Chapter 2.
Fourier Analysis

... Untwisting all the chains that tie
the hidden soul of harmony.
— JOHN MILTON, *L'Allegro (1631)*

The last chapter dealt with time dependence, and this one is motivated by space dependence. Later chapters will combine the two.

Fourier analysis touches almost every aspect of partial differential equations and their numerical solution. Sometimes Fourier ideas enter into the analysis of a numerical algorithm derived from other principles—especially in the stability analysis of finite-difference formulas. Sometimes they underlie the design of the algorithm itself—spectral methods. And sometimes the situation is a mixture of both, as with iterative and multigrid methods for elliptic equations. For one reason or another, Fourier analysis will appear in all of the remaining chapters of this book.

The impact of Fourier analysis is also felt in many fields besides differential equations and their numerical solution, such as quantum mechanics, crystallography, signal processing, statistics, and information theory.

There are four varieties of Fourier transform, depending on whether the spatial domain is unbounded or bounded, continuous or discrete:

Name	Space variable	Transform variable
Fourier transform	unbounded, continuous	continuous, unbounded
Fourier series	bounded, continuous	discrete, unbounded
semidiscrete Fourier transform or *z-transform*	unbounded, discrete	continuous, bounded
discrete Fourier transform (DFT)	bounded, discrete	discrete, bounded

(The second and third varieties are mathematically equivalent.) This chapter will describe the essentials of these operations, emphasizing the parallels between them. In discrete methods for partial differential equations, one looks for a representation that will converge to a solution of the continuous problem as the mesh is refined. Our definitions are chosen so that the same kind of convergence holds also for the transforms.

Rigorous Fourier analysis is a highly technical and highly developed area of mathematics, which depends heavily on the theory of Lebesgue measure and integration. We shall make use of L^2 and ℓ^2 spaces, but for the most part this chapter avoids the technicalities. In particular, a number of statements made in this chapter hold not at every point of a domain, but "almost everywhere" — everywhere but on a set of measure zero.

2.1. The Fourier transform

If $u(x)$ is a (Lebesgue-measurable) function of $x \in \mathbb{R}$, the L^2**-norm** of u is the nonnegative or infinite real number

$$\|u\| = \Big[\int_{-\infty}^{\infty} |u(x)|^2 dx\Big]^{1/2}. \tag{2.1.1}$$

The symbol L^2 ("L-two") denotes the set of all functions for which this integral is finite:

$$L^2 = \{u \colon \|u\| < \infty\}. \tag{2.1.2}$$

Similarly, L^1 and L^∞ are the sets of functions having finite L^1- and L^∞-norms, defined by

$$\|u\|_1 = \int_{-\infty}^{\infty} |u(x)| dx, \qquad \|u\|_\infty = \sup_{-\infty<x<\infty} |u(x)|. \tag{2.1.3}$$

Note that since the L^2 norm is the norm used in most applications, because of its many desirable properties, we have reserved the symbol $\|\cdot\|$ without a subscript for it.

The **convolution** of two functions u, v is the function $u*v$ defined by

$$(u*v)(x) = (u*v)(x) = \int_{-\infty}^{\infty} u(x-y)v(y)dy = \int_{-\infty}^{\infty} u(y)v(x-y)dy, \tag{2.1.4}$$

assuming these integrals exist. One way to think of $u*v$ is as a weighted moving average of values $u(x)$ with weights defined by $v(x)$, or vice versa.

For any $u \in L^2$, the **Fourier transform** of u is the function $\hat{u}(\xi)$ defined by

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x) dx, \qquad \xi \in \mathbb{R}.$$

The quantity ξ is known as the **wave number,** the spatial analog of frequency. For many functions $u \in L^2$, this integral converges in the usual sense for all $\xi \in \mathbb{R}$, but there are situations where this is not true, and in these cases one must interpret the integral as a limit in a certain L^2-norm sense of integrals $\int_{-M}^{M}$ as $M \to \infty$. The reader interested in such details should consult the various books listed in the references.*

*If $u \in L^1$, then $\hat{u}(\xi)$ exists for every ξ and is continuous with respect to ξ. According to the **Riemann-Lebesgue Lemma,** it also satisfies $|\hat{u}(\xi)| \to 0$ as $\xi \to \infty$.

Figure 2.1.1. Space and wave number domains for the Fourier transform (compare Figures 2.2.1 and 2.4.1).

The following theorem summarizes some of the fundamental properties of Fourier transforms.

THE FOURIER TRANSFORM

Theorem 2.1.* *If $u \in L^2$, then the Fourier transform*

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x)\,dx, \qquad \xi \in \mathbb{R} \tag{2.1.5}$$

belongs to L^2 also, and u can be recovered from $\hat{u}$ by the **inverse Fourier transform**

$$u(x) = (\mathcal{F}^{-1}\hat{u})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\hat{u}(\xi)\,d\xi, \qquad x \in \mathbb{R}. \tag{2.1.6}$$

The L^2-norms of u and $\hat{u}$ are related by **Parseval's equality**,

$$\|\hat{u}\| = \sqrt{2\pi}\,\|u\|. \tag{2.1.7}$$

If $u \in L^2$ and $v \in L^1$ (or vice versa), then $u*v \in L^2$, and $\widehat{u*v}$ satisfies

$$\widehat{u*v}(\xi) = \hat{u}(\xi)\hat{v}(\xi). \tag{2.1.8}$$

These four equations are of such fundamental importance that they are worth commenting on individually, although it is assumed the reader has already been exposed to Fourier analysis.

*As mentioned in the introduction to this chapter, some of these properties—namely equations (2.1.6) and (2.1.8)—hold merely for "almost every" value of x or ξ. In fact even if $f(z)$ is a continuous function in L^2, its Fourier transform may fail to converge at certain points x. To ensure pointwise convergence one needs additional assumptions such as that f is of **bounded variation** (defined below before Theorem 2.4) and belongs to L^1. These assumptions also ensure that at any point x where f has a jump discontinuity, its Fourier transform converges to the average value $(f(x^-)+f(x^+))/2$.

First of all, (2.1.5) indicates that $\hat{u}(\xi)$ is a measure of the correlation of $u(x)$ with the function $e^{i\xi x}$. The idea behind Fourier analysis is to interpret $u(x)$ as a superposition of monochromatic waves $e^{i\xi x}$ with various wave numbers ξ, and $\hat{u}(\xi)$ represents the complex amplitude (more precisely: amplitude density with respect to ξ) of the component of u at wave number ξ.

Conversely, (2.1.6) expresses the synthesis of $u(x)$ as a superposition of its components $e^{i\xi x}$, each multiplied by the appropriate factor $\hat{u}(\xi)$. The factor 2π is a nuisance that could have been put in various places in our formulas, but is hard to eliminate entirely.

Equation (2.1.7), Parseval's equality, is a statement of energy conservation: the L^2 energy of any signal $u(x)$ is equal to the sum of the energies of its component vibrations (except for the factor $\sqrt{2\pi}$). By "energy" we mean the square of the L^2 norm.

Finally, the convolution equation (2.1.8) is perhaps the most subtle of the four. The left side, $\widehat{u*v}(\xi)$, represents the strength of the wave number ξ component that results when u is convolved with v—in other words, the degree to which u and v beat in and out of phase with each other at wave number ξ when multiplied together in reverse order with a varying offset. Such beating is caused by a quadratic interaction of the wave number component ξ in u with the same component of v—hence the right-hand side $\hat{u}(\xi)\hat{v}(\xi)$.

All of the assertions of Theorem 2.1 can be verified in the following example, which the reader should study carefully.

EXAMPLE 2.1.1. *B-splines.* Suppose u is the function

$$u(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \le x \le 1, \\ 0 & \text{otherwise} \end{cases} \qquad (2.1.9)$$

(Figure 2.1.2). Then by (2.1.1) we have $\|u\| = 1/\sqrt{2}$, and (2.1.5) gives

$$\hat{u}(\xi) = \tfrac{1}{2}\int_{-1}^{1} e^{-i\xi x}dx = \left.\frac{e^{-i\xi x}}{-2i\xi}\right|_{-1}^{1} = \frac{\sin\xi}{\xi}. \qquad (2.1.10)$$

(This function $\hat{u}(\xi)$ is called a **sinc function;** more on these in §2.3.) From (2.1.1) and the indispensable identity*

$$\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = \pi, \qquad (2.1.11)$$

which can be derived by complex contour integration, we calculate $\|\hat{u}\| = \sqrt{\pi}$, which confirms (2.1.7).

From the definition (2.1.4) it is readily verified that in this example

$$(u*u)(x) = \begin{cases} \frac{1}{2}(1-|x|/2) & \text{for } -2 \le x \le 2, \\ 0 & \text{otherwise} \end{cases} \qquad (2.1.12)$$

* worth memorizing!

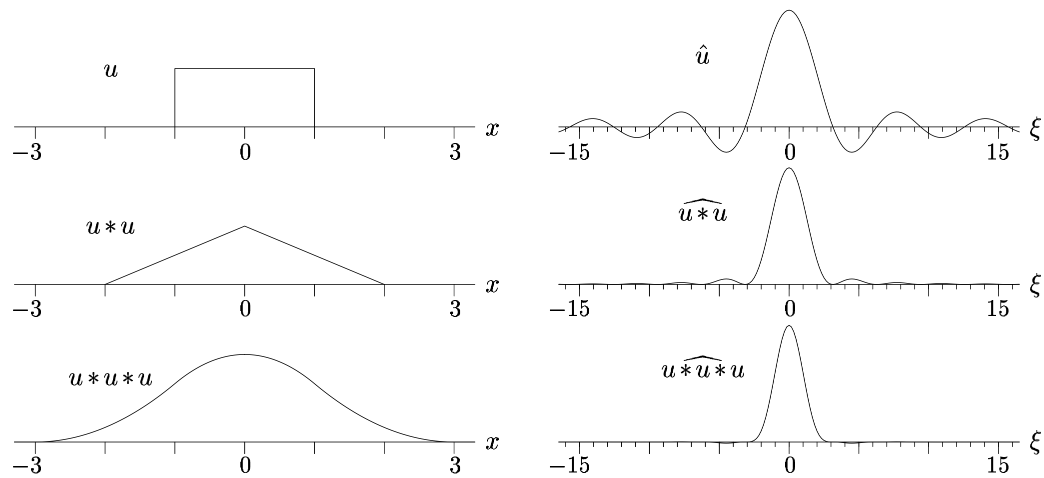


Figure 2.1.2. The first three B-splines of Example 2.1.1 and their Fourier transforms.

and

$$(u*u*u)(x) = \begin{cases} \frac{3}{4} - \frac{1}{4}x^2 & \text{for } -1 \le x \le 1, \\ \frac{1}{8}(9-6|x|+x^2) & \text{for } 1 \le |x| \le 3, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1.13}$$

and by (2.1.8) and (2.1.10), the corresponding Fourier transforms must be

$$\widehat{u*u}(\xi) = \frac{\sin^2\xi}{\xi^2}, \qquad \widehat{u*u*u}(\xi) = \frac{\sin^3\xi}{\xi^3}. \tag{2.1.14}$$

See Figure 2.1.2. In general, a convolution $u_{(p)}$ of p copies of u has the Fourier transform

$$\widehat{u_{(p)}}(\xi) = \mathcal{F}\{u*u*\cdots*u\}(\xi) = \left(\frac{\sin\xi}{\xi}\right)^p. \tag{2.1.15}$$

Note that whenever $u_{(p)}$ or any other function is convolved with the function u of (2.1.9), it becomes smoother, since the convolution amounts to a local moving average. In particular, u itself is piecewise continuous, $u*u$ is continuous and has a piecewise continuous first derivative, $u*u*u$ has a continuous derivative and a piecewise continuous second derivative, and so on. In general $u_{(p)}$ is a piecewise polynomial of degree $p-1$ with a continuous $(p-2)$nd derivative and a piecewise continuous $(p-1)$st derivative, and is known as a **B-spline**. (See, for example, C. de Boor, *A Practical Guide to Splines,* Springer, 1978.)

Thus convolution with u makes a function smoother, while the effect on the Fourier transform is to multiply it by $\sin\xi/\xi$ and thereby make it decay more rapidly $\xi\to\infty$. This relationship is evident in Figure 2.1.2.

For applications to numerical methods for partial differential equations, there are two properties of the Fourier transform that are most important.

One is equation (2.1.8): the Fourier transform converts convolution into multiplication. The second can be derived by integration by parts:

$$\widehat{u_x}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u_x(x)dx = -\int_{-\infty}^{\infty} (-i\xi) e^{-i\xi x} u(x)dx = i\xi\, \hat{u}(\xi), \qquad (2.1.16)$$

assuming $u(x)$ is smooth and decays at ∞. That is, the Fourier transform converts differentiation into multiplication by $i\xi$. This result is rigorously valid for any absolutely continuous function $u \in L^2$ whose derivative belongs to L^2. Note that differentiation makes a function less smooth, so the fact that it makes the Fourier transform decay less rapidly fits the pattern mentioned above for convolution.

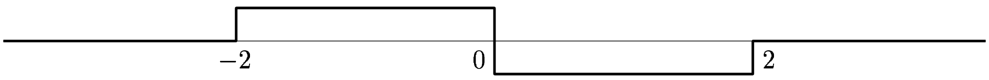


Figure 2.1.3.

EXAMPLE 2.1.2. The function

$$u(x) = \begin{cases} \frac{1}{4} & \text{for } -2 \le x < 0, \\ -\frac{1}{4} & \text{for } 0 < x \le 2, \\ 0 & \text{otherwise,} \end{cases} \qquad (2.1.17)$$

illustrated in Figure 2.1.3, has Fourier transform

$$\begin{aligned} \hat{u}(\xi) &= \tfrac{1}{4} \int_{-2}^{0} e^{-i\xi x} dx - \tfrac{1}{4} \int_{0}^{2} e^{-i\xi x} dx \\ &= \frac{1}{-4i\xi}(1 - e^{2i\xi} - e^{-2i\xi} + 1) = \frac{1}{4i\xi}(e^{i\xi} - e^{-i\xi})^2 = \frac{i \sin^2 \xi}{\xi}, \end{aligned} \qquad (2.1.18)$$

which is $i\xi$ times the Fourier transform (2.1.14) of the triangular hat function (2.1.12). In keeping with (2.1.16), (2.1.17) is the derivative of (2.1.12).

The following theorem collects (2.1.16) together with a number of additional properties of the Fourier transform:

PROPERTIES OF THE FOURIER TRANSFORM

Theorem 2.2. *Let $u, v \in L^2$ have Fourier transforms $\hat{u} = \mathcal{F}u$, $\hat{v} = \mathcal{F}v$. Then:*

(a) Linearity. $\mathcal{F}\{u+v\}(\xi) = \hat{u}(\xi) + \hat{v}(\xi)$; $\mathcal{F}\{cu\}(\xi) = c\hat{u}(\xi)$.

(b) Translation. If $x_0 \in \mathbb{R}$, *then* $\mathcal{F}\{u(x+x_0)\}(\xi) = e^{i\xi x_0}\hat{u}(\xi)$.

(c) Modulation. If $\xi_0 \in \mathbb{R}$, *then* $\mathcal{F}\{e^{i\xi_0 x}u(x)\}(\xi) = \hat{u}(\xi - \xi_0)$.

(d) Dilation. If $c \in \mathbb{R}$ *with* $c \neq 0$, *then* $\mathcal{F}\{u(cx)\}(\xi) = \hat{u}(\xi/c)/|c|$.

(e) Conjugation. $\mathcal{F}\{\overline{u}\}(\xi) = \overline{\hat{u}(-\xi)}$.

(f) Differentiation. If $u_x \in L^2$, *then* $\mathcal{F}\{u_x\}(\xi) = i\xi\hat{u}(\xi)$.

(g) Inversion. $\mathcal{F}^{-1}\{u\}(\xi) = \dfrac{1}{2\pi}\hat{u}(-\xi)$.

Proof. See Exercise 2.1.2. ∎

In particular, taking $c = -1$ in part *(d)* above gives $\mathcal{F}\{u(-x)\} = \hat{u}(-\xi)$. Combining this result with part *(e)* leads to the following elementary but useful results. Definitions: $u(x)$ is even, odd, real, or imaginary if $u(x) = u(-x)$, $u(x) = -u(-x)$, $u(x) = \overline{u(x)}$, or $u(x) = -\overline{u(x)}$, respectively; $u(x)$ is hermitian or skew-hermitian if $u(x) = \overline{u(-x)}$ or $u(x) = -\overline{u(-x)}$, respectively.

SYMMETRIES OF THE FOURIER TRANSFORM

Theorem 2.3. *Let $u \in L^2$ have Fourier transform $\hat{u} = \mathcal{F}u$. Then*

(a) $u(x)$ is even (odd) $\Longleftrightarrow$ $\hat{u}(\xi)$ is even (odd);

(b) $u(x)$ is real (imaginary) $\Longleftrightarrow$ $\hat{u}(\xi)$ is hermitian (skew-hermitian);

and therefore

(c) $u(x)$ is real and even $\Longleftrightarrow$ $\hat{u}(\xi)$ is real and even;

(d) $u(x)$ is real and odd $\Longleftrightarrow$ $\hat{u}(\xi)$ is imaginary and odd;

(e) $u(x)$ is imaginary and even $\Longleftrightarrow$ $\hat{u}(\xi)$ is imaginary and even;

(f) $u(x)$ is imaginary and odd $\Longleftrightarrow$ $\hat{u}(\xi)$ is real and odd.

Proof. See Exercise 2.1.3. ∎

In the discussion above we have twice observed the following relationships between the smoothness of a function and the decay of its Fourier transform:

$u(x)$		$\hat{u}(\xi)$
smooth	$\leftrightarrow$	decays rapidly as $\|\xi\| \to \infty$
decays rapidly as $\|x\| \to \infty$	$\leftrightarrow$	smooth

(Of course since the Fourier transform is essentially the same as the inverse Fourier transform, by Theorem 2.2g, the two rows of this summary are equivalent.) The intuitive explanation is that if a function is smooth, then it can be accurately represented as a superposition of slowly-varying waves, so one does not need much energy in the high wave number components. Conversely, a non-smooth function requires a considerable amplitude of high wave number components to be represented accurately. These relationships are the bedrock of analog and digital signal processing, where all kinds of smoothing operations are effected by multiplying the Fourier transform by a "windowing function" that decays suitably rapidly.

The following theorem makes these connections between u and $\hat{u}$ precise. This theorem may seem forbidding at first, but it is worth studying carefully. Each of the four parts of the theorem makes a stronger smoothness assumption on u than the last, and reaches a correspondingly stronger conclusion about the rate of decay of $\hat{u}(\xi)$ as $|\xi| \to \infty$. Parts *(c)* and *(d)* are known as the **Paley-Wiener theorems**.

First, a standard definition. A function u defined on $\mathbb{R}$ is said to have **bounded variation** if there is a constant M such that for any finite m and any points $x_0 < x_1 < \cdots < x_m$, $\sum_{j=1}^{m} |u(x_j) - u(x_{j-1})| \le M$.

SMOOTHNESS OF u AND DECAY OF $\hat{u}$

Theorem 2.4. *Let u be a function in L^2.*

(a) If u has $p-1$ continuous derivatives in L^2 for some $p \geq 0$, and a pth derivative in L^2 that has bounded variation, then

$$\hat{u}(\xi) = O(|\xi|^{-p-1}) \qquad \text{as} \quad |\xi| \to \infty. \tag{2.1.19}$$

(b) If u has infinitely many continuous derivatives in L^2, then

$$\hat{u}(\xi) = O(|\xi|^{-M}) \qquad \text{as} \quad |\xi| \to \infty \quad \text{for all} \quad M, \tag{2.1.20}$$

and conversely.

(c) If u can be extended to an analytic function of $z = x+iy$ in the complex strip $|\operatorname{Im} z| < a$ for some $a > 0$, with $\|u(x+iy)\| \leq$ const uniformly for each constant $-a < y < a$, then

$$e^{a|\xi|}\hat{u}(\xi) \in L^2, \tag{2.1.21}$$

and conversely.

(d) If u can be extended to an entire function of $z = x+iy$ with $|u(z)| = O(e^{a|z|})$ as $|z| \to \infty$ $(z \in \mathbb{C})$ for some $a > 0$, then $\hat{u}$ has compact support contained in $[-a,a]$, i.e.*

$$\hat{u}(\xi) = 0 \qquad \text{for all} \quad |\xi| > a, \tag{2.1.22}$$

and conversely.

Proof. See, for example, §VI.7 of Y. Katznelson, *An Introduction to Harmonic Analysis,* Dover, 1976. [Also see Rudin (p. 407), Paley & Wiener, Reed & Simon v. 2, Hoïmander v. 1 (p. 181), entire functions books...] ∎

A function of the kind described in *(d)* is said to be **band-limited**, since only a finite band of wave numbers are represented in it.

Since the Fourier transform and its inverse are essentially the same, by Theorem 2.2g, Theorem 2.4 also applies if the roles of $u(x)$ and $\hat{u}(\xi)$ are interchanged.

EXAMPLE 2.1.1, CONTINUED. The square wave u of Example 2.1.1 (Figure 2.1.2) satisfies condition *(a)* of Theorem 2.4 with $p = 0$, so its Fourier transform should satisfy

* An entire function is a function that is analytic throughout the complex plane $\mathbb{C}$.

$|\hat{u}(\xi)| = O(|\xi|^{-1})$, as is verified by (2.1.10). On the other hand, suppose we interchange the roles of u and $\hat{u}$ and apply the theorem again. The function $u(\xi) = \sin\xi/\xi$ is entire, and since $\sin(\xi) = (e^{i\xi} - e^{-i\xi})/2i$, it satisfies $u(\xi) = O(e^{|\xi|})$ as $|\xi| \to \infty$ (with ξ now taking complex values). By part *(d)* of Theorem 2.4, it follows that $u(x)$ must have compact support contained in $[-1,1]$, as indeed it does.

Repeating the example for $u*u$, condition *(a)* now applies with $p=1$, and the Fourier transform (2.1.14) is indeed of magnitude $O(|\xi|^{-2})$, as required. Interchanging u and $\hat{u}$, we note that $\sin^2\xi/\xi^2$ is an entire function of magnitude $O(e^{2|\xi|})$ as $|\xi| \to \infty$, and $u*u$ has support contained in $[-2,2]$.

EXERCISES

▷ *2.1.1.* Show that the two integrals in the definition (2.1.4) of $u*v$ are equivalent.

▷ *2.1.2.* Derive conditions *(a)–(g)* of Theorem 2.2. (Do not worry about justifying the usual operations on integrals.)

▷ *2.1.3.* Prove Theorem 2.3.

▷ *2.1.4.*
(a) Which functions $u \in L^2 \cap L^1$ satisfy $u*u = 0$?
(b) How about $u*u = u$?

▷ *2.1.5. Integration.*
(a) What does part *(f)* of Theorem 2.2 suggest should be the Fourier transform of the function $U(x) = \int_{-\infty}^{x} u(s)ds$?
(b) Obviously $U(x)$ cannot belong to L^2 unless $\int_{-\infty}^{\infty} u(x)dx = 0$, so by Theorem 2.1, this is a necessary condition for $\hat{U}$ to be in L^2 also. Explain how the condition $\int_{-\infty}^{\infty} u(x)dx = 0$ relates to your formula of *(a)* for $\hat{U}$ in terms of $\hat{u}$.

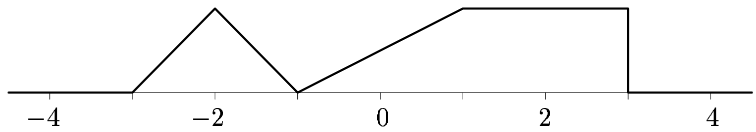
▷ *2.1.6.*
(a) Calculate the Fourier transform of $u(x) = (1+x^2)^{-1}$. (*Hint:* use a complex contour integral if you know how. Otherwise, look the result up in a table of integrals.)
(b) How does this example fit into the framework of Theorem 2.4? Which parts of the theorem apply to u?
(c) If the roles of u and $\hat{u}$ are interchanged, how does the example now fit Theorem 2.4? Which parts of the theorem apply to $\hat{u}$?

▷ *2.1.7.* The **autocorrelation function** of a function $u \in L^2 \cap L^1$ may be defined by

$$\phi(c) = \frac{1}{\|u\|^2} \int_{-\infty}^{\infty} u(x)u(x+c)dx.$$

Find an expression for $\phi(c)$ as an inverse Fourier transform of a product of Fourier transforms involving u. This expression is the basis of some algorithms for computing $\phi(c)$.

▷ *2.1.8.* Without evaluating any integrals, use Theorem 2.2 and (2.1.10) to determine the Fourier transform of the following function:



▷ *2.1.9. The uncertainty principle.* Show by using Theorem 2.4 that if $u(x)$ and $\hat{u}(\xi)$ both have compact support, with $u \in L^2$, then $u(x) \equiv 0$.

2.2. The semidiscrete Fourier transform

The semidiscrete Fourier transform is the reverse of the more familiar Fourier series: instead of a bounded, continuous spatial domain and an unbounded, discrete transform domain, it involves the opposite. This is just what is needed for the analysis of numerical methods for partial differential equations, where we are perpetually concerned with functions defined on discrete grids. For many analytical purposes it is simplest to think of these grids as infinite in extent.

Let $h>0$ be a real number, the **space step**, and let $\dots,x_{-1},x_0,x_1,\dots$ be defined by $x_j=jh$. Thus $\{x_j\}=h\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. We are concerned now with spatial **grid functions** $v=\{v_j\}$, which may or may not be approximations to a continuous function u,

$$v_j \approx u(x_j).$$

As in the last chapter, it will be convenient to write $v(x_j)$ sometimes for v_j.

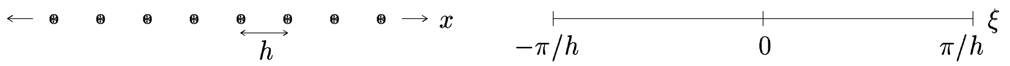


Figure 2.2.1. Space and wave number domains for the semidiscrete Fourier transform.

For functions defined on discrete domains it is standard to replace the upper-case letter L by a lower-case script letter ℓ. (Both symbols honor Henri Lebesgue, the mathematician who laid the foundations of modern functional analysis early in the twentieth century.) The ℓ_h^2**-norm** of a grid function v is the nonnegative or infinite real number

$$\|v\| = \left[h \sum_{j=-\infty}^{\infty} |v_j|^2\right]^{1/2}. \tag{2.2.1}$$

Notice the h in front of the summation. One can think of (2.2.1) as a discrete approximation to the integral (2.1.1) by the trapezoid rule or the rectangle rule for quadrature. (On an unbounded domain these two are equivalent.) The symbol ℓ_h^2 ("little L-two sub h") denotes the set of grid functions of finite norm,

$$\ell_h^2 = \{v : \|v\| < \infty\},$$

and similarly with ℓ_h^1 and ℓ_h^∞. In contrast to the situation with L^1, L^2, and L^∞, these spaces are nested:

$$\ell_h^1 \subseteq \ell_h^2 \subseteq \ell_h^\infty. \tag{2.2.2}$$

(See Exercise 2.1.1.)

The **convolution** $v * w$ of two functions v, w is the function $v * w$ defined by

$$(v*w)_m = h \sum_{j=-\infty}^{\infty} v_{m-j} w_j = h \sum_{j=-\infty}^{\infty} v_j w_{m-j}, \tag{2.2.3}$$

provided that these sums exist. This formula is a trapezoid or rectangle rule approximation to (2.1.4).

For any $v \in \ell_h^2$, the **semidiscrete Fourier transform** of v is the function $\hat{v}(\xi)$ defined by

$$\hat{v}(\xi) = (\mathcal{F}_h v)(\xi) = h \sum_{j=-\infty}^{\infty} e^{-i\xi x_j} v_j, \qquad \xi \in [-\pi/h, \pi/h],$$

a discrete approximation to (2.1.5). A priori, this sum defines a function $\hat{v}(\xi)$ for all $\xi \in \mathbb{R}$. However, notice that for any integer m, the exponential $e^{2\pi i m x_j/h} = e^{2\pi i m j}$ is exactly 1 at all of the grid points x_j. More generally, any wave number ξ is indistinguishable on the grid from all other wave numbers $\xi + 2\pi m/h$, where m is any integer—a phenomenon called **aliasing**. This means that the function $\hat{v}(\xi)$ is $2\pi/h$-periodic on $\mathbb{R}$. To make sense of the idea of analyzing v into component oscillations, we shall normally restrict attention to one period of $\hat{v}$ by looking only at wave numbers in the range $[-\pi/h, \pi/h]$, and it is in this sense that the Fourier transform of a grid function is defined on a bounded domain. But the reader should bear in mind that the restriction of ξ to any particular interval is a matter of labeling, not mathematics; in principle e^0 and $e^{100\pi i j}$ are equally valid representations of the grid function $v_j \equiv 1$.

Thus for discretized functions v, the transform $\hat{v}(\xi)$ inhabits a bounded domain. On the other hand the domain is still continuous. This reflects the fact that arbitrarily fine gradations of wave number are distinguishable on an unbounded grid.

Since x and ξ belong to different sets, it is necessary to define an additional vector space for functions $\hat{v}$. The L_h^2**-norm** of a function $\hat{v}$ is the number

$$\|\hat{v}\| = \left[\int_{-\pi/h}^{\pi/h} |\hat{v}(\xi)|^2 d\xi \right]^{1/2}. \tag{2.2.4}$$

One can think of this as an approximation to (2.1.1) in which the wave number components with $|\xi| > \pi/h$ have been replaced by zero. The symbol L_h^2 denotes the set of (Lebesgue-measurable) functions on $[-\pi/h, \pi/h]$ of finite norm,

$$L_h^2 = \{\hat{v} : \|\hat{v}\| < \infty\}. \tag{2.2.5}$$

Now we can state a theorem analogous to Theorem 2.1:

THE SEMIDISCRETE FOURIER TRANSFORM

Theorem 2.5. *If $v \in \ell_h^2$, then the semidiscrete Fourier transform*

$$\hat{v}(\xi) = (\mathcal{F}_h v)(\xi) = h \sum_{j=-\infty}^{\infty} e^{-i\xi x_j} v_j, \qquad \xi \in [-\pi/h, \pi/h] \tag{2.2.6}$$

belongs to L_h^2, and v can be recovered from $\hat{v}$ by the **inverse semidiscrete Fourier transform**

$$v_j = (\mathcal{F}_h^{-1}\hat{v})(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x_j}\hat{v}(\xi)d\xi, \qquad j \in \mathbb{Z}. \tag{2.2.7}$$

The ℓ_h^2-norm of v and the L_h^2-norm of $\hat{v}$ are related by **Parseval's equality**,

$$\|\hat{v}\| = \sqrt{2\pi}\|v\|. \tag{2.2.8}$$

If $u \in \ell_h^2$ and $v \in \ell_h^1$ (or vice versa), then $v * w \in \ell_h^2$, and $\widehat{v*w}$ satisfies

$$\widehat{v*w}(\xi) = \hat{v}(\xi)\hat{w}(\xi). \tag{2.2.9}$$

As in the continuous case, the following properties of the semidiscrete Fourier transform will be useful. In *(c)*, and throughout this book wherever convenient, we take advantage of the fact that $\hat{v}(\xi)$ can be treated as a periodic function defined for all $\xi \in \mathbb{R}$.

PROPERTIES OF THE SEMIDISCRETE FOURIER TRANSFORM

Theorem 2.6. *Let $v, w \in \ell_h^2$ have Fourier transforms $\hat{v}, \hat{w}$. Then:*

(a) Linearity. $\mathcal{F}_h\{v+w\}(\xi) = \hat{v}(\xi) + \hat{w}(\xi)$; $\mathcal{F}_h\{cv\}(\xi) = c\hat{v}(\xi)$.

(b) Translation. If $j_0 \in \mathbb{Z}$, *then* $\mathcal{F}_h\{v_{j+j_0}\}(\xi) = e^{i\xi x_{j_0}}\hat{v}(\xi)$.

(c) Modulation. If $\xi_0 \in \mathbb{R}$, *then* $\mathcal{F}_h\{e^{i\xi_0 x_j} v_j\}(\xi) = \hat{v}(\xi - \xi_0)$.

(d) Dilation. If $m \in \mathbb{Z}$ *with* $m \neq 0$, *then* $\mathcal{F}_h\{v_{mj}\}(\xi) = \hat{v}(\xi/m)/|m|$.

(e) Conjugation. $\mathcal{F}_h\{\overline{v}\}(\xi) = \overline{\hat{v}(-\xi)}$.

The symmetry properties of the Fourier transform summarized in Theorem 2.3 apply to the semidiscrete Fourier transform too; we shall not repeat the list here.

We come now to a fundamental result that describes the relationship of the Fourier transform of a continuous function u to that of a discretization v of u—or if x and ξ are interchanged, the relationship of Fourier series to Fourier transforms. Recall that because of the phenomenon of **aliasing**, all wave numbers $\xi + 2\pi j/h$, $j \in \mathbb{Z}$, are indistinguishable on the grid $h\mathbb{Z}$. Suppose that $u \in L^2$ is a sufficiently smooth function defined on $\mathbb{R}$, and let $v \in \ell_h^2$ be the discretization obtained by sampling $u(x)$ at the points x_j. The aliasing principle implies that $\hat{v}(\xi)$ must consist of the sum of all of the values $\hat{u}(\xi + 2\pi j/h)$. This result is known as the **Poisson summation formula** or the **aliasing formula**:

ALIASING FORMULA

Theorem 2.7. *Let $u \in L^2$ be sufficiently smooth [?], with Fourier transform $\hat{u}$, and let $v \in \ell_h^2$ be the restriction of u to the grid $h\mathbb{Z}$. Then*

$$\hat{v}(\xi) = \sum_{j=-\infty}^{\infty} \hat{u}(\xi + 2\pi j/h), \qquad \xi \in [-\pi/h, \pi/h]. \tag{2.2.10}$$

Proof. Not yet written. See P. Henrici, *Applied and Computational Complex Analysis, v. 3,* Wiley, 1986. ∎

In applications, we are very often concerned with functions v obtained by discretization, and it will be useful to know how much the Fourier transform is affected in the process. Theorems 2.4 and 2.7 combine to give the following answers to this question:

EFFECT OF DISCRETIZATION ON THE FOURIER TRANSFORM

Theorem 2.8. *Let v be the restriction to the grid $h\mathbb{Z}$ of a function $u \in L^2$. The following estimates hold uniformly for all $\xi \in [-\pi/h, \pi/h]$, or a forteriori, for ξ in any fixed interval $[-A, A]$.*

(a) If u has $p-1$ continuous derivatives in L^2 for some $p \geq 1$ [?], and a pth derivative in L^2 that has bounded variation, then

$$|\hat{v}(\xi) - \hat{u}(\xi)| = O(h^{p+1}) \qquad \text{as } h \to 0. \tag{2.2.11}$$

(b) If u has infinitely many continuous derivatives in L^2, then

$$|\hat{v}(\xi) - \hat{u}(\xi)| = O(h^{M}) \qquad \text{as } h \to 0 \quad \text{for all } M. \tag{2.2.12}$$

(c) If u can be extended to an analytic function of $z = x+iy$ in the complex strip $|\operatorname{Im} z| < a$ for some $a > 0$, with $\|u(\cdot + iy)\| \leq$ const uniformly for each $-a < y < a$, then for any $\epsilon > 0$,

$$|\hat{v}(\xi) - \hat{u}(\xi)| = O(e^{-\pi(a-\epsilon)/h}) \qquad \text{as } h \to 0. \tag{2.2.13}$$

(d) If u can be extended to an entire function of $z = x+iy$ with $u(z) = O(e^{a|z|})$ as $|z| \to \infty$ ($z \in \mathbb{C}$) for some $a > 0$, then

$$\hat{v}(\xi) = \hat{u}(\xi) \qquad \text{provided } h < \pi/a. \tag{2.2.14}$$

In part *(c)*, $u(\cdot + iy)$ denotes a function of x, namely $u(x+iy)$ with x interpreted as a variable and y as a fixed parameter.

Proof. In each part of the theorem, $u(x)$ is smooth enough for Theorem 2.7 to apply, which gives the identity

$$\hat{v}(\xi) - \hat{u}(\xi) = \sum_{j=1}^{\infty} \hat{u}(\xi + 2\pi j/h) + \hat{u}(\xi - 2\pi j/h). \tag{2.2.15}$$

Note that since $\xi \in [-\pi/h, \pi/h]$, the arguments of $\hat{u}$ in this series have magnitudes at least π/h, $2\pi/h$, $3\pi/h, \ldots$.

For part *(a)*, Theorem 2.4*(a)* asserts that $|\hat{u}(\xi)| \leq C_1 |\xi|^{-p-1}$ for some constant C_1. With (2.2.15) this implies

$$|\hat{v}(\xi) - \hat{u}(\xi)| \leq C_1 \sum_{j=1}^{\infty} (j\pi/h)^{-p-1} = C_2 h^{p+1} \sum_{j=1}^{\infty} j^{-p-1}.$$

Since $p \geq 1$ this sum converges to a constant, which implies (2.2.11) as required.

Part *(b)* follows from part *(a)*.

For part *(c)*, ... [?]

For part *(d)*, note that if $h < \pi/a$, then $\pi/h > a$. Thus (2.2.15) reduces to 0 for all $\xi \in [-\pi/h, \pi/h]$, as claimed. ∎

Note that part *(d)* of Theorem 2.8 asserts that on a grid of size h, the semidiscrete Fourier transform is exact for band-limited functions containing energy only at wave numbers

$|\xi|$ smaller than π/h—the **Nyquist wave number**, corresponding to two grid points per wavelength. This two-points-per-wavelength restriction is famous among engineers, and has practical consequences in everything from fighter planes to compact disc players. When we come to discretize solutions of partial differential equations, two points per wavelength will be the coarsest resolution we can hope for under normal circumstances.

EXERCISES

▷ *2.2.1.*

(a) Prove (2.2.2): $\ell_h^1 \subseteq \ell_h^2 \subseteq \ell_h^\infty$.

(b) Give examples to show that these inclusions are proper: $\ell_h^2 \not\subseteq \ell_h^1$ and $\ell_h^\infty \not\subseteq \ell_h^2$.

(c) Give examples to show that neither inclusion in *(a)* holds for functions on continuous domains: $L^1 \not\subseteq L^2$ and $L^2 \not\subseteq L^\infty$.

▷ *2.2.2.* Let $\delta, \mu : \ell_h^2 \to \ell_h^2$ be the discrete differentiation and smoothing operators defined by

$$(\delta v)_j = \frac{1}{2h}(v_{j+1} - v_{j-1}), \qquad (\mu v)_j = \tfrac{1}{2}(v_{j-1} + v_{j+1}). \tag{2.2.16}$$

(a) Show that δ and μ are equivalent to convolutions with appropriate sequences $d, m \in \ell_h^2$. (Be careful with factors of h.)

(b) Compute the Fourier transforms $\hat{d}$ and $\hat{m}$. How does $\hat{d}$ compare to the transform of the exact differentiation operator for functions defined on $\mathbb{R}$ (Theorem 2.2f)? Illustrate this comparison with a sketch of $\hat{d}(\xi)$ against ξ.

(c) Compute $\|d\|$, $\|\hat{d}\|$, $\|m\|$, and $\|\hat{m}\|$, and verify Parseval's equality.

(d) Compute the Fourier transforms of the convolution sequences corresponding to the iterated operators δ^p and μ^p $(p \ge 2)$. Discuss how these results relate to the rule of thumb discussed in the last section: the smoother the function, the more rapidly its Fourier transform decays as $|\xi| \to \infty$. What imperfection in μ does this analysis bring to light?

▷ *2.2.3. Continuation of Exercise 2.1.6.* Let v be the discretization on the grid $h\mathbb{Z}$ of the function $u(x) = (1+x^2)^{-1}$.

(a) Determine $\hat{v}(\xi)$. (*Hint:* calculating it from the definition (2.2.6) is very difficult.)

(b) How fast does $\hat{v}(\xi)$ approach $\hat{u}(\xi)$ as $h \to 0$? Give a precise answer based on *(a)*, then compare your answer with the prediction of Theorem 2.8.

(c) What would the answer to *(b)* have been if the roles of u and $\hat{u}$ had been interchanged—that is, if v had been the discretization not of $u(x)$ but of its Fourier transform?

▷ *2.2.4. Integration by the trapezoid rule.* A function $u \in L^2 \cap L^1$ can be integrated approximately by the trapezoid rule:

$$I = \int_{-\infty}^{\infty} u(x)\,dx \quad \approx \quad I_h = h \sum_{j=-\infty}^{\infty} u(x_j). \tag{2.2.17}$$

This is an infinite sum, but in practice one might delete the tails if u decays sufficiently rapidly as $|x| \to \infty$. (This idea leads to excellent quadrature algorithms even for finite intervals, which are first transformed to the real axis by a change of variables; for a survey see M. Mori, "Quadrature formulas obtained by variable transformation and the DE-rule," *J. Comp. Appl. Math. 12 & 13* (1985), 119–130.)

As $h \to 0$, how good an approximation is I_h to the exact integral I? Of course the answer will depend on the smoothness of $u(x)$.

(*a*) State how I_h is related to the semidiscrete Fourier transform.

(*b*) Give a bound for $|I_h - I|$ based on the theorems of this section.

(*c*) In particular, what can you say about $|I_h - I|$ for the function $u(x) = e^{-x^2}$?

(*d*) Show that your bound can be improved in a certain sense by a factor of 2.

▶ *2.2.5.* Draw a plot of $\sin n$ as a function of n, where n ranges over the integers (*not* the real numbers) from 1 to 5000. (That is, your plot should contain 5000 dots; in Matlab this can be done in one line.) Explain why the plot looks the way it does to the human eye, and what this has to do with aliasing. Make your explanation precise and quantitative. (See G. Strang, *Calculus,* Wellesley-Cambridge Press, 1991.)

2.3. Interpolation and sinc functions

[This section is not written yet, but here's a hint as to what will be in it.]

If δ_j is the Kronecker delta function

$$\delta_j = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0, \end{cases} \tag{2.3.1}$$

then (2.2.6) gives the semidiscrete Fourier transform

$$\hat{\delta}_j(\xi) = h \qquad (\text{for all } \xi).$$

If we now apply the inverse transform formula (2.2.7), we find after a little algebra

$$\delta_j = \frac{\sin(\pi x_j/h)}{\pi x_j/h}, \tag{2.3.2}$$

at least for $j \neq 0$. Since x_j/h is a nonzero integer for each $j \neq 0$, the sines are zero and this formula matches (2.3.1).

Suppose, however, that we evaluate (2.3.2) not just for $x = x_j$ but for all values $x \in \mathbb{R}$. Then we've got a sinc function again, one that can be called a **grid sinc function**:

$$S_h(x) = \frac{\sin(\pi x/h)}{\pi x/h}. \tag{2.3.3}$$

The plot of $S_h(x)$ is the same as the upper-right plot of Figure 2.1.2, except scaled so that the zeros are on the grid (i.e. at integer multiples of h). Obviously $S_h(x)$ is a continuous interpolant to the discrete delta function δ_j. Which one? It is the unique **band-limited interpolant**, band-limited in the sense that its Fourier transform $\widehat{S_h}(\xi)$ is zero for $\xi \notin [-\pi/h, \pi/h]$. (Proof: by construction it's band-limited in that way, and uniqueness can be proved via an argument by contradiction, making use of Parseval's equality (2.2.8).)

More generally, suppose we have an arbitrary grid function v_j (well, not quite arbitrary; we'll need certain integrability assumptions, but let's forget that for now). Then the **band-limited interpolant** to v_j is the unique function $v(x)$ defined for $x \in \mathbb{R}$ with $v(x_j) = v_j$ and $\hat{v}(\xi) = 0$ for $\xi \notin [-\pi/h, \pi/h]$. It can be derived in two equivalent ways:

Method 1: Fourier transform. Given v_j, compute the semidiscrete Fourier transform $\hat{v}(\xi)$. Then invert that transform, and evaluate the resulting formula for all x rather than just on the grid.

Method 2: linear combination of sinc functions. Write

$$v_j = \sum_{m=-\infty}^{\infty} v_m \delta_{m-j},$$

and then set

$$v(x) = \sum_{m=-\infty}^{\infty} v_m S_h(x - x_m).$$

The equivalence of Methods 1 and 2 is trivial; it follows from the linearity and translation-invariance of all the processes in question.

The consideration of band-limited interpolation is a good way to get insight into the Aliasing Formula presented as Theorem 2.7. (In fact, maybe that should go in this section.) The following schema summarizes everything. Study it!

$$\begin{array}{lcl}
u(x) & \xrightarrow{F.T.} & \hat{u}(\xi) \\
\downarrow \textit{DISCRETIZE} & & \updownarrow \begin{array}{l}\textit{ALIASING} \\ \textit{FORMULA}\end{array} \\
v_j & \xrightarrow{F.T.} & \hat{v}(\xi) \\
\downarrow \begin{array}{l}\textit{BAND-LIMITED} \\ \textit{INTERPOLATION}\end{array} & & \downarrow \begin{array}{l}\textit{ZERO HIGH} \\ \textit{WAVE NOS.}\end{array} \\
v(x) & \xrightarrow{F.T.} & \underline{\hat{v}}(\xi)
\end{array}$$

The **Gibbs phenomenon** is a particular phenomenon of band-limited interpolation that has received much attention. After an initial discovery by Wilbraham in 1848, it was made famous by Michelson in 1898 in a letter to *Nature*, and then by an analysis by Gibbs in *Nature* the next year. Gibbs showed that if the step function

$$u(x) = \begin{cases} +1 & x < 0, \\ -1 & x > 0 \end{cases}$$

is sampled on a grid and then interpolated in the band-limited manner, then the resulting function $v(x)$ exhibits a ringing effect: it overshoots the limits ± 1 by about 9%, achieving a maximum amplitude

$$\int_{-1}^{1} \frac{\sin(\pi y)}{\pi y} dy \approx 1.089490. \qquad (2.3.4)$$

The ringing is scale-invariant; it does not go away as $h \to 0$. In the final text I will illustrate the Gibbs phenomenon and include a quick derivation of (2.3.4).

2.4. The discrete Fourier transform

Note: although the results of the last two sections will be used throughout the remainder of the book, the material of the present section will not be needed until Chapters 8 and 9.

For the discrete Fourier transform, both x and ξ inhabit discrete, bounded domains—or if your prefer, they are periodic functions defined on discrete, unbounded domains. Thus there is a pleasing symmetry here, as with the Fourier transform, that was missing in the semidiscrete case.

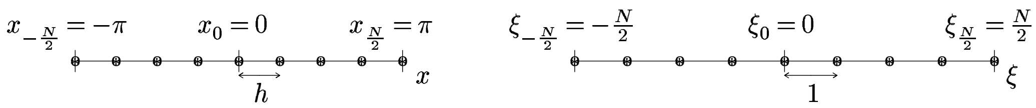


Figure 2.4.1. Space and wave number domains for the discrete Fourier transform.

For the fundamental spatial domain we shall take $[-\pi,\pi)$, as illustrated in Figure 2.4.1. Let N be a positive even integer, set

$$h = \frac{2\pi}{N} \qquad (N \text{ even}), \tag{2.4.1}$$

and define $x_j = jh$ for any j. The grid points in the fundamental domain are

$$x_{-N/2} = -\pi, \dots, \quad x_0 = 0, \dots, \quad x_{N/2-1} = \pi - h.$$

An invaluable identity to keep in mind is this:

$$\frac{N}{2} = \frac{\pi}{h}. \tag{2.4.2}$$

Let ℓ_N^2 denote the set of functions on $\{x_j\}$ that are N-periodic with respect to j, i.e, 2π-periodic with respect to x, with the norm

$$\|v\| = \Big[h \sum_{j=-N/2}^{N/2-1} |v_j|^2 \Big]^{1/2}. \tag{2.4.3}$$

(Since the sum is finite, the norm is finite, so every function of the required type is guaranteed to belong to ℓ_N^2—and to ℓ_N^1 and ℓ_N^∞.) The **discrete Fourier transform (DFT)** of a function $v \in \ell_N^2$ is defined by

$$\hat{v}(\xi) = (\mathcal{F}_N v)(\xi) = h \sum_{j=-N/2}^{N/2-1} e^{-i\xi x_j} v_j, \qquad \xi \in \mathbb{Z}.$$

Since the spatial domain is periodic, the set of wave numbers ξ is discrete, and in fact ξ ranges precisely over the set of integers $\mathbb{Z}$. Thus it is natural to use ξ as a subscript,

$$\hat{v}_\xi = (\mathcal{F}_N v)_\xi = h \sum_{j=-N/2}^{N/2-1} e^{-i\xi jh} v_j, \qquad \xi \in \mathbb{Z},$$

and since $h = 2\pi/N$, $\hat{v}_\xi$ is N-periodic as a function of ξ. We shall take $[-N/2, N/2]$ as the fundamental domain of wave numbers, and let L_N^2 denote the set of N-periodic functions on the grid $\mathbb{Z}$, with norm

$$\|\hat{v}\| = \Big[\sum_{\xi=-N/2}^{N/2-1} |\hat{v}_\xi|^2 \Big]^{1/2}. \tag{2.4.4}$$

This is nonstandard notation, for an upper case L is normally reserved for a family of functions defined on a continuum. We use it here to highlight the relationship of the discrete Fourier transform with the semidiscrete Fourier transform.

The convolution of two functions in ℓ_N^2 is defined by

$$(v * w)_m = h \sum_{j=-N/2}^{N/2-1} v_{m-j} w_j = h \sum_{j=-N/2}^{N/2-1} v_j w_{m-j}. \tag{2.4.5}$$

Again, since the sum is finite, there is no question of convergence.

Here is a summary of the discrete Fourier transform:

THE DISCRETE FOURIER TRANSFORM

Theorem 2.9. *If $v \in \ell_N^2$, then the discrete Fourier transform*

$$\hat{v}_\xi = (\mathcal{F}_N v)_\xi = h \sum_{j=-N/2}^{N/2-1} e^{-i\xi jh} v_j, \qquad -\frac{N}{2} \le \xi \le \frac{N}{2} - 1 \tag{2.4.6}$$

belongs to L_N^2, and v can be recovered from $\hat{v}$ by the **inverse discrete Fourier transform**

$$v_j = (\mathcal{F}_N^{-1}\hat{v})_j = \frac{1}{2\pi} \sum_{\xi=-N/2}^{N/2-1} e^{i\xi jh} \hat{v}_\xi. \tag{2.4.7}$$

The ℓ_N^2-norm of v and the L_N^2-norm of $\hat{v}$ are related by **Parseval's equality**,

$$\|\hat{v}\| = \sqrt{2\pi}\|v\|. \tag{2.4.8}$$

If $v, w \in \ell_N^2$, then $\widehat{v*w}$ satisfies

$$(\widehat{v*w})_\xi = \hat{v}_\xi \hat{w}_\xi. \tag{2.4.9}$$

As with the other Fourier transforms we have considered, the following properties of the discrete Fourier transform will be useful. Once again we take advantage of the fact that $\hat{v}(\xi)$ can be treated as a periodic function defined for all $\xi \in \mathbb{Z}$.

PROPERTIES OF THE DISCRETE FOURIER TRANSFORM

Theorem 2.10. *Let $v, w \in \ell_N^2$ have discrete Fourier transforms $\hat{v}, \hat{w}$. Then:*

(a) Linearity. $\mathcal{F}_N\{v+w\}(\xi) = \hat{v}(\xi) + \hat{w}(\xi)$; $\mathcal{F}_N\{cv\}(\xi) = c\hat{v}(\xi)$.

(b) Translation. If $j_0 \in \mathbb{Z}$, then $\mathcal{F}_N\{v_{j+j_0}\}(\xi) = e^{i\xi x_{j_0}}\hat{v}(\xi)$.

(c) Modulation. If $\xi_0 \in \mathbb{Z}$, then $\mathcal{F}_N\{e^{i\xi_0 x_j} v_j\}(\xi) = \hat{v}(\xi - \xi_0)$.

(e) Conjugation. $\mathcal{F}_N\{\overline{v}\}(\xi) = \overline{\hat{v}(-\xi)}$.

(g) Inversion. $\mathcal{F}_N^{-1}\{v\}(\xi) = \dfrac{1}{2\pi h}\hat{v}(-\xi)$.

An enormously important fact about discrete Fourier transforms is that they can be computed rapidly by the recursive algorithm known as the **fast Fourier transform** (FFT).* A direct implementation of (2.4.6) or (2.4.7) requires $\Theta(N^2)$ arithmetic operations, but the

* The fast Fourier transform was discovered by Gauss in 1805 at the age of 28, but although he wrote a paper on the subject, he did not publish it, and the idea was more or less lost until its celebrated

FFT is based upon a recursion that reduces this figure to $\Theta(N \log N)$. We shall not describe the details of the FFT here, but refer the reader to various books in numerical analysis, signal processing, or other fields. However, to illustrate how simple an implementation of this idea may be, Figure 2.4.2 reproduces the original Fortran program that appeared in a 1965 paper by Cooley, Lewis, and Welch.† Assuming that N is a power of 2, it computes $2\pi\mathcal{F}_N^{-1}$, in our notation: the vector $A(1\!:\!N)$ represents $\hat{v}_0, \dots, \hat{v}_{N-1}$ on input and $2\pi v_0, \dots, 2\pi v_{N-1}$ on output.

```
      subroutine fft(a,m)
      complex a(1),u,w,t
      n = 2**m
      nv2 = n/2
      nm1 = n-1
      j=1
      do 7 i = 1,nm1
        if (i.ge.j) goto 5
        t = a(j)
        a(j) = a(i)
        a(i) = t
5       k = nv2
6       if (k.ge.j) goto 7
        j = j-k
        k = k/2
        goto 6
7       j = j+k
      do 20 l = 1,m
        le = 2**l
        le1 = le/2
        u = 1.
        ang = 3.14159265358979/le1
        w = cmplx(cos(ang),sin(ang))
        do 20 j = 1,le1
          do 10 i = j,n,le
            ip = i+le1
            t = a(ip)*u
            a(ip) = a(i)-t
10          a(i) = a(i)+t
20        u = u*w
      return
      end
```

Figure 2.4.2. Complex inverse FFT program of Cooley, Lewis, and Welch (1965).

As mentioned above, this program computes the *inverse* Fourier transform according to our definitions, times 2π. The same program can be used for the forward transform by making use of the following identity:

rediscovery by Cooley and Tukey in 1965. (See M. T. Heideman, et al., "Gauss and the history of the fast Fourier transform," *IEEE ASSP Magazine*, October 1984.) Since then, fast Fourier transforms have changed prevailing computational practices in many areas.

†Before publication, permission to print this program will be secured.

$$\hat{v}_\xi = \overline{\mathcal{F}_N\{\bar{v}\}(-\xi)} = 2\pi h \overline{\mathcal{F}_N^{-1}\{\bar{v}\}(\xi)}. \tag{2.4.10}$$

These equalities follow from parts *(e)* and *(g)* of Theorem 2.10, respectively.

2.5. Vectors and multiple space dimensions

Fourier analysis generalizes with surprising ease to situations where the independent variable x and/or the dependent variable u are vectors. We shall only sketch the essentials, which are based on the following two ideas:

- If x is a d-vector, then the dual variable ξ is a d-vector too, and the Fourier integral is a multiple integral involving the inner product $x \cdot \xi$;
- If u is an N-vector, then its Fourier transform $\hat{u}$ is an N-vector too, and is defined componentwise.

As these statements suggest, our notation will be as follows:

$$\begin{aligned} d = & \text{ number of space dimensions: } x = (x_1, \dots, x_d)^T, \\ N = & \text{ number of dependent variables: } u = (u_1, \dots, u_N)^T. \end{aligned}$$

Both ξ and $\hat{u}$ become vectors of the same dimensions,

$$\xi = (\xi_1, \dots, \xi_d)^T, \qquad \hat{u} = (\hat{u}_1, \dots, \hat{u}_N)^T,$$

and ξx becomes the dot product $\xi \cdot x = \xi_1 x_1 + \cdots + \xi_d x_d$. The formulas for the Fourier transform and its inverse read

$$\begin{aligned} \hat{u}(\xi) = (\mathcal{F}u)(\xi) &= \int e^{-i\xi \cdot x} u(x)\, dx \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i\xi \cdot x} u(x)\, dx_1 \cdots dx_d \end{aligned} \tag{2.5.1}$$

for $\xi \in \mathbb{R}^d$, and

$$\begin{aligned} u(x) = (\mathcal{F}^{-1}\hat{u})(x) &= (2\pi)^{-d} \int e^{i\xi \cdot x} \hat{u}(\xi) d\xi \\ &= (2\pi)^{-d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\xi \cdot x} \hat{u}(\xi)\, d\xi_1 \cdots d\xi_d \end{aligned} \tag{2.5.2}$$

for $x \in \mathbb{R}^d$. In other words, u and $\hat{u}$ are related componentwise:

$$\hat{u}(\xi) = (\hat{u}^{(1)}(\xi), \dots, \hat{u}^{(N)}(\xi))^T. \tag{2.5.3}$$

If the vector L^2-norm is defined by

$$\|u\|^2 = \int \|u(x)\|^2 dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \|u(x)\|^2 dx_1 \cdots dx_d, \tag{2.5.4}$$

where the symbol $\|\cdot\|$ in the integrand denotes the 2-norm on vectors of length N, then **Parseval's equality** for vector Fourier transforms takes the form

$$\|\hat{u}\| = (2\pi)^{d/2}\|u\|. \tag{2.5.5}$$

The set of vector functions with bounded vector 2-norms can be written simply as $(L^2)^N$.

Before speaking of convolutions, we have to go a step further and allow $u(x)$ and $\hat{u}(\xi)$ to be $M\times N$ matrices rather than just N-vectors. The definitions above extend to this further case unchanged, if the symbol $\|\cdot\|$ in the integrand of (2.5.4) now represents the 2-norm (largest singular value) of a matrix. If $u(x)$ is an $M\times P$ matrix and $v(x)$ is a $P\times N$ matrix, then the convolution $u*v$ is defined by

$$\begin{aligned}(u*v)(x) &= \int u(x-y)v(y)\,dy\\ &= \int u(y)v(x-y)\,dy\\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} u(x-y)v(y)\,dy_1\cdots dy_d,\end{aligned} \tag{2.5.6}$$

and it satisfies

$$\widehat{u*v}(\xi) = \hat{u}(\xi)\hat{v}(\xi). \tag{2.5.7}$$

Since matrices do not commute in general, it is no longer possible to exchange u and v as in (2.1.4).

This generalization of Fourier transforms and convolutions to matrix functions is far from idle, for we shall need it for the Fourier analysis of multistep finite difference approximations such as the leap frog formula.

Similar generalizations of our scalar results hold for semidiscrete and discrete Fourier transforms.

EXERCISES

▷ *2.5.1.* What is the Fourier transform of the vector function

$$u(x) = \left(\frac{\sin x}{x}, \frac{\sin 2x}{2x}\right)^T,$$

defined for $x\in\mathbb{R}$?

▷ *2.5.2.* What is the Fourier transform of the scalar function

$$u(x) = e^{-\frac{1}{2}(x_1^2+x_2^2)},$$

defined for $x=(x_1,x_2)^T\in\mathbb{R}^2$?