{(2n+\alpha+\beta)\sqrt{2n+\alpha+\beta-1}}\times\psi_{n-1}(x)+(2n+\alpha+\beta+1)\left\{\frac{2N+\alpha-\beta-2}{4}+\frac{(\beta^2-\alpha^2)(2N+\alpha+\beta)}{4(2n+\alpha+\beta)(2n+\alpha+\beta+2)}-x\right\}\psi_n(x)=0 \tag{NHa2}$$

$$L^{+}(x,n)\psi_n(x)=\sqrt{x(N+\alpha-x)(\beta+x)(N-x)}\psi_n(x-1)-\left[\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2}\times\{(\beta+1)(N-1)-(\alpha+\beta+2+2n)x+(N-n-\beta-2)n\}+x(N+\alpha-x)\right]\psi_n(x)=\frac{\sqrt{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}}{(2n+\alpha+\beta+2)\sqrt{2n+\alpha+\beta+3}}\times\sqrt{(2n+\alpha+\beta+1)(N+n+\alpha+\beta+1)(N-n-1)}\cdot\psi_{n+1}(x) \tag{NHa3}$$

$$L^{-}(x,n)\psi_n(x)=\sqrt{(x+1)(N+\alpha-x-1)(x+\beta+1)(N-x-1)}\psi_n(x+1)+\left[\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2}\{(\beta+1)(N-1)-(\alpha+\beta+2+2n)\,x+(N-n-\beta-2)n\}+n(n+\alpha+\beta+1)+(2n+\alpha+\beta+1)\times\left(x-\frac{2N+\alpha-\beta-2}{4}-\frac{(\beta^2-\alpha^2)(\alpha+\beta+2N)}{4(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)-(N-x-1)(x+\beta+1)\right]\psi_n(x)=\frac{\sqrt{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}}{(2n+\alpha+\beta)\sqrt{2n+\alpha+\beta-1}}\times\sqrt{(2n+\alpha+\beta+1)(N+n+\alpha+\beta)(N-n)}\cdot\psi_{n-1}(x) \tag{NHa4}$$

$$\psi_0(x)=\sqrt{\frac{(\alpha+\beta+1)(N-1)!\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(N+\alpha+\beta+1)}}\sqrt{\frac{\Gamma(N+\alpha-x)\Gamma(x+\beta+1)}{\Gamma(N-x)\Gamma(x+1)}}$$

$$\psi_n(x) = \prod_{k=0}^{n-1} \left\{ \frac{(2k+\alpha+\beta+2)}{\sqrt{(k+1)(k+\alpha+1)(k+\beta+1)(k+\alpha+\beta+1)}} \times \sqrt{\frac{(2k+\alpha+\beta+3)}{(2k+\alpha+\beta+1)(N+k+\alpha+\beta+1)(N-k-1)}} L^+(x, n-1-k) \right\} \cdot \psi_0$$

$$L^+(x, n-1)L^-(x, n)\psi_n(x) = \frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)(N+n+\alpha+\beta)(N-n)}{(2n+\alpha+\beta)^2}\psi_n(x)$$

$$L^-(x, n+1)L^+(x, n)\psi_n(x) = \frac{(n+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+2)^2} \times (n+\alpha+\beta+1)(N+n+\alpha+\beta+1)(N-n-1)\psi_n(x).$$

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