MAPA xxxxA Special topics in approximation theory.

2000-2001: Asymptotic estimates in complex rational approximation.

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Abstract: several ways to build rational approximations to various species of analytic functions are examined. Special emphasis is put on strong asymptotic estimates of the form $f(z) - R_n(z) \sim \sigma(z)\rho^n(z)$, when such estimates are available.

CONTENTS

1. Complex approximation theory and potential theory.

2. The exponential function.

2.1. Padé.

for e^z,

$$[m/n] = \frac{1 + \frac{m}{m+n}\frac{z}{1!} + \frac{m(m-1)}{(m+n)(m+n-1)}\frac{z^2}{2!} + \cdots + \frac{m(m-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(n+1)}\frac{z^m}{m!}}{1 - \frac{n}{m+n}\frac{z}{1!} + \frac{n(n-1)}{(m+n)(m+n-1)}\frac{z^2}{2!} - \cdots + (-1)^n\frac{n(n-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(m+1)}\frac{z^n}{n!}}$$

[27]

Exponential behaviour of numerator and denominator has been much worked, especially the distribution of zeros and poles. Saff & Varga remark [] that, when $m \sim n$, these distributions had already been examined by Olver [26, ,] in a study of Bessel polynomials. For orthogonal polynomials, see Chen & Ismail [5], Gawronski & Shawyer [8], also Perron

(much to fill here)

2.2. **Rational interpolation.**

2.2.1. *Equidistant points.* [12, 13]

As far as we only need e^{Az} at $z = z_0, z_0+h, \dots, z_0+(m+n)h$,

$$\begin{aligned} e^{Az} &= (\boldsymbol{I}+\boldsymbol{\Delta})^{(z-z_0)/h} e^{Az_0} \\ &= \sum_{k=0}^{m+n} \binom{(z-z_0)/h}{k} \boldsymbol{\Delta}^k e^{Az_0} \\ &= \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{1}{k!}(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \end{aligned}$$

which we multiply by the denominator $Q(z) = \sum_{j=0}^{n} q_j (z-z_0)\cdots(z-z_0-(j-1)h)$, using

$$(z-z_0)(z-z_0-h)\cdots(z-z_0-(j-1)h)e^{Az} = \\ e^{A(z_0+jh)} \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^{k-j} \frac{1}{(k-j)!}(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

$$Q(z)e^{Az} = e^{Az_0} \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{C(k)}{k!}(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

where $C(k) = \sum_{j=0}^{n} q_j e^{Ajh} \left(\frac{e^{Ah}-1}{h}\right)^{-j} \frac{1}{(k-j)!}$ is a polynomial of degree n in k, which must vanish at $k = m+1, m+2, \dots, m+n$,

$$\begin{aligned} P(z) &= e^{Az_0} \sum_{k=0}^{m} \left(\frac{e^{Ah}-1}{h}\right)^k \binom{m}{k} (m+n-k)!(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \\ Q(z) &= \sum_{k=0}^{n} \left(\frac{e^{-Ah}-1}{h}\right)^k \binom{n}{k} (m+n-k)!(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \end{aligned}$$

and, formally:

$$Q(z)e^{Az} - P(z) = e^{Az_0} m!(-1)^n \sum_{k=m+n+1}^{\infty} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{1}{k!}(k-m-1)(k-m-2)\cdots(k-m-n)(z-z_0)(z-z_0-h)\cdots(z \tag{1}$$

Rough asymptotics. Let $E := \exp(Ah)$ and $\zeta := \frac{z-z_0}{nh} - 1$. Then, when $m \sim n$, each new term in the expansions of P and Q is the former one times $\frac{(E^{\pm 1}-1)(n-k)(n\zeta+n-k)}{k(2n-k)}$.

We intend to follow things at constant E and ζ, i.e., a fixed exponential and z expanding linearly with n, or A increasing linearly with n, and $2n$ interpolating points filling a fixed segment $[z_0, z_0+2nh]$. Remark that this segment of interpolation points is $-1 \leqslant \zeta \leqslant 1$.

At least when each term has the same phase, the sum is roughly given by the term reached when the preceding ratio has unit value:

$$\frac{(E^{\pm 1}-1)\kappa(\zeta+\kappa)}{1-\kappa^2}=1, \tag{2}$$

where $\kappa := 1-k/n$. Remark that the ratio $\to 0$ when $\kappa\to 0$, and $\to\infty$ when $\kappa\to 1$.

Of course, an integral formula gives more weight to these asymptotic constructions (to be continued)

The dominant term (there will sometimes be two dominant terms, see later), is, still roughly,

$$\frac{(E^{\pm 1}-1)^{n-n\kappa}n!(n+n\kappa)!\Gamma(n\zeta+n)}{(n-n\kappa)!(n\kappa)!\Gamma(n\zeta+n\kappa)},$$

where κ is one of the roots of (2). Tedious application of Stirling's formula yields, keeping only the wildest factors,

$$\begin{aligned}
&e^{-2n}n^{2n}\left[\frac{(E^{\pm 1}-1)^{1-\kappa}(1+\kappa)^{1+\kappa}(1+\zeta)^{1+\zeta}}{(1-\kappa)^{1-\kappa}\kappa^{\kappa}(\zeta+\kappa)^{\zeta+\kappa}}\right]^n\\
=&e^{-2n}n^{2n}\left[\underbrace{\left(\frac{1-\kappa^2}{(E^{\pm 1}-1)\kappa(\zeta+\kappa)}\right)^{\kappa}}_{1}\frac{(E^{\pm 1}-1)(1+\kappa)(1+\zeta)^{1+\zeta}}{(1-\kappa)(\zeta+\kappa)^{\zeta}}\right]^n\\
=&e^{-2n}n^{2n}\left[\frac{(E^{\pm 1}-1)(1+\kappa)(1+\zeta)^{1+\zeta}}{(1-\kappa)(\zeta+\kappa)^{\zeta}}\right]^n\\
=&e^{-2n}n^{2n}\left[\frac{(1+E^{\pm 1}\kappa)^{1+\zeta}(1+\kappa)^{1-\zeta}}{\kappa}\right]^n,
\end{aligned} \tag{3}$$

using $\zeta=\dfrac{\kappa^{-1}-E^{\pm 1}\kappa}{E^{\pm 1}-1}$, from (2).

Check: for large z, we should have $(E^{\pm 1}-1)^n n!(z/h)^n\approx(E^{\pm 1}-1)^n e^{-n}n^{2n}\zeta^n$. Indeed, with the root ≈ 0 of (2), i.e., $\kappa\sim[(E^{\pm 1}-1)\zeta]^{-1}$, one finds $e^{-2n}n^{2n}\{\kappa^{-1}[(1+(E^{\pm 1}-1)\kappa]^{\zeta}\}^n$.

3. **Simplest potential problems.**

3.1. **Conditions on a single arc.**

Let the function (often associated to a distribution of poles) $\mathcal{V}_p'(z)=\displaystyle\int_\alpha^\beta\frac{d\mu_p(t)}{z-t}$. Suppose that we know that

$$=\!\!\!\!\!\!\int_\alpha^\beta\frac{d\mu_p(t)}{z-t}=g(z), \tag{4}$$

with g analytic in some domain (the arc $[\alpha,\beta]$ is not yet known). The trick is to multiply $\mathcal{V}_p'$ by a function $[(z-\alpha)(z-\beta)]^{\gamma/2}$ taking *opposite* values on the two sides of $[\alpha,\beta]$. We consider only $\gamma=1$ and $\gamma=-1$. Also, $[(z-\alpha)(z-\beta)]^{\gamma/2}$ is defined to be continuous outside the arc, and behaves like z^γ for large z. As $\mathcal{V}_p'(z)[(z-\alpha)(z-\beta)]^{\gamma/2}-\delta_{\gamma,1}$ has a Laurent expansion with only negative powers at ∞,

$$\mathcal{V}_p'[(z-\alpha)(z-\beta)]^{\gamma/2}-\delta_{\gamma,1}=\frac{1}{2\pi i}\oint\frac{\mathcal{V}_p'(t)[(t-\alpha)(t-\beta)]^{\gamma/2}-\delta_{\gamma,1}}{z-t}\,dt$$

on a big counterclockwise contour having the arc $[\alpha,\beta]$ inside and z outside. Making the contour shrink to a neighbourhood of the arc $[\alpha,\beta]$, we get $\dfrac{1}{2\pi i}\displaystyle\int_\alpha^\beta\frac{\Delta\{\mathcal{V}_p'(t)[(t-\alpha)(t-\beta)]^{\gamma/2}-\delta_{\gamma,1}\}}{z-t}\,dt$, where $\Delta\{F\}$ means the difference $F_- - F_+$ between the limit values of F on the "lower" side of the arc (from which the arc is seen at left), and the "upper" side. The difference is here $2g(t)[(t-\alpha)(t-\beta)]_-^{\gamma/2}$, whence quite explicit solutions

$$\mathcal{V}_p''[(z-\alpha)(z-\beta)]^{\gamma/2}-\delta_{\gamma,1}=\frac{1}{\pi i}\int_\alpha^\beta \frac{g(t)[(t-\alpha)(t-\beta)]_-^{\gamma/2}}{z-t}\,dt, \qquad \gamma=\pm 1. \tag{5}$$

It may help to realize that the phase of $\dfrac{\beta-\alpha}{[(t-\alpha)(t-\beta)]_-^{1/2}}$ is exactly the one of $+i$ on the rectilinear segment $[\alpha,\beta]$.

Some questions: the -1 in the left-hand side of (5) when $\gamma=1$ is needed from $\mathcal{V}_p''(z)=1/z+O(1/z^2)$ for large z. But the two sides of (5) when $\gamma=-1$ should be $\sim 1/z^2$ for large z, everything works only if

$$\int_\alpha^\beta \frac{g(t)\,dt}{[(t-\alpha)(t-\beta)]_-^{1/2}}=0, \quad \int_\alpha^\beta \frac{tg(t)\,dt}{[(t-\alpha)(t-\beta)]_-^{1/2}}=\pi i. \tag{6}$$

The two forms of (5) then agree, either with $\gamma=-1$, or $\gamma=1$. It will also be useful to check that, as (4) is a plain integral when $z=\alpha$ and $z=\beta$, one has $\mathcal{V}_p''(\alpha)=g(\alpha)$, and $\mathcal{V}_p''(\beta)=g(\beta)$.

3.1.1. *A little bit of Chebyshev polynomials calculus.* Let us consider the Chebyshev polynomials expansion of a generic function F on $[\alpha,\beta]$:

$$F(t)=\frac{c_0}{2}+\sum_1^\infty c_nT_n\left(\frac{2t-\alpha-\beta}{\beta-\alpha}\right).$$

Then, we have the integral

$$\int_\alpha^\beta \frac{F(t)\,dt}{[(t-\alpha)(t-\beta)]_-^{1/2}}=\pi i\frac{c_0}{2}.$$

Therefore, from (5) with $\gamma=-1$, $\mathcal{V}_p''[(z-\alpha)(z-\beta)]^{-1/2}$ is the constant term of the Chebyshev expansion of $g(t)/(z-t)$.

Let $g_0/2+\sum_1^\infty g_nT_n$ be the expansion of g. Remark that (6) becomes

$$g_0=0, \qquad g_1=\frac{4}{\beta-\alpha}. \tag{7}$$

We need the expansion of $1/(z-t)=X_0/2+\sum_1^\infty X_nT_n$, which we multiply by $\dfrac{2(z-t)}{\beta-\alpha}=\dfrac{2z-\alpha-\beta}{\beta-\alpha}-\dfrac{2t-\alpha-\beta}{\beta-\alpha}$:

$$\frac{2}{\beta-\alpha}=\frac{X_0}{2}\left(\frac{2z-\alpha-\beta}{\beta-\alpha}-T_1((2t\alpha-\beta)/(\beta-\alpha))\right)+\sum_{n=1}^\infty X_n\left(\frac{2z-\alpha-\beta}{\beta-\alpha}T_n-\frac{T_{n-1}+T_{n+1}}{2})\right)$$

whence the recurrence $X_{n+1}-2\dfrac{2z-\alpha-\beta}{\beta-\alpha}X_n+X_{n-1}=0$ for $n=1,2,\dots$ solved by $X_n=X_0\rho^n$, where ρ is a root of

$$\frac{\rho+\rho^{-1}}{2}=\frac{2z-\alpha-\beta}{\beta-\alpha}, \tag{8}$$

normally with $|\rho|<1$, but this will have to be discussed later. The value of X_0 comes from $n=0$: $4/(\beta-\alpha)=X_0((\rho+\rho^{-1})/2)-X_1=X_0(\rho^1-\rho)/2$, so

$$X_0=\frac{8}{(\beta-\alpha)(\rho^{-1}-\rho)}.$$

Remark that $[(z-\alpha)(z-\beta)]^{1/2}=(\beta-\alpha)^2(1-\rho^2)^2/(16\rho^2)$, so that

$$\mathcal{V}_p''(z)=\sum_{n=1}^\infty g_n\rho^n. \tag{9}$$

The two determinations of $\mathcal{V}_p''$ on the two sides of the cut $[\alpha,\beta]$ are obtained with the *two* roots ρ and $1/\rho$ of (8). One checks that the arithmetic mean is indeed

$$(\mathcal{V}_{p,+}''(z)+\mathcal{V}_{p,-}''(z))/2=\sum_1^\infty g_n(\rho^n+\rho^{-n})/2=\sum_1^\infty g_n T_n=g(z).$$

As for the discontinuity along the cut,

$$\pm\pi i\mu_p'(z)=\mathcal{V}_{p,-}''(z)+\mathcal{V}_{p,+}''(z)=\sum_1^\infty g_n(\rho^{-n}-\rho^n)=\frac{4}{\beta-\alpha}[(z-\alpha)(z-\beta)]^{1/2}\sum_1^\infty g_n U_{n-1}(z), \qquad (10)$$

it appears as a kind of harmonic conjugate to g.

3.1.2. *Check with rational approximation to* $\exp(Az)$.

From (3),

$$\mathcal{V}_p(z)=\lim_{n\to\infty}\frac{\log Q(z)}{n}=(1+\zeta)\log(1+E^{-1}\kappa)+(1-\zeta)\log(1+\kappa)-\log(\kappa),$$

where ζ is basically our z (managed so that the interpolation points are in $[-1,1]$). Then, the derivative in ζ simplifies into

$$\frac{d\mathcal{V}_p(z)}{d\zeta}=\log\frac{1+E^{-1}\kappa}{1+\kappa},$$

where κ is related to ζ through $\zeta=\dfrac{\kappa^{-1}-E^{-1}\kappa}{E^{-1}-1}$, from (2). This matches (8) provided

$$\rho=iE^{-1/2}\kappa,\alpha=-\beta=\frac{2i}{E^{1/2}-E^{-1/2}},$$

and

$$\frac{d\mathcal{V}_p(z)}{d\zeta}=\log\frac{1-iE^{-1/2}\rho}{1-iE^{1/2}\rho}=-\sum_{n=1}^\infty\frac{i^n(E^{-n/2}-E^{n/2})}{n}\rho^n.$$

Remark that $g_1=-i(E^{-1/2}-E^{1/2})=4/(\beta-\alpha)$ as it should.

3.1.3. *Rational interpolation to* $\exp(nB_1z+nB_2z^2)$.

This very interesting rational interpolation appears in special nonlinear Schrödinger problems ([19] and remarks by J. Nuttall)

Let the interpolation points be equidistant on $[I_1,I_2]$. Then,

$$g(z)=\int_{I_1}^{I_2}\frac{(I_2-I_1)^{-1}dt}{z-t}-\frac{B_1}{2}-B_2z=\frac{\log\dfrac{z-I_1}{z-I_2}}{I_2-I_1}-\frac{B_1}{2}-B_2\left(\frac{\beta-\alpha}{2}\frac{2z-\alpha-\beta}{\beta-\alpha}+\frac{\alpha+\beta}{2}\right) \qquad (11)$$

The logarithms have the expansions

$$\log(z-I_k)=\log\frac{\alpha-\beta}{4\rho_k}-2\sum_1^\infty\frac{\rho_k^n}{n}T_n,$$

where ρ_k is now a root of

$$\frac{\rho_k+\rho_k^{-1}}{2}=\frac{2I_k-\alpha-\beta}{\beta-\alpha}\,,k=1,2, \qquad (12)$$

where $|\rho_k|<1$ should be the orthodox choice, but which will not be kept in the final formula. Precisely, the closed form is now

$$\mathcal{V}_p''(z)=\sum_1^\infty g_n\rho^n=\frac{2}{I_2-I_1}\log\frac{1-\rho_1\rho}{1-\rho_2\rho}-B_2\frac{\beta-\alpha}{2}\rho, \qquad (13)$$

with the conditions (7) on g_0 and g_1

$$\frac{\log(\rho_2/\rho_1)}{I_2-I_1}-\frac{B_1}{2}-B_2\frac{\alpha+\beta}{2}=0, \tag{14}$$

$$g_1=2\frac{\rho_2-\rho_1}{I_2-I_1}-B_2\frac{\beta-\alpha}{2}=\frac{4}{\beta-\alpha}, \tag{15}$$

If ρ_1 and ρ_2 are known, α and β are got by (12):

$$\alpha=\frac{\rho_1(1+\rho_2)^2I_1-\rho_2(1+\rho_1)^2I_2}{(\rho_2-\rho_1)(\rho_1\rho_2-1)},\beta=\frac{\rho_1(1-\rho_2)^2I_1-\rho_2(1-\rho_1)^2I_2}{(\rho_2-\rho_1)(\rho_1\rho_2-1)}. \tag{16}$$

(As for ρ_1 and ρ_2, they are simply found to be, if $B_2=0$, $\rho_1=i\exp(-B_1(I_2-I_1)/4)$ and $\rho_2=i\exp(B_1(I_2-I_1)/4)$. Remark that $\rho_1\rho_2=-1$: no chance to have the comfortable $|\rho_k|<1\ldots$)

Also,

$$\frac{\beta-\alpha}{2}=\frac{2\rho_1\rho_2(I_2-I_1)}{(\rho_2-\rho_1)(\rho_1\rho_2-1)},\ \frac{\alpha+\beta}{2}=\frac{I_1+I_2}{2}-\frac{(\rho_1+\rho_2)(\rho_1\rho_2+1)(I_2-I_1)}{2(\rho_2-\rho_1)(\rho_1\rho_2-1)}.$$

and (14) and (15) become

$$\frac{\log(\rho_2/\rho_1)}{I_2-I_1}-\frac{B_1}{2}-B_2\frac{I_1+I_2}{2}+\frac{B_2}{2}\frac{(\rho_2+\rho_1)(\rho_1\rho_2+1)}{(\rho_2-\rho_1)(\rho_1\rho_2-1)}(I_2-I_1)=0, B_2(I_2-I_1)^2=\frac{(\rho_2-\rho_1)^2(\rho_1^2\rho_2^2-1)}{2\rho_1^2\rho_2^2},$$

or this form emphasizing $\rho_1\rho_2$ and ρ_2/ρ_1:

$$2B_2(I_2-I_1)^2=\left(\frac{\rho_2}{\rho_1}-2+\frac{\rho_1}{\rho_2}\right)\left(\rho_1\rho_2-\frac{1}{\rho_1\rho_2}\right), \tag{17}$$

$$\log\left(\frac{\rho_2}{\rho_1}\right)+\frac{1}{4}\left(\frac{\rho_2}{\rho_1}-\frac{\rho_1}{\rho_2}\right)\left(\rho_1\rho_2+2+\frac{1}{\rho_1\rho_2}\right)=\frac{B_1}{2}(I_2-I_1)+B_2\frac{I_2^2-I_1^2}{2}. \tag{18}$$

For a given B_2 and various ratios ρ_2/ρ_1, we find valid values for B_1, etc. For instance, $I_1=-Ai$, $I_2=Ai$, B_2 negative imaginary and ρ_2/ρ_1 negative real, which is of interest in [19]:

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                   GP/PARI CALCULATOR Version 2.0.12 (alpha)
                           Copyright (C) 1989-1998 by
          C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

? \r expr1r2
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$A=1, B_1=\pi-2ix, x=0.5, B_2=-2iAt,$

At	$-\rho_2/\rho_1$	Q	ρ_1	ρ_2	$\alpha=\overline{\beta}$
0.10	2.758	0.02455	$0.6002+0.04717i$	$-1.655-0.1301i$	$0.03677+0.8865i$
0.10	0.439	3.971	$0.1284+1.502i$	$-0.05646-0.6606i$	$-4.554+10.81i$
0.20	2.888	0.09570	$0.5812+0.09100i$	$-1.679-0.2629i$	$0.07604+0.8847i$
0.20	0.429	3.883	$0.2604+1.504i$	$-0.1117-0.6454i$	$-2.307+5.374i$
0.30	3.138	0.2038	$0.5499+0.1274i$	$-1.725-0.3998i$	$0.1197+0.8788i$
0.30	0.411	3.737	$0.3997+1.507i$	$-0.1643-0.6197i$	$-1.574+3.545i$
0.40	3.553	0.3275	$0.5082+0.1518i$	$-1.806-0.5394i$	$0.1674+0.8641i$
0.40	0.384	3.533	$0.5507+1.514i$	$-0.2119-0.5830i$	$-1.221+2.622i$
0.50	4.176	0.4363	$0.4618+0.1616i$	$-1.928-0.6749i$	$0.2147+0.8365i$
0.50	0.351	3.277	$0.7171+1.527i$	$-0.2517-0.5363i$	$-1.023+2.064i$
0.60	5.006	0.5082	$0.4175+0.1593i$	$-2.090-0.7976i$	$0.2544+0.7974i$
0.60	0.311	2.988	$0.9009+1.548i$	$-0.2805-0.4824i$	$-0.9027+1.692i$
0.70	6.007	0.5430	$0.3792+0.1503i$	$-2.278-0.9031i$	$0.2832+0.7524i$
0.70	0.270	2.692	$1.099+1.578i$	$-0.2972-0.4266i$	$-0.8243+1.431i$
0.80	7.139	0.5518	$0.3474+0.1390i$	$-2.480-0.9924i$	$0.3017+0.7071i$
0.80	0.232	2.414	$1.307+1.612i$	$-0.3033-0.3742i$	$-0.7690+1.241i$
0.90	8.372	0.5452	$0.3211+0.1275i$	$-2.689-1.068i$	$0.3124+0.6643i$
0.90	0.198	2.166	$1.517+1.650i$	$-0.3019-0.3282i$	$-0.7263+1.099i$
1.0	9.692	0.5303	$0.2991+0.1169i$	$-2.899-1.133i$	$0.3178+0.6252i$
1.0	0.171	1.953	$1.727+1.687i$	$-0.2962-0.2893i$	$-0.6908+0.9883i$
1.1	11.08	0.5113	$0.2804+0.1073i$	$-3.109-1.190i$	$0.3195+0.5899i$
1.1	0.149	1.770	$1.933+1.723i$	$-0.2882-0.2568i$	$-0.6599+0.9000i$
1.2	12.55	0.4906	$0.2643+0.09885i$	$-3.318-1.240i$	$0.3187+0.5582i$
1.2	0.130	1.613	$2.136+1.757i$	$-0.2791-0.2296i$	$-0.6323+0.8278i$

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Good bye!

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We integrate (13) along the lines suggested by the exercises of section 3.1.2, p. 5:

$$\mathcal{V}_p(z)=\frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho_1\rho)-(z-I_2)\log(1-\rho_2\rho)+X(\rho)],$$

which yields indeed, using from (8) and (12) $z-I_k=\dfrac{\beta-\alpha}{4}\left(1-\dfrac{\rho_k}{\rho}\right)\left(\rho-\dfrac{1}{\rho_k}\right)$,

$$\frac{d\mathcal{V}_p(z)}{dz}=\frac{2}{I_2-I_1}\left\{\log\frac{1-\rho_1\rho}{1-\rho_2\rho}+\left[\frac{z-I_1}{\rho-\rho_1^{-1}}-\frac{z-I_2}{\rho-\rho_1^{-2}}+\frac{dX(\rho)}{d\rho}\right]\frac{d\rho}{dz}\right\}$$

$$=\frac{2}{I_2-I_1}\left\{\log\frac{1-\rho_1\rho}{1-\rho_2\rho}+\left[\frac{\beta-\alpha}{4}\frac{\rho_2-\rho_1}{\rho}+\frac{dX(\rho)}{d\rho}\right]\frac{4}{(\beta-\alpha)(1-\rho^{-2})}\right\}$$

One must have $\dfrac{dX}{d\rho}=-\dfrac{\beta-\alpha}{4}\dfrac{\rho_2-\rho_1}{\rho}-B_2\dfrac{(\beta-\alpha)^2}{16}(I_2-I_1)\left(\rho-\dfrac{1}{\rho}\right)$, finally:

$$\frac{dX}{d\rho}=-\frac{I_2-I_1}{2}\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\rho+\frac{1}{\rho},$$

$$\mathcal{V}_p(z)=\frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho_1\rho)-(z-I_2)\log(1-\rho_2\rho)]-\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2}{2}-\log\rho. \tag{19}$$

The two determinations of $\mathcal{V}_p$ on the two sides of the cut are found with the two roots ρ and $1/\rho$ of (8). In particular, the *arithmetic mean* of the two values of the derivative must give (4) again, with g given by

(11). Indeed, one finds

$$\frac{1}{I_2-I_1}\{\log[(1-\rho_1\rho)(1-\rho_1/\rho)]-\log[(1-\rho_2\rho)(1-\rho_2/\rho)]\}-B_2\frac{\beta-\alpha}{2}\frac{\rho+\rho^{-1}}{2},$$

which is

$$\frac{1}{I_2-I_1}\left[\log\frac{z-I_1}{z-I_2}+\log\frac{\rho_1}{\rho_2}\right]-B_2 z+B_2\frac{\alpha+\beta}{2},$$

The *difference* of the two determinations of $\mathcal{V}_p''$ must be $\pm 2\pi i\mu'$:

$$\pm 2\pi i\mu'(z)=\frac{2}{I_2-I_1}\left[\log\frac{1-\rho_1\rho}{1-\rho_1/\rho}-\log\frac{1-\rho_2\rho}{1-\rho_2/\rho}\right]-B_2\frac{\beta-\alpha}{2}(\rho-\rho^{-1}), \tag{20}$$

(Nuttall's $\Delta\Psi_2$)

and the cut itself is the locus $\{z:\mu'(z)\,dz \text{ real }\}$, which is integrated as $\{z:\mathcal{V}_{p,+}(z)-\mathcal{V}_{p,-}(z)$ pure imaginary $\}$,

$$\frac{2}{I_2-I_1}\left[(z-I_1)\log\frac{1-\rho_1\rho}{1-\rho_1/\rho}-(z-I_2)\log\frac{1-\rho_2\rho}{1-\rho_2/\rho}\right]-\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2-\rho^{-2}}{2}-2\log\rho \text{ pure imaginary.} \tag{21}$$

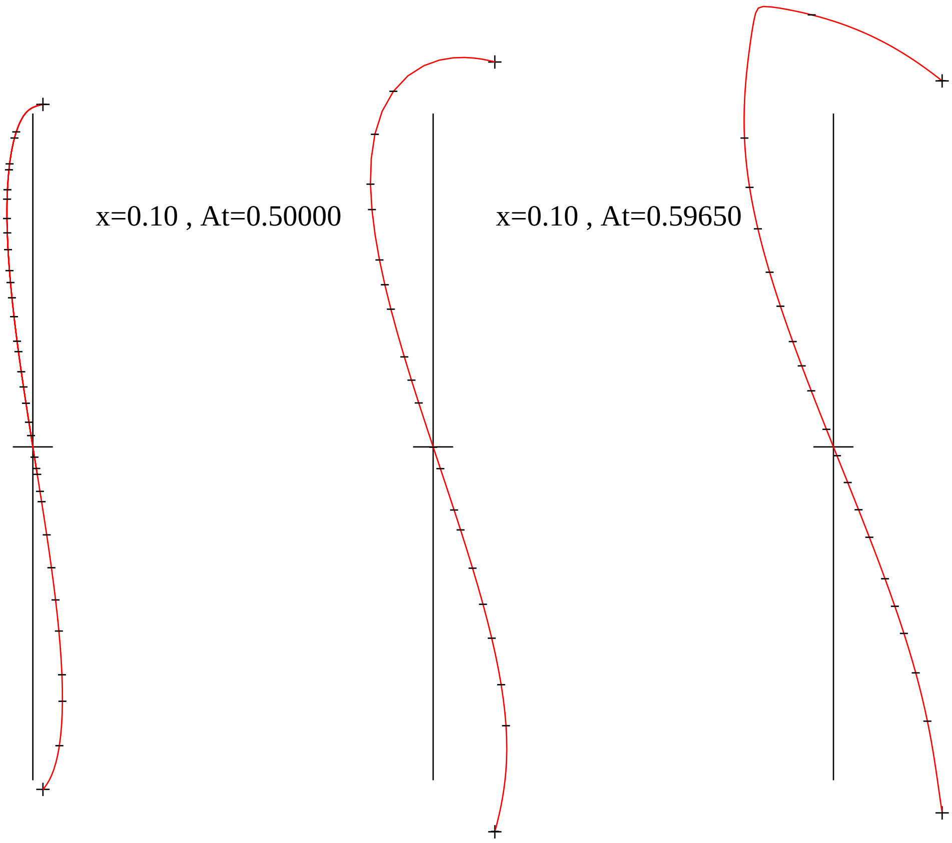
Writing (21) as a function of ρ (using (8) and (12)), we have

$$F(\rho)=\frac{2}{(\rho_2-\rho_1)(1-1/(\rho_1\rho_2))}\left[(\rho-\rho_1)\left(1-\frac{1}{\rho\rho_1}\right)L_1-(\rho-\rho_2)\left(1-\frac{1}{\rho\rho_2}\right)L_2\right]-\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2-\rho^{-2}}{2},$$

with $L_1=\log\dfrac{1-\rho_1\rho}{\rho-\rho_1}$,$L_2=\log\dfrac{1-\rho_2\rho}{\rho-\rho_2}$, and where, for given B_1,B_2,I_1,I_2, one must determine ρ_1 and ρ_2 from (17) and (18).

Caustic.

The present setting of the limit set of poles as a single arc joining $z=\alpha$ to $z=\beta$ (or $\rho=-1$ to $\rho=1$) holds as long as $\mu_p'(z)dz$ remains positive on the cut. A critical situation occurs when μ_p' happens to vanish right on the cut, i.e., if dF/dz vanishes at a point where the real part of F vanishes too.



The locus of (x, At) with $B_1 = \pi - 2ix$, $B_2 = -2iAt$, where this happens is called the (first) caustic in [19]. We then have $\rho_1 = R^{-1/2}e^{i\theta}$, $\rho_2 = -R^{1/2}e^{i\theta}$, with real R and θ. For a trial value of At, we look for R and θ such that $(R+1/R)/2 = 2At/\sin 2\theta - 1$ (from (17)) and $2x = \log R + (1/R - R)\sin^2\theta$ (from (18)). Knowing ρ_1 and ρ_2, one looks for the zero of the analytic function dF/dz, or $dF/d\rho$. This yields

$$L_1 - L_2 = \left(1 + \frac{1}{\rho_1\rho_2}\right)(\rho_2 - \rho_1)\frac{\rho - \rho^{-1}}{2}.$$

One then manages to have the real part of $F = 0$ as well.

Some values:

x	At	$\sin\theta$	ρ_2	$R=-\rho_2/\rho_1$	ρ	F
0.001	0.500973	0.6990	-0.749 -0.733 i	1.098	-1.27846 -0.24352 i	6.282 i
0.010	0.509711	0.6781	-0.854 -0.788 i	1.350	-1.52506 -0.51989 i	6.261 i
0.050	0.548383	0.6288	-1.095 -0.885 i	1.982	-1.84975 -0.94099 i	6.123 i
0.100	0.596697	0.5838	-1.324 -0.952 i	2.660	-2.06555 -1.23283 i	5.922 i
0.250	0.744301	0.4821	-1.928 -1.061 i	4.842	-2.48476 -1.76294 i	5.301 i
0.500	1.009193	0.3636	-2.919 -1.139 i	9.817	-2.98988 -2.26747 i	4.371 i
0.750	1.312145	0.2794	-4.017 -1.169 i	17.504	-3.41534 -2.56407 i	3.587 i
1.000	1.672677	0.2167	-5.311 -1.179 i	29.600	-3.78112 -2.72755 i	2.923 i
1.250	2.117750	0.1686	-6.897 -1.180 i	48.954	-4.06939 -2.79596 i	2.359 i
1.500	2.682341	0.1314	-8.888 -1.178 i	80.381	-4.27061 -2.81082 i	1.884 i
1.750	3.408138	0.1024	-11.423 -1.176 i	131.867	-4.39913 -2.80582 i	1.493 i
2.000	4.344519	0.0798	-14.669 -1.174 i	216.553	-4.47798 -2.79719 i	1.175 i
2.500	7.107960	0.0484	-24.179 -1.172 i	585.997	-4.55461 -2.78484 i	0.721 i
3.000	11.684073	0.0294	-39.858 -1.171 i	1590.003	-4.58262 -2.77939 i	0.439 i

We see that $\theta\to\pi/4$ when $x\to 0$, and that $\theta\to 0$ when $x\to\infty$, but many features are still unexplained ...

Numerator and interpolation.

Remind that $\mathcal{V}_p(z)$ is the limit when $n\to\infty$ of $n^{-1}\log Q_n(z) = n^{-1}\sum\log(z-\text{ poles})$. It must behave like $\log(z)+O(1/z)$ for large z, compatible with (19) if one adds a constant:

$$\mathcal{V}_p(z) = \frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho_1\rho)-(z-I_2)\log(1-\rho_2\rho)] - \frac{\rho^2}{2}\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1} - \log\rho + C_p,$$

which behaves for large z as $\log z - log\dfrac{\beta-\alpha}{4} + \dfrac{(\beta-\alpha)(\rho_2-\rho_1)}{2(I_2-I_1)} + C_p$, from $\rho\sim(\beta-\alpha)/(4z)$. Therefore,

$$C_p = log\frac{\beta-\alpha}{4} - \frac{(\beta-\alpha)(\rho_2-\rho_1)}{2(I_2-I_1)}.$$

The numerator of the interpolant to $\exp(n(B_1z+B_2z^2))$ is the denominator of the interpolant to $\exp(n(-B_1z-B_2z^2))$, so that the calculations made before apply with $(B_1,B_2)\longrightarrow(-B_1,-B_2)$. The equations (14) and (15) are now satisfied by $(\rho_1,\rho_2)\longrightarrow(1/\rho_1,1/\rho_2)$. And the values for α and β are the same as before. Let $\mathcal{V}_{\text{num}}(z)$ be the (presumed to exist) limit when $n\to\infty$ of $n^{-1}\log P_n(z)$, where P_n is the numerator. We expect a formula similar to (19), but with another constant:

$$\mathcal{V}_{\text{num}}(z) = \frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho/\rho_1)-(z-I_2)\log(1-\rho/\rho_2)] + \frac{\rho^2}{2}\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1} - \log\rho + C_{\text{num}}.$$

The remaining constant C_{num} is determined by $\mathcal{V}_{\text{num}}(z)-\mathcal{V}_p(z) = B_1z+B_2z^2$ in a neighbourhood of the set of the interpolation points. Everything works if one determination, say with ρ, is used for $\mathcal{V}_p$, while the determination with $1/\rho$ is used for $\mathcal{V}_{\text{num}}$:

$$\mathcal{V}_{\text{num}}(z)-\mathcal{V}_p(z) = \frac{2}{I_2-I_1}\left[(z-I_1)\log\frac{1-1/(\rho\rho_1)}{1-\rho\rho_1} - (z-I_2)\log\frac{1-1/(\rho/\rho_2)}{1-\rho\rho_2}\right] + \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^{-2}+\rho^2}{2} + 2\log\rho + C_{\text{num}}$$

$$= \frac{2}{I_2-I_1}[(z-I_2)\log(-\rho_2)-(z-I_1)\log(-\rho_1)] + \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\left[2\left(\frac{2z-\alpha-\beta}{\beta-\alpha}\right)^2-1\right] + C_{\text{num}} - C_p,$$

$$= [B_1+B_2(\alpha+\beta)]z + 2\frac{I_1\log(-\rho_1)-I_2\log(-\rho_2)}{I_2-I_1} + \frac{B_2}{4}(2z-\alpha-\beta)^2 - \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1} + C_{\text{num}} - C_p,$$

whence

$$C_{\text{num}} - C_p = 2\frac{I_2\log(-\rho_2) - I_1\log(-\rho_1)}{I_2 - I_1} - \frac{B_2}{4}(\alpha+\beta)^2 + \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}. \tag{22}$$

3.2. **Best rational approximation on a real interval.**

Best rational approximation to $\exp(-z)$ on a given real interval, say $[0,c]$ shows the now familiar linear scale with respect to the degree n, as seen with the sequence of real poles:

	1/1	3/3	5/5
$c=0$	-2	-4.644	-7.293
$c=1$	-1.572	-4.176	$-6.80(?)$
$c=2$	-1.274	-3.768	-6.37
$c=5$	-0.862	-2.875	-5.277
$c=10$	-0.697	-2.141	-4.086
$c=\infty$	-0.579	-1.369	-2.155

We therefore get a stabilized picture by taking z/n as new variable, so that we approximate $\exp(-nz)$ on a shrinking interval $[0,c/n]$, and just reproduce , for any finite c, the Padé performance ($c=0$). [More data, for $n=7,9\ldots$ welcome]

It is much more interesting to approximate $\exp(-nz)$ on $[0,c]$:

	1/1	3/3	5/5
$c=0$	-2	-1.548	-1.459
$c=1$	-1.572	-1.140	-1.055
$c=2$	-1.274	-0.890	-0.817
$c=5$	-0.862	-0.615	-0.570
$c=10$	-0.697	-0.526	-0.49
$c=\infty$	-0.579	-0.456	-0.431

We expect the poles to tend to be distributed on a fixed arc F with a limit distribution $d\mu_p$, and the interpolation points on $E=[0,c]$ with a limit distribution $d\mu_i$, so that the complex potential

$$\mathcal{V}(z) := \int_F \log\frac{1}{z-t}\,d\mu_p(t) - \int_E \log\frac{1}{z-t}\,d\mu_i(t) \tag{23}$$

satisfies

$$V := \operatorname{Re}\mathcal{V} = \text{ a constant } = \rho \text{ on } E, \tag{24}$$

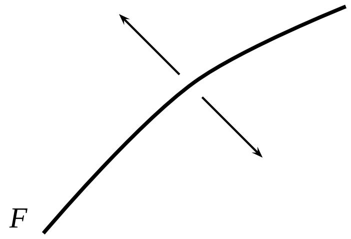
$$V(z) + \frac{\operatorname{Re} z}{2} = \text{ a constant } = \sigma \text{ on } F, \tag{25}$$

$$V(z) + \frac{\operatorname{Re} z}{2} \text{ has equal normal derivatives on the two sides of } F, \tag{26}$$

$$\int_E d\mu_i(t) = \int_F d\mu_p(t) = 1$$

(charges on E and F), equivalent to $\mathcal{V}$ bounded at ∞, actually, $\mathcal{V}'(z) \sim$ constant z^{-2} for large z, and $\int_C \frac{\partial V(t)}{\partial n}|dt| = -2\pi$ on any contour containing F but not E, or also, that the imaginary part of $\mathcal{V}$ increases by π on $[0,c]$.

Conditions (25) and (26) amount to the realization that $V+\operatorname{Re}z/2$ has opposite *gradients* along the normal on the two sides of F:



As the derivative of an analytic function has real and imaginary parts building the gradient of its real part (Cauchy-Riemann: grad Re$F = \overline{F'}$), it follows that $\mathcal{V}' + 1/2$ takes opposite values on the two sides of F.

Now, limit values of such functions are given by ***Sokhotskyi-Plemelj*** formulas [11], chap. 14, etc.

$$\mathcal{V}'(z) = -\int_{E\cup F} \frac{d\mu(t)}{z-t} = -⨍_{E\cup F} \frac{d\mu(t)}{z-t} \pm \pi i \mu'(z) \tag{27}$$

when z tends to a point of E or F, and where $⨍$ is the Cauchy principal value. We therefore have

$$⨍_{E\cup F} \frac{d\mu(t)}{z-t} = ⨍_F \frac{d\mu_p(t)}{z-t} - \int_E \frac{d\mu_i(t)}{z-t} = \frac{1}{2},\ z \in F, \tag{28}$$

which is an integral equation for the distribution μ_p, to be considered with (24) as another equation for μ_p and μ_i ...

We get rid of the condition (24) by using complex Green functions [1] of E: first, let

$$\varphi(z) := \frac{2z}{c} - 1 + \sqrt{\left(\frac{2z}{c} - 1\right)^2 - 1} \tag{29}$$

with the square root such that $|\varphi(z)| > 1$ for $z \notin E$: φ maps $\mathbb{C} \setminus E$ on the exterior of the unit disk, with $\varphi(\infty) = \infty$.

Remark that $\varphi(z) + \dfrac{1}{\varphi(z)} = \dfrac{4z}{c} - 2$.

We now build $\varphi(z,t)$, with $\varphi(t,t) = \infty$:

$$\varphi(z,t) = \frac{\varphi(z)\overline{\varphi(t)} - 1}{\varphi(z) - \varphi(t)}, \qquad t \notin E, \tag{30}$$

and reconsider a formula for $\mathcal{V}$:

$$\mathcal{V}(z) := \int_F \log \varphi(z,t)\, d\mu(t), \tag{31}$$

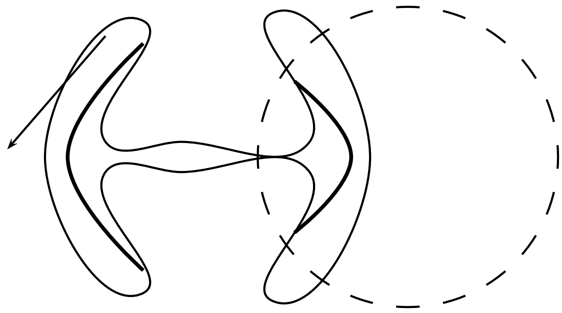
which automatically satisfies (24), with $\rho = 0$, as Re$\log\varphi(z,t) = \log|\varphi(z,t)| = 0$ when $z \in E$.

As $\dfrac{d}{dz}\log\varphi(z,t) = \dfrac{\varphi'(z)}{\varphi(z) - 1/\overline{\varphi(t)}} - \dfrac{\varphi'(z)}{\varphi(z) - \varphi(t)}$,

$$\frac{d\mathcal{V}(z)}{d\varphi(z)} = \int_F \frac{d\mu(t)}{\varphi(z) - 1/\overline{\varphi(t)}} - \int_F \frac{d\mu(t)}{\varphi(z) - \varphi(t)} \tag{32}$$

corresponding to charges and their images spread on $\varphi(F)$ and $1/\overline{\varphi(F)}$ in the $\varphi-$plane.

[1] used by Gonchar in several works ...



?? As an argument of validity of the form (31), let us show how to recover, at least partially, the derivative of (31):

$\mathcal{V}'(z) = -\int_{E\cup F} \frac{d\mu(t)}{z-t}$ is analytic in a neighbourhood of ∞, and can be written for large z as $\mathcal{V}'(z) = \alpha z^{-1} + \beta z^{-2} + \cdots$ (actually, $\alpha = 0$), with the contour integrals $\alpha = (2\pi i)^{-1}\int_C \mathcal{V}'(t)\,dt$, $\beta = (2\pi i)^{-1}\int_C t\mathcal{V}'(t)\,dt$, ... so, $\mathcal{V}'(z) = \frac{1}{2\pi i}\int_C \frac{\mathcal{V}'(t)}{z-t}\,dt$ on a large contour C containing the singular loci E and F, for z outside C. We may as well consider the Laurent series in powers of $\varphi(z)$

$$\mathcal{V}'(z) = \mathcal{V}'\left(c\frac{\varphi+1/\varphi+2}{4}\right) = \frac{1}{2\pi i}\int_D \frac{\mathcal{V}'((c/4)(u+1/u+2)}{\varphi-u}\,du\,,$$

and we make the contour D shrink about the singular loci $\varphi(F)$ and $1/\varphi(F)$

The contributions about $\varphi(F)$ and $1/\varphi(F)$ sum as

$$\mathcal{V}'((c/4)(\varphi+1/\varphi+2)) = \frac{1}{2\pi i}\int_{\varphi(F)} \mathcal{V}'\left[\frac{1}{\varphi-u} - \frac{1/u^2}{\varphi-1/u}\right] du$$

As remarked by J. Nuttall (22 Oct. 1999), we may adapt Gonchar and Rakhmanov [9], and find that all these requirements are met by an expression

$$\mathcal{V}''(z) = \frac{\text{constant} + \text{constant } z}{\sqrt{z^3(z-c)^3(z-a)(z-b)}}$$

where a and $b = \overline{a}$ are the (still unknown) endpoints of F. One must have $\mathcal{V}'(a) = \mathcal{V}'(b) = -1/2$

$$\begin{aligned} \mathcal{V}'(z) &= \int_\infty^z \frac{(C+Dt)dt}{\sqrt{t^3(t-c)^3(t-a)(t-b)}}\,, \\ \mathcal{V}(z) &= \text{constant} + \int_\infty^z \frac{(z-t)(C+Dt)dt}{\sqrt{t^3(t-c)^3(t-a)(t-b)}} \end{aligned} \tag{33}$$

are elliptic integrals of first and second kind.

The standard forms of the elliptic integrals of 1$^{\text{st}}$ and 2$^{\text{nd}}$ kinds are [20]

$$F(\varphi\backslash\alpha) = \int_0^\varphi (1-\sin^2\alpha\sin^2\theta)^{-1/2}d\theta = \int_0^{x=\sin\varphi}[(1-u^2)(1-k^2u^2)]^{-1/2}du,$$

$$E(\varphi\backslash\alpha) = \int_0^\varphi (1-\sin^2\alpha\sin^2\theta)^{1/2}d\theta = \int_0^{x=\sin\varphi}(1-u^2)^{-1/2}(1-k^2u^2)^{1/2}du.$$

Elementary change of variable will not easily lead to these forms, but what is closest to our needs appears to be [20, 17.4.51]

$$\begin{aligned} F(\varphi\backslash\alpha) &= (A^2+B^2)^{1/2}\int_0^x \frac{dv}{[(v^2+A^2)(B^2-v^2)]^{1/2}}, \\ E(\varphi\backslash\alpha) &= A^2(A^2+B^2)^{1/2}\int_0^x \frac{1}{v^2+A^2}\,\frac{dv}{[(v^2+A^2)(B^2-v^2)]^{1/2}}, \end{aligned} \tag{34}$$

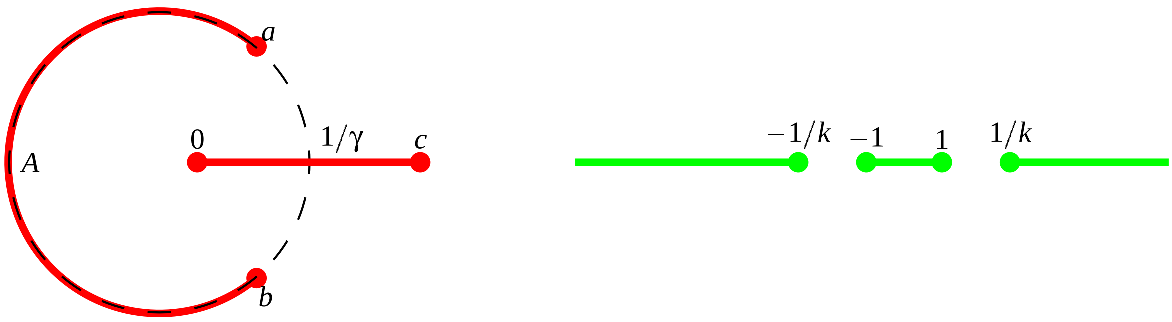
where $\tan\alpha = B/A$, $k^2 = B^2/(A^2+B^2)$, $\sin^2\varphi = \dfrac{x^2(A^2+B^2)}{B^2(A^2+x^2)}$. One may check that $dF/d\sin\varphi$ and $dE/d\sin\varphi$ are what they should be (using $\cos^2\varphi = \dfrac{A^2(B^2-x^2)}{B^2(A^2+x^2)}$ and $1-k^2\sin^2\varphi = \dfrac{A^2}{A^2+x^2}$). We will also need

$$\begin{aligned} \int_0^x \frac{dv}{(v\pm iA)\sqrt{(v^2+A^2)(B^2-v^2)}} &= \int_0^x \frac{(v\mp iA)\,dv}{(v^2+A^2)\sqrt{(v^2+A^2)(B^2-v^2)}} = \\ & -\frac{1}{A^2+B^2}\sqrt{\frac{B^2-x^2}{x^2+A^2}} + \frac{B}{A(A^2+B^2)} \mp \frac{iE(\varphi\backslash\alpha)}{A\sqrt{A^2+B^2}} \end{aligned} \tag{35}$$

In order to translate the conditions on $\mathcal{V}'(a)$ and $\mathcal{V}'(b)$ as complete elliptic integrals ($\varphi = \pi/2$, i.e., $x = \pm B$), we have to map $t = a$ and $t = b = \overline{a}$ in (33) on $v = \pm B$ in (34), and $t = 0$, $t = c$ on $v = \pm iA$. If c is twice the common real part of a and b, the very elementary $t = c/2 + iv$ does the trick, with $B = \operatorname{Im} a$ and $A = c/2$. Let us try

$$t = \frac{A+iv}{1+i\gamma v}.$$

So, $v = iA$ is mapped on $t = 0$, one must have, for $v = -iA$, $\dfrac{2A}{1+\gamma A} = c$, and $\dfrac{A\pm iB}{1\pm i\gamma B} = a, b$. As neither a nor b is known, we may as well take A and B, keeping in mind that $\gamma = \dfrac{2}{c} - \dfrac{1}{A}$ (for given A, a and b are on a circle of diametral points A and $1/\gamma$).



The $z-$ plane and the $\sin\varphi-$plane.

We need integrals

$$\int_\infty^z R(t)\frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}},$$

where R is a rational function with at most simple poles at 0 and/or c. One finds

$$\sqrt{(1+\gamma^2B^2)\frac{1+\gamma A}{1-\gamma A}}\int_{i/\gamma}^{v(z)} R\left(\frac{A+iv}{1+i\gamma v}\right)\frac{dv}{\sqrt{(A^2+v^2)(B^2-v^2)}},$$

where $x = v(z) = i\dfrac{A-z}{1-\gamma z}$.

So, the constant and the simple fractions $1/t$ and $1/(t-c)$ lead to the (indefinite) integrals

$$R(t)=1:\sqrt{\frac{(1+\gamma^2B^2)(1+\gamma A)}{(1-\gamma A)(A^2+B^2)}}F(\varphi\backslash\alpha),$$

$$R(t)=\frac{1}{t}:\sqrt{\frac{(1+\gamma^2B^2)(1+\gamma A)}{(1-\gamma A)(A^2+B^2)}}\left\{\gamma F(\varphi\backslash\alpha)+i\frac{1-A\gamma}{\sqrt{A^2+B^2}}\sqrt{\frac{B^2-x^2}{x^2+A^2}}+(1-A\gamma)E(\varphi\backslash\alpha)/A\right\},$$

$$R(t)=\frac{1}{t-c}:\sqrt{\frac{(1+\gamma^2B^2)(1+\gamma A)^3}{(1-\gamma A)^3(A^2+B^2)}}\left\{\gamma F(\varphi\backslash\alpha)+i\frac{1+A\gamma}{\sqrt{A^2+B^2}}\sqrt{\frac{B^2-x^2}{x^2+A^2}}-(1+A\gamma)E(\varphi\backslash\alpha)/A\right\},$$

For $\mathcal{V}''(z), R(t)=\dfrac{C+Dt}{t(t-c)}=\dfrac{-C/c}{t}+\dfrac{C/c+D}{t-c}$,

$$\begin{aligned}\mathcal{V}''(z)=\text{constant }&+\sqrt{\frac{(1+\gamma^2B^2)(1+\gamma A)}{(1-\gamma A)^3(A^2+B^2)}}\\ &\left\{\gamma\left[-\frac{C}{c}(1-\gamma A)+\left(\frac{C}{c}+D\right)(1+\gamma A)\right]F(\varphi\backslash\alpha)\right.\\ &+\frac{i}{\sqrt{A^2+B^2}}\left[-\frac{C}{c}(1-\gamma A)^2+\left(\frac{C}{c}+D\right)(1+\gamma A)^2\right]\sqrt{\frac{B^2-x^2}{x^2+A^2}}\\ &\left.-\left[\frac{C}{c}(1-\gamma A)^2+\left(\frac{C}{c}+D\right)(1+\gamma A)^2\right]E(\varphi\backslash\alpha)/A\right\}.\end{aligned}\tag{36}$$

The function $\mathcal{V}''$ must be single-valued in $\mathbb{C}\setminus\{E\bigcup F\}$, i.e., have vanishing periods about the sets E and F.

$$\gamma\left[-\frac{C}{c}(1-\gamma A)+\left(\frac{C}{c}+D\right)(1+\gamma A)\right]\mathsf{K}-\left[\frac{C}{c}(1-\gamma A)^2+\left(\frac{C}{c}+D\right)(1+\gamma A)^2\right]\mathsf{E}/A=0,\tag{37}$$

or

$$[\gamma(1-\gamma A)\mathsf{K}+(1-\gamma A)^2\mathsf{E}/A](-C/c)+[\gamma(1+\gamma A)\mathsf{K}-(1+\gamma A)^2\mathsf{E}/A](C/c+D)=0$$

Remark that everything but the constant vanishes in (36) at $x=\pm B$, i.e., at $z=a$ and $z=b$, so that this constant is $\mathcal{V}''(a)=\mathcal{V}''(b)=-1/2$.

With $X=-C(1-\gamma A)/c$ and $Y=(C/c+D)(1+\gamma A)$,

$$\begin{aligned}\mathcal{V}''(z)=-\frac{1}{2}+&\sqrt{\frac{(1+\gamma^2B^2)(1+\gamma A)}{(1-\gamma A)^3(A^2+B^2)}}\\ &\left\{\gamma(X+Y)F(\varphi\backslash\alpha)+ik[(1-\gamma A)X+(1+\gamma A)Y]\frac{\cos\varphi}{A}+[(1-\gamma A)X-(1+\gamma A)Y]\frac{E(\varphi\backslash\alpha)}{A}\right\},\end{aligned}\tag{38}$$

$$\begin{aligned}\mathcal{V}(z)=\text{constant }&+\sqrt{\frac{(1+\gamma^2B^2)(1+\gamma A)}{(1-\gamma A)^3(A^2+B^2)}}\\ &\left\{[-D(1-\gamma A)+\gamma(zX+(z-c)Y)]F(\varphi\backslash\alpha)+ik[(1-\gamma A)zX+(1+\gamma A)(z-c)Y]\frac{\cos\varphi}{A}\right.\\ &\left.+[(1-\gamma A)zX-(1+\gamma A)(z-c)Y]\frac{E(\varphi\backslash\alpha)}{A}\right\}.\end{aligned}\tag{39}$$

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