Heun differential equation satisfied by some classical biorthogonal rational functions.

Preliminary notes.

14th International Symposium on Orthogonal Polynomials, Special Functions and Applications
(OPSFA-14), University of Kent, Canterbury, UK

July 3-7, 2017

Alphonse P. Magnus
Math. Université catholique de Louvain,
Louvain-La-Neuve, Belgium
alphonse.magnus@uclouvain.be

François Ndayiragije
Math. University of Burundi
Bujumbura (Burundi)
ndayiragijefrancois@yahoo.fr

André Ronveaux
retired from U. Namur (Belgium)
andre.michel.ronveaux@gmail.com

May 19, 2017 (incomplete and unfinished)

'We must always remember that we are part [sic] of the Continent,
but we must never forget that we are neighbours to it.
Bolingbroke, quoted in André Maurois’s History of England,
the English translation of 1937. The original French text gives the right translation
"voisins, mais non partie".

Abstract. We consider a special family of classical biorthogonal rational functions and
their differential equations. They are NOT hypergeometric, but neighbours to it, actually,
Heun’s differential equations!

1. Biorthogonal rational functions.

1.1. Orthogonality.

Let two sequences \{a_0, a_1, \ldots\}, \{b_0, b_1, \ldots\} be given. With polynomials \(P_n\) and \(Q_m\) of
degrees \(\leq n\) and \(m\), let

\[
 p_n(x) = \frac{P_n(x)}{(x - a_0) \cdots (x - a_n)} \quad \text{and} \quad q_m(x) = \frac{Q_m(x)}{(x - b_0) \cdots (x - b_{m+1})}
\]

be orthogonal when \(m \neq n\) with respect to the bilinear form

\[
\langle f, g \rangle = \int_c^d f(t)g(t)w(t)dt.
\]
This also means that, when \( n \geq 1 \), \( P_n \) is orthogonal to all polynomials of degree \( < n \) with respect to \( \frac{w(t)}{(t-x_0)(t-x_1)\cdots(t-x_{2n+1})} \), where \( x_{2k} = a_k, x_{2k+1} = b_k, k = 0,1,\ldots \) So, \( P_0 = 1, P_1(x) = x-x_3 - \frac{\mu_2}{\mu_3} \), where \( \mu_\nu = \int_c^d \frac{w(t)dt}{(t-x_0)\cdots(t-x_{\nu})} \).

Existence, and unicity up to multiplication by constants, depends on the nonvanishing of some determinants of the moments \( \mu_\nu \)'s [17, 18], see also, of course, introductions to biorthogonality in general [3], [4, § 2.6].

Of course, \( Q_m \) satisfies similar, but not exactly the same, orthogonality conditions. Orthogonality now holds with respect to \( \frac{w(t)}{(t-a_0)(t-a_1)\cdots(t-a_{m-1})(t-b_0)(t-b_1)\cdots(t-b_{m+1})} \) = \( \frac{w(t)}{(t-x_0)(t-x_1)\cdots(t-x_{2m-2})(t-x_{2m-1})(t-x_{2m+1})(t-x_{2m+3})} \). The polynomials \( Q_m \) will not be needed further here.

1.2. Recurrence relations.

One has

\[
P_{n+1}(x) = (r_n x + r'_n) P_n(x) + s_n(x-x_{2n})(x-x_{2n+1}) P_{n-1}(x), \quad P_{-1}(x) \equiv 0, P_0(x) \equiv 1. \quad (2)
\]

cf. \( R_{ff} \)-type in Ismail & Masson [9, p. 14].

Indeed, with \( r_n x + r'_n \) being the first degree polynomial interpolating \( P_{n+1}(x)/P_n(x) \) at \( x = x_{2n} \) and \( x_{2n+1} \), see that \( \frac{P_{n+1}(x) - (r_n x + r'_n) P_n(x)}{(x-x_{2n})(x-x_{2n+1})} \) is orthogonal to any polynomial of degree \( < n - 1 \) with respect to \( w(t)/[(t-x_0)(t-x_1)\cdots(t-x_{n-1})] \).

2. Rational interpolation.

There is a polynomial \( N_n \) of degree \( n + 1 \) such that \( N_n/P_n \) interpolates the Stieltjes, or Markov, function of \( w \):

\[
S(x) = \int_c^d \frac{w(t)dt}{x-t}
\]

at \( x_0, x_1, \ldots, x_{2n+1} \) [12, § 3].

Polynomial interpolation of \( P_n(x)S(x) \) at \( x_0, \ldots, x_{2n+1} \) has normally a degree \( 2n+1 \), but the actual degree is found by describing the interpolant from

\[
\begin{align*}
P_n(x)S(x) &= \int_c^d \frac{P_n(x) - P_n(t)}{x-t} w(t)dt + \int_c^d \frac{P_n(t)w(t)dt}{x-t} \text{, where we replace } 1/(x-t) \text{ by a polynomial} \\
\end{align*}
\]

its polynomial interpolant in \( x \) at \( x = x_0, \ldots, x_{2n+1} \). This interpolant, for a fixed \( t \), is \( \frac{(t-x_0)\cdots(t-x_{2n+1}) - (x-x_0)\cdots(x-x_{2n+1})}{(t-x_0)\cdots(t-x_{2n+1})(x-t)} \), and is a linear combination of ratios \( \frac{t^r - x^r}{x-t} \) with \( r \leq 2n+2 \), so of terms \( t^u x^v \) with \( u + v = r - 1 \leq 2n + 1 \). From the orthogonality properties of \( P_n \), the integrals of \( P_n(t)t^u \) vanish when \( u < n \), only \( x^v \) remain when \( u \geq n \), so \( v = r - 1 - u \leq r - 1 - n \leq 2n + 1 - n = n + 1 \).
\[
\frac{N_n(x)}{P_n(x)} = r_{-1}x + r'_1 + \frac{s_0(x-x_0)(x-x_1)}{r_0x + r'_0 + \cdots} + \frac{s_1(x-x_2)(x-x_3)}{r_{-2}x + r'_1 + \cdots} + \frac{s_{n-1}(x-x_{2n-2})(x-x_{2n-1})}{r_{n-1}x + r'_{n-1}}
\]

\(P_{-1} = 0, P_0 = 1, P_1(x) = r_0x + r'_0, P_2(x) = (r_0x + r'_0)(r_1x + r'_1) + s_1(x-x_0)(x-x_1).\)

\(N_{-1} = 1, N_0(x) = r_{-1}x + r'_1, N_1 = (r_0x + r'_0)(r_{-1}x + r'_1) + s_0(x-x_0)(x-x_1),\) etc.

\(N_0\) interpolates \(S\) at \(x_0\) and \(x_1\), so \(r_{-1} = \frac{S(x_1) - S(x_0)}{x_1 - x_0} = \mu_1.\)

Casonati:

\[N_{n-1}(x)P_n(x) - N_n(x)P_{n-1}(x) = (-1)^ns_0 \cdots s_{n-1}(x-x_0) \cdots (x-x_{2n-1}) \quad (3)\]

**Two-Points Padé Approximants** occur when \(a_k = a, b_k = b, \forall k.\) The interpolation property is then an order property \(\frac{N_n(x)}{P_n(x)} - S(x) = O([(x-a)(x-b)]^{n+1}),\) and (3) above is now

\[N_{n-1}(x)P_n(x) - N_n(x)P_{n-1}(x) = (-1)^ns_0 \cdots s_{n-1}(x-a)^n(x-b)^n \quad (4)\]

The rather strange choice in (1) (why \(b_{m+1}\)?) ensures same powers of \((x-a)\) and \((x-b)\) in (1).

Laurent expansions setting. Let \(z = \frac{x-a}{x-b}, \) so, \(x = \frac{bz - a}{z - 1}.\) Taylor series in powers of \(x-a\) and \(x-b\) are expansions in \(x-a = \frac{(a-b)z}{1-z} = (a-b)(z + z^2 + z^3 + \cdots)\) and \(x-b = \frac{a-b}{z-1} = (a-b)(z^{-1} + z^{-2} + \cdots).\)

### 3. Classical families.

Among the various ways to define a classical family of orthogonal functions [2, § 5], consider the Sonine-Hahn criterion: the derivatives \(P_n\) are themselves involved in some family of biorthogonal rational functions, so they satisfy recurrence relations leading to differential relations and equations for \(P_n.\)

Zhedanov [19] finds linear recurrence relations for the coefficients of the Laurent expansion of \(S\) in powers of \(z = \frac{x-a}{x-b},\) leading to a differential equation of the form \(AdS/dz = BS,\) with polynomials \(A\) and \(B\) of degrees 2 and 1.

Returning to \(x\) such that \(z = (x-a)/(x-b),\) one finds \(X(x)S'(x) = Y(x)S(x),\) and we may suspect that its rational approximation \(N_n(x)/P_n(x) = S(x) + O([(x-a)/(x-b)]^{n+1})\) satisfies almost the same differential equation, and this also leads to differential relations and equations for \(P_n\) (Laguerre [11]).

Most general classical weight function for orthogonal polynomials is essentially the Jacobi \((d-t)^{\alpha}(t-c)^{\beta}.\) Then,
4.1. Biorthogonal relations.

We even take a simpler case $\alpha = -1/2$, $d = -c$, $b = -a$, so to have a completely symmetric situation. In order to have $S(a) = 1$, let

$$S(x) = \sqrt{\frac{a^2 - c^2}{x^2 - c^2}},$$

(5)

where the square root is a continuous function outside $x \in [c, -c]$, so that $S$ is an odd function.

4.1. Recurrence relations.

In particular, $S(b) = S(-a) = -1$, whence the first approximation $N_0(x)/P_0(x) = x/a$ with $r_{-1} = 1/a$. All the $r_n = 0$.

We start the continued fraction expansion:

$$S(x) = \frac{x}{a} + \frac{x^2 - a^2}{a^2 - x^2} \frac{\frac{\sqrt{a^2 - c^2}}{x} + \frac{x}{a}}{x^2 - c^2 + \frac{x}{a}},$$

yielding the ratio $s_0/r_0$. A convenient choice is $r_0 = 2(1 - a^2/c^2)$ and $s_0 = -1 + 2a^2/c^2$. More details on $r_n$ and $s_n$ are not given here, they involve values of the Chebyshev polynomials $T_n$ and $U_n$ at the argument $a/c$, see the method suggested in §.

4.2. Differential relations.

We expect the approximation $N_n/P_n$ to satisfy differential relations close to the differential equation $(x^2 - a^2)dS(x)/dx + xS(x) = 0$: $(x^2 - a^2)(N_n(x)/P_n(x))' + xN_n(x)/P_n(x) = O((x^2 - a^2)^n)$. Multiply by $P_n(x)^2$, the left-hand side being a polynomial, so is the right-hand side:

$$(x^2 - a^2)[N_n(x)P_n(x) - N_n(x)P_n'(x)] + xN_n(x)P_n(x) = (x^2 - a^2)^n\Theta_n(x),$$
where $\Theta_n$ is a polynomial of degree 2 (Laguerre’s [11] notations!)
$\Theta_0(x) = (2x^2 - c^2)/a$, etc.
We also have
\[
[(x^2 - c^2)^{1/2}N_n(x)]'P_n(x) - (x^2 - c^2)^{1/2}N_n(x)P_n(x)' = (x^2 - c^2)^{-1/2}(x^2 - a^2)^n\Theta_n(x).
\] (6)
With $P_n(x) = C_n x^n + \cdots$ and $(x^2 - c^2)^{1/2}N_n(x) = x^{n+2}/a + \cdots$ for large $x$, we find the coeff. of $x^2$ to be $2C_n$: $\Theta_n(x) = 2C_n x^2 + \Theta_n(0)$.
To get rid of the numerator polynomials $N_n$, subtract
\[
(x^2 - c^2) \left( \frac{N_{n+1}(x)}{P_{n+1}(x)} - \frac{N_n(x)}{P_n(x)} \right) + x \left( \frac{N_{n+1}(x)}{P_{n+1}(x)} - \frac{N_n(x)}{P_n(x)} \right) = \frac{(x^2 - a^2)^{n+1}\Theta_{n+1}(x)}{P_{n+1}^2(x)} - \frac{(x^2 - a^2)^n\Theta_n(x)}{P_n^2(x)}.
\]
and derivate the Casorati identity (4) in the left-hand side, so
\[
(-1)^n s_0 \cdots s_n \left[ (x^2 - c^2) \left( \frac{(x^2 - a^2)^{n+1}}{P_n(x)P_{n+1}(x)} \right) + x \frac{(x^2 - a^2)^n\Theta_n(x)}{P_n(x)P_{n+1}(x)} \right] = \frac{(x^2 - a^2)^{n+1}\Theta_{n+1}(x)}{P_{n+1}(x)} - \frac{(x^2 - a^2)^n\Theta_n(x)}{P_n(x)}
\]
\[
(\Omega_n, W_n) = \Omega_n(x)P_n(x)
\]
which is our **differential relation**
\[
UP'_n = V_n P_n + W_n P_{n+1}; \tag{7a}
\]
with $U(x) = (x^2 - a^2)(x^2 - c^2)$, $V_n(x) = 2(n+1)x(x^2 - c^2) + x(x^2 - a^2) - \Omega_n(x)$, and
\[
W_n(x) = \frac{(-1)^n \Theta_n(x)}{s_0 \cdots s_n}
\]
The second equation is
\[
-(x^2 - c^2)(x^2 - a^2)P'_{n+1}(x) - (x^2 - a^2) \frac{(-1)^n \Theta_{n+1}(x)}{s_0 \cdots s_n} P_n(x) = -\Omega_n(x)P_{n+1}(x),
\]
\[
U(x)P'_{n+1}(x) = [2(n+1)x(x^2 - c^2) + x(x^2 - a^2) - \Omega_n(x)]P_{n+1}(x) + s_{n+1}(x^2 - a^2)W_{n+1}(x)P_n(x) \tag{7b}
\]
\[
n \to n + 1 \text{ in (7a) and subtraction:}
\]
\[
V_{n+1}(x)P_{n+1}(x) + W_{n+1}(x) [P_{n+2}(x) - s_{n+1}(x^2 - a^2)P_n(x)] - [2(n+1)x(x^2 - c^2) + x(x^2 - a^2) - \Omega_n(x)]P_{n+1}(x) = 0,
\]
so
\[
V_{n+1}(x) + V_n(x) + x s_{n+1}x W_{n+1}(x) - 2(n+1)x(x^2 - c^2) - x(x^2 - a^2) = 0,
\]
or
\[
2(n+2)x(x^2 - c^2) + x(x^2 - a^2) + x s_{n+1}x W_{n+1}(x) = \Omega_{n+1}(x) + \Omega_n(x),
\]
confirming by the way that the $\Omega_n$s are polynomials, here: odd cubics.
When $x$ is a zero of $U$, i.e., when $x = \pm a$ or $x = \pm c$, (7a) [7b] show
\[
\frac{P_{n+1}(x)}{P_n(x)} = -\frac{V_n(x)}{W_n(x)} = \frac{s_{n+1}(x^2 - a^2)W_{n+1}(x)}{2(n+1)x(x^2 - c^2) + x(x^2 - a^2) - \Omega_n(x)} = \frac{s_{n+1}(x^2 - a^2)W_{n+1}(x)}{V_{n+1}(x) + r_{n+1}xW_{n+1}(x)}.
\]
so, \( V_n(x) [V_{n+1}(x) + r_{n+1}x W_{n+1}(x)] = s_{n+1}(x^2 - a^2) W_n(x) W_n(x) \) at the zeros of \( U \).

### 4.3. Differential equation.

From \( UP''_n = V_n P_n + W_n P_{n+1} \), put \( P'_{n+1} = \{ V_{n+1} P_{n+1} + W_{n+1} r_{n+1} x P_{n+1}(x) + s_{n+1} (x^2 - a^2) P_{n+1}(x) \} / U \), into the derivative of \( P_{n+1} = [UP'n - V_n P_n] / W_n \):

\[
\frac{U P' - V_n P_n}{W_n} = \frac{V_{n+1} + r_{n+1} x W_{n+1}}{U} \frac{U P' - V_n P_n}{W_n} + \frac{s_{n+1} (x^2 - a^2) W_{n+1} P_n}{U},
\]

\[
U W_n P'' + \left\{ U' W_n - U W' - W_n \{ V_n + V_{n+1} + r_{n+1} x W_{n+1} \right\} x [2(n+1)(x^2 - c^2) + x^2 - a^2] \right\} P_n

+ \left\{ V_n W'_n - V W'_n + W_n(x) \frac{V_n(V_{n+1} + r_{n+1} x W_{n+1}) - s_{n+1} (x^2 - a^2) W_{n+1} W_n(x)}{U(x)} \right\} P_n = 0
\]

Remark that the coefficient of \( P' \) inside the curly braces is \(-U W_n\) times the logarithmic derivative of the \textbf{Wronskian} which is therefore \( W_n(x) = (x^2 - c^2)^{-1/2}(x^2 - a^2)^n W_n(x) \), and that is exactly \([6]\) ! So that the solutions of the differential equation are \( y = P_n(x) \) AND \( y = (x^2 - c^2)^{1/2} N_n(x) \) \([10]\) in

\[
U(x) W_n(x) \left\{ y'' + \left[ x \frac{x}{x^2 - c^2} - \frac{2nx}{x^2 - a^2} - \frac{W'_n(x)}{W_n(x)} \right] y' \right\} + K_n(x) y = 0,
\]

where \( K_n \) is a polynomial of degree \( \leq 4 \).

### 4.4. Heun.

Let \( W_n(x) = (W_n''/2)(x^2 - \theta_n) \). The equation \([\S]\) has three singular points \( a^2, c^2, \) and \( \theta_n \) in the bounded \( x^2 \)-plane. This suggests that we are close to the Heun equation

\[
(u - a)(u - b)(u - c) \left\{ \frac{d^2 Y}{du^2} + \left[ \frac{\gamma}{u - a} + \frac{\delta}{u - b} + \frac{\epsilon}{u - c} \right] \frac{d Y}{du} \right\} + (\alpha \beta u - Q) Y = 0.
\]

When \( u \to \infty \), a \( u^\rho \) behaviour implies \( \rho(\rho - 1) + (\gamma + \delta + \epsilon) \rho + \alpha \beta = 0 \Rightarrow \rho = -\alpha \) and \( -\beta \), with \( \alpha + \beta = \gamma + \delta + \epsilon - 1 \) \([5, 8, 13, 15, 16]\). It seems obvious that \([\S]\) becomes \([\S]\) by taking \( u = x^2 \), but a new singularity rises then at the origin. Considering that the expansions of the two solutions of \([\S]\) for large \( x \) are \( P_n(x) = C_n x^n + O(x^{n-2}) \) and \( (x^2 - c^2)^{1/2} N_n(x) = x^{n+2}/a + O(x^n) \), we kill the singularity at \( x = \infty \) by building the differential equation for \( x^{-n-2} y \), and by taking the new variable \( t = 1/x^2 \).

#### 4.4.1. First. , the equation for \( y = x^{-n-2} y \):

\[
U(x) W_n(x) \left\{ x^2 \tilde{y}'' + 2(n+2)x \tilde{y}' + (n+2)(n+1) \tilde{y} + \left[ x \frac{2nx}{x^2 - a^2} - \frac{W'_n(x)}{W_n(x)} \right] \right\} x^2 \tilde{y}' + (n+2)x \tilde{y}
\]

+ \( K_n(x) \tilde{y} = 0 \).

\[
U(x) W_n(x) \left\{ x^2 \tilde{y}'' + x \left[ 3 + \frac{c^2}{x^2 - c^2} - \frac{2n a^2}{x^2 - a^2} - \frac{2\theta_n}{x^2 - \theta_n} \right] \right\} \tilde{y}' + \tilde{K}_n(x) \tilde{y} = 0, \text{ with}
\]

\[
\tilde{K}_n(x) = (n+1)(n+2)U W_n + (n+2)x U W_n \left[ x \frac{x}{x^2 - c^2} - \frac{2nx}{x^2 - a^2} - \frac{W'_n(x)}{W_n(x)} \right] + x^2 K_n(x),
\]

an even polynomial of degree \( \leq 6 \).
Seems worse than before, but the degree of $\tilde{K}_n$ is only 2! Indeed, put the solution $\tilde{y} = ax^{-n-2}(x^2 - c^2)^{1/2}N_n(x) = 1 + \zeta_n/x^2 + \cdots$ for large $x$ in the differential equation:

$$\tilde{K}_n(x) = -U(x)W_n(x) \left\{ x^2 \tilde{y}'' + x \left[ 3 + \frac{c^2}{x^2 - c^2} - \frac{2na^2}{x^2 - a^2} - \frac{2\theta_n}{x^2 - \theta_n} \right] \tilde{y}' \right\} = O(x^2).$$

$\tilde{K}_n(0) = (n + 1)(n + 2)\tilde{U}(0)W_n(0) = -(n + 1)(n + 2)a^2c^2\theta_n$ is especially interesting, $\tilde{K}_n(x) = (\tilde{K}_n'/2)x^2 - (n + 1)(n + 2)a^2c^2\theta_n$.

4.4.2. Finally, $t = 1/x^2 : x = t^{-1/2}$, $Y(t) = \tilde{y}(x)$,

$$\left(1/t - a^2\right)\left(1/t - c^2\right)(1/t - \theta_n) \left\{ t^{-1/2} \frac{d^3}{dt^3} \frac{dY}{dt} - 2t^{-1/2} \left[ 3 + \frac{c^2}{1/t - c^2} - \frac{2na^2}{1/t - a^2} - \frac{2\theta_n}{1/t - \theta_n} \right] t^{3/2} \frac{dY}{dt} \right\} +$$

$\left(\tilde{K}_n'/2\right)t^{-1} - (n + 1)(n + 2)a^2c^2\theta_n Y = 0$,

$$(1 - a^2t)(1 - c^2t)(1 - \theta_nt) \left\{ t^{1/2} \frac{d^2Y}{dt^2} - 2 \left[ \frac{c^2}{1-c^2t} - \frac{2na^2}{1-a^2t} - \frac{2\theta_n}{1-\theta_nt} \right] \frac{dY}{dt} \right\} + t\tilde{K}_n(t^{-1/2})Y = 0$$

is our Heun equation. It has two Frobenius-polynomial solutions $t^{(n+2)/2}P_n(t^{-1/2})$, $(t - 1/c^2)^{1/2}t^{(n+1)/2}N_n(t^{-1/2})$, a much studied property [5, 7, 8, 13, 15, 16].

5. Another square root story.

There is of course an $\infty$ simpler continued fraction expansion of our square root function, with constant coefficients:

$$\sqrt{\frac{a^2 - c^2}{x^2 - c^2}} = \frac{1}{x/a + \frac{c^2(x^2 - a^2)/[a^2(a^2 - c^2)]}{2x/a + \frac{c^2(x^2 - a^2)/[a^2(a^2 - c^2)]}{2x/a + \cdots}}}$$

see also Ismail & Masson [9, Example 3.1, p. 18]. Let $\tilde{P}_n$ be the polynomials of interest here.

They are now true Chebyshev polynomials $\tilde{P}_n(x) = \left( \frac{c^2(x^2 - a^2)}{a^2(c^2 - a^2)} \right)^{n/2} T_n \left( \frac{x}{\sqrt{c^2 - a^2}} \right)$, orthogonal with respect to $w(t)/(t^2 - a^2)^n$ instead of $w(t)/(t^2 - a^2)^{n+1}$. This means that the $P_n$ of before is related to a modified weight function involving multiplication or division by a polynomial (Christoffel, Uvarov): $\frac{P_{n+2}(x)P_n(a) - P_{n+2}(a)P_n(x)}{x^2 - a^2} = \text{constant } \tilde{P}_n(x)$.

Recurrence relations of the $P_n$s involve special values

$$\tilde{P}_n(a) = \lim_{x \to a} 2^{n-1} \left( \frac{c^2(x^2 - a^2)}{a^2(c^2 - a^2)} \right)^{n/2} \left( \frac{x}{\sqrt{c^2 - a^2}} \right)^n = 2^{n-1}.$$ 

Coefficient of $x^n$ in $\tilde{P}_n = \left( \frac{c^2}{a^2(c^2 - a^2)} \right)^{n/2} T_n \left( \sqrt{1 - a^2/c^2} \right)$. 
This may lead to recurrence coefficients, and even differential properties and equations, by particular methods [13, 14], but we plan to study other classical biorthogonal rational functions, when these modified weight function techniques do not apply.

References


