# Elliptic hypergeometric solutions to elliptic difference equations.

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*Nacht und Stürme werden Licht*
*Choral Fantasy, Op. 80*

Abstract. It is shown how to define difference equations on particular lattices $\{x_n\}$, $n \in \mathbb{Z}$, where the $x_n$s are values of an elliptic function at a sequence of arguments in arithmetic progression (*elliptic lattice*). Solutions to special difference equations have remarkable simple interpolatory expansions. Unfortunately, only linear difference equations of first order are considered here.

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## 1. Difference equations on elliptic lattices.

We consider functional equations involving the difference operator

$$(\mathcal{D}f)(x) = \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)}, \tag{1}$$

most instances [17] are $(\varphi(x), \psi(x)) = (x, x+h)$, or the more symmetric $(x - h/2, x + h/2)$, or also $(x, qx)$ in $q-$difference equations [10, 12, 13]. Recently, more complicated forms $(r(x) - \sqrt{s(x)}, r(x) + \sqrt{s(x)})$ have appeared [1, 2, 12, 13, 14, 15, 18, 19, 16], where $r$ and $s$ are rational functions.

This latter trend will be examined here: we need, for each $x$, two values $f(\varphi(x))$ and $f(\psi(x))$ for $f$.
A first-order difference equation is $\mathcal{F}(x, f(\varphi(x)), f(\psi(x))) = 0$, or $f(\varphi(x)) - f(\psi(x)) = \mathcal{G}(x, f(\varphi(x)), f(\psi(x)))$ if we want to emphasize the difference of $f$. There is of course some freedom in this latter writing. Only *symmetric* forms in $\varphi$ and $\psi$ will be considered here:

$$(\mathcal{D}f)(x) = \mathscr{F}(x, f(\varphi(x)), f(\psi(x))), \tag{2}$$

where $\mathcal{D}$ is the divided difference operator (1) and where $\mathscr{F}$ is a symmetric function of its two last arguments.

For instance, a linear difference equation of first order may be written as $a(x)f(\varphi(x)) + b(x)f(\psi(x)) + c(x) = 0$,
as well as $\alpha(x)(\mathcal{D}f)(x) = \beta(x)[f(\varphi(x)) + f(\psi(x))] + \gamma(x)$,
with $\alpha(x) = [b(x) - a(x)][\psi(x) - \varphi(x)]/2$, $\beta(x) = -[a(x) + b(x)]/2$, and $\gamma(x) = -c(x)$.

The simplest choice for $\varphi$ and $\psi$ is to take the two determinations of an algebraic function of degree 2, i.e., the two $y-$roots of

$$F(x,y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \tag{3a}$$

where $X_0, X_1$, and $X_2$ are rational functions.

Remark that the sum and the product of $\varphi$ and $\psi$ are the rational functions

$$\varphi + \psi = -X_1/X_2, \quad \varphi\psi = X_0/X_2. \tag{3b}$$

When the divided difference operator $\mathcal{D}$ of (1) is applied to a rational function, the result is still a rational function.

Difference equations must allow the recovery of $f$ on a whole set of points. An initial-value problem for a first order difference equation starts with a value for $f(y_0)$ at $x = x_0$, where $y_0$ is one root of (3a) at $x = x_0$. The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where $y_1$ is the second root of (3a) at $x_0$. We need $x_1$ such that $y_1$ is one of the two roots of (3a) at $x_1$, so for one of the roots of $F(x, y_1) = 0$ which is not $x_0$. Here again, the simplest case is when $F$ is of degree 2 in $x$:

$$F(x,y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \tag{3c}$$

Both forms (3a) and (3c) hold simultaneously when $F$ is ***biquadratic***:

$$F(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}x^i y^j. \tag{4}$$

The construction where successive points on the curve $F(x,y) = 0$ are $(x_n, y_n)$, $(x_n, y_{n+1})$, $(x_{n+1}, y_{n+1})$, is called "T-algorithm" in [23, Theorem 6], see also the Fritz John's algorithm in [3, 4] . The sequence $\{x_n\}$ is then an instance of elliptic lattice, or grid.

Of course, the sequence $\{y_n\}$ is elliptic too, $x_n$ and $y_n$ have elliptic functions representations

$$x_n = \mathcal{E}_1(t_0 + nh), \quad y_n = \mathcal{E}_2(t_0 + nh), \tag{5}$$

where $(x = \mathcal{E}_1(t), y = \mathcal{E}_2(t))$ is a parametric representation of the biquadratic curve $F(x,y) = 0$ with the $F$ of (4).

Note that the names of the $x-$ and $y-$ lattices are sometimes inverted, as in [23, eq. (1.2)]

As $y_n$ and $y_{n+1}$ are the two roots in $t$ of $F(x_n, t) = X_0(x_n) + X_1(x_n)t + X_2(x_n)t^2 = 0$, useful identities are

$$y_n + y_{n+1} = -\frac{X_1(x_n)}{X_2(x_n)}, \quad y_n y_{n+1} = \frac{X_0(x_n)}{X_2(x_n)}, \tag{6}$$

and the direct formula

$$y_n \text{ and } y_{n+1} = \frac{-X_1(x_n) \pm \sqrt{P(x_n)}}{2X_2(x_n)}, \tag{7}$$

where

$$P = X_1^2 - 4X_0X_2 \tag{8}$$

is a polynomial of degree 4.

Also, as $x_{n+1}$ and $x_n$ are the two roots in $t$ of $F(t, y_{n+1}) = 0$,

$$x_n + x_{n+1} = -\frac{Y_1(y_{n+1})}{Y_2(y_{n+1})}, \quad x_n x_{n+1} = \frac{Y_0(y_{n+1})}{Y_2(y_{n+1})}. \tag{9}$$

As the operators considered here are symmetric in $\varphi(x)$ and $\psi(x)$, we do not need to define precisely what $\varphi$ and $\psi$ are. However, once a starting point $(x_0, y_0)$ is chosen, it will be convenient to define $\varphi(x_n) = y_n$ and $\psi(x_n) = y_{n+1}$, $n \in \mathbb{Z}$.

The difference operator applied to a simple rational function is of special interest. Let $f(x) = \dfrac{1}{x-A}$, then $\mathcal{D}\dfrac{1}{x-A} = \dfrac{1}{\psi(x)-\varphi(x)}\left[\dfrac{1}{\psi(x)-A} - \dfrac{1}{\varphi(x)-A}\right] = -\dfrac{1}{(\psi(x)-A)(\varphi(x)-A)} = -\dfrac{X_2(x)}{X_0(x)+AX_1(x)+A^2X_2(x)}$,
and let $\{(x'_n, y'_n), (x'_n, y'_{n+1})\}$ be the elliptic sequence on the biquadratic curve $F(x,y) = 0$ such that $y'_0 = A$, then

$$\mathcal{D}\frac{1}{x-A} = -\frac{X_2(x)}{Y_2(A)(x-x'_0)(x-x'_{-1})} \tag{10}$$

*Special cases.* We already encountered the usual difference operators $(\varphi(x), \psi(x)) = (x, x+h)$ or $(x-h, x)$ or $(x-h/2, x+h/2)$ corresponding to $X_2(x) \equiv 1$, $X_1$ of degree 1, $X_0$ of degree 2 with $P = X_1^2 - 4X_0X_2$ of degree 0. For the geometric difference operator, $P$ is the square of a first degree polynomial. For the Askey-Wilson operator [1, 2, 11, 12, 14, 15], $P$ is an arbitrary second degree polynomial.

## 2. Rational interpolatory elliptic expansions.

Let $\{(x_n, y_n), (x_n, y_{n+1})\}$ be a first elliptic sequence on the biquadratic curve $F(x,y) = 0$, and $\{(x'_n, y'_n), (x'_n, y'_{n+1})\}$ be another elliptic sequence on the same curve. The two sequences have the same formula (5), but with different starting values $t_0$ and $t'_0$.

Rational interpolants of some functions $f$ at $y_0, y_1, \dots$, with poles at $y'_1, y'_2, \dots$, are successive sums

$$c_0 = f(y_0), c_0 + c_1\frac{x-y_0}{x-y'_1}, \dots, \qquad \sum c_k \mathcal{Y}_k(x), \tag{11}$$

where

$$\mathcal{Y}_n(x) = \frac{(x-y_0)\cdots(x-y_{n-1})}{(x-y'_1)\cdots(x-y'_n)}, \mathcal{X}_n(x) = \frac{(x-x_0)\cdots(x-x_{n-1})}{(x-x'_1)\cdots(x-x'_n)}.$$

If, by chance, $c_k$ shows a similar form of ratio of products, we see special cases of hypergeometric expansions!

See that

$$\mathcal{D}\mathcal{Y}_n(x) = C_n X_2(x)\frac{\mathcal{X}_{n-1}(x)}{(x-x'_0)(x-x'_n)} \tag{12}$$

Indeed, $(\varphi(x)-y_0)(\varphi(x)-y_1)\cdots(\varphi(x)-y_{n-1})$ and $(\psi(x)-y_0)(\psi(x)-y_1)\cdots(\psi(x)-y_{n-1})$ both vanish at $x = x_0, x_1, \dots, x_{n-2}$; $(\varphi(x)-y'_1)(\varphi(x)-y'_2)\cdots(\varphi(x)-y'_n)$ vanishes at $x = x'_1, \dots, x'_n$, whereas $(\psi(x)-y'_1)(\psi(x)-y'_2)\cdots(\psi(x)-y'_n)$ vanishes at $x = x'_0, \dots, x'_{n-1}$.

Simple fractions give $\mathcal{D}\dfrac{1}{x-y'_k} = -\dfrac{X_2(x)}{Y_2(y'_k)(x-x'_{k-1})(x-x'_k)}$, as seen earlier in (10).

The constant $C_n$ is found through particular values of $x$, either $x_{-1}$, where $\mathcal{Y}_n(\psi(x)) = 0$ but $\mathcal{Y}_n(\varphi(x)) \neq 0$, or $x_{n-1}$, where $\mathcal{Y}_n(\varphi(x)) = 0$ but $\mathcal{Y}_n(\psi(x)) \neq 0$ :

$$C_n = -\frac{\mathcal{Y}_n(\varphi(x_{-1}) = y_{-1})(x_{-1}-x'_0)(x_{-1}-x'_n)}{(y_0-y_{-1})X_2(x_{-1})\mathcal{X}_{n-1}(x_{-1})} \tag{13a}$$

$$C_n = \frac{\mathcal{Y}_n(\psi(x_{n-1}) = y_n)(x_{n-1}-x'_0)(x_{n-1}-x'_n)}{(y_n-y_{n-1})X_2(x_{n-1})\mathcal{X}_{n-1}(x_{n-1})} \tag{13b}$$

(Of course, $C_0 = 0$). Or through residues at $x'_0$, where $\mathcal{Y}_n(\psi(x)) = \infty$, or $x'_n$ where $\mathcal{Y}_n(\varphi(x)) = \infty$,

$$C_n = \frac{(y'_1-y_0)\cdots(y'_1-y_{n-1})}{\dfrac{d\psi(x'_0)}{dx}(y'_1-y'_2)\cdots(y'_1-y'_n)}\frac{x'_0-x'_n}{(y'_1-y'_0)X_2(x'_0)\mathcal{X}_{n-1}(x'_0)} \tag{13c}$$

$$C_n = -\frac{(y'_n-y_0)\cdots(y'_n-y_{n-1})}{(y'_n-y'_1)\cdots(y'_n-y'_{n-1})\dfrac{d\varphi(x'_n)}{dx}}\frac{x'_n-x'_0}{(y'_{n+1}-y'_n)X_2(x'_n)\mathcal{X}_{n-1}(x'_n)} \tag{13d}$$

Let $(\mathcal{M}f)(x) = [f(\varphi(x)) + f(\psi(x))]/2$.

Also, $2(\mathcal{M}\mathcal{Y}_n)(x) =$

$$\frac{(\varphi(x)-y_0)(\varphi(x)-y_1)\cdots(\varphi(x)-y_{n-1})}{(\varphi(x)-y'_1)(\varphi(x)-y'_2)\cdots(\varphi(x)-y'_n)} + \frac{(\psi(x)-y_0)(\psi(x)-y_1)\cdots(\psi(x)-y_{n-1})}{(\psi(x)-y'_1)(\psi(x)-y'_2)\cdots(\psi(x)-y'_n)}$$

$$= 2D_n(x)\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-2})}{(x-x'_0)(x-x'_1)\cdots(x-x'_n)} = 2D_n(x)\frac{\mathcal{X}_{n-1}(x)}{(x-x'_0)(x-x'_n)},$$

where $D_n$ is a polynomial of degree 2.

Interesting values are found at the same point as in (13a)-(13d):

$$D_n(x_{-1}) = -\frac{C_n X_2(x_{-1})(y_0-y_{-1})}{2}, \tag{14a}$$

$$D_n(x_{n-1}) = \frac{C_n X_2(x_{n-1})(y_n-y_{n-1})}{2}, \tag{14b}$$

$$D_n(x'_0) = \frac{C_n X_2(x'_0)(y'_1-y'_0)}{2}, \tag{14c}$$

$$D_n(x'_n) = -\frac{C_n X_2(x'_n)(y'_{n+1}-y'_n)}{2}, \tag{14d}$$

when $n > 0$. Of course, $D_0 = 1$.

## 3. Linear $1^{\text{st}}$ order difference equations.

$$a(x)(\mathcal{D}f)(x) = c(x)(\mathcal{M}f)(x) + d(x) \tag{15}$$

Where is $b$? The full flexibility of first order difference equations is achieved with the Riccati form [16]

$a(x)(\mathcal{D}f)(x) = b(x)f(\varphi(x))f(\psi(x)) + c(x)[f(\varphi(x)) + f(\psi(x))] + d(x)$

but only linear equations will be considered here. However, (15) already allows elliptic exponentials ($c(x) \equiv a(x)$) or logarithms ($c(x) \equiv 0$).

We now try to expand a solution to (15) as an interpolatory series. If the initial condition is $f(y_0)$ at $x = x_0$, the difference equation allows to find $f(y_1) = \dfrac{[a(x_0)/(y_1 - y_0) + c(x_0)/2]f(y_0) + d(x_0)}{a(x_0)/(y_1 - y_0) - c(x_0)/2}$, $f(y_2), \ldots$ This works fine if no division by zero is encountered. Let us call $x'_0$ one of the roots of the algebraic equation

$$\frac{a(x)}{\psi(x) - \varphi(x)} - \frac{c(x)}{2} = 0, \text{ at } x = x'_0 \tag{16}$$

and let, as usual, $\psi(x'_0) = y'_1$, $\varphi(x'_0) = y'_0$. This shows that $y'_1$ is a singular point of $f$, as trying to compute $f(y'_1)$ from $f(y'_0)$ requires a division by zero. Then $y'_2$, $y'_3, \ldots$ are poles as well. That's why the expansion in (11) starts with poles at $y'_1, y'_2, \ldots$ We also see that such expansions represent meromorphic functions with a natural boundary made of poles. At least, if the poles are spread on a curve, this will be discussed in § 4

We also manage to have the initial value $f(y_0)$ completely determined by the equation, i.e., independent of $f(y_{-1})$, so, considering

$f(y_0) = \dfrac{[a(x_{-1})/(y_0 - y_{-1}) + c(x_{-1})/2]f(y_{-1}) + d(x_{-1})}{a(x_{-1})/(y_0 - y_{-1}) - c(x_{-1})/2}$,

we ask $x_{-1}$ to be a root of

$$\frac{a(x)}{\psi(x) - \varphi(x)} + \frac{c(x)}{2} = 0, \text{ at } x = x_{-1}. \tag{17}$$

Finally, we shall need the polynomials $c$ and $d$ to be of degree 3, with $X_2$ as factor:

$$c(x) = (\beta x + \gamma)X_2(x), \quad d(x) = (\delta x + \epsilon)X_2(x). \tag{18}$$

We now have enough information for understanding the

**Theorem.** *The difference equation* (15) *on the elliptic lattice* $F(x_n, y_n) = 0$ *of* (3a)-(4), *where* $a$, $c$, *and* $d$ *are polynomials of degree* $\leqslant 3$, $X_2$ *being a factor of* $c$ *and* $d$ *as in* (18), *has a solution with the formal expansion* (11), *where* $x_{-1}$ *is a root of* (17) *and* $x'_0$ *is a root of* (16), *with* $c_0 = f(y_0) = \dfrac{d(x_{-1})}{a(x_{-1})/(y_0 - y_{-1}) - c(x_{-1})/2} = -\dfrac{d(x_{-1})}{c(x_{-1})} = -\dfrac{\delta x_{-1} + \epsilon}{\beta x_{-1} + \gamma}$,

$c_1 = \dfrac{(\delta + \beta c_0)(x_0 - x'_1)}{C_1(a(x_0) - c(x_0)(y_1 - y_0)/2)} = \dfrac{(\gamma\delta - \beta\epsilon)(y_1 - y'_1)X_2(x'_0)}{(y_1 - y'_0)(x_0 - x'_0)[a(x_0) - c(x_0)(y_1 - y_0)/2]}$, *and when* $n \geqslant 1$,

$$c_n = c_1 \frac{C_1}{x'_1 - x_0} \frac{x'_n - x_{n-1}}{C_n} \prod_{k=1}^{n-1} \frac{a(x'_k) + c(x'_k)(y'_{k+1} - y'_k)/2}{a(x_k) - c(x_k)(y_{k+1} - y_k)/2} \frac{(x_k - x_{-1})(x_k - x'_0)}{(x'_k - x_{-1})(x'_k - x'_0)}$$
$$= -c_1 \frac{C_1}{x'_1 - x_0} (x'_n - x_{n-1}) \frac{(y_{-1} - y'_1) \cdots (y_{-1} - y'_{n-1}) X_2(x_{-1})(x_{-1} - x_0) \cdots (x_{-1} - x_{n-2})}{(y_{-1} - y_1) \cdots (y_{-1} - y_{n-2})(x_{-1} - x'_0) \cdots (x_{-1} - x'_n)}$$
$$\prod_{k=0}^{n-1} \frac{a(x'_k) + c(x'_k)(y'_{k+1} - y'_k)/2}{a(x_k) - c(x_k)(y_{k+1} - y_k)/2} \frac{(x_k - x_{-1})(x_k - x'_0)}{(x'_k - x_{-1})(x'_k - x'_0)} \qquad (19)$$

**Proof:** put the expansion (11) in

$$d(x) = a(x)\mathcal{D}f(x) - c(x)\mathcal{M}f(x) = \sum_0^\infty c_n \left[a(x)\mathcal{D}\mathcal{Y}_n(x) - c(x)(\mathcal{M}\mathcal{Y}_n(x)\right]$$
$$= -c_0 c(x) + \sum_1^\infty c_n \left[a(x)C_n X_2(x) - c(x)D_n(x)\right] \frac{\mathcal{X}_{n-1}(x)}{(x - x'_0)(x - x'_n)},$$

The polynomial $a(x)C_nX_2(x) - c(x)D_n(x) = [a(x)C_n - (\beta x + \gamma)D_n(x)]X_2(x)$ already has $X_2$ as factor from (18). A factor of degree $\leqslant 3$ remains. Complete factoring follows:

at $x_{-1}$, $a(x)C_nX_2(x) - c(x)D_n(x) = C_nX_2(x_{-1})[a(x_{-1}) + (y_0 - y_{-1})c(x_{-1})/2] = 0$ from (14a) and (17),

at $x'_0$, $a(x)C_nX_2(x) - c(x)D_n(x) = C_nX_2(x'_0)[a(x'_0) - (y'_1 - y'_0)c(x'_0)/2] = 0$ from (14c) and (16),

therefore we have three factors of first degree $a(x)C_nX_2(x) - c(x)D_n(x) = X_2(x)(x - x_{-1})(x - x'_0)[\xi_n(x - x_{n-1}) + \eta_n(x - x'_n)]$, where

$$\xi_n = \frac{a(x'_n)C_nX_2(x'_n) - c(x'_n)D_n(x'_n)}{X_2(x'_n)(x'_n - x_{-1})(x'_n - x'_0)(x'_n - x_{n-1})} = C_n \frac{a(x'_n) + c(x'_n)(y'_{n+1} - y'_n)/2}{(x'_n - x_{-1})(x'_n - x'_0)(x'_n - x_{n-1})} \qquad (20)$$

from (14d),

$$\eta_n = \frac{a(x_{n-1})C_nX_2(x_{n-1}) - c(x_{n-1})D_n(x_{n-1})}{X_2(x_{n-1})(x_{n-1} - x_{-1})(x_{n-1} - x'_0)(x_{n-1} - x'_n)} = C_n \frac{a(x_{n-1}) - c(x_{n-1})(y_n - y_{n-1})/2}{(x_{n-1} - x_{-1})(x_{n-1} - x'_0)(x_{n-1} - x'_n)} \qquad (21)$$

from (14b)

$$0 = a(x)\mathcal{D}f(x) - c(x)\mathcal{M}f(x) - d(x) = -c_0c(x) - d(x) + \sum_1^\infty c_n\left[a(x)C_nX_2(x) - c(x)D_n(x)\right]\frac{\mathcal{X}_{n-1}(x)}{(x-x'_0)(x-x'_n)}$$

$$= -c_0c(x) - d(x) + \sum_1^\infty c_nX_2(x)\left[\xi_n(x-x_{n-1}) + \eta_n(x-x'_n)\right]\frac{(x-x_{-1})(x-x_0)\cdots(x-x_{n-2})}{(x-x'_1)\cdots(x-x'_n)}$$

$$= -c_0c(x) - d(x) + X_2(x)\sum_1^\infty c_n\xi_n\frac{(x-x_{-1})(x-x_0)\cdots(x-x_{n-2})(x-x_{n-1})}{(x-x'_1)\cdots(x-x'_n)}$$

$$+ X_2(x)\sum_1^\infty c_n\eta_n\frac{(x-x_{-1})(x-x_0)\cdots(x-x_{n-2})}{(x-x'_1)\cdots(x-x'_{n-1})}$$

$$= -c_0c(x) - d(x) + c_1X_2(x)\eta_1(x-x_{-1})$$

$$+ X_2(x)\sum_1^\infty(c_n\xi_n + c_{n+1}\eta_{n+1})\frac{(x-x_{-1})(x-x_0)\cdots(x-x_{n-2})(x-x_{n-1})}{(x-x'_1)\cdots(x-x'_n)}$$

$$= (x-x_{-1})X_2(x)\left[-c_0\beta - \delta + c_1\eta_1 + \sum_1^\infty(c_n\xi_n + c_{n+1}\eta_{n+1})\mathcal{X}_n(x)\right]$$

$X_2$ is a factor everywhere, from (18), so $0 = -c_0(\beta x+\gamma) - (\delta x+\epsilon) + c_1C_1\dfrac{a(x_0) - c(x_0)(y_1-y_0)/2}{x_0 - x'_1}(x - x_{-1} + \displaystyle\sum_1^\infty(c_n\xi_n + c_{n+1}\eta_{n+1})\mathcal{X}_n(x)$: $c_0 = f(y_0) = \dfrac{d(x_{-1})}{a(x_{-1})/(y_0-y_{-1}) - c(x_{-1})/2} = -\dfrac{d(x_{-1})}{c(x_{-1})} = -(\delta x_{-1}+\epsilon)/(\beta x_{-1}+\gamma)$,

$$c_1 = \frac{(\delta+\beta c_0)(x_0-x'_1)}{C_1(a(x_0) - c(x_0)(y_1-y_0)/2)} = \frac{(\gamma\delta - \beta\epsilon)(y_1-y'_1)X_2(x'_0)}{(y_1-y'_0)(x_0-x'_0)(a(x_0)-c(x_0)(y_1-y_0)/2)},$$

as $$\frac{c_{n+1}}{c_n} = -\frac{\xi_n}{\eta_{n+1}} = -\frac{C_n}{C_{n+1}}\frac{a(x'_n) + c(x'_n)(y'_{n+1}-y'_n)/2}{a(x_n) - c(x_n)(y_{n+1}-y_n)/2}\frac{(x_n-x_{-1})(x_n-x'_0)(x_n-x'_{n+1})}{(x'_n-x_{-1})(x'_n-x'_0)(x'_n-x_{n-1})}$$

$$c_n = \ldots\frac{x'_n - x_{n-1})}{C_n}\prod^{n-1}\frac{a(x'_k) + c(x'_k)(y'_{k+1}-y'_k)/2}{a(x_k) + c(x_k)(y_{k+1}-y_k)/2}\mathcal{X}_n(x_{-1})\mathcal{X}_n(x'_0)$$

The formula (19) achieves a construction of hypergeometric type, as each term is a product of values of elliptic functions with arguments in arithmetic progression. The exact order of each term, i.e., the number of zeros and poles in a minimal parallelogram, is not obvious [22]. Of course, a factor like, say, $x_{-1} - x_k$ is an elliptic function of order 2 of $t_0 + kh$ from (5). The same order holds for the ratio $\dfrac{x_{-1}-x_k}{y_{-1}-y_k} = \dfrac{x_{-1} - \mathcal{E}_1(t_0+kh)}{y_{-1} - \mathcal{E}_2(t_0+kh)}$, as zeros of the numerator and the denominator cancel each other.

Similar effects probably hold in other ratios encountered in (19), such as $\dfrac{a(x_k) - c(x_k)(y_{k+1}-y_k)/2}{(x_k - x_{-1})(x_k - x'_0)}$ but it is not clear if more can be obtained by keeping elementary means, or if more elliptic function machinery (theta functions) is needed.

## 4. A word on convergence.

We expect products occurring in (11) to behave like powers, like $\prod_1^n (x - x_k) = \prod_1^n (x - \mathcal{E}(t_0 + kh)) \approx \Phi_+(x)^n$. What is $\Phi_+(x) = \exp \mathcal{V}_+(x)$, where $\mathcal{V}_+$ is the complex potential of the distributions of the $x_k$s. For the $x'_k$s, we write $\mathcal{V}_-(x)$.

Let $h$ be a ***real*** irrational multiple of a period $\omega$, then the same factors reappear approximately in the product after $N$ steps if $Nh$ is close to an integer times $\omega$. $\Phi(x)$ is the limit of the $N^{\text{th}}$ roots of such products. The various $kh$, for $k = 1, 2, \ldots, N$, modulo $\omega$, fill uniformly the segment $[0, \omega]$, and the $x_k$s fill a curve which is the set of $\mathcal{E}(t_0 + u), u \in [0, \omega]$:

for any $j$ in $\{1, 2, \ldots, N\}$, there is a $k$ such that $kh$ is close to $j\omega/N$ modulo $\omega$. Indeed, let $Nh$ be close to $M_N\omega$, with $\gcd(N, M_N) = 1$.Then,

$$kh - \frac{j\omega}{N} = \omega \left( \frac{h}{\omega} - \frac{M_N}{N} \right) k + \omega \frac{kM_N - j}{N},$$

to any $j$, there are integers $k$ and $m$ such that $kM_N - mN = j$ (Bezout).

So, we rearrange the product as $\Phi(x) \sim \left[ \prod_{j=1}^N (x - \mathcal{E}(j\omega/N + t_0)) \right]^{1/N} \sim \exp \left[ \frac{1}{\omega} \int_0^\omega \log(x - \mathcal{E}(u + t_0))\, du \right]$.

As $\mathcal{E}$ is the inversion of an elliptic integral of the first kind, $u + t_0 = \int^{\mathcal{E}} \frac{dv}{\sqrt{P(v)}}$, we have $\Phi(x) = \exp \left[ \frac{1}{\omega} \int_{\{x_n\}} \frac{\log(x - v)\, dv}{\sqrt{P(v)}} \right]$, where $\{x_n\}$ is the locus of the $x_n$s $= \{\mathcal{E}(u + t_0)\}, u \in [0, \omega]$. The constant $1/\omega$ is such that $\Phi(x) \sim x$ for large $x$: $\omega = \int_{\{x_n\}} \frac{dv}{\sqrt{P(v)}}$.

So, let the complex potential $\mathcal{V}_+(x) = \frac{1}{\omega} \int_{\{x_n\}} \frac{\log(x - v)\, dv}{\sqrt{P(v)}}$,

($\mathcal{V}_-$ will be used with the $x'_n$s)

The formula for the potential will be linear after a convenient conformal map.

derivative: $\mathcal{V}'_+(x) = \frac{1}{\omega} \int_{\{x_n\}} \frac{dv}{(x - v)\sqrt{P(v)}}$,

$P(x)\mathcal{V}'_+(x) = \text{polynomial } + \frac{1}{\omega} \int_{\{x_n\}} \frac{\sqrt{P(v)}\, dv}{x - v}$,

$(P(x)\mathcal{V}'_+(x))' = \text{ (another) polynomial } + \frac{1}{\omega} \int_{\{x_n\}} \frac{P'(v)\, dv}{2(x - v)\sqrt{P(v)}}$,

Finally

$$P(x)\mathcal{V}''_+(x) + P'(x)\mathcal{V}'_+(x)/2 = \text{(still another) pol.}$$

$$(\sqrt{P(x)}\, \mathcal{V}'_+(x)\, )' = \frac{\text{pol.}}{\sqrt{P(x)}}$$

so, $\mathcal{V}'_+(x) =$ an incomplete elliptic integral of the second(?) kind divided by $\sqrt{P(x)}$.

With $\xi$ such that $x = \mathcal{E}(\xi), dx/d\xi = \sqrt{P(x)}$:

$$\frac{d^2\mathcal{V}_+(x)}{d\xi^2} = \text{a pol. (in } x = \mathcal{E}(\xi))$$

What is this polynomial, by the way? $\mathcal{V}'_+(x) = x^{-1} + (\mu_{+,1}/\mu_{+,0})x^{-2} + (\mu_{+,2}/\mu_{+,0})x^{-3} + \cdots$, $\sqrt{P(x)} = \pi_0 x^2 + \pi_1 x + \pi_2 + \cdots$,

the pol. is

$$(\pi_0 x^2 + \pi_1 x + \pi_2 + \cdots)[\pi_0 - (\pi_2 + \pi_1(\mu_{+,1}/\mu_{+,0}) + \pi_0(\mu_{+,2}/\mu_{+,0}))x^{-2} + \cdots]$$
$$= \pi_0^2 x^2 + \pi_1\pi_0 x + \pi_2\pi_0 - \pi_0(\pi_2 + \pi_1(\mu_{+,1}/\mu_{+,0}) + \pi_0(\mu_{+,2}/\mu_{+,0}))$$

where $\mu_{+,k} = \displaystyle\int_{\{x_n\}} \frac{v^k\,dv}{\sqrt{P(v)}}$ is the $k^{\text{th}}$ moment of the contour drawn by the $x_n$s. The result is

$$\pi_0^2 x^2 + \pi_1\pi_0 x + \pi_2\pi_0 - \frac{\pi_0}{\mu_{+,0}} \int_{\{x_n\}} \frac{\pi_2 + \pi_1 v + \pi_0 v^2 = \sqrt{P(v)} - O(1/v)}{\sqrt{P(v)}}\,dv$$

The contour integrals on the $x'_n$s are the same(yes, see later on), so, at last

$$\frac{d^2\mathcal{V}(x)}{d\xi^2} = 0,$$

where $\mathcal{V} = \mathcal{V}_+ - \mathcal{V}_-$.

Jump of $\mathcal{V}'(x)$ when $x$ crosses the $x_n$ line: $\mathcal{V}'(x)_{\text{average}} \pm \pi i \dfrac{1}{\omega}\dfrac{1}{\sqrt{P(x)}}$, or in $\xi$:

$$\left(\frac{d\mathcal{V}}{d\xi}\right)_{\text{average}} \pm \pi i \frac{1}{\omega}$$

A much faster and more complete derivation: $\mathcal{V}_+(x)$ and $\mathcal{V}_-(x)$ are contour integrals on the locii filled by $\{x_n\}$ and $\{x'_n\}$ drawn by $\mathcal{E}(nh + t_0)$ and $\mathcal{E}(nh + t'_0)$. If $x$ is between these two locii, the two contour integrals of $\dfrac{dv}{(x-v)\sqrt{P(v)}}$ are the same for $\mathcal{V}'_+(x)$ and $\mathcal{V}'_-(x)$, up to the residue at $v = x$:

$$\mathcal{V}'(x) = \mathcal{V}'_+(x) - \mathcal{V}'_-(x) = \frac{2\pi i}{\omega\sqrt{P(x)}} \Rightarrow \frac{d\mathcal{V}(x)}{d\xi} = \frac{2\pi i}{\omega}$$

Rate of approximation has already been related to potential problems by Walsh [24], in papers and books going back to the 1930s! See also Ganelius [5]. For more recent surveys and papers, the works by Gončar and colleagues are recommended [6, 7, 8, 9].

The properties of the irrational number relating the step $h$ to a period $\omega$ must also be considered [21].

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