On the construction of a basis of $L^2_\omega(\mathbb{R})$ formed by pole-free rational functions.

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Abstract

In this article it is presented a general method to construct an orthonormal basis of the space $L^2_\omega(a,b)$, where ω represents a continuous, nonnegative and integrable weight function. This method is firstly described through an example: we present an orthonormal basis of $L^2_\omega(\mathbb{R})$, where $\omega(x) = 2\pi^{-1}\left(1+x^2\right)^{-1}$. This basis is formed by pole-free rational functions whose behaviour over $\mathbb{R}$ is very close to that of the basis trigonometric functions over $(-1,1)$. The concrete behaviour of these functions and their main properties are described and some applications are given; also, a connection with the Sturm-Liouville theory is presented. Normally it's very difficult, when not impossible, to calculate the explicit Fourier coefficients with respect to this basis; however, in some cases it's possible to find them by using technics of complex analysis; in order to illustrate this idea some concrete expansions are presented. The last part of the work is devoted to the study of the general method.

1 Introduction.

It's a very well known fact that the functions

$$\left\{\frac{1}{\sqrt{2}}, \cos\pi\theta, \sin\pi\theta, \ldots, \cos n\pi\theta, \sin n\pi\theta, \ldots\right\} \tag{1}$$

form an orthonormal basis of the space $L^2(-1,1)$, if we consider the change of variable given by the expression

$$\theta \in (-1,1) \longmapsto x(\theta) = \tan\pi\frac{\theta}{2} \in \mathbb{R}, \tag{2}$$

we find the also well known equalities

$$\cos\theta(x) = \frac{1-x^2}{1+x^2}, \qquad \sin\theta(x) = \frac{2x}{1+x^2}, \qquad \frac{d\theta}{dx}(x) = \frac{2}{\pi(1+x^2)}.$$

These relations allow us to show immediately that if we define

$$\phi_1(x) := \frac{1-x^2}{1+x^2}, \qquad \psi_1(x) := \frac{2x}{1+x^2}, \qquad \omega(x) := \frac{2}{\pi(1+x^2)}, \tag{3}$$

then we will have

$$\int_{\mathbb{R}} \phi_1(x)\,\omega(x)\,dx = \int_{\mathbb{R}} \psi_1(x)\,\omega(x)\,dx = \int_{\mathbb{R}} \phi_1(x)\,\psi_1(x)\,\omega(x)\,dx = 0,$$

$$\int_{\mathbb{R}} \phi_1^2(x)\,\omega(x)\,dx = \int_{\mathbb{R}} \psi_1^2(x)\,\omega(x)\,dx = 1, \qquad \int_{\mathbb{R}} \omega(x)\,dx = 2;$$

in other words, the set

$$\left\{\frac{1}{\sqrt{2}}, \phi_1, \psi_1\right\} = \left\{\frac{1}{\sqrt{2}}, \frac{1-x^2}{1+x^2}, \frac{2x}{1+x^2}\right\}$$

is an orthonormal system of the Hilbert space $L^2_\omega(\mathbb{R})$, where the inner product is defined by

$$\langle f, g\rangle_\omega := \int_{\mathbb{R}} f(x)\,g(x)\,\omega(x)\,dx.$$

By extending this idea it's easy to show that if we define, for $n \geq 2$,

$$\begin{cases} \phi_n(x) := \cos n\pi\theta(x) = \cos n\pi\left(\dfrac{2}{\pi}\arctan x\right) \\ \\ \psi_n(x) := \sin n\pi\theta(x) = \sin n\pi\left(\dfrac{2}{\pi}\arctan x\right), \end{cases}$$

then the sequence

$$\left\{\frac{1}{\sqrt{2}},\phi_1,\psi_1,\ldots,\phi_n,\psi_n,\ldots\right\} \tag{4}$$

is an orthonormal basis of $L^2_\omega(\mathbb{R})$; in fact it's just the usual trigonometric basis of $L^2(-1,1)$ transformed by the diffeomorphism defined in (2).

In this work this basis is presented and studied, and explicit expressions for these functions are found. The article is organized as follows: in section two we find these explicit expressions by generalizing a fundamental trigonometric identity; these expressions allow us to describe some basic properties of these functions and some of their applications, a connection with the Sturm-Liouville theory is also presented. Section three is devoted to the construction of some Fourier series with respect to this basis, including those of the trigonometric functions $\sin x$ and $\cos x$ and also that of Heaviside's function; a simple proof of the density of rational functions in $L^2(\mathbb{R})$ is also presented in this section. Finally, a more general version of this theorem is given in section four, which is devoted to generalize the previous ideas; this generalization allows us to construct a kind of trigonometric series which might be called, with the biggest caution, "weighted trigonometric series".

2 The basis of $L^2_\omega(\mathbb{R})$.

2.1 Construction via a recursion formula.

We begin by considering the orthonormal system $\left\{\frac{1}{\sqrt{2}},\phi_1,\psi_1\right\}$ described in the introduction; we saw there that ϕ_1 "plays the role" of the function $\cos\theta$ and ψ_1 "plays the role" of the function $\sin\theta$ in the usual trigonometric basis of $L^2(-1,1)$; roughly speaking, what we are going to do is to "extend" this orthonormal system to a basis by finding the corresponding functions which will "play the role" of the trigonometric functions $\cos 2\theta$, $\sin 2\theta$, ... etc. In order to achieve this, and taking into account the identities

$$\begin{cases} \cos(n+1)\pi\theta = \cos n\pi\theta\cos\pi\theta - \sin n\pi\theta\sin\pi\theta, \\ \\ \sin(n+1)\pi\theta = \sin n\pi\theta\cos\pi\theta + \cos n\pi\theta\sin\pi\theta, \end{cases}$$

it seems natural to define, for $n \geq 1$ and $x \in \mathbb{R}$:

$$\begin{cases} \phi_{n+1}(x) := \phi_n(x)\,\phi_1(x) - \psi_n(x)\,\psi_1(x) \\ \psi_{n+1}(x) := \psi_n(x)\,\phi_1(x) + \phi_n(x)\,\psi_1(x), \end{cases} \tag{5}$$

with

$$\phi_1(x) = \frac{1-x^2}{1+x^2}, \qquad \psi_1(x) = \frac{2x}{1+x^2}.$$

According to our previous discussion, this sequence completes the initial orthonormal system to an orthonormal basis of $L^2_\omega(\mathbb{R})$: since (1) is an orthonormal basis of $L^2(-1,1)$, this is just an immediate consequence of the equalities

$$\phi_n(x) = \cos n\pi\theta(x), \qquad \psi_n(x) = \sin n\pi\theta(x), \qquad n \geq 1,$$

and

$$d\theta(x) = \omega(x)\,dx,$$

all of them being clear.

Relations (5) give us a first idea of how to find the functions of the basis of $L^2_\omega(\mathbb{R})$: we have, for instance,

$$\phi_2(x) = \phi_1^2(x) - \psi_1^2(x)$$

$$= \left(\frac{1-x^2}{1+x^2}\right)^2 - \left(\frac{2x}{1+x^2}\right)^2 = \frac{1-6x^2+x^4}{(1+x^2)^2},$$

and

$$\psi_2(x) = 2\phi_1(x)\,\psi_1(x)$$

$$= 2\cdot\frac{1-x^2}{1+x^2}\cdot\frac{2x}{1+x^2} = \frac{4x-4x^3}{(1+x^2)^2}.$$

The necessary calculations to construct ϕ_n and ψ_n become much more complicated as n grows; direct application of this idea becomes impractical and an alternative way to find the basis functions is then necessary. In the next sections we give closed expressions for the basis functions which are based on the generating functions.

2.2 A fundamental equality.

This equality will allow us to construct the generating functions, which will be a key for finding easy expressions for the basis functions. It comes directly from (5), and can be stated as follows: if we define, for $n \geq 1$ and $x \in \mathbb{R}$,

$$\gamma_n(x) := \phi_n(x) + i\psi_n(x),$$

then we have

$$\gamma_n = \gamma_1^n,$$

in other words,

$$\phi_n + i\psi_n = (\phi_1 + i\psi_1)^n. \tag{6}$$

The proof goes as follows: the property being true for $n = 1$, let's suppose that it's true up to the index n; we have, by (5),

$$\begin{aligned}\phi_{n+1} + i\psi_{n+1} &= \phi_n\phi_1 - \psi_n\psi_1 + i(\psi_n\phi_1 + \phi_n\psi_1) \\ &= (\phi_n + i\psi_n)(\phi_1 + i\psi_1) = (\phi_1 + i\psi_1)^n (\phi_1 + i\psi_1) \\ &= (\phi_1 + i\psi_1)^{n+1}.\end{aligned}$$

2.3 Extension of the sequence of basis functions to $\mathbb{Z}$.

By considering again the equalities

$$\phi_n(x) = \cos n\pi\theta(x), \qquad \psi_n(x) = \sin n\pi\theta(x),$$

one finds

$$\phi_n^2 + \psi_n^2 = 1,$$

and this yields

$$(\phi_n + i\psi_n)^{-1} = \overline{\phi_n + i\psi_n} = \phi_n - i\psi_n.$$

This simple relation, together with (6), will allow us to define functions ϕ_k and ψ_k not only for the positive integers but for all the integers $k \in \mathbb{Z}$: let's consider a natural number $n \geq 1$ and let's suppose that (6) is to be valid also for the negative integers, we will have

$$\begin{aligned}\phi_{-n} + i\psi_{-n} &= (\phi_1 + i\psi_1)^{-n} \\ &= \left((\phi_1 + i\psi_1)^{-1}\right)^n = \left(\overline{\phi_1 + i\psi_1}\right)^n\end{aligned}$$

$$\overline{(\phi_1 + i\psi_1)^n} = \overline{(\phi_n + i\psi_n)}$$
$$= \phi_n - i\psi_n;$$

this forces, for $n \geq 1$, the definition

$$\begin{aligned} \phi_{-n}(x) &:= \phi_n(x), \quad x \in \mathbb{R}, \\ \psi_{-n}(x) &:= -\psi_n(x), \quad x \in \mathbb{R}. \end{aligned}$$

According to the last equality we are obliged to define

$$\psi_0(x) := 0, \quad x \in \mathbb{R};$$

taking this into account, and if the relation

$$\phi_0(x) + i\psi_0(x) = [\phi_1(x) + i\psi_1(x)]^0 = 1$$

is also to be valid, we must define

$$\phi_0(x) = 1, \quad x \in \mathbb{R};$$

also, with these definitions it turns out that

$$\gamma_{-n}(x) = \phi_{-n}(x) + i\psi_{-n}(x) = \phi_n(x) - i\psi_n(x).$$

This is the unique coherent extension of the sequence of basis functions to $\mathbb{Z}$ which ensures that (6) is preserved. Of course, we find an obvious parallelism with the relations

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta, \quad \cos 0 = 1, \quad \sin 0 = 0,$$

and

$$\exp(-i\theta) = \cos\theta - i\sin\theta.$$

More similarities with the trigonometric system are now found: according to this extension, the equalities

$$\phi_{p+q} = \phi_p\phi_q - \psi_p\psi_q, \qquad \psi_{p+q} = \psi_p\phi_q + \phi_p\psi_q,$$

make now sense for any $p, q \in \mathbb{Z}$, and it follows that

$$\phi_p\phi_q = \frac{1}{2}\left(\phi_{p-q} + \phi_{p+q}\right), \quad \psi_p\psi_q = \frac{1}{2}\left(\phi_{p-q} - \phi_{p+q}\right), \quad \psi_p\phi_q = \frac{1}{2}\left(\psi_{p+q} + \psi_{p-q}\right).$$

Also, taking into account (6) and the equalities

$$\phi_1' = -\pi\omega\psi_1, \qquad \psi_1' = \pi\omega\phi_1,$$

which come directly from (3), one finds

$$\gamma_n' = in\pi\omega\gamma_n,$$

in other words,

$$\phi_n' = -n\pi\omega\psi_n, \qquad \psi_n' = n\pi\omega\phi_n.$$

2.4 Construction of the generating functions and applications.

We begin by stating the following property, which is an immediate consequence of (5): given a natural number $n > 0$, a polynomial p_n of degree $2n$ and a polynomial q_n of degree $2n-1$ exist in such a way that

$$\phi_n(x) = \frac{p_n(x)}{(1+x^2)^n}, \qquad \psi_n(x) = \frac{q_n(x)}{(1+x^2)^n}. \tag{7}$$

The problem of finding ϕ_n and ψ_n is then equivalent to that of finding p_n and q_n. On the other hand, equation (6) allows us to construct immediately the generating functions of ϕ_n and ψ_n; if we define formally, for $x, r \in \mathbb{R}$,

$$\begin{aligned} \Phi(x,r) \ &:= \sum_{k=0}^{\infty} \frac{\phi_k(x)}{k!} r^k, \\ \Psi(x,r) \ &:= \sum_{k=0}^{\infty} \frac{\psi_k(x)}{k!} r^k, \end{aligned}$$

then we have

$$\Phi(x,r) + i\Psi(x,r) = \sum_{k=0}^{\infty} \frac{\phi_k(x) + i\psi_k(x)}{k!} r^k$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{\gamma_k(x)}{k!} r^k = \sum_{k=0}^{\infty} \frac{\gamma_1^k(x)}{k!} r^k = \exp(\gamma_1(x) r) \\ &= \exp(i\psi_1(x) r) \exp(\phi_1(x) r) \end{aligned}$$

$$= \cos\left(\psi_1(x)\, r\right) \exp\left(\phi_1(x)\, r\right) + i \sin\left(\psi_1(x)\, r\right) \exp\left(\phi_1(x)\, r\right),$$

this yields

$$\Phi(x, r) = \cos\left(\psi_1(x)\, r\right) \exp\left(\phi_1(x)\, r\right)$$
$$\Psi(x, z) = \sin\left(\psi_1(x)\, r\right) \exp\left(\phi_1(x)\, r\right),$$

or, in other words,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\phi_k(x)}{k!} r^k &= \cos\left(\frac{2xr}{1+x^2}\right) \exp\left(r \cdot \frac{1-x^2}{1+x^2}\right), \\ \sum_{k=0}^{\infty} \frac{\psi_k(x)}{k!} r^k &= \sin\left(\frac{2xr}{1+x^2}\right) \exp\left(r \cdot \frac{1-x^2}{1+x^2}\right). \end{aligned}$$

Taking into account the equalities

$$\phi_k(x) = \cos k\theta(x), \qquad \psi_k(x) = \sin k\theta(x),$$

one finds

$$|\gamma_k|^2 = |\phi_k + i\psi_k|^2 = \phi_k^2 + \psi_k^2 = 1,$$

and this proves convergence everywhere of the series defined above.

Now, by differentiating the preceding expressions with respect to r and evaluating at $r = 0$ one finds

$$\begin{aligned} \phi_k(x) &= \sum_{r=0}^{\left[\frac{k}{2}\right]} \binom{k}{2r} (-1)^r \left(\frac{2x}{1+x^2}\right)^{2r} \left(\frac{1-x^2}{1+x^2}\right)^{k-2r} \\ \psi_k(x) &= \sum_{r=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2r+1} (-1)^r \left(\frac{2x}{1+x^2}\right)^{2r+1} \left(\frac{1-x^2}{1+x^2}\right)^{k-2r-1}, \end{aligned}$$

where $[x]$ stands for the largest integer smaller or equal to x; these expressions can also be written as

$$\begin{aligned} \phi_k(x) &= \left(1+x^2\right)^{-k} \cdot \sum_{r=0}^{\left[\frac{k}{2}\right]} \binom{k}{2r} (-1)^r (2x)^{2r} (1-x)^{2k-2r}, \\ \psi_k(x) &= \left(1+x^2\right)^{-k} \cdot \sum_{r=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2r+1} (-1)^r (2x)^{2r+1} \left(1-x^2\right)^{k-2r-1}, \end{aligned}$$

since these formulae are not recursive, they can be shown as a first improvement of (5). However, a much easier expression can be found as follows: from the last equalities, and taking into account (7) we deduce

$$
\begin{aligned}
p_k(x) &= \sum_{r=0}^{\left[\frac{k}{2}\right]} \binom{k}{2r} (-1)^r (2x)^{2r} \left(1-x^2\right)^{k-2r}, \\
q_k(x) &= \sum_{r=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2r+1} (-1)^r (2x)^{2r+1} \left(1-x^2\right)^{k-2r-1},
\end{aligned}
$$

which can be also written as

$$
\begin{aligned}
p_k(x) + iq_k(x) &= \sum_{r=0}^{\left[\frac{k}{2}\right]} \binom{k}{2r} (2ix)^{2r} \left(1-x^2\right)^{k-2r} \\
&+ \sum_{r=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2r+1} (2ix)^{2r+1} \left(1-x^2\right)^{k-2r-1} \\
&= \sum_{r=0}^{k} \binom{k}{r} (2ix)^r \left(1-x^2\right)^{k-r} \\
&= \left(1-x^2+2ix\right)^k = (1+ix)^{2k};
\end{aligned}
$$

binomial formula applied here yields

$$
\begin{aligned}
(1+ix)^{2k} &= \sum_{r=0}^{2k} \binom{2k}{r} i^r x^r \\
&= \sum_{r=0}^{k} \binom{2k}{2r} i^{2r} x^{2r} + \sum_{r=0}^{k-1} \binom{2k}{2r+1} i^{2r+1} x^{2r+1} \\
&= \sum_{r=0}^{k} \binom{2k}{2r} (-1)^r x^{2r} + i \sum_{r=0}^{k-1} \binom{2k}{2r+1} (-1)^r x^{2r+1},
\end{aligned}
$$

and we can then conclude

$$p_k(x) = \sum_{r=0}^{k} \binom{2k}{2r} (-1)^r x^{2r},$$
$$q_k(x) = \sum_{r=0}^{k-1} \binom{2k}{2r+1} (-1)^r x^{2r+1}. \tag{8}$$

These are the easiest expressions we have found for the polynomials p_k and q_k. Of course, we have the corresponding expressions for ϕ_k and ψ_k:

$$\phi_k(x) = (1+x^2)^{-k} \cdot \sum_{r=0}^{k} \binom{2k}{2r} (-1)^r x^{2r},$$

$$\psi_k(x) = (1+x^2)^{-k} \cdot \sum_{r=0}^{k-1} \binom{2k}{2r+1} (-1)^r x^{2r+1}.$$

In order to illustrate the behaviour of the basis functions we give the graphic representation of ϕ_3 and ψ_3:

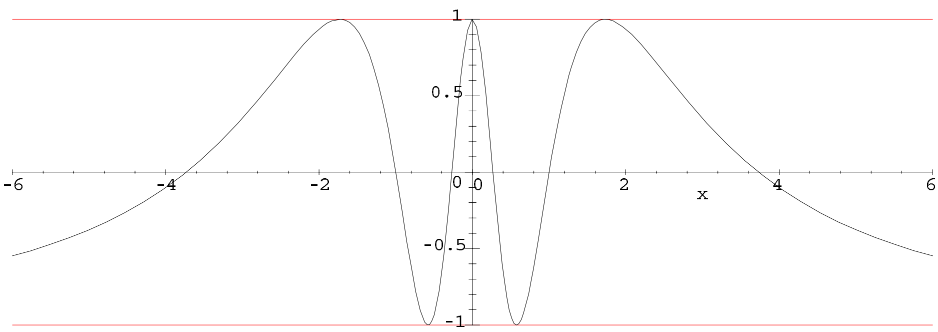

Fig. 1: The function ϕ_3.

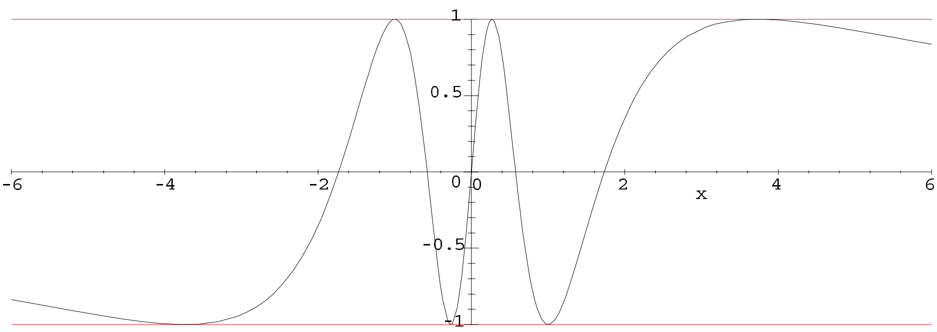

Fig. 2: The function ψ_3.

2.5 Extension of the basis functions to $\mathbb{C}$.

If we define $\phi_n(z)$, $\psi_n(z)$ and $\gamma_n(z)$ to be the respective extensions of $\phi_n(x)$, $\psi_n(x)$ and $\gamma_n(x)$ to the whole complex plane $\mathbb{C}$, we immediately find

$$\gamma_n(z) = \phi_n(z) + i\psi_n(z) = \left(\frac{i-z}{i+z}\right)^n;$$

in other words, γ_n is the n-th power of the unique Möbius transformation f satisfying

$$f(\infty) = -1, \qquad f(1) = i, \qquad f(0) = 1;$$

it's known that such a transformation applies the upper half plane onto the unit disk $D(0,1)$ and the real line into the edge of this disk. It's also an easy matter to show that

$$\phi_n(z) - i\psi_n(z) = \left(\frac{i+z}{i-z}\right)^n,$$

and this yields

$$\frac{\phi_n(z) - i\psi_n(z)}{1+z^2} = (-1)^n \frac{(z+i)^{n-1}}{(z-i)^{n+1}};$$

this equality is going to be useful later, for the computation of the Fourier coefficients of the functions of $L^2_\omega(\mathbb{R})$ with respect to the basis we have just found.

2.6 An application to differential equations.

Let's choose an integer $n \in \mathbb{Z} - \{0\}$; it's known that the unique solution of the Cauchy problem

$$\begin{cases} u''(\theta) + n^2\pi^2 u(\theta) = 0, & \theta \in \mathbb{R}, \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases} \tag{9}$$

is given by

$$u(\theta) = u_0 \cos n\pi\theta + \frac{u_0'}{n\pi} \sin n\pi\theta, \quad \theta \in \mathbb{R};$$

if we take now (2) into account we find

$$\frac{dx}{d\theta} = \frac{\pi}{2}\left(1 + \tan^2 \pi\frac{\theta}{2}\right) = \frac{\pi}{2}\left(1 + x^2\right)$$

and

$$\frac{d^2x}{d\theta^2} = \frac{\pi^2}{2} \tan \pi\frac{\theta}{2}\left(1 + \tan^2 \pi\frac{\theta}{2}\right) = \frac{\pi^2}{2} x\left(1 + x^2\right),$$

these relations transform the differential equation in problem (9) into

$$\left(1 + x^2\right)^2 u''(x) + 2x\left(1 + x^2\right) u'(x) + 4n^2 u(x) = 0. \tag{10}$$

Now, if we consider the equalities $\phi_n(0) = 1$, $\psi_n(0) = 0$, $\phi_n'(0) = 0$ and $\psi_n'(0) = 2n$, we deduce that the unique solution of (10) satisfying the initial conditions

$$u(0) = u_0, \qquad u'(0) = u_0',$$

is given by the expression

$$u_n(x) = u_0\phi_n(x) + \frac{u_0'}{2n}\psi_n(x), \quad x \in \mathbb{R};$$

according to (7) and (8) this is

$$u_n(x) = \left(1 + x^2\right)^{-n} \cdot \left(\sum_{k=0}^{n} \binom{2n}{2k} (-1)^k u_0 x^{2k} + \sum_{k=0}^{n-1} \binom{2n}{2k+1} (-1)^k \frac{u_0'}{2n} x^{2k+1}\right),$$

solution that can be written in a more compact way as

$$u_n(x) = \left(1 + x^2\right)^{-n} \cdot \sum_{k=0}^{n} \binom{2n}{k} (-1)^k c_k x^k,$$

where

$$c_k = \begin{cases} u_0, & k \quad even, \\ u_0'/2n, & k \quad odd. \end{cases}$$

Equation (10) can be presented in the following alternative way: multiplying (10) by $(1+x^2)^{-1}$ we find

$$\left(1+x^2\right) u''(x) + 2xu'(x) + \frac{4n^2}{(1+x^2)} u(x) = 0,$$

this can also be written as

$$\frac{d}{dx}\left(\left(1+x^2\right)\frac{d}{dx}u(x)\right) + 2\pi n^2 \omega(x)\, u(x) = 0,$$

since $(1+x^2) = \dfrac{\pi}{2\omega(x)}$, a more elegant and compact way to write this equation is

$$\left(\omega^{-1}u'\right)' + \pi^2 n^2 \omega u = 0,$$

and this is just a very particular case of a Sturm-Liouville equation. According to 2.3 we have

$$\phi_n' = -\pi\omega\psi_n, \qquad \psi_n' = \pi\omega\phi_n,$$

it follows that if we define the spaces

$$V = \left\{ f : \mathbb{R} \to \mathbb{R} : \limsup_{|x| \longrightarrow \infty} |f(x)| < \infty, f' = O\left(\frac{1}{x^3}\right) \right\},$$

$$W = \left\{ f : \mathbb{R} \to \mathbb{R} : f = O\left(\frac{1}{x}\right), f' = O\left(\frac{1}{x^2}\right) \right\},$$

then we can regard the functions $\{\phi_n\}_{n=1}^{\infty}$ as the set of orthonormal solutions of the Sturm-Liouville problem

$$\begin{cases} (\omega^{-1}u')' + \lambda_n \omega u = 0, & x \in \mathbb{R} \\ u \in V \end{cases}$$

and the functions $\{\psi_n\}_{n=1}^{\infty}$ as the orthonormal solutions of the analogous problem

$$\begin{cases} (\omega^{-1}u')' + \lambda_n \omega u = 0, & x \in \mathbb{R} \\ u \in W \end{cases}$$

The relations $u \in V$, $u \in W$ contain asimptotic behaviours corresponding to the "boundary conditions", and in both cases $\{\lambda_n = \pi^2 n^2\}_{n=1}^{\infty}$ is the corresponding sequence of eingenvalues.

3 Some examples of Fourier series.

Let's consider a function $f \in L^2_\omega(\mathbb{R})$, according to the previous sections and to the elementary theory of Hilbert spaces, we know that f admits an expansion of the type

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \phi_n + b_n \psi_n\right),$$

where

$$\begin{aligned} a_n &= \langle f, \phi_n \rangle_\omega = \int_{\mathbb{R}} f(x)\, \phi_n(x)\, \omega(x)\, dx, \quad n \geq 0; \\ b_n &= \langle f, \psi_n \rangle_\omega = \int_{\mathbb{R}} f(x)\, \psi_n(x)\, \omega(x)\, dx, \quad n \geq 1. \end{aligned}$$

Some concrete expansions of this kind can be found in a direct way: for instance, the series

$$\theta = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi\theta$$

in $L^2(-1,1)$ gives immediately, via (2),

$$\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \psi_n(x)$$

in $L^2_\omega(\mathbb{R})$; the next figure shows this expansion with five terms:

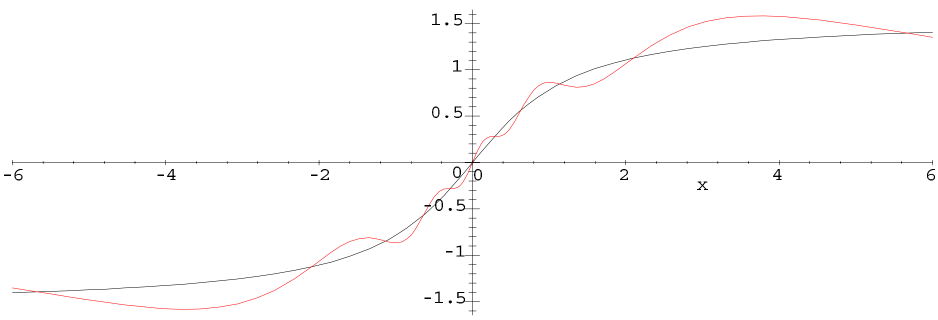

Fig. 3: Approximation with five terms.

A more interesting example can be found by considering the function

$$\widetilde{f}(\theta) = \begin{cases} 0, & -1 \leq \theta < 0 \\ 1, & 0 \leq \theta \leq 1 \end{cases}$$

which has the trigonometric expansion

$$\widetilde{f}(\theta) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(2n+1)\pi\theta \tag{11}$$

in $L^2(-1,1)$, we deduce that if we define f to be the Heaviside's function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

then f will admit the expansion

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \psi_{2n+1}(x)$$

in $L^2_\omega(\mathbb{R})$; the following figure corresponds to this series taken up to the term $n = 2$:

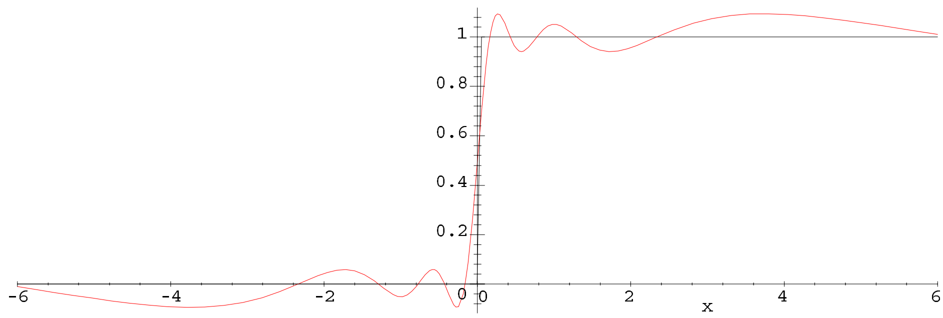

Fig. 4: Approximation up to $n = 2$.

In general, from a Fourier series

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\pi\theta + b_n \sin n\pi\theta\right)$$

in $L^2(-1,1)$ one obtains directly the expansion

$$f\left(\frac{2}{\pi} \arctan x\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \phi_n(x) + b_n \psi_n(x)\right)$$

in $L^2_\omega(\mathbb{R})$.

For arbitrary functions, the Fourier coefficients are normally impossible to be found; however, in certain cases the real integrals can be transformed into complex integrals and these coefficients can be computed by means of the residue theorem. The following series were found by making use of this method.

3.1 Expansion of the function $f(x) = \dfrac{1}{c^2 + x^2}$.

Let's fix a real number $c \neq 0$, $|c| \neq 1$; we need to calculate the coefficients

$$a_n = \int_{\mathbb{R}} f(x)\, \phi_n(x)\, \omega(x)\, dx, \quad n \geq 0;$$

in order to achieve this, we begin by defining $z \mapsto f(z) = \dfrac{1}{c^2+z^2}$ the extension of f to the whole complex plane; we can then write

$$f(z) = \frac{i}{2c} \cdot \frac{1}{z+ic} - \frac{i}{2c} \cdot \frac{1}{z-ic}$$

or, in other words,

$$f(z) = \frac{1}{2c} \sum_{k=0}^{\infty} i^k \left[\frac{1}{(c+1)^{k+1}} + \frac{(-1)^k}{(c-1)^{k+1}} \right] (z-i)^k .$$

On the other hand, according to section 2.5 we have

$$\frac{\phi_n(z) - i\psi_n(z)}{1+z^2} = (-1)^n \frac{(z+i)^{n-1}}{(z-i)^{n+1}},$$

combining this with our previous expression we find

$$Res \left\{ f(z) \cdot \frac{\phi_n(z) - i\psi_n(z)}{1+z^2}; ic \right\} = \frac{i(-1)^n (c+1)^{n-1}}{2c(c-1)^{n+1}}.$$

Now, let n be a natural number, $n \geq 1$, the equality

$$(z+i)^{n-1} = (z-i+2i)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} (2i)^k (z-i)^{n-1-k}$$

yields

$$\frac{\phi_n(z) - i\psi_n(z)}{1+z^2} = (-1)^n \sum_{k=0}^{n-1} \binom{n-1}{k} (2i)^k \frac{1}{(z-i)^{k+2}},$$

taking into account these expansions we find

$$Res \left\{ f(z) \cdot \frac{\phi_n(z) - i\psi_n(z)}{1+z^2}; i \right\}$$

$$= \frac{(-1)^n i}{2c} \sum_{k=0}^{n-1} \binom{n-1}{k} (-2)^k \left(\frac{1}{(c+1)^{k+2}} - \frac{(-1)^k}{(c-1)^{k+2}} \right).$$

Let r be a real number, $r > 1$, and let's define the path $\gamma_r : [0,1] \longrightarrow \mathbb{C}$ by

$$\gamma_r(t) = \begin{cases} r \exp(2\pi i t), & 0 \leq t \leq 1/2 \\ (4t-3) r, & 1/2 \leq t \leq 1 \end{cases} \tag{12}$$

After applying the residue theorem and after simplifications one finds, for $n \geq 1$:

$$\begin{aligned} a_n &= \int_{\mathbb{R}} f(x)\,\phi_n(x)\,\omega(x)\,dx \\ &= \frac{(-1)^{n+1}\,2}{c}\left(\frac{(c+1)^{n-1}}{(c-1)^{n+1}} + \frac{s_n(c)}{(c^2-1)^2}\right), \end{aligned}$$

where we have defined

$$s_n(c) := \sum_{k=0}^{n-1} \binom{n-1}{k} (-2)^k \frac{(1-c)^{k+2} - (1+c)^{k+2}}{(1-c^2)^k}.$$

On the other hand we have

$$\int_{\gamma_r} f(z) \cdot \frac{1}{1+z^2} dz = \int_{\gamma_r} \frac{1}{c^2+z^2} \cdot \frac{1}{1+z^2} dz = \frac{\pi}{c(1+c)},$$

combining this with the equality

$$a_0 = \int_{\mathbb{R}} f(x)\,\omega(x)\,dx = \frac{2}{\pi} \lim_{r\to\infty} \int_{\gamma_r} f(z) \cdot \frac{1}{1+z^2} dz$$

we find

$$a_0 = \frac{2}{c(1+c)};$$

we can then deduce

$$f(x) = \frac{1}{c(1+c)} + \frac{2}{c}\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(c+1)^{n-1}}{(c-1)^{n+1}} + \frac{s_n(c)}{(c^2-1)^2}\right) \phi_n(x),$$

which can also be written as

$$\frac{c}{c^2+x^2} = \frac{1}{1+c} + 2\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(c+1)^{n-1}}{(c-1)^{n+1}} + \frac{s_n(c)}{(c^2-1)^2}\right) \phi_n(x).$$

The following figures show, in red, the expansions of $x \mapsto \frac{5}{25+x^2}$ with five and ten terms respectively:

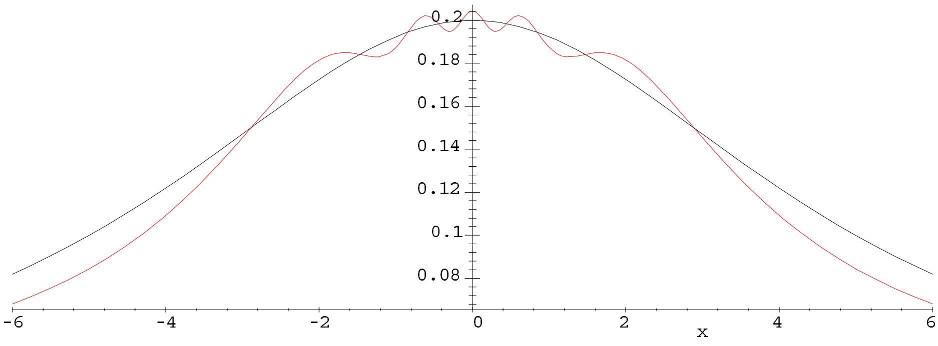

Fig. 5: Approximation with five terms.

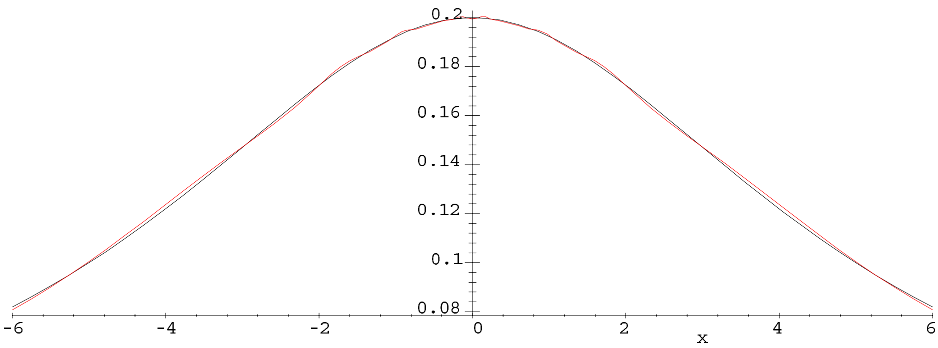

Fig. 10: Approximation with ten terms.

3.2 Expansions of the trigonometric functions $\cos \alpha x$ and $\sin \alpha x$.

Let α be a real number, $\alpha \neq 0$; to find the expansion of the function $\cos \alpha x$ we need to compute

$$a_n = \int_{\mathbb{R}} \cos(\alpha x)\, \phi_n(x)\, \omega(x)\, dx, \quad n \geq 0;$$

in order to achieve this we call $p_n(z)$ the extension of $p_n(x)$ to the whole complex plane $\mathbb{C}$; the expression (8) allows us to form the power series

$$p_n(z) = \sum_{k=0}^{n} \binom{2n}{2k} (-1)^k z^{2k}$$

$$= \sum_{j=0}^{2n} (-i)^j \left[\sum_{k=\left[\frac{j+1}{2}\right]}^{n} \binom{2n}{2k} \binom{2k}{j} \right] (z-i)^j ,$$

this can be written as

$$p_n(z) = \sum_{j=0}^{2n} (-i)^j s_{nj} (z-i)^j ,$$

where

$$s_{nj} := \sum_{k=\left[\frac{j+1}{2}\right]}^{n} \binom{2n}{2k} \binom{2k}{j};$$

on the other hand,

$$\frac{1}{(z+i)^{n+1}} = \frac{1}{(2i)^{n+1}} \sum_{l=0}^{\infty} \binom{n+l}{l} \left(\frac{i}{2}\right)^l (z-i)^l ,$$

after simplifications one finds

$$\frac{p_n(z)}{(z+i)^{n+1}} = \frac{1}{(2i)^{n+1}} \sum_{k=0}^{\infty} i^k \sigma_{nk} (z-i)^k ,$$

where

$$\sigma_{nk} := \sum_{j+l=k} \binom{n+l}{l} \frac{(-1)^j s_{nj}}{2^l},$$

and this yields, finally,

$$\frac{p_n(z)}{(1+z^2)^{n+1}} = \frac{1}{(2i)^{n+1}} \sum_{k=0}^{\infty} \frac{i^k \sigma_{nk}}{(z-i)^{n+1-k}}.$$

We will also take into account the following expansion:

$$\exp(i\alpha z) = \exp(-\alpha) \sum_{k=0}^{\infty} \frac{i^k \alpha^k}{k!} (z-i)^k .$$

Let r be a real number, $r > 1$; if we consider again the path (12) we find

$$a_n = \int_{\mathbb{R}} \cos(\alpha x) \phi_n(x) \omega(x) dx$$

$$= \frac{2}{\pi} \int_{\mathbb{R}} \cos(\alpha x) \frac{p_n(x)}{(1+x^2)^{n+1}} dx + i \frac{2}{\pi} \int_{\mathbb{R}} \sin(\alpha x) \frac{p_n(x)}{(1+x^2)^{n+1}} dx$$

$$= \frac{2}{\pi} \int_{\mathbb{R}} \exp(i\alpha x) \frac{p_n(x)}{(1+x^2)^{n+1}} dx$$

$$= \frac{2}{\pi} \lim_{r\to\infty} \int_{\gamma_r} \exp(i\alpha z) \frac{p_n(z)}{(1+z^2)^{n+1}} dz$$

$$= 4i Res \left\{ \exp(i\alpha z) \frac{p_n(z)}{(1+z^2)^{n+1}} ; i \right\},$$

but, following the preceding expansions,

$$Res \left\{ \exp(i\alpha z) \frac{p_n(z)}{(1+z^2)^{n+1}} ; i \right\}$$

$$= \frac{\exp(-\alpha)}{(2i)^{n+1}} \sum_{k=0}^{n} i^k \sigma_{nk} \frac{i^{n-k} \alpha^{n-k}}{(n-k)!}$$

$$= \frac{-i\alpha^n \exp(-\alpha)}{2^{n+1}} \sum_{k=0}^{n} \frac{\sigma_{nk}}{\alpha^k (n-k)!},$$

this yields

$$a_n = \frac{\alpha^n \exp(-\alpha)}{2^{n-1}} \sum_{k=0}^{n} \frac{\sigma_{nk}}{\alpha^k (n-k)!}$$

and then we can deduce, finally,

$$\cos(\alpha x) = e^{-\alpha} + \alpha e^{-\alpha} \sum_{n=1}^{\infty} \left[\left(\frac{\alpha}{2}\right)^{n-1} \sum_{k=0}^{n} \frac{\sigma_{nk}}{\alpha^k (n-k)!} \right] \phi_n(x) .$$

As a particular case we find the formula

$$\cos x = \frac{1}{e} + \frac{1}{e} \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} \sum_{k=0}^{n} \frac{\sigma_{nk}}{(n-k)!} \right) \phi_n (x) \, .$$

The following figures show this expansion with ten and twenty terms respectively:

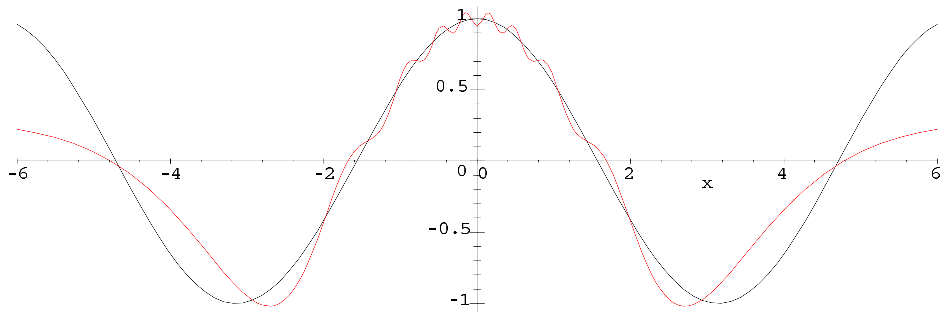

Fig. 6: Approximation with ten terms.

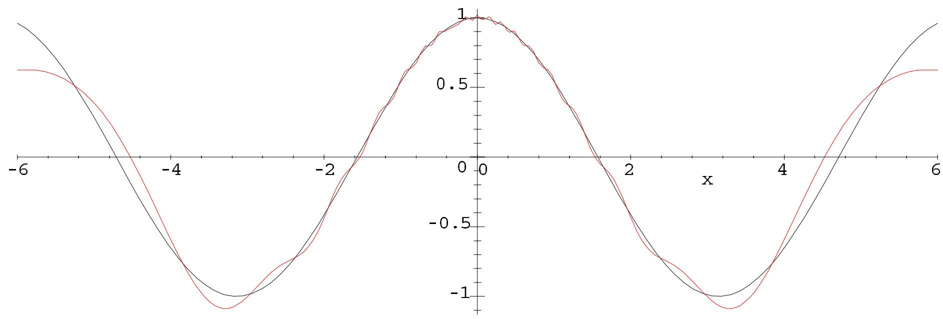

Fig. 7: Approximation with twenty terms.

The expansion of the function $\sin \alpha x$ can be found exactly in the same way: we have to compute

$$b_n = \int_{\mathbb{R}} \sin(\alpha x) \psi_n(x) \omega(x) \, dx, \quad n \geq 1;$$

now we have

$$q_n(z) = i \sum_{j=0}^{2n-1} t_{nj} (-i)^j (z-i)^j ,$$

where

$$t_{nj} := \sum_{k=\left[\frac{j}{2}\right]}^{n-1} \binom{2n}{2k+1} \binom{2k+1}{j},$$

this yields

$$\frac{q_n(z)}{(1+z^2)^{n+1}} = \frac{i}{(2i)^{n+1}} \sum_{k=0}^{\infty} \frac{i^k \tau_{nk}}{(z-i)^{n+1-k}},$$

with

$$\tau_{nk} := \sum_{j+l=k} \binom{n+l}{l} \frac{(-1)^j t_{nj}}{2^{k-j}}.$$

By applying the same arguments as before one finds

$$\begin{aligned} b_n &= \int_{\mathbb{R}} \sin(\alpha x) \psi_n(x) \omega(x) \, dx \\ &= \frac{-2i}{\pi} \int_{\mathbb{R}} \exp(i\alpha x) \frac{q_n(x)}{(1+x^2)^{n+1}} dx \\ &= \frac{-2i}{\pi} \lim_{r\to\infty} \int_{\gamma_r} \exp(i\alpha z) \frac{q_n(z)}{(1+z^2)^{n+1}} dz \\ &= 4 Res \left\{ \exp(i\alpha z) \frac{q_n(z)}{(1+z^2)^{n+1}}; i \right\}; \end{aligned}$$

in this case the residue is found to be

$$Res \left\{ \exp(i\alpha z) \frac{q_n(z)}{(1+z^2)^{n+1}}; i \right\} = \frac{\alpha^n \exp(-\alpha)}{2^{n+1}} \sum_{k=0}^{n} \frac{\tau_{nk}}{\alpha^k (n-k)!},$$

from this we deduce

$$b_n = \frac{\alpha^n \exp(-\alpha)}{2^{n-1}} \sum_{k=0}^{n} \frac{\tau_{nk}}{\alpha^k (n-k)!},$$

and from here we can finally conclude

$$\sin(\alpha x) = \alpha \exp(-\alpha) \sum_{n=1}^{\infty} \left[\left(\frac{\alpha}{2}\right)^{n-1} \sum_{k=0}^{n} \frac{\tau_{nk}}{\alpha^k (n-k)!} \right] \psi_n(x).$$

As a particular case we have the corresponding formula for the function $\sin x$:

$$\sin x = \frac{1}{e} \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} \sum_{k=0}^{n} \frac{\tau_{nk}}{(n-k)!} \right) \psi_n(x).$$

The following figure shows this expansion with twenty terms:

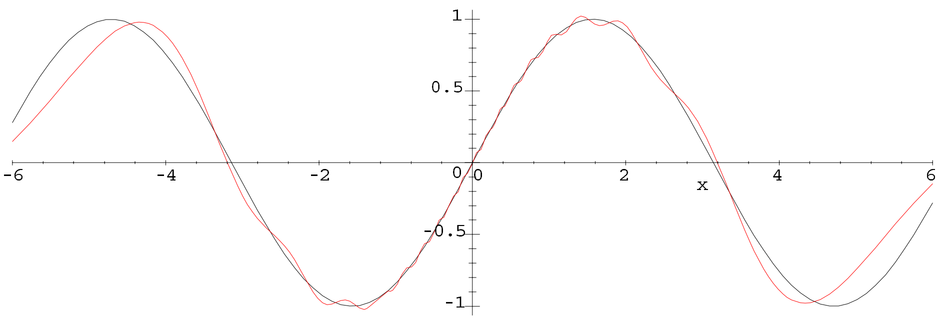

Fig. 8: Approximation with twenty terms.

3.3 A density theorem

In this section we present a proof of the density of rational functions in $L^2(\mathbb{R})$; we begin by a simple result:

Proposition 1 *Let $f \in L^2_\omega(\mathbb{R})$; for each real number $\varepsilon > 0$, it exists $n \equiv n(\varepsilon) \in \mathbb{N}$ and a polynomial $r_m(x)$ of degree $m \leq 2n$ such that*

$$\int_{\mathbb{R}} \left| f(x) - \frac{r_m(x)}{(1+x^2)^n} \right|^2 \omega(x)\, dx < \varepsilon.$$

Proof. Since (4) is an basis of $L^2_\omega(\mathbb{R})$, this is an immediate consequence of (7).

Remark 1 *This property being true for any* $f \in L^2_\omega(\mathbb{R})$, *it's remains true for any* $f \in L^2(\mathbb{R})$ *since we clearly have* $L^2(\mathbb{R}) \subset L^2_\omega(\mathbb{R})$.

We also need the following

Lemma 2 *With the usual notation, we have* $L^2_\omega(\mathbb{R}) \subset L^2_{\omega^2}(\mathbb{R})$*; moreover,* $L^2_\omega(\mathbb{R})$ *is a dense vector subspace of the Hilbert space* $L^2_{\omega^2}(\mathbb{R})$.

Proof. The first assertion comes directly from $0 < \omega < 1$; in fact one has

$$\|f\|_{L^2_{\omega^2}} \leq \|f\|_{L^2_\omega} \quad \forall f \in L^2_{\omega^2}(\mathbb{R}). \tag{13}$$

The second affirmation is an immediate consequence of the inclusion $C_c(\mathbb{R}) \subset L^2_\omega(\mathbb{R})$, where $C_c(\mathbb{R})$ stands for the set of continuous functions with compact support.

Now we can prove the main result in this section.

Theorem 3 *Let* $f \in L^2(\mathbb{R})$*; for each real number* $\varepsilon > 0$, *it exists* $n \equiv n(\varepsilon) \in \mathbb{N}$ *and a polynomial* r_m *of degree* $m \leq 2n$ *such that*

$$\int_{\mathbb{R}} \left| f(x) - \frac{r_m(x)}{(1+x^2)^{n+1}} \right|^2 dx < \varepsilon.$$

Proof. Since $f \in L^2(\mathbb{R})$, it's clear that $\omega^{-1} f \in L^2_{\omega^2}(\mathbb{R})$; according to the preceding lemma, a function $\widetilde{f} \in L^2_\omega(\mathbb{R})$ can be found such that

$$\left\| \omega^{-1} f - \widetilde{f} \right\|_{L^2_{\omega^2}} < \frac{\varepsilon}{2};$$

on the other hand, applying the result in proposition 1 to $\widetilde{f}$, it exists $n \equiv n(\varepsilon) \in \mathbb{N}$ and a polynomial $\widetilde{r}_m$ of degree $m \leq 2n$ such that

$$\left\| \widetilde{f} - \frac{\widetilde{r}_m(x)}{(1+x^2)^n} \right\|_{L^2_\omega} < \frac{\varepsilon}{2};$$

now, if we define $r_m := 2\pi^{-1}\widetilde{r}_m$ we find, after applying the triangle inequality and (13) :

$$\left\| f - \frac{r_m(x)}{(1+x^2)^{n+1}} \right\|_{L^2} = \left\| \omega^{-1} f - \frac{\tilde{r}_m(x)}{(1+x^2)^n} \right\|_{L^2_{\omega^2}} \leq$$

$$\leq \left\| \omega^{-1} f - \tilde{f} \right\|_{L^2_{\omega^2}} + \left\| \tilde{f} - \frac{\tilde{r}_m(x)}{(1+x^2)^n} \right\|_{L^2_{\omega^2}} \leq$$

$$\leq \frac{\varepsilon}{2} + \left\| \tilde{f} - \frac{\tilde{r}_m(x)}{(1+x^2)^n} \right\|_{L^2_{\tilde{\omega}}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

4 Weighted trigonometric series.

This section is devoted to generalize the preceding methods and ideas; we will find here some expansions that could be called "weighted trigonometric series", they lie on the following

Theorem 4 *Let's consider an interval* $(a,b) \subset \mathbb{R}$. *Let* $\tilde{\omega} : \overline{(a,b)} \to \mathbb{R}$ *a continuous and nonnegative function such that:*

(i) $mes\{x : \tilde{\omega}(x) = 0\} = 0$,

(ii) $c := \int_a^b \tilde{\omega}(t)\,dt < \infty$.

Let's define $\omega := 2c^{-1}\tilde{\omega}$ and

$$\theta : x \in (a,b) \longmapsto \int_a^x \omega(t)\,dt - 1 \in (-1,1),$$

$$\phi_1(x) := \cos \pi\theta(x), \qquad \psi_1(x) := \sin \pi\theta(x);$$

then, the sequence of functions $\left\{\frac{1}{\sqrt{2}}, \phi_1, \psi_1, \ldots, \phi_n, \psi_n, \ldots\right\}$ defined by (5) or, equivalently, by the expressions

$$\phi_n \ : \ = \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n}{2r} (-1)^r \phi_1^{n-2r} \psi_1^{2r},$$

$$\psi_n \ : \ = \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2r+1} (-1)^r \phi_1^{n-2r-1} \psi_1^{2r+1}.$$

is an orthonormal basis of $L^2_\omega(a,b)$, where the inner product is given by

$$\langle f,g\rangle_\omega = \int_a^b f(x)\, g(x)\, \omega(x)\, dx.$$

Proof. First, we prove the equivalence between (5) and the preceding formulae: we have

$$\phi_n + i\psi_n = \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n}{2r} (-1)^r \phi_1^{n-2r} \psi_1^{2r}$$

$$+ i \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2r+1} (-1)^r \phi_1^{n-2r-1} \psi_1^{2r+1}$$

$$= \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n}{2r} i^{2r} \psi_1^{2r} \phi_1^{n-2r} + \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2r+1} i^{2r+1} \psi_1^{2r+1} \phi_1^{n-2r-1}$$

$$= \sum_{r=0}^{n} \binom{n}{r} i^r \psi_1^r \phi_1^{n-r} = (\phi_1 + i\psi_1)^n,$$

and from here we deduce (5); the other implication was shown in section 2.4. In order to prove orthonormality, let's choose $n,m \in \mathbb{N}$, we need to compute

$$\int_a^b \phi_n(x)\, \psi_m(x)\, \omega(x)\, dx,$$

if we define the change of variable $\theta(x) := \int_a^x \omega(t)\, dt - 1$ then we have $\phi_n(x) = \cos n\pi\theta$, $\psi_m(x) = \sin m\pi\theta$ and, of course, $d\theta = \omega(x)\, dx$; this yields

$$\int_a^b \phi_n(x)\, \psi_m(x)\, \omega(x)\, dx = \int_{-1}^{1} \cos n\pi\theta \sin m\pi\theta d\theta = 0,$$

and the rest of the proof clearly follows. ■

As a first consequence, we have a generalization of theorem 3 concerning rational approximation:

Corollary 5 *Let σ be a polynomial of degree $\nu \geq 1$ such that $\sigma' \geq 0$, and let's consider a function $f \in L^2(\mathbb{R})$. For each $\varepsilon > 0$, a natural number $n \equiv n(\varepsilon)$ and polynomial r_m of degree $m \leq 2\nu n$ can be found such that*

$$\int_{\mathbb{R}} \left| f - \frac{r_m}{(1+\sigma^2)^{n+1}} \right|^2 dx < \varepsilon.$$

Proof. Define

$$\omega = \frac{2\sigma'}{\pi\left(1+\sigma^2\right)},$$

this yields

$$\sigma = \tan \pi \frac{\theta}{2},$$

and

$$\phi_1 = \frac{1-\sigma^2}{1+\sigma^2}, \qquad \psi_1 = \frac{2\sigma}{1+\sigma^2};$$

the rest of the proof comes from 2.4 and, with very little change, from that of theorem 3.

Now, let's present some concrete expansions found by using this method. First, consider the weight function

$$\tilde{\omega} : t \in (-1,1) \longmapsto \tilde{\omega}(t) = \exp t;$$

since $c = \int_{-1}^{1} \tilde{\omega}(t)\, dt = e - e^{-1}$, we define $\omega = \dfrac{2\tilde{\omega}}{e - e^{-1}}$ and

$$\theta : x \in (-1,1) \longmapsto \theta(x) = \int_{-1}^{x} \omega(t)\, dt - 1 \in (-1,1),$$

in this case an explicit expression can be given:

$$\theta(x) = \frac{2e^x - e - e^{-1}}{e - e^{-1}};$$

let's plot the functions $\phi_1 = \cos \pi\theta$, $\psi_1 = \sin \pi\theta$, $\phi_3 = \cos 3\pi\theta$ and $\psi_3 = \sin 3\pi\theta$:

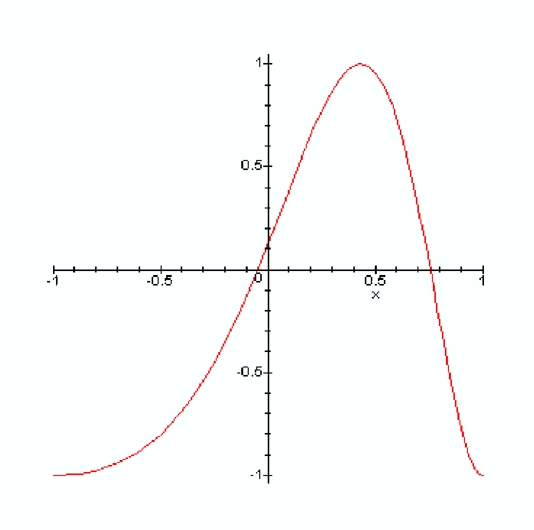

Fig. 9: The function ϕ_1.

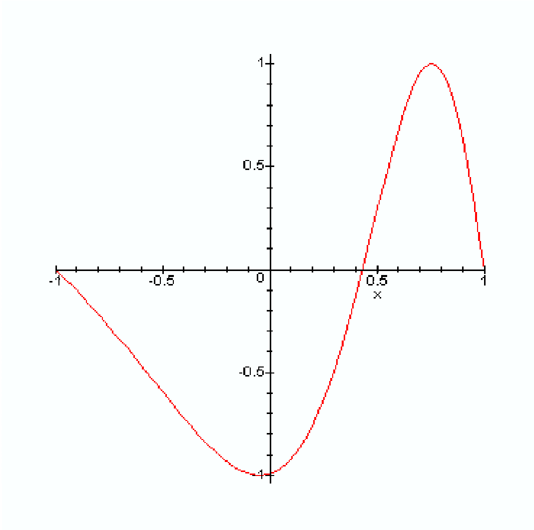

Fig 10: The function ψ_1.

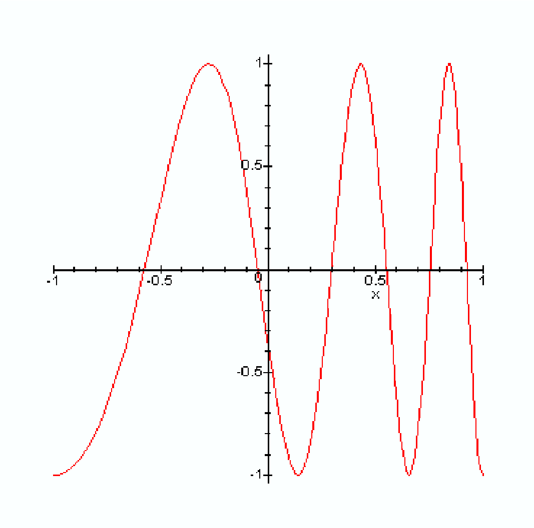

Fig 11: The function ϕ_3.

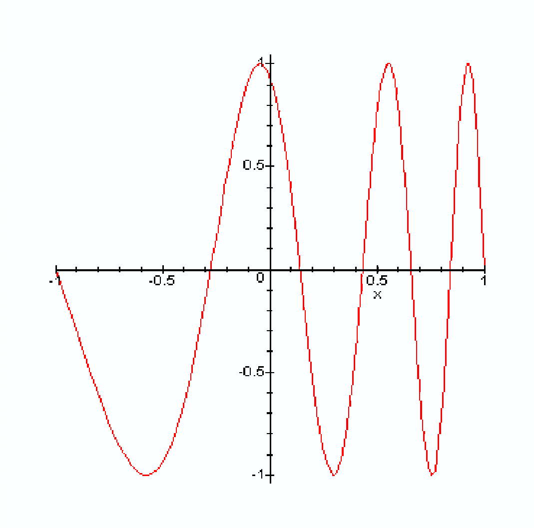

Fig 12: The function ψ_3.

These figures reveal how the basis functions are affected by the weight function. Now, we define the function $x \in \mathbb{R} \mapsto f(x) = x \in \mathbb{R}$; in order to illustrate how ω affects its expansion we plot the corresponding Fourier series with five and ten terms respectively:

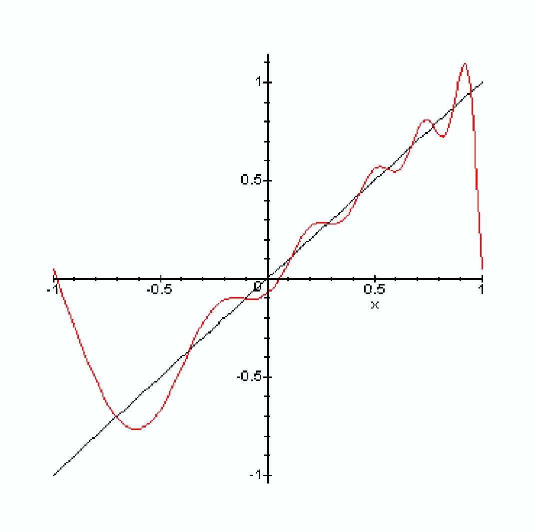

Fig 13: Approximation with five terms.

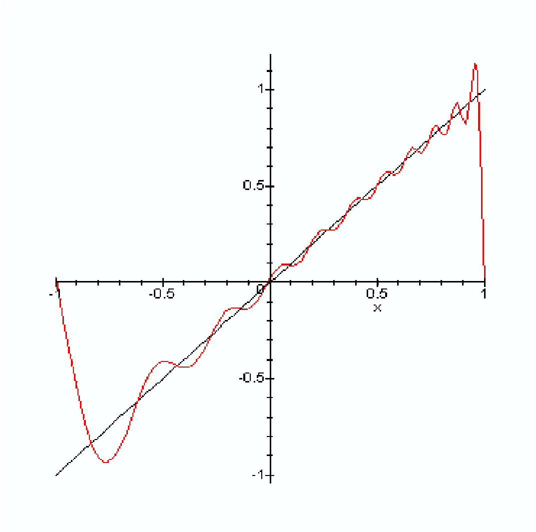

Fig 14: Approximation with ten terms.

The effect of the weight function is clear again; it's found that the general aspect of the approximations is very similar to that of trigonometric series, but we obtain more accurate approximations, as one might expect, in the areas where a higher weight is assigned.

The preceding theorem can be partially generalized to the following situation: $mes\,\{x : \tilde{\omega}(x) = 0\} > 0$, but a subinterval $(\alpha, \beta) \subset (a, b)$ exits in such a way that:

(i) $\tilde{\omega}_{|\overline{(\alpha,\beta)}}$ satisfy the hypothesis of the theorem in $\overline{(\alpha, \beta)}$ and

(ii) $\tilde{\omega}_{|\overline{(\alpha,\beta)}^c} \equiv 0$.

In this case, ω does not define an inner product and the resulting sequence can be shown not to be an orthonormal basis of $L^2_\omega(a, b)$ but of the subspace

$$\left\{ f \in L^2_\omega(a, b) : f_{|\overline{(\alpha,\beta)}^c} \equiv 0 \right\}.$$

By making use of this fact, we can find out more about the effect of the weight function by "taking the things to the extreme": let's define

$$\tilde{\omega} : t \in (-1, 1) \longmapsto \tilde{\omega}(t) = \begin{cases} 0 & -1 \leq x < -1/2 \\ 1 & -1/2 \leq x < 1/2 \\ 0 & 1/2 \leq x \leq 1 \end{cases}$$

since $c = \int_{-1}^{1} \tilde{\omega}(t)\, dt = 1$, we set $\omega = 2\tilde{\omega}$, and define θ, ϕ_1 and ψ_1 as in the theorem. The next figures show the graphic representation of the functions ϕ_1 and ψ_1:

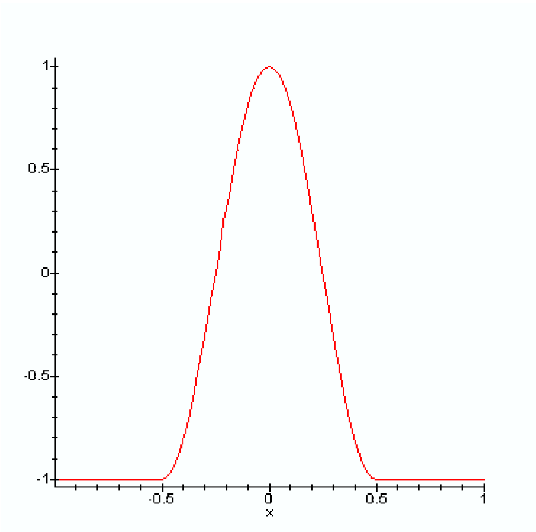

Fig. 15: The function ϕ_1.

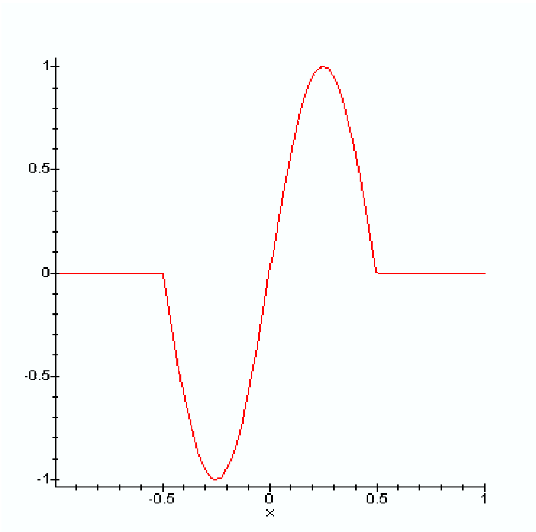

Fig. 16: The function ψ_1.

Now, let's plot the expansions corresponding to the same function as before calculated with two and five terms respectively:

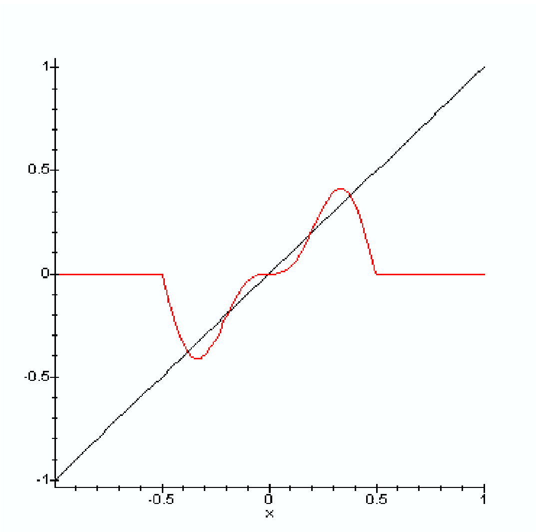

Fig 17: Approximation with two terms.

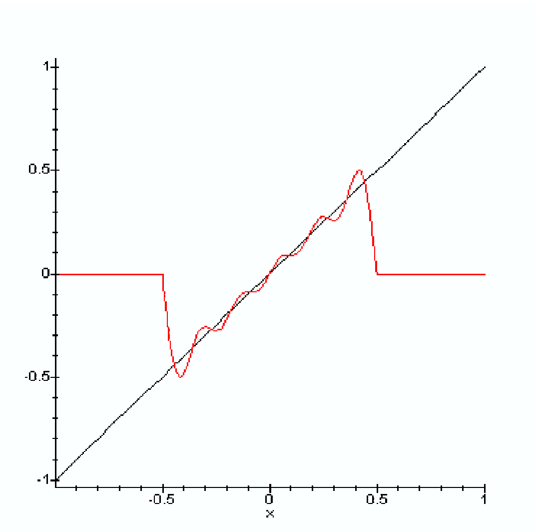

Fig 18: Approximation with five terms.

Basically, what we get is an approximation of f by means of a weighted trigonometric series exactly in the "window" prescribed by the weight function ω and out of this window the approximating function is constant; this corresponds to the "extremal" behaviour we could have expected.

We cannot evaluate in what measure this is an interesting method. The problem is clear: even in very simple cases we don't dispose of analytic expressions for the basis functions; however, given both the recursive relations (5) and the analytic expressions of the preceding theorem, it should be easy to design and develop programs to make this method useful at least from a computational and numerical point of view.

5 Conclusions.

We have presented a basis of $L^2_\omega(\mathbb{R}) \supset L^2(\mathbb{R})$, where $\omega(x) = 2\pi^{-1}(1+x^2)^{-1}$, it comes directly from the trigonometric basis of $L^2(-1,1)$ via a change of variable; in fact, given its properties, this basis could be regarded in some sense as the "trigonometric" basis of $L^2_\omega(\mathbb{R})$. We have found some expansions with respect to this basis that allowed us to give some beautiful formulae, such as

$$\cos x = \frac{1}{e} + \frac{1}{e}\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}\sum_{k=0}^{n}\frac{\sigma_{nk}}{(n-k)!}\right)\phi_n(x),$$

$$\sin x = \frac{1}{e}\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}\sum_{k=0}^{n}\frac{\tau_{nk}}{(n-k)!}\right)\psi_n(x),$$

and

$$H(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\psi_{2n+1}(x).$$

As an application of our results, we have given a proof of the density of rational functions in $L^2(\mathbb{R})$; also an application in ordinary differential equations has been described and a connection with the Sturm-Liouville theory has been presented. In the last part of the article a generalization of this ideas has been discussed; a general application of the trigonometric recursions

$$\begin{cases} \cos(n+1)\theta = \cos n\theta\cos\theta - \sin n\theta\sin\theta, \\ \sin(n+1)\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta, \end{cases}$$

has allowed us to present what could be called (still with the biggest caution!) "weighted trigonometric series"; a kind of approximation that might offer an alternative to orthogonal polynomials for weighted approximation in some contexts. To our knowledge, all the ideas presented here seem to be original, and it will correspond to the experts in Approximation Theory to decide in what measure they may be of some interest in the future.