Heun differential equation satisfied by some classical biorthogonal rational functions.

Preliminary notes.

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'We must always remember that we are part [sic] *of the Continent,*
but we must never forget that we are neighbours to it.
Bolingbroke, quoted in André Maurois's *History of England*,
the English translation of 1937. The original French text gives the right translation
"voisins, mais non partie".

Abstract. We consider a special family of classical biorthogonal rational functions and their differential equations. They are NOT hypergeometric, but neighbours to it, actually, ***Heun***'s differential equations!

1. Biorthogonal rational functions.

1.1. Orthogonality.

Let two sequences $\{a_0, a_1, \dots\}, \{b_0, b_1, \dots\}$ be given. With polynomials P_n and Q_m of degrees $\leqslant n$ and m, let

$$p_n(x) = \frac{P_n(x)}{(x-a_0)\cdots(x-a_n)} \quad \text{and} \quad q_m(x) = \frac{Q_m(x)}{(x-b_0)\cdots(x-b_{m+1})} \tag{1}$$

be orthogonal when $m \neq n$ with respect to the bilinear form

$$\langle f, g\rangle = \int_c^d f(t)g(t)w(t)dt.$$

This also means that, when $n \geqslant 1$, P_n is orthogonal to all polynomials of degree $< n$ with respect to $\dfrac{w(t)}{(t-x_0)(t-x_1)\cdots(t-x_{2n+1})}$, where $x_{2k}=a_k, x_{2k+1}=b_k, k=0,1,\dots$ So, $P_0=1, P_1(x)=x-x_3-\dfrac{\mu_2}{\mu_3}$, where $\mu_r=\displaystyle\int_c^d \frac{w(t)dt}{(t-x_0)\cdots(t-x_r)}$.

Existence, and unicity up to multiplication by constants, depends on the nonvanishing of some determinants of the moments μ_rs [17, 18], see also, of course, introductions to biorthogonality in general [3], [4, § 2.6].

Of course, Q_m satisfies similar, but not exactly the same, orthogonality conditions. Orthogonality now holds with respect to $\dfrac{w(t)}{(t-a_0)(t-a_1)\cdots(t-a_{m-1})(t-b_0)(t-b_1)\cdots(t-b_{m+1})} = \dfrac{w(t)}{(t-x_0)(t-x_1)\cdots(t-x_{2m-2})(t-x_{2m-1})(t-x_{2m+1})(t-x_{2m+3})}$. The polynomials Q_m will not be needed further here.

1.2. **Recurrence relations.**

One has

$$P_{n+1}(x)=(r_nx+r_n')P_n(x)+s_n(x-x_{2n})(x-x_{2n+1})P_{n-1}(x), \quad P_{-1}(x)\equiv 0, P_0(x)\equiv 1. \tag{2}$$

cf. R_{II}-type in Ismail & Masson [9, p. 14].

Indeed, with r_nx+r_n' being the first degree polynomial interpolating $P_{n+1}(x)/P_n(x)$ at $x=x_{2n}$ and x_{2n+1}, see that $\dfrac{P_{n+1}(x)-(r_nx+r_n')P_n(x)}{(x-x_{2n})(x-x_{2n+1})}$ is orthogonal to any polynomial of degree $< n-1$ with respect to $w(t)/[(t-x_0)(t-x_1)\cdots(t-x_{2n-1})]$.

2. Rational interpolation.

There is a polynomial N_n of degree $n+1$ such that N_n/P_n interpolates the Stieltjes, or Markov, function of w:

$$S(x)=\int_c^d \frac{w(t)dt}{x-t}$$

at $x_0, x_1, \dots, x_{2n+1}$ [12, § 3].

Polynomial interpolation of $P_n(x)S(x)$ at $x_0,\dots,x_{2n+1}$ has normally a degree $2n+1$, but the actual degree is found by describing the interpolant from

$P_n(x)S(x) = \underbrace{\displaystyle\int_c^d \frac{P_n(x)-P_n(t)}{x-t}w(t)dt}_{\text{a polynomial}} + \displaystyle\int_c^d \frac{P_n(t)w(t)dt}{x-t}$, where we replace $1/(x-t)$ by its polynomial interpolant in x at $x=x_0,\dots,x_{2n+1}$. This interpolant, for a fixed t, is $\dfrac{(t-x_0)\cdots(t-x_{2n+1})-(x-x_0)\cdots(x-x_{2n+1})}{(t-x_0)\cdots(t-x_{2n+1})(x-t)}$, and is a linear combination of ratios $\dfrac{t^r-x^r}{x-t}$ with $r\leqslant 2n+2$, so of terms t^ux^v with $u+v=r-1\leqslant 2n+1$. From the orthogonality properties of P_n, the integrals of $P_n(t)t^u$ vanish when $u<n$, only x^v remain when $u\geqslant n$, so $v=r-1-u\leqslant r-1-n\leqslant 2n+1-n=n+1$.

$$\frac{N_n(x)}{P_n(x)} = r_{-1}x + r'_{-1} + \cfrac{s_0(x-x_0)(x-x_1)}{r_0x + r'_0 + \cfrac{s_1(x-x_2)(x-x_3)}{\ddots \over r_{n-2}x + r'_{n-2} + \cfrac{s_{n-1}(x-x_{2n-2})(x-x_{2n-1})}{r_{n-1}x + r'_{n-1}}}}$$

$P_{-1} = 0, P_0 = 1, P_1(x) = r_0x + r'_0, P_2(x) = (r_0x + r'_0)(r_1x + r'_1) + s_1(x - x_0)(x - x_1)$.
$N_{-1} = 1, N_0(x) = r_{-1}x + r'_{-1}, N_1 = (r_0x + r'_0)(r_{-1}x + r'_{-1}) + s_0(x - x_0)(x - x_1)$, etc.
N_0 interpolates S at x_0 and x_1, so $r_{-1} = \dfrac{S(x_1) - S(x_0)}{x_1 - x_0} = \mu_1$.
Casorati:

$$N_{n-1}(x)P_n(x) - N_n(x)P_{n-1}(x) = (-1)^n s_0 \cdots s_{n-1}(x - x_0)\cdots(x - x_{2n-1}) \tag{3}$$

Two-Points Padé Approximants occur when $a_k = a, b_k = b, \forall k$. The interpolation property is then an order property $\dfrac{N_n(x)}{P_n(x)} - S(x) = O([(x-a)(x-b)]^{n+1})$, and (3) above is now

$$N_{n-1}(x)P_n(x) - N_n(x)P_{n-1}(x) = (-1)^n s_0 \cdots s_{n-1}(x - a)^n(x - b)^n \tag{4}$$

The rather strange choice in (1) (why b_{m+1}?) ensures same powers of $(x-a)$ and $(x-b)$ in (4).

Laurent expansions setting. Let $z = \dfrac{x-a}{x-b}$, so, $x = \dfrac{bz-a}{z-1}$. Taylor series in powers of $x-a$ and $x-b$ are expansions in $x - a = \dfrac{(a-b)z}{1-z} = (a-b)(z + z^2 + z^3 + \cdots)$ and $x - b = \dfrac{a-b}{z-1} = (a-b)(z^{-1} + z^{-2} + \cdots)$.

3. Classical families.

Among the various ways to define a classical family of orthogonal functions [2, § 5], consider the Sonine-Hahn criterion: the derivatives P'_n are themselves involved in some family of biorthogonal rational functions, so they satisfy recurrence relations leading to differential relations and equations for P_n.

Zhedanov [19] finds linear recurrence relations for the coefficients of the Laurent expansion of S in powers of $z = \dfrac{x-a}{x-b}$, leading to a differential equation of the form $AdS/dz = BS$, with polynomials A and B of degrees 2 and 1.

Returning to x such that $z = (x-a)/(x-b)$, one finds $X(x)S'(x) = Y(x)S(x)$, and we may suspect that its rational approximation $N_n(x)/P_n(x) = S(x) + O([(x-a)(x-b)]^{n+1})$ satisfies almost the same differential equation, and this also leads to differential relations and equations for P_n (Laguerre [11]).

Most general classical weight function for orthogonal polynomials is essentially the Jacobi $(d-t)^\alpha(t-c)^\beta$. Then,

$(x-c)(x-d)S(x) = \int_c^d \frac{(x-c)(x-d)-(t-c)(t-d)}{x-t} w(t)dt - \int_c^d \frac{(d-t)^{\alpha+1}(t-c)^{\beta+1}}{x-t} dt,$

$d[(x-c)(x-d)S(x)]/dx = \mu_{0,0} - \int_c^d \frac{d[(d-t)^{\alpha+1}(t-c)^{\beta+1}]/dt}{x-t} dt,$

$(x-c)(x-d)S'(x) + (2x-c-d)S(x) = \mu_{0,0} + (\alpha+1)[-\mu_{0,0} + (x-c)S(x)] - (\beta+1)[\mu_{0,0} - (x-d)S(x)]$:

$(x-c)(x-d)S'(x) = [(\alpha+\beta)x - \alpha c - \beta d]S(x) - (\alpha+\beta+1)\mu_{0,0}$.

Of special interest is $\alpha+\beta+1=0$, we then have $S(x) =$ a constant times $(x-d)^\alpha(x-c)^{-1-\alpha}$, studied by Grosjean [6], by Komlov and Suetin [10].

4. Chebyshev family of biorthogonal rational functions.

We even take a simpler case $\alpha=\beta=-1/2$, $d=-c$, $b=-a$, so to have a completely symmetric situation. In order to have $S(a)=1$, let

$$S(x) = \sqrt{\frac{a^2-c^2}{x^2-c^2}}, \tag{5}$$

where the square root is a continuous function outside $x \in [c,-c]$, so that S is an odd function.

4.1. Recurrence relations.

In particular, $S(b) = S(-a) = -1$, whence the first approximation $N_0(x)/P_0(x) = x/a$ with $r_{-1} = 1/a$. All the $r'_n = 0$.

We start the continued fraction expansion:

$$S(x) = \sqrt{\frac{a^2-c^2}{x^2-c^2}} = \frac{x}{a} + \sqrt{\frac{a^2-c^2}{x^2-c^2}} - \frac{x}{a} = \frac{x}{a} + \frac{\dfrac{a^2-c^2}{x^2-c^2} - \dfrac{x^2}{a^2}}{\sqrt{\dfrac{a^2-c^2}{x^2-c^2}} + \dfrac{x}{a}}$$

$= \dfrac{x}{a} + \dfrac{x^2-a^2}{a^2\dfrac{x^2-c^2}{c^2-a^2-x^2}\left[\sqrt{\dfrac{a^2-c^2}{x^2-c^2}} + \dfrac{x}{a}\right] \;=\; 2a\dfrac{c^2-a^2}{2a^2-c^2}x + O(x^2-a^2)}$, yielding the ratio s_0/r_0. A convenient choice is $r_0 = 2(1-a^2/c^2)$ and $as_0 = -1+2a^2/c^2$. More details on r_n and s_n are not given here, they involve values of the Chebyshev polynomials T_n and U_n at the argument a/c, see the method suggested in § 5.

4.2. Differential relations.

We expect the approximation N_n/P_n to satisfy differential relations close to the differential equation $(x^2-c^2)dS(x)/dx + xS(x) = 0$: $(x^2-c^2)(N_n(x)/P_n(x))' + xN_n(x)/P_n(x) = O((x^2-a^2)^n)$. Multiply by $P_n^2(x)$, the left-hand side being a polynomial, so is the right-hand side:

$$(x^2-c^2)[N'_n(x)P_n(x) - N_n(x)P'_n(x)] + xN_n(x)P_n(x) = (x^2-a^2)^n\Theta_n(x),$$

where Θ_n is a polynomial of degree 2 (Laguerre's [11] notations!)
$\Theta_0(x) = (2x^2 - c^2)/a$, etc.
We also have

$$[(x^2-c^2)^{1/2}N_n(x)]'P_n(x) - (x^2-c^2)^{1/2}N_n(x)P_n(x)' = (x^2-c^2)^{-1/2}(x^2-a^2)^n\Theta_n(x). \quad (6)$$

With $P_n(x) = C_nx^n + \cdots$ and $(x^2-c^2)^{1/2}N_n(x) = x^{n+2}/a + \cdots$ for large x, we find the coeff. of x^2 to be $2C_n$: $\Theta_n(x) = 2C_nx^2 + \Theta_n(0)$.

To get rid of the numerator polynomials N_n, subtract

$$(x^2-c^2)\left(\frac{N_{n+1}(x)}{P_{n+1}(x)} - \frac{N_n(x)}{P_n(x)}\right)' + x\left(\frac{N_{n+1}(x)}{P_{n+1}(x)} - \frac{N_n(x)}{P_n(x)}\right) = \frac{(x^2-a^2)^{n+1}\Theta_{n+1}(x)}{P^2_{n+1}(x)} - \frac{(x^2-a^2)^n\Theta_n(x)}{P^2_n(x)}.$$

and derivate the Casorati identity (4) in the left-hand side, so

$$(-1)^n s_0\cdots s_n\left[(x^2-c^2)\left(\frac{(x^2-a^2)^{n+1}}{P_n(x)P_{n+1}(x)}\right)' + x\frac{(x^2-a^2)^{n+1}}{P_n(x)P_{n+1}(x)}\right]$$
$$= \frac{(x^2-a^2)^{n+1}\Theta_{n+1}(x)}{P^2_{n+1}(x)} - \frac{(x^2-a^2)^n\Theta_n(x)}{P^2_n(x)}$$

or $(-1)^n s_0\cdots s_n\{(x^2-c^2)[2(n+1)xP_n(x)P_{n+1}(x) - (x^2-a^2)(P_n'(x)P_{n+1}(x) + P_n(x)P_{n+1}'(x))] + x(x^2-a^2)P_n(x)P_{n+1}(x)\} = (x^2-a^2)\Theta_{n+1}(x)P_n^2(x) - \Theta_n(x)P^2_{n+1}(x)$

Now, from $AP_n + BP_{n+1} \equiv 0$, form $A \equiv \Omega_nP_{n+1}$ and $B \equiv -\Omega_nP_n$, with a polynomial Ω_n (Laguerre [11, § 5, eq. (8)] again!).

$(x^2-c^2)[2(n+1)xP_n(x) - (x^2-a^2)P_n'(x)] + x(x^2-a^2)P_n(x) + \dfrac{(-1)^n\Theta_n(x)}{s_0\cdots s_n}P_{n+1}(x) = \Omega_n(x)P_n(x)$,

which is our ***differential relation***

$$UP_n' = V_nP_n + W_nP_{n+1}, \quad (7a)$$

with $U(x) = (x^2-a^2)(x^2-c^2)$, $V_n(x) = 2(n+1)x(x^2-c^2) + x(x^2-a^2) - \Omega_n(x)$, and $W_n(x) = \dfrac{(-1)^n\Theta_n(x)}{s_0\cdots s_n}$.

The second equation is

$-(x^2-c^2)(x^2-a^2)P_{n+1}'(x) - (x^2-a^2)\dfrac{(-1)^n\Theta_{n+1}(x)}{s_0\cdots s_n}P_n(x) = -\Omega_n(x)P_{n+1}(x)$, or

$$U(x)P_{n+1}'(x) = [2(n+1)x(x^2-c^2) + x(x^2-a^2) - V_n(x)]P_{n+1}(x) + s_{n+1}(x^2-a^2)W_{n+1}(x)P_n(x) \quad (7b)$$

$n \to n+1$ in (7a) and subtraction:

$V_{n+1}(x)P_{n+1}(x) + W_{n+1}(x)[\overbrace{P_{n+2}(x) - s_{n+1}(x^2-a^2)P_n(x)}^{r_{n+1}xP_{n+1}(x)}] - [2(n+1)x(x^2-c^2) + x(x^2-a^2) - V_n(x)]P_{n+1}(x) = 0$,

so $V_{n+1}(x) + V_n(x) + r_{n+1}xW_{n+1}(x) - 2(n+1)x(x^2-c^2) - x(x^2-a^2) = 0$, or
$2(n+2)x(x^2-c^2) + x(x^2-a^2) + r_{n+1}xW_{n+1}(x) = \Omega_{n+1}(x) + \Omega_n(x)$, confirming by the way that the Ω_ns are polynomials, here: odd cubics.

When x is a zero of U, i.e., when $x = \pm a$ or $x = \pm c$, (7a-7b) show $\dfrac{P_{n+1}(x)}{P_n(x)} = -\dfrac{V_n(x)}{W_n(x)} = -\dfrac{s_{n+1}(x^2-a^2)W_{n+1}(x)}{2(n+1)x(x^2-c^2) + x(x^2-a^2) - V_n(x)} = -\dfrac{s_{n+1}(x^2-a^2)W_{n+1}(x)}{V_{n+1}(x) + r_{n+1}xW_{n+1}(x)}$, so, $V_n(x)[V_{n+1}(x) + r_{n+1}xW_{n+1}(x)] = s_{n+1}(x^2 - a^2)W_n(x)W_n(x)$ at the zeros of U.

4.3. **Differential equation.**

From $UP_n' = V_nP_n + W_nP_{n+1}$, put $P_{n+1}' = \{V_{n+1}P_{n+1} + W_{n+1}[r_{n+1}xP_{n+1}(x) + s_{n+1}(x^2 - a^2)P_n(x)]\}/U$, into the derivative of $P_{n+1} = [UP_n' - V_nP_n]/W_n$:

$$\left[\frac{UP_n' - V_nP_n}{W_n}\right]' = \frac{V_{n+1} + r_{n+1}xW_{n+1}}{U}\,\frac{UP_n' - V_nP_n}{W_n} + \frac{s_{n+1}(x^2 - a^2)W_{n+1}P_n}{U},$$

$$UW_nP_n'' + \left\{U'W_n - UW_n' - W_n[\underbrace{V_n + V_{n+1} + r_{n+1}xW_{n+1}}_{x[2(n+1)(x^2-c^2)+x^2-a^2]}]\right\}P_n'$$

$$+\left[V_nW_n' - V_n'W_n + W_n(x)\frac{V_n(V_{n+1} + r_{n+1}xW_{n+1}) - s_{n+1}(x^2 - a^2)W_{n+1}W_n(x)}{U(x)}\right]P_n = 0$$

Remark that the coefficient of P_n' inside the curly braces is $-UW_n$ times the logarithmic derivative of the ***Wronskian*** which is therefore $\mathscr{W}_n(x) = (x^2 - c^2)^{-1/2}(x^2 - a^2)^nW_n(x)$, and that is exactly (6)! So that the solutions of the differential equation are $y = P_n(x)$ AND $y = (x^2 - c^2)^{1/2}N_n(x)$ [10] in

$$U(x)W_n(x)\left\{y'' + \left[\frac{x}{x^2 - c^2} - \frac{2nx}{x^2 - a^2} - \frac{W_n'(x)}{W_n(x)}\right]y'\right\} + K_n(x)y = 0, \tag{8}$$

where K_n is a polynomial of degree $\leqslant 4$.

4.4. **Heun.**

Let $W_n(x) = (W_n''/2)(x^2 - \theta_n)$. The equation (8) has three singular points a^2, c^2, and θ_n in the bounded x^2-plane. This suggests that we are close to the Heun equation

$$(u - \mathfrak{a})(u - \mathfrak{b})(u - \mathfrak{c})\left\{\frac{d^2Y}{du^2} + \left[\frac{\gamma}{u - \mathfrak{a}} + \frac{\delta}{u - \mathfrak{b}} + \frac{\epsilon}{u - \mathfrak{c}}\right]\frac{dY}{du}\right\} + (\alpha\beta u - Q)Y = 0. \tag{9}$$

When $u \to \infty$, a u^ρ behaviour implies $\rho(\rho - 1) + (\gamma + \delta + \epsilon)\rho + \alpha\beta = 0 \Rightarrow \rho = -\alpha$ and $-\beta$, with $\alpha + \beta = \gamma + \delta + \epsilon - 1$ [5, 8, 13, 15, 16]. It seems obvious that (8) becomes (9) by taking $u = x^2$, but a new singularity rises then at the origin. Considering that the expansions of the two solutions of (8) for large x are $P_n(x) = C_nx^n + O(x^{n-2})$ and $(x^2 - c^2)^{1/2}N_n(x) = x^{n+2}/a + O(x^n)$, we kill the singularity at $x = \infty$ by building the differential equation for $x^{-n-2}y$, and by taking the new variable $t = 1/x^2$.

4.4.1. *First.* , the equation for $\tilde{y} = x^{-n-2}y$:

$$U(x)W_n(x)\left\{x^2\tilde{y}'' + 2(n+2)x\tilde{y}' + (n+2)(n+1)\tilde{y} + \left[\frac{x}{x^2 - c^2} - \frac{2nx}{x^2 - a^2} - \frac{W_n'(x)}{W_n(x)}\right][x^2\tilde{y}' + (n+2)x\tilde{y}]\right\}$$
$$+ K_n(x)x^2\tilde{y} = 0.$$

$$U(x)W_n(x)\left\{x^2\tilde{y}'' + x\left[3 + \frac{c^2}{x^2 - c^2} - \frac{2na^2}{x^2 - a^2} - \frac{2\theta_n}{x^2 - \theta_n}\right]\tilde{y}'\right\} + \tilde{K}_n(x)\tilde{y} = 0, \text{ with}$$

$$\tilde{K}_n(x) = (n+1)(n+2)UW_n + (n+2)xUW_n\left[\frac{x}{x^2 - c^2} - \frac{2nx}{x^2 - a^2} - \frac{W_n'(x)}{W_n(x)}\right] + x^2K_n(x),$$

an even polynomial of degree $\leqslant 6$.

Seems worse than before, but the degree of $\tilde{K}_n$ is only 2! Indeed, put the solution $\tilde{y} = ax^{-n-2}(x^2-c^2)^{1/2}N_n(x) = 1 + \zeta_n/x^2 + \cdots$ for large x in the differential equation:
$\tilde{K}_n(x) = -U(x)W_n(x)\left\{x^2\frac{\tilde{y}''}{\tilde{y}} + x\left[3 + \frac{c^2}{x^2-c^2} - \frac{2na^2}{x^2-a^2} - \frac{2\theta_n}{x^2-\theta_n}\right]\frac{\tilde{y}'}{\tilde{y}}\right\} = O(x^2).$

$\tilde{K}_n(0) = (n+1)(n+2)U(0)W_n(0) = -(n+1)(n+2)a^2c^2\theta_n$ is especially interesting, $\tilde{K}_n(x) = (\tilde{K}_n''/2)x^2 - (n+1)(n+2)a^2c^2\theta_n$.

4.4.2. *Finally.* , $t = 1/x^2 : x = t^{-1/2}$, $Y(t) = \tilde{y}(x)$,

$$(1/t-a^2)(1/t-c^2)(1/t-\theta_n)\left\{t^{-1}2t^{3/2}\frac{d}{dt}\left[2t^{3/2}\frac{dY}{dt}\right] - 2t^{-1/2}\left[3 + \frac{c^2}{1/t-c^2} - \frac{2na^2}{1/t-a^2} - \frac{2\theta_n}{1/t-\theta_n}\right]t^{3/2}\frac{dY}{dt}\right\} +$$
$[(\tilde{K}_n''/2)t^{-1} - (n+1)(n+2)a^2c^2\theta_n]Y = 0$,

$$(1-a^2t)(1-c^2t)(1-\theta_n t)\left\{4\frac{d^2Y}{dt^2} - 2\left[\frac{c^2}{1-c^2t} - \frac{2na^2}{1-a^2t} - \frac{2\theta_n}{1-\theta_n t}\right]\frac{dY}{dt}\right\} + t\tilde{K}_n(t^{-1/2})Y = 0$$

$$(t-1/a^2)(t-1/c^2)(t-1/\theta_n)\left\{\frac{d^2Y}{dt^2} + \left[\frac{1/2}{t-1/c^2} - \frac{n}{t-1/a^2} - \frac{1}{t-1/\theta_n}\right]\frac{dY}{dt}\right\} + \left[\frac{(n+1)(n+2)}{4}t - \frac{\tilde{K}_n''/2}{4a^2c^2\theta_n}\right]Y = 0 \quad (10)$$

is our Heun equation. It has two Frobenius-polynomial solutions $t^{(n+2)/2}P_n(t^{-1/2})$, $(t-1/c^2)^{1/2}t^{(n+1)/2}N_n(t^{-1/2})$, a much studied property [5, 7, 8, 13, 15, 16].

5. Another square root story.

There is of course an ∞ simpler continued fraction expansion of our square root function, with constant coefficients:

$$\sqrt{\frac{a^2-c^2}{x^2-c^2}} = \cfrac{1}{x/a + \cfrac{c^2(x^2-a^2)/[a^2(a^2-c^2)]}{2x/a + \cfrac{c^2(x^2-a^2)/[a^2(a^2-c^2)]}{2x/a + \ddots}}}$$

see also Ismail & Masson [9, Example 3.1, p. 18]. Let $\tilde{P}_n$ be the polynomials of interest here. They are now true Chebyshev polynomials $\tilde{P}_n(x) = \left(\frac{c^2(x^2-a^2)}{a^2(c^2-a^2)}\right)^{n/2} T_n\left(\frac{x}{c}\sqrt{\frac{c^2-a^2}{x^2-a^2}}\right)$, orthogonal with respect to $w(t)/(t^2-a^2)^n$ instead of $w(t)/(t^2-a^2)^{n+1}$. This means that the P_n of before is related to a modified weight function involving multiplication or division by a polynomial (Christoffel, Uvarov): $\frac{P_{n+2}(x)P_n(a) - P_{n+2}(a)P_n(x)}{x^2-a^2} = \text{constant } \tilde{P}_n(x)$.

Recurrence relations of the P_ns involve special values
$\tilde{P}_n(a) = \lim_{x\to a} 2^{n-1}\left(\frac{c^2(x^2-a^2)}{a^2(c^2-a^2)}\right)^{n/2}\left(\frac{x}{c}\sqrt{\frac{c^2-a^2}{x^2-a^2}}\right)^n = 2^{n-1}$.

Coefficient of x^n in $\tilde{P}_n = \left(\frac{c^2}{a^2(c^2-a^2)}\right)^{n/2} T_n\left(\sqrt{1-a^2/c^2}\right)$.

This may lead to recurrence coefficients, and even differential properties and equations, by particular methods [13, 14], but we plan to study other classical biorthogonal rational functions, when these modified weight function techniques do not apply.

References

[1] M. Abramowitz,I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series, **55**, 1964 = Dover.

[2] W.A. Al-Salam, Characterization theorems for orthogonal polynomials, pp. 1-24 *in Orthogonal Polynomials: Theory and Practice*, (P. Nevai, ed.), NATO ASI Series C: Math. and Phys. Sci. **294**, Kluwer, 1990.

[3] C. Brezinski, *Biorthogonality and its Applications to Numerical Analysis*, M. Dekker, 1992.

[4] P.J. Davis, *Interpolation and Approximation*, Blaisdell, Waltham, 1963 = Dover, New York, 1975.

[5] A. Erdélyi *et al.*, *Higher Transcendental Functions*, vol. **3**, McGraw-Hill, New York, 1955. § 15.3

[6] C.C. Grosjean, The weight functions, generating functions and miscellaneous properties of the sequences of orthogonal polynomials of the second kind associated with the Jacobi and the Gegenbauer polynomials, *J. Comput. Appl. Math.* **16** (1986), no. 3, 259-307.

[7] André Hautot, Sur les solutions polynomiales de l'équation différentielle $z(1-z)(\alpha-z)P_n''(z)+(az^2+bz+c)P_n'+(d+ez+fz^2)P_n=0$, *Bull. Soc. Roy. Sci. Liège* **40** (1971), 7-12.

[8] Mahouton Norbert Hounkonnou, André Ronveaux, About derivatives of Heun's functions from polynomial transformations of hypergeometric equations, *Applied Mathematics and Computation* **209** (2009) 421-424.

[9] M.E.H. Ismail, D.R. Masson, Generalized orthogonality and continued fractions, *J. Approx. Th.* **83** (1995) 1-40.

[10] A. V. Komlov and S. P. Suetin, Strong Asymptotics of Two-Point Padé Approximants for Power-Like Multivalued Functions, *Doklady Akademii Nauk* Vol. **455** (2014), No. 1, pp. 138-142. (in Russian) = *Doklady Mathematics*, Vol. **89** (2014), No. 2, pp. 1-4.

[11] E. Laguerre, Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels, *J. Math. Pures Appl. (4)* **1** (1885), 135-165 = pp. 685-711 *in Oeuvres*, Vol.II, Chelsea, New-York 1972.

[12] G. López Lagomasino, Survey on multipoint Padé approximation to Markov-type meromorphic functions and asymptotic properties of the orthogonal polynomials generated by them, pp. 309-316 in C. Brezinski & *al.*, eds: *Polynômes Orthogonaux et Applications Proceedings of the Laguerre Symposium held at Bar-le-Duc, October 15-18, 1984*, Lecture Notes in Mathematics **1171**, Springer, 1985.

[13] A. Ronveaux, Sur l'équation différentielle du second ordre satisfaite par une classe de polynômes orthogonaux semi-classiques, *C. R. Acad. Sci. Paris* **305** série I, p. 163-166, 1987.

[14] A. Ronveaux and F. Marcellan, Differential equation for classical-type orthogonal polynomials, *Canad. Math. Bull.* **32** (4), 1989, 404-411.

[15] A. Ronveaux (Ed.) *Heun's Differential Equations*, The Clarendon Press Oxford University Press, 1995.

[16] B. D. Sleeman and V. B. Kuznetsov, Heun Functions, Chapter 31 in Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, Charles W. Clark, editors: *NIST Handbook of Mathematical Functions*, National Institute of Standards and Technology and Cambridge University Press, 2010.

[17] V. P. Spiridonov and A. S. Zhedanov, Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, pp. 365-388 *in* J. Bustoz *et al.* (eds), *Special functions 2000*, Kluwer, 2001.

[18] V. P. Spiridonov and A. S. Zhedanov, To the theory of biorthogonal rational functions, *RIMS Kokyouroku* **1302** (2003), 172-192.

[19] A. Zhedanov, The "classical" Laurent biorthogonal polynomials, *Journal of Computational and Applied Mathematics* **98** (1998) 121-147