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ELLIPTIC HYPERGEOMETRIC FUNCTIONS

V. P. SPIRIDONOV

Introduction. The wonderful book by Andrews, Askey, and Roy [2] is mainly devoted to special functions of hypergeometric type – to the plain and q-hypergeometric series and integrals. Shortly before its publication, examples of a third type of such functions, related to elliptic curves, began to appear. A systematic theory of elliptic hypergeometric functions was constructed in 2000-2004 over a short period of time. The present complement reviews briefly the status of this theory to the end of 2006. It repeats where possible the structure of the complemented book [2], and it is based mainly on the material [44].

The theory of quantum and classical completely integrable systems played a crucial role in the discovery of these new special functions. An elliptic extension of the terminating very well poised balanced q-hypergeometric series ${}_{10}\varphi_9$ with discrete values of parameters appeared for the first time in elliptic solutions of the Yang-Baxter equation [14] associated with the exactly solvable models of statistical mechanics [9]. The same terminating series with arbitrary parameters appeared in [47] as a particular solution of a pair of linear finite difference equations, the compatibility condition of which yields the most general known $(1+1)$-dimensional nonlinear integrable chain analogous to the discrete time Toda chain. In [35], the generalized gamma functions were investigated in detail, including one of the elliptic analogues of Euler's gamma function, which appeared implicitly already in Baxter's eight vertex model. The appearance of such mathematical objects was quite unexpected, since no handbook or textbook of special functions contained any hint of their existence. Some relatives of these functions were considered only in the old original papers by Barnes [6] and Jackson [19].

Generalized gamma functions. In the beginning of the XX-th century Barnes [6] suggested a multiple zeta function depending on m quasiperiods $\omega_j \in \mathbb{C}$:

$$\zeta_m(s,u;\omega) = \sum_{n_1,\ldots,n_m\in\mathbb{N}} \frac{1}{(u+\Omega)^s}, \quad \Omega = n_1\omega_1 + \ldots + n_m\omega_m, \quad \mathbb{N} = \{0,1,\ldots\},$$

where the series converges for $\mathrm{Re}(s) > m$ and under the condition that if $\omega_j/\omega_k \in \mathbb{R}$, then $\omega_j/\omega_k > 0$. Using an integral representation for ζ_m, Barnes also defined the multiple gamma function $\Gamma_m(u;\omega) = \exp(\partial\zeta_m(s,u;\omega)/\partial s)|_{s=0}$, which has the

This is a complement to the book by G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Math. Appl. **71**, Cambridge Univ. Press, Cambridge, 1999, written for its Russian edition (to be published by the Independent University press, Moscow, 2007).

infinite product representation

$$\frac{1}{\Gamma_m(u;\omega)} = e^{\sum_{k=0}^m \gamma_{mk}\frac{u^k}{k!}}\, u \prod_{n_1,\ldots,n_m=0}^{\infty}{}' \Big(1+\frac{u}{\Omega}\Big)\, e^{\sum_{k=1}^m (-1)^k \frac{u^k}{k\Omega^k}}, \tag{1}$$

where γ_{mk} are some constants analogous to Euler's constant (in [6], the normalization $\gamma_{m0}=0$ was used). The primed product means that the point $n_1=\ldots=n_m=0$ is excluded from it. The function $\Gamma_m(u;\omega)$ satisfies m finite difference equations of the first order

$$\Gamma_m(u+\omega_j;\omega) = \Gamma_{m-1}^{-1}(u;\omega(j))\,\Gamma_m(u;\omega), \qquad j=1,\ldots,m, \tag{2}$$

where $\omega(j)=(\omega_1,\ldots,\omega_{j-1},\omega_{j+1},\ldots,\omega_m)$ and $\Gamma_0(u;\omega):=1/u$. The function $\Gamma_1(\omega_1 x;\omega_1)$ essentially coincides with Euler's gamma function $\Gamma(x)$. The plain, q-, and elliptic hypergeometric functions are respectively connected to $\Gamma_m(u;\omega)$ for $m=1,2,3$.

We define three base variables $p,q,r\in\mathbb{C}$ related to the pairwise incommensurate quasiperiods $\omega_{1,2,3}$ as follows:

$$q=e^{2\pi i\frac{\omega_1}{\omega_2}},\quad p=e^{2\pi i\frac{\omega_3}{\omega_2}},\quad r=e^{2\pi i\frac{\omega_3}{\omega_1}},$$
$$\tilde{q}=e^{-2\pi i\frac{\omega_2}{\omega_1}},\quad \tilde{p}=e^{-2\pi i\frac{\omega_2}{\omega_3}},\quad \tilde{r}=e^{-2\pi i\frac{\omega_1}{\omega_3}},$$

where $\tilde{q},\tilde{p},\tilde{r}$ denote the modular transformed bases. For $|p|,|q|<1$, the infinite products

$$(z;q)_\infty=\prod_{j=0}^\infty(1-zq^j),\quad (z;p,q)_\infty=\prod_{j,k=0}^\infty(1-zp^jq^k)$$

are well defined. It is easy to derive equalities [19]

$$\frac{(z;q)_\infty}{(qz;q)_\infty}=1-z,\qquad \frac{(z;q,p)_\infty}{(qz;q,p)_\infty}=(z;p)_\infty,\qquad \frac{(z;q,p)_\infty}{(pz;q,p)_\infty}=(z;q)_\infty. \tag{3}$$

The odd Jacobi theta function (see formula (10.7.1) in [2]) can be written as

$$\begin{aligned}\theta_1(u|\tau) &= -i\sum_{n=-\infty}^{\infty}(-1)^n e^{\pi i\tau(n+1/2)^2}e^{\pi i(2n+1)u}\\ &= ip^{1/8}e^{-\pi iu}\,(p;p)_\infty\,\theta(e^{2\pi iu};p),\quad u\in\mathbb{C},\end{aligned}$$

where $p=e^{2\pi i\tau}$. The shortened theta function (see Theorem 10.4.1 in [2])

$$\theta(z;p):=(z;p)_\infty(pz^{-1};p)_\infty=\frac{1}{(p;p)_\infty}\sum_{k\in\mathbb{Z}}(-1)^kp^{k(k-1)/2}z^k \tag{4}$$

plays a crucial role in the following. It obeys the following properties:

$$\theta(pz;p)=\theta(z^{-1};p)=-z^{-1}\theta(z;p) \tag{5}$$

and $\theta(z;p)=0$ for $z=p^k,\ k\in\mathbb{Z}$. We denote

$$\theta(a_1,\ldots,a_k;p):=\theta(a_1;p)\cdots\theta(a_k;p),\quad \theta(at^{\pm};p):=\theta(at;p)\theta(at^{-1};p).$$

Then the Riemann identity for products of four theta functions takes the form

$$\theta(xw^{\pm}, yz^{\pm}; p) - \theta(xz^{\pm}, yw^{\pm}; p) = yw^{-1}\theta(xy^{\pm}, wz^{\pm}; p) \tag{6}$$

(the ratio of the left- and right-hand sides is a bounded function of $x \in \mathbb{C}^*$, and it does not depend on x due to the Liouville theorem, but for $x = w$ the equality is evident).

Euler's gamma function can be defined as a special meromorphic solution of the functional equation $f(u + \omega_1) = uf(u)$. q-Gamma functions are connected to solutions of the equation $f(u+\omega_1) = (1-e^{2\pi iu/\omega_2})f(u)$ with $q = e^{2\pi i\omega_1/\omega_2}$. For $|q| < 1$, one of its solutions has the form $\Gamma_q(u) = 1/(e^{2\pi iu/\omega_2}; q)_\infty$ defining the standard q-gamma function (it differs from function (10.3.3) in [2] by the substitution $u = \omega_1 x$ and some elementary multiplier). The modified q-gamma function ("the double sine", "hyperbolic gamma function", etc), which remains well defined even for $|q| = 1$, has the form

$$\gamma(u; \omega) = \exp\left(-\int_{\mathbb{R}+i0} \frac{e^{ux}}{(1 - e^{\omega_1 x})(1 - e^{\omega_2 x})} \frac{dx}{x}\right), \tag{7}$$

where the contour $\mathbb{R} + i0$ coincides with the real axis deformed to pass clockwise the point $x = 0$ in an infinitesimal way. If $\mathrm{Re}(\omega_1), \mathrm{Re}(\omega_2) > 0$, then the integral converges for $0 < \mathrm{Re}(u) < \mathrm{Re}(\omega_1 + \omega_2)$. Under appropriate restrictions upon u and $\omega_{1,2}$, the integral can be computed as a convergent sum of the residues of poles in the upper half plane. When $\mathrm{Im}(\omega_1/\omega_2) > 0$, this yields the expression $\gamma(u; \omega) = (e^{2\pi iu/\omega_1}\tilde{q}; \tilde{q})_\infty/(e^{2\pi iu/\omega_2}; q)_\infty$, which can be extended analytically to the whole complex u-plane. This function, serving as a key building block of the q-hypergeometric functions for $|q| = 1$, was not considered in [2] and [16]; for its detailed description see [20, 23, 35, 50] and the literature cited therein.

In an analogous way, elliptic gamma functions are connected to the equation

$$f(u + \omega_1) = \theta(e^{2\pi iu/\omega_2}; p)f(u). \tag{8}$$

Using the factorization (4) and equalities (3), it is not difficult to see that the ratio

$$\Gamma(z; p, q) = \frac{(pqz^{-1}; p, q)_\infty}{(z; p, q)_\infty} \tag{9}$$

satisfies the equations

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \quad \Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q).$$

Therefore the function $f(u) = \Gamma(e^{2\pi iu/\omega_2}; p, q)$ defines a solution of equation (8) valid for $|q|, |p| < 1$, which is called the (standard) elliptic gamma function [35]. It can be defined uniquely as a meromorphic solution of three equations: (8) and

$$f(u + \omega_2) = f(u), \qquad f(u + \omega_3) = \theta(e^{2\pi iu/\omega_2}; q)f(u)$$

with the normalization $f(\sum_{m=1}^{3} \omega_m/2) = 1$, since there do not exist non-trivial triply periodic functions. The reflection formula for it has the form $\Gamma(z; p, q)$ $\Gamma(pq/z; p, q) = 1$. For $p = 0$, we have $\Gamma(z; 0, q) = 1/(z; q)_\infty$.

The modified elliptic gamma function, which is well defined for $|q|=1$, has the form [41]

$$G(u;\omega)=\Gamma(e^{2\pi i\frac{u}{\omega_2}};p,q)\Gamma(re^{-2\pi i\frac{u}{\omega_1}};\tilde{q},r). \tag{10}$$

It yields the unique solution of three equations: (8) and

$$f(u+\omega_2)=\theta(e^{2\pi iu/\omega_1};r)f(u), \qquad f(u+\omega_3)=e^{-\pi iB_{2,2}(u;\omega)}f(u)$$

with the normalization $f(\sum_{m=1}^3\omega_m/2)=1$. Here

$$B_{2,2}(u;\omega)=\frac{u^2}{\omega_1\omega_2}-\frac{u}{\omega_1}-\frac{u}{\omega_2}+\frac{\omega_1}{6\omega_2}+\frac{\omega_2}{6\omega_1}+\frac{1}{2}$$

denotes the second Bernoulli polynomial appearing in the modular transformation law for the theta function

$$\theta\left(e^{-2\pi i\frac{u}{\omega_1}};e^{-2\pi i\frac{\omega_2}{\omega_1}}\right)=e^{\pi iB_{2,2}(u;\omega)}\theta\left(e^{2\pi i\frac{u}{\omega_2}};e^{2\pi i\frac{\omega_1}{\omega_2}}\right). \tag{11}$$

One can check [12] that the same three equations and normalization are satisfied by the function

$$G(u;\omega)=e^{-\pi iP(u)}\Gamma(e^{-2\pi i\frac{u}{\omega_3}};\tilde{r},\tilde{p}), \tag{12}$$

where $|\tilde{p}|,|\tilde{r}|<1$, and the polynomial of the third degree $P(u)$ has the form

$$P\left(u+\sum_{m=1}^3\frac{\omega_m}{2}\right)=\frac{u(u^2-\frac{1}{4}\sum_{m=1}^3\omega_m^2)}{3\omega_1\omega_2\omega_3}.$$

Functions (10) and (12) therefore coincide, and their equality defines one of the laws of the $SL(3;\mathbb{Z})$-group of modular transformations for the elliptic gamma function [13]. From expression (12), the function $G(u;\omega)$ is easily seen to remain meromorphic when $\omega_1/\omega_2>0$, i.e. $|q|=1$. The reflection formula for this function has the form $G(a;\omega)G(b;\omega)=1$, $a+b=\sum_{k=1}^3\omega_k$. In the regime $|q|<1$ and $p,r\to 0$ (i.e., $\mathrm{Im}(\omega_3/\omega_1)$, $\mathrm{Im}(\omega_3/\omega_2)\to+\infty$), expression (10) obviously degenerates to the modified q-gamma function $\gamma(u;\omega)$. Representation (12) yields an alternative way of reduction to $\gamma(u;\omega)$; a rigorous limiting connection of such a type was built for the first time in a different way by Ruijsenaars [35].

As shown by Barnes, the q-gamma function $1/(z;q)_\infty$ with $z=e^{2\pi iu/\omega_2}$ and $q=e^{2\pi i\omega_1/\omega_2}$, $\mathrm{Im}(\omega_1/\omega_2)>0$, equals in essence to the product $\Gamma_2(u;\omega_1,\omega_2)\Gamma_2(u-\omega_2;\omega_1,-\omega_2)$. Analogously, the modified q-gamma function $\gamma(u;\omega)$ coincides in essence with the ratio $\Gamma_2(\omega_1+\omega_2-u;\omega)/\Gamma_2(u;\omega)$. Since $\theta(z;q)=(z;q)_\infty(qz^{-1};q)_\infty$, the $\Gamma_2(u;\omega)$-function represents "a quarter" of the $\theta_1(u/\omega_2|\omega_1/\omega_2)$ Jacobi theta function (in the sense of the number of divisor points). Correspondingly, one can consider equation (8) as a composition of four equations for $\Gamma_3(u;\omega)$ with different arguments and quasiperiods and represent the elliptic gamma functions as ratios of four Barnes gamma functions of the third order with some simple exponential multipliers [15, 41]. For some other important results for the generalized gamma functions, see [25, 30].

The elliptic beta integral. It is convenient to use the compact notation

$$\Gamma(a_1,\ldots,a_k;p,q) := \Gamma(a_1;p,q)\cdots\Gamma(a_k;p,q),$$
$$\Gamma(tz^{\pm};p,q) := \Gamma(tz;p,q)\Gamma(tz^{-1};p,q), \quad \Gamma(z^{\pm 2};p,q) := \Gamma(z^2;p,q)\Gamma(z^{-2};p,q)$$

for working with elliptic hypergeometric integrals. We start consideration from the elliptic beta integral discovered by the author in [38].

Theorem 1. *We take eight complex parameters $t_j,\ j=1,\ldots,6$, and p,q, satisfying the constraints $|p|,|q|,|t_j|<1$ and $\prod_{j=1}^6 t_j = pq$. Then the following equality is valid*

$$\kappa\int_{\mathbb{T}} \frac{\prod_{j=1}^6 \Gamma(t_j z^{\pm};p,q)}{\Gamma(z^{\pm 2};p,q)}\frac{dz}{z} = \prod_{1\leq j<k\leq 6}\Gamma(t_jt_k;p,q), \tag{13}$$

where $\mathbb{T}$ denotes the positively oriented unit circle and $\kappa = (p;p)_\infty(q;q)_\infty/4\pi i$.

The first proof of this formula was based on the elliptic extension of Askey's method [3]. A rather short proof was given in [43], but still it does not fit the margins of this page and we skip it. The elliptic beta integral (13) defines the most general known univariate exact integration formula generalizing Euler's beta integral. For $p\to 0$, one obtains the Rahman integral [26] (see Theorem 10.8.2 in [2]), which goes to the well known Askey-Wilson q-beta integral [4] (see Theorem 10.8.1 in [2]), if one of the parameters vanishes.

We replace $\mathbb{T}$ by a contour C which separates sequences of poles converging to zero along the points $z = t_jq^kp^m$, $k,m\in\mathbb{N}$, from their reciprocals obtained by the change $z\to 1/z$, which diverge to infinity. This allows us to lift the constraints $|t_j|<1$ without changing the right-hand side of (13). We substitute $t_6 = pq/A$, $A=\prod_{s=1}^5 t_s$, and suppose that $|t_m|<1$, $m=1,\ldots,4$, $|pt_5|<1<|t_5|$, $|pq|<|A|$, and that the arguments of t_s, $s=1,\ldots,5$, and p,q are linearly independent over $\mathbb{Z}$. Then the following equality takes place [10]:

$$\kappa\int_C \Delta_E(z,\underline{t})\frac{dz}{z} = \kappa\int_{\mathbb{T}}\Delta_E(z,\underline{t})\frac{dz}{z} + c_0(\underline{t})\sum_{|t_5q^n|>1,\, n\geq 0}\nu_n(\underline{t}), \tag{14}$$

where $\Delta_E(z,\underline{t}) = \prod_{m=1}^5\Gamma(t_mz^{\pm};p,q)/\Gamma(z^{\pm 2},Az^{\pm};p,q)$ and

$$c_0(\underline{t}) = \frac{\prod_{m=1}^4\Gamma(t_mt_5^{\pm};p,q)}{\Gamma(t_5^{-2},At_5^{\pm};p,q)}, \qquad \nu_n(\underline{t}) = \frac{\theta(t_5^2q^{2n};p)}{\theta(t_5^2;p)}\prod_{m=0}^5\frac{\theta(t_mt_5)_n}{\theta(qt_m^{-1}t_5)_n}\,q^n.$$

We have introduced here the new parameter t_0 with the help of the relation $\prod_{m=0}^5 t_m = q$ and used the elliptic Pochhammer symbol

$$\theta(t)_n = \prod_{j=0}^{n-1}\theta(tq^j;p) = \frac{\Gamma(tq^n;p,q)}{\Gamma(t;p,q)}, \qquad \theta(t_1,\ldots,t_k)_n := \prod_{j=1}^k\theta(t_j;p)_n$$

(the indicated ratio of elliptic gamma functions defines $\theta(t)_n$ for arbitrary $n\in\mathbb{C}$). The multiplier κ is absent in the coefficient c_0 due to the relation $\lim_{z\to 1}(1-$

$z)\Gamma(z;p,q) = 1/(p;p)_\infty(q;q)_\infty$ and doubling of the number of residues because of the symmetry $z \to z^{-1}$.

In the limit $t_5t_4 \to q^{-N}$, $N \in \mathbb{N}$, the integral value in the left-hand side of (14), i.e. the right-hand side of (13), and the multiplier $c_0(\underline{t})$ in front of the sum of residues on the right-hand side diverge, whereas the integral over the unit circle $\mathbb{T}$ remains finite. After dividing all the terms by $c_0(\underline{t})$ and going to the limiting relation, we obtain the Frenkel-Turaev summation formula

$$\sum_{n=0}^{N} \nu_n(\underline{t}) = \frac{\theta(qt_5^2, \frac{q}{t_1t_2}, \frac{q}{t_1t_3}, \frac{q}{t_2t_3})_N}{\theta(\frac{q}{t_1t_2t_3t_5}, \frac{qt_5}{t_1}, \frac{qt_5}{t_2}, \frac{qt_5}{t_3})_N}, \tag{15}$$

which was established for the first time in [14] by a completely different method. For $p \to 0$ and fixed parameters, equality (15) reduces to the Jackson sum for a terminating ${}_8\varphi_7$-series (see formula (12.3.5) in [2]).

General elliptic hypergeometric functions. Definitions of the general elliptic hypergeometric series and integrals were respectively given and investigated in detail in [39] and [41]. So, formal series $\sum_{n\in\mathbb{Z}} c_n$ are called the elliptic hypergeometric series, if $c_{n+1} = h(n)c_n$, where $h(n)$ is some elliptic function of $n \in \mathbb{C}$. It is well known [5], that an arbitrary elliptic function $h(u)$ of order $s+1$ with the periods ω_2/ω_1 and ω_3/ω_1 can be represented in the form

$$h(u) = y \prod_{k=1}^{s+1} \frac{\theta(t_kz;p)}{\theta(w_kz;p)}, \tag{16}$$

where $z = q^u$. The equality $h(u+\omega_2/\omega_1) = h(u)$ is evident, and the periodicity $h(u+\omega_3/\omega_1) = h(u)$ brings in the balancing condition $\prod_{k=1}^{s+1} t_k = \prod_{k=1}^{s+1} w_k$. Because of the factorization of $h(n)$, in order to determine the coefficients c_n it suffices to solve the equation $a_{n+1} = \theta(tq^n;p)\, a_n$, which leads to the elliptic Pochhammer symbol $a_n = \theta(t)_n\, a_0$. The explicit form of the bilateral elliptic hypergeometric series is now easily found to be

$${}_{s+1}G_{s+1}\left(\begin{matrix} t_1,\ldots,t_{s+1} \\ w_1,\ldots,w_{s+1}\end{matrix}; q,p;y\right) := \sum_{n\in\mathbb{Z}} \prod_{k=1}^{s+1} \frac{\theta(t_k)_n}{\theta(w_k)_n}\, y^n,$$

where we have chosen the normalization $c_0 = 1$. By setting $w_{s+1} = q$, $t_{s+1} \equiv t_0$, we obtain the one sided series

$${}_{s+1}E_s\left(\begin{matrix} t_0,t_1,\ldots,t_s \\ w_1,\ldots,w_s\end{matrix}; q,p;y\right) := \sum_{n\in\mathbb{N}} \frac{\theta(t_0,t_1,\ldots,t_s)_n}{\theta(q,w_1,\ldots,w_s)_n}\, y^n. \tag{17}$$

For fixed t_j and w_j, the function ${}_{s+1}E_s$ degenerates in the limit $p \to 0$ to the basic q-hypergeometric series ${}_{s+1}\varphi_s$ with the condition $\prod_{k=0}^{s} t_s = q\prod_{k=1}^{s} w_s$. There are some problems with the convergence of infinite series (17), and therefore we assume that they terminate due to the condition $t_k = q^{-N}p^M$ for some k and $N \in \mathbb{N}$, $M \in \mathbb{Z}$. For consideration of some questions, the additive system of notation for the elliptic hypergeometric series is more convenient (see, e.g., Ch. 11 in [16] or [44]), but we skip it here.

Series (17) is called well poised, if $t_0q = t_1w_1 = \ldots = t_sw_s$. In this case the balancing condition takes the form $t_1\cdots t_s = \pm q^{(s+1)/2}t_0^{(s-1)/2}$, and the functions $h(u)$ and ${}_{s+1}E_s$ become invariant under the changes $t_j \to pt_j$, $j = 1,\ldots,s-1$, and $t_0 \to p^2t_0$. For odd s and the balancing condition of the form $t_1\cdots t_s = +q^{(s+1)/2}t_0^{(s-1)/2}$, there appears the symmetry $t_0 \to pt_0$ and ${}_{s+1}E_s$ becomes an elliptic function of all free parameters $\log t_j$, $j = 0,\ldots,s-1$, with equal periods (such functions were called in [39, 44] the totally elliptic functions). Under the four additional constraints $t_{s-3} = q\sqrt{t_0}$, $t_{s-2} = -q\sqrt{t_0}$, $t_{s-1} = q\sqrt{t_0/p}$, $t_s = -q\sqrt{pt_0}$, connected with the doubling of the argument of theta functions, the series are called very well poised. In [40], a special notation was introduced for the very well poised elliptic hypergeometric series:

$$
\begin{aligned}
&{}_{s+1}E_s\left(\begin{matrix} t_0, t_1, \ldots, t_{s-4}, q\sqrt{t_0}, -q\sqrt{t_0}, q\sqrt{t_0/p}, -q\sqrt{pt_0} \\ qt_0/t_1, \ldots, qt_0/t_{s-4}, \sqrt{t_0}, -\sqrt{t_0}, \sqrt{pt_0}, -\sqrt{t_0/p} \end{matrix}; q, p; -y\right) \qquad (18)\\
&\quad = \sum_{n=0}^{\infty} \frac{\theta(t_0q^{2n};p)}{\theta(t_0;p)} \prod_{m=0}^{s-4} \frac{\theta(t_m)_n}{\theta(qt_0t_m^{-1})_n} (qy)^n =: {}_{s+1}V_s(t_0; t_1, \ldots, t_{s-4}; q, p; y),
\end{aligned}
$$

where the balancing condition has the form $\prod_{k=1}^{s-4} t_k = \pm t_0^{(s-5)/2} q^{(s-7)/2}$, and for odd s we assume the positive sign choice for preserving the symmetry $t_0 \to pt_0$. For the y argument value $y = 1$, it is omitted in the series notation. Summation formula (15) thus gives a closed form expression for the terminating ${}_{10}V_9(t_0; t_1, \ldots, t_5; q, p)$-series.

Contour integrals $\int_C \Delta(u)du$ are called the elliptic hypergeometric integrals, if the function $\Delta(u)$ satisfies the system of three equations

$$
\Delta(u + \omega_k) = h_k(u)\Delta(u), \quad k = 1, 2, 3, \qquad (19)
$$

where $\omega_{1,2,3} \in \mathbb{C}$ are some pairwise incommensurate parameters and $h_k(u)$ – some elliptic functions with periods ω_k, ω_{k+1} (we set $\omega_{k+3} = \omega_k$). One can weaken requirement (19) by leaving only one equation, but then there appears a functional freedom in the choice of $\Delta(u)$, which should be fixed in some other way.

Omitting the details of consideration [41, 44], we present the general form of permissible $\Delta(u)$. We suppose that this function satisfies equations (19) for $k = 1, 2$, where

$$
h_1(u) = y_1 \prod_{j=1}^{s} \frac{\theta(t_je^{2\pi iu/\omega_2};p)}{\theta(w_je^{2\pi iu/\omega_2};p)}, \quad h_2(u) = y_2 \prod_{j=1}^{\ell} \frac{\theta(\tilde{t}_je^{-2\pi iu/\omega_1};r)}{\theta(\tilde{w}_je^{-2\pi iu/\omega_1};r)},
$$

with $|p|, |r| < 1$ and $\prod_{j=1}^s t_j = \prod_{j=1}^s w_j$, $\prod_{j=1}^\ell \tilde{t}_j = \prod_{j=1}^\ell \tilde{w}_j$. If we take $|q| < 1$, then the most general meromorphic function $\Delta(u)$ has the form

$$\Delta(u) = \prod_{j=1}^s \frac{\Gamma(t_j e^{2\pi i \frac{u}{\omega_2}}; p, q)}{\Gamma(w_j e^{2\pi i \frac{u}{\omega_2}}; p, q)} \prod_{j=1}^\ell \frac{\Gamma(\tilde{t}_j e^{-2\pi i \frac{u}{\omega_1}}; \tilde{q}, r)}{\Gamma(\tilde{w}_j e^{-2\pi i \frac{u}{\omega_1}}; \tilde{q}, r)} \prod_{k=1}^m \frac{\theta(a_k e^{2\pi i \frac{u}{\omega_2}}; q)}{\theta(b_k e^{2\pi i \frac{u}{\omega_2}}; q)} e^{cu+d}, \quad (20)$$

where the parameters d and m are arbitrary, and a_k, b_k, c are connected with y_1 and y_2 by the relations $y_2 = e^{c\omega_2}$ and $y_1 = e^{c\omega_1} \prod_{k=1}^m b_k a_k^{-1}$. It appears that the function $h_3(u)$ cannot be arbitrary – it is determined from expression (20).

For $|q| = 1$, it is necessary in (20) to choose $\ell = s$ and to fix parameters in such a way that the Γ-functions are combined to the modified elliptic gamma function $G(u;\omega)$ (it is precisely in this way that this function was built in [41]):

$$\Delta(u) = \prod_{j=1}^s \frac{G(u + g_j; \omega)}{G(u + v_j; \omega)} e^{cu+d}, \quad (21)$$

where the parameters g_j, v_j are connected with t_j, w_j by the relations $t_j = e^{2\pi i g_j/\omega_2}$, $w_j = e^{2\pi i v_j/\omega_2}$, and $y_{1,2} = e^{c\omega_{1,2}}$. The integrals $\int_C \Delta(u) du$ with kernels of the indicated form define elliptic analogues of the Meijer function. A more general theta hypergeometric analogue of the Meijer function was constructed in [41], but we skip it here.

We limit consideration to the choice $\ell = m = 0$ in (20). The corresponding integrals are called well poised, if $t_1 w_1 = \ldots = t_s w_s = pq$. The additional condition of very well poisedness fixes eight parameters $t_{s-7}, \ldots, t_s = \{\pm(pq)^{1/2}, \pm q^{1/2} p, \pm p^{1/2} q, \pm pq\}$ and doubles the argument of the elliptic gamma function: $\prod_{j=s-7}^s \Gamma(t_j z; p, q) = 1/\Gamma(z^{-2}; p, q)$. The most interesting are the very well poised elliptic hypergeometric integrals with even number of parameters

$$I^{(m)}(t_1, \ldots, t_{2m+6}) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^{2m+6} \Gamma(t_j z^{\pm}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}, \qquad \prod_{j=1}^{2m+6} t_j = (pq)^{m+1}, \quad (22)$$

with $|t_j| < 1$ and the "correct" choice of the sign in the balancing condition. They represent integral analogues of the ${}_{s+1}V_s$-series with odd s, "correct" balancing condition and the argument $y = 1$, in the sense that these series appear as residue sums of particular pole sequences of the kernel of $I^{(m)}$. We note that $I^{(0)}$ coincides with the elliptic beta integral.

Properties of the elliptic functions explain the origins of the old hypergeometric notions of balancing, well poisedness, and very well poisedness. However, strictly speaking these notions are consistently defined only at the elliptic level, because there are limits to such q-hypergeometric identities in which they are not conserved any more [28, 40]! The fact of unique determination of the balancing condition for series (18) with odd s and integrals (22) (precisely these objects emerge in the main part of interesting applications) illustrates a deep internal tie between the "elliptic" and "hypergeometric" classes of special functions. Multivariable elliptic hypergeometric series and integrals are defined analogously to the univariate case

– it is necessary to use systems of finite difference equations for kernels with the coefficients which are elliptic functions of all summation or integration variables [39, 41], which is a natural generalization of the approach of Pochhammer and Horn to the functions of hypergeometric type [2, 17].

An elliptic analogue of the Gauss hypergeometric function. We take eight parameters $t_1, \ldots, t_8 \in \mathbb{C}$ and two base variables $p, q \in \mathbb{C}$ satisfying the constraints $|p|, |q| < 1$ and $\prod_{j=1}^{8} t_j = p^2 q^2$ (the balancing condition). For $|t_j| < 1$, $j = 1, \ldots, 8$, an elliptic analogue of the Gauss hypergeometric function ${}_2F_1(a, b; c; x)$ is defined by the integral representation [44]

$$V(t_1, \ldots, t_8; p, q) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^{8} \Gamma(t_j z^{\pm}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}, \tag{23}$$

i.e. by the choice $m = 1$ in expression (22). For other admissible values of parameters, the V-function is defined by the analytical continuation of expression (23). From this continuation one can see that the V-function is meromorphic in all parameters for arbitrary $t_j \in \mathbb{C}^*$ – it is sufficient for this to compute residues of the integrand and to define analytically continued function as a sum of the integral over some fixed contour and residues of the poles crossing this contour. For a more detailed analysis of the meromorphic character of the elliptic hypergeometric integrals and an interesting role of the "correct" sign choice in the balancing condition played in that, see [27].

The first nontrivial property of function (23) consists in its reduction to the elliptic beta integral under the condition $t_j t_k = pq$ for an arbitrary pair of parameters t_j and t_k (expression (13) appears from $t_7 t_8 = pq$). The V-function is evidently symmetric in p and q. It is invariant also under the S_8-group of permutations of parameters t_j isomorphic to the A_7 Weyl group. We consider the double integral

$$\kappa \int_{\mathbb{T}^2} \frac{\prod_{j=1}^{4} \Gamma(a_j z^{\pm}, b_j w^{\pm}; p, q)\, \Gamma(c z^{\pm} w^{\pm}; p, q)}{\Gamma(z^{\pm 2}, w^{\pm 2}; p, q)} \frac{dz}{z} \frac{dw}{w},$$

where $a_j, b_j, c \in \mathbb{C}$, $|a_j|, |b_j|, |c| < 1$, and $c^2 \prod_{j=1}^{4} a_j = c^2 \prod_{j=1}^{4} b_j = pq$. Using formula (13) for integrations over z or w (the permutation of the order of integrations is permitted), we obtain the following transformation formula:

$$V(\underline{t}) = \prod_{1 \leq j < k \leq 4} \Gamma(t_j t_k, t_{j+4} t_{k+4}; p, q)\, V(\underline{s}), \tag{24}$$

where we used the compact notation $V(\underline{t}) = V(t_1, \ldots, t_8; p, q)$,

$$\left\{ \begin{array}{ll} s_j = \rho^{-1} t_j, & j = 1, 2, 3, 4 \\ s_j = \rho t_j, & j = 5, 6, 7, 8 \end{array} \right. ; \quad \rho = \sqrt{\frac{t_1 t_2 t_3 t_4}{pq}} = \sqrt{\frac{pq}{t_5 t_6 t_7 t_8}},$$

and $|t_j|, |s_j| < 1$. This fundamentally important relation was obtained by the author in [41], where the function $V(\underline{t})$ appeared for the first time. It represents an elliptic analogue (moreover, its integral generalization) of the Bailey transformation for four non terminating ${}_{10}\varphi_9$-series [16].

We repeat transformation (24) once more with the parameters $s_{3,4,5,6}$, playing the role of $t_{1,2,3,4}$, and permute parameters t_3, t_4 with t_5, t_6 in the result. This yields the relation

$$V(\underline{t}) = \prod_{j,k=1}^{4} \Gamma(t_j t_{k+4}; p, q)\; V(T^{\frac{1}{2}}/t_1, \ldots, T^{\frac{1}{2}}/t_4, U^{\frac{1}{2}}/t_5, \ldots, U^{\frac{1}{2}}/t_8), \qquad (25)$$

where $T = t_1 t_2 t_3 t_4$, $U = t_5 t_6 t_7 t_8$ and $|T|^{1/2} < |t_j| < 1$, $|U|^{1/2} < |t_{j+4}| < 1$, $j = 1, 2, 3, 4$. We equate now the right-hand sides of relations (24) and (25), express parameters t_j in terms of s_j and obtain the third relation

$$V(\underline{s}) = \prod_{1 \le j < k \le 8} \Gamma(s_j s_k; p, q)\, V(\sqrt{pq}/s_1, \ldots, \sqrt{pq}/s_8), \qquad (26)$$

where $|pq|^{1/2} < |s_j| < 1$ for all j.

We consider Euclidean space $\mathbb{R}^8$ with the scalar product $\langle x, y\rangle$ and an orthonormal basis $e_i \in \mathbb{R}^8$, $\langle e_i, e_j\rangle = \delta_{ij}$. The root system A_7 consists of the vectors $v = \{e_i - e_j,\, i \neq j\}$. Its Weyl group is composed from the reflections $x \to S_v(x) = x - 2v\langle v, x\rangle/\langle v, v\rangle$ acting in the hyperplane orthogonal to the vector $\sum_{i=1}^{8} e_i$ (i.e., the coordinates of the vectors $x = \sum_{i=1}^{8} x_i e_i$ satisfy the constraint $\sum_{i=1}^{8} x_i = 0$), and it coincides with the permutation group S_8.

We connect parameters of the $V(\underline{t})$-function with the coordinates x_j by the relations $t_j = e^{2\pi i x_j}(pq)^{1/4}$. Then the balancing condition assumes that $\sum_{i=1}^{8} x_i = 0$. The first coordinate transformation (24) is now easily seen to correspond to the reflection $S_v(x)$ for the vector $v = (\sum_{i=5}^{8} e_i - \sum_{i=1}^{4} e_i)/2$ having the canonical length $\langle v, v\rangle = 2$. This reflection extends the group A_7 to the exceptional Weyl group E_7. Relations (25) and (26) were proved in a different way by Rains in [27], where it was indicated that these transformations belong to the group E_7.

We denote by $V(qt_j, q^{-1}t_k)$ elliptic hypergeometric functions contiguous to $V(\underline{t})$ in the sense that t_j and t_k are respectively replaced by qt_j and $q^{-1}t_k$. The following contiguous relation for the V-functions is valid

$$t_7 \theta\left(t_8 t_7^{\pm}/q; p\right) V(qt_6, q^{-1}t_8) - (t_6 \leftrightarrow t_7) = t_7 \theta\left(t_6 t_7^{\pm}; p\right) V(\underline{t}), \qquad (27)$$

where $(t_6 \leftrightarrow t_7)$ denotes the permutation of parameters in the preceding expression (such a relation was used already in [38]). Indeed, for $y = t_6, w = t_7$, and $x = q^{-1}t_8$ the Riemann identity (6) is equivalent to the q-difference equation for V-function's integrand $\Delta(z, \underline{t}) = \prod_{k=1}^{8} \Gamma(t_k z^{\pm}; p, q)/\Gamma(z^{\pm 2}; p, q)$ coinciding with (27) after replacement of V-functions by $\Delta(z, \underline{t})$ with appropriate parameters. Integration of this equation over the contour $\mathbb{T}$ yields (27). We now substitute symmetry transformation (26) in (27) and obtain the second contiguous relation

$$t_6 \theta\Big(\frac{t_7}{qt_8}; p\Big) \prod_{k=1}^{5} \theta\Big(\frac{t_6 t_k}{q}; p\Big) V(q^{-1}t_6, qt_8) - (t_6 \leftrightarrow t_7) = t_6 \theta\Big(\frac{t_7}{t_6}; p\Big) \prod_{k=1}^{5} \theta(t_8 t_k; p)) V(\underline{t}).$$

An appropriate combination of these two equalities yields the equation

$$\mathcal{A}(\underline{t})\Big(U(qt_6, q^{-1}t_7) - U(\underline{t})\Big) + (t_6 \leftrightarrow t_7) + U(\underline{t}) = 0, \tag{28}$$

where we have denoted $U(\underline{t}) = V(\underline{t})/\Gamma(t_6t_8^{\pm}, t_7t_8^{\pm}; p, q)$ and

$$\mathcal{A}(\underline{t}) = \frac{\theta(t_6/qt_8, t_6t_8, t_8/t_6; p)}{\theta(t_6/t_7, t_7/qt_6, t_6t_7/q; p)} \prod_{k=1}^{5} \frac{\theta(t_7t_k/q; p)}{\theta(t_8t_k; p)}. \tag{29}$$

Substituting $t_j = e^{2\pi i g_j/\omega_2}$, one can check that the potential $\mathcal{A}(\underline{t})$ is a modular invariant elliptic function of the variables $g_1, \ldots, g_7$, i.e. it does not change after the replacements $g_j \to g_j + \omega_{2,3}$ and $(\omega_2, \omega_3) \to (-\omega_3, \omega_2)$.

We introduce now the new notation for parameters: $t_6 = cx, t_7 = c/x$, and

$$\varepsilon_k = \frac{q}{ct_k},\ k = 1, \ldots, 5, \quad \varepsilon_8 = \frac{c}{t_8}, \quad \varepsilon_7 = \frac{\varepsilon_8}{q}.$$

Setting $c = \sqrt{\varepsilon_6\varepsilon_8}/p^2$, we find that the balancing condition in terms of ε_k takes the form $\prod_{k=1}^{8} \varepsilon_k = p^2q^2$. After the replacement of $U(\underline{t})$ in (28) by some unknown function $f(x)$, we obtain a q-difference equation of the second order which is called *the elliptic hypergeometric equation*:

$$A(x)\left(f(qx) - f(x)\right) + A(x^{-1})\left(f(q^{-1}x) - f(x)\right) + \nu f(x) = 0, \tag{30}$$

$$A(x) = \frac{\prod_{k=1}^{8} \theta(\varepsilon_k x; p)}{\theta(x^2, qx^2; p)}, \qquad \nu = \prod_{k=1}^{6} \theta\left(\frac{\varepsilon_k\varepsilon_8}{q}; p\right). \tag{31}$$

We have already one functional solution of this equation

$$f_1(x) = \frac{V(q/c\varepsilon_1, \ldots, q/c\varepsilon_5, cx, c/x, c/\varepsilon_8; p, q)}{\Gamma(c^2x^{\pm}/\varepsilon_8, x^{\pm}\varepsilon_8; p, q)}, \tag{32}$$

where, for the choice $C = \mathbb{T}$ of the integration contour in the definition of the V-function, it is necessary to impose the constraints (in the previous parametrization) $\sqrt{|pq|} < |t_j| < 1$, $j = 1, \ldots, 5$, and $\sqrt{|pq|} < |q^{\pm 1}t_6|, |q^{\pm 1}t_7|, |q^{\pm 1}t_8| < 1$, which can be relaxed by analytical continuation. Other independent solutions can be obtained by the multiplication of one of the parameters $\varepsilon_1, \ldots, \varepsilon_5$, and x by some powers of p or by permutations of $\varepsilon_1, \ldots, \varepsilon_5$ with ε_6.

We denote $\varepsilon_k = e^{2\pi i a_k/\omega_2}$, $x = e^{2\pi i u/\omega_2}$, and $f_1(x) =: F_1(u; \underline{a}; \omega_1, \omega_2, \omega_3)$. After these changes one can check that equation (30) is invariant with respect to the modular transformation $(\omega_2, \omega_3) \to (-\omega_3, \omega_2)$. One of the linear independent solutions of (30) therefore has the form $F_2(u; \underline{a}; \omega_1, \omega_2, \omega_3) := F_1(u; \underline{a}; \omega_1, -\omega_3, \omega_2)$. The same solution would be obtained if we repeat the derivation of equation (30) and its solution (32) after replacing the Γ-functions by the modified elliptic gamma function $G(u; \omega)$. The F_2-function is therefore well defined even for $|q| = 1$. Different limiting transitions from the V-function and other elliptic hypergeometric integrals to q-hypergeometric integrals of the Mellin-Barnes or Euler type are described in [44, 45] and much more systematically in [8, 30].

Biorthogonal functions of the hypergeometric type. In analogy with the residue calculus for the elliptic beta integral (14), one can consider the sum of residues for a particular geometric progression of poles of the V-function kernel for one of the parameters. This leads to the very well poised ${}_{12}V_{11}$-elliptic hypergeometric series the termination condition of which is guaranteed by a special discretization of the chosen parameter. In this way one can derive contiguous relations for the terminating ${}_{12}V_{11}$-series [47, 48] out of the contiguous relations for the V-function, which we omit due to the lack of space. For instance, this yields the following particular solution of the elliptic hypergeometric equation (30):

$$R_n(x;q,p) = {}_{12}V_{11}\left(\frac{\varepsilon_6}{\varepsilon_8}; \frac{q}{\varepsilon_1\varepsilon_8}, \frac{q}{\varepsilon_2\varepsilon_8}, \frac{q}{\varepsilon_3\varepsilon_8}, \frac{qp}{\varepsilon_4\varepsilon_8}, \frac{qp}{\varepsilon_5\varepsilon_8}, \varepsilon_6 x, \frac{\varepsilon_6}{x}; q, p\right), \tag{33}$$

valid under the condition $pq/\varepsilon_4\varepsilon_8 = q^{-n}$, $n \in \mathbb{N}$ (we remind that $\prod_{k=1}^8 \varepsilon_k = p^2q^2$). We describe below properties of the R_n-function found in [41] in the notation passing to our after the replacements $t_{0,1,2} \to \varepsilon_{1,2,3}$, $t_3 \to \varepsilon_6$, $t_4 \to \varepsilon_8$, $\mu \to \varepsilon_4\varepsilon_8/pq$, and $A\mu/qt_4 \to pq/\varepsilon_5\varepsilon_8$.

Equation (30) is symmetric in $\varepsilon_1, \ldots, \varepsilon_6$. Since series (18) are elliptic in all parameters, function (33) is symmetric in $\varepsilon_1, \ldots, \varepsilon_5$ and each of them can be used for the termination of series. A permutation of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_5$ with ε_6 yields $R_n(z;q,p)$ up to some multiplier independent on x due to an elliptic analogue of the Bailey transformation for terminating ${}_{12}V_{11}$-series [14], which can be obtained by degeneration from equality (24).

The same contiguous relations for the ${}_{12}V_{11}$-series yield the following three term recurrence relation for $R_n(x;q,p)$ in the index n:

$$\begin{aligned}
&(z(x)-\alpha_{n+1})\rho(Aq^{n-1}/\varepsilon_8)\left(R_{n+1}(x;q,p)-R_n(x;q,p)\right)+(z(x)-\beta_{n-1}) \\
&\quad\times\rho(q^{-n})\left(R_{n-1}(x;q,p)-R_n(x;q,p)\right)+\delta(z(x)-z(\varepsilon_6))R_n(x;q,p)=0, \\
&\quad z(x)=\frac{\theta(x\xi^{\pm};p)}{\theta(x\eta^{\pm};p)}, \qquad \alpha_n = z(q^n/\varepsilon_8), \qquad \beta_n = z(Aq^{n-1}), \\
&\quad \rho(t)=\frac{\theta\left(t, \frac{\varepsilon_6}{\varepsilon_8 t}, \frac{q\varepsilon_6}{\varepsilon_8 t}, \frac{qt}{\varepsilon_1\varepsilon_2}, \frac{qt}{\varepsilon_2\varepsilon_3}, \frac{qt}{\varepsilon_1\varepsilon_3}, \frac{q^2t\eta^{\pm}}{A}; p\right)}{\theta\left(\frac{qt^2\varepsilon_8}{A}, \frac{q^2t^2\varepsilon_8}{A}; p\right)}, \\
&\quad \delta=\theta\left(\frac{q^2\varepsilon_6}{A}, \frac{q}{\varepsilon_1\varepsilon_8}, \frac{q}{\varepsilon_2\varepsilon_8}, \frac{q}{\varepsilon_3\varepsilon_8}, \varepsilon_6\eta^{\pm}; p\right),
\end{aligned} \tag{34}$$

where $A = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_6\varepsilon_8$, and ξ and η are arbitrary gauge parameters, $\xi \neq \eta^{\pm 1}p^k$, $k \in \mathbb{Z}$. The initial conditions $R_{-1} = 0$ and $R_0 = 1$ guarantee that all the dependence on the variable x enters only through $z(x)$, and that $R_n(x)$ is a rational function of $z(x)$ with poles at the points $\alpha_1, \ldots, \alpha_n$.

The elliptic hypergeometric equation for the R_n-function can be rewritten in the form of a generalized eigenvalue problem $\mathcal{D}_1 R_n = \lambda_n \mathcal{D}_2 R_n$ for some q-difference operators of the second order $\mathcal{D}_{1,2}$ and discrete spectrum λ_n [41]. We denote by ϕ_λ solutions of an abstract spectral problem $\mathcal{D}_1\phi_\lambda = \lambda\mathcal{D}_2\phi_\lambda$, and by ψ_λ solutions of the equation $\mathcal{D}_1^T\psi_\lambda = \lambda\mathcal{D}_2^T\psi_\lambda$, where $\mathcal{D}_{1,2}^T$ are the operators conjugated with respect to some inner product $\langle\psi|\phi\rangle$, i.e. $\langle\mathcal{D}_{1,2}^T\psi|\phi\rangle = \langle\psi|\mathcal{D}_{1,2}\phi\rangle$. Then $0 = \langle\psi_\mu|(\mathcal{D}_1 - \lambda\mathcal{D}_2)\phi_\lambda\rangle = (\mu-\lambda)\langle\mathcal{D}_2^T\psi_\mu|\phi_\lambda\rangle$, i.e. the function $\mathcal{D}_2^T\psi_\mu$ is orthogonal to ϕ_λ for $\mu\neq\lambda$. This simple observation was put by Zhedanov into the basis of the theory of biorthogonal rational functions generalizing orthogonal polynomials [54] (see also [47, 48]). Analogues of the functions $\mathcal{D}_2^T\psi_\mu$ for $R_n(z;q,p)$ have the form

$$T_n(x;q,p) = {}_{12}V_{11}\left(\frac{A\varepsilon_6}{q}; \frac{A}{\varepsilon_1}, \frac{A}{\varepsilon_2}, \frac{A}{\varepsilon_3}, \varepsilon_6 x, \frac{\varepsilon_6}{x}, \frac{qp}{\varepsilon_4\varepsilon_8}, \frac{qp}{\varepsilon_5\varepsilon_8}; q, p\right), \qquad (35)$$

which are rational functions of $z(x)$ with poles at the points $\beta_1, \ldots, \beta_n$.

We denote $R_{nm}(x) \equiv R_n(x;q,p)R_m(x;p,q)$ and $T_{nm}(x) \equiv T_n(x;q,p)T_m(x;p,q)$, where all the ${}_{12}V_{11}$-series terminate simultaneously because of the modified termination condition $\varepsilon_4\varepsilon_8 = p^{m+1}q^{n+1}$, $n, m \in \mathbb{N}$. The functions R_{nm} solve now not one but two generalized eigenvalue problems which differ from each other by the permutation of p and q.

Theorem 2. *The following two-index biorthogonality relation is valid:*

$$\kappa\int_{C_{mn,kl}} T_{nl}(x)R_{mk}(x)\frac{\prod_{j\in S}\Gamma(\varepsilon_j x^{\pm};p,q)}{\Gamma(x^{\pm 2}, Ax^{\pm};p,q)}\frac{dx}{x} = h_{nl}\,\delta_{mn}\,\delta_{kl}, \qquad (36)$$

where $S = \{1,2,3,6,8\}$, $C_{mn,kl}$ *denotes the contour separating sequences of points* $\varepsilon_j p^a q^b$ $(j = 1,2,3,6)$, $\varepsilon_8 p^{a-k}q^{b-m}, p^{a+1-l}q^{b+1-n}/A$, $a, b \in \mathbb{N}$, *from their* $x \to x^{-1}$ *reciprocals, and the normalization constants have the form*

$$\begin{aligned} h_{nl} &= \frac{\prod_{j<k,\,j,k\in S}\Gamma(\varepsilon_j\varepsilon_k;p,q)}{\prod_{j\in S}\Gamma(A\varepsilon_j^{-1};p,q)}\,h_n(q,p)\cdot h_l(p,q), \\ h_n(q,p) &= \frac{\theta(A/q\varepsilon_8;p)\theta(q, q\varepsilon_6/\varepsilon_8, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3, A\varepsilon_6)_n\, q^{-n}}{\theta(Aq^{2n}/q\varepsilon_8;p)\theta(1/\varepsilon_6\varepsilon_8, \varepsilon_1\varepsilon_6, \varepsilon_2\varepsilon_6, \varepsilon_3\varepsilon_6, A/q\varepsilon_6, A/q\varepsilon_8)_n}. \end{aligned}$$

This theorem was proved in [41] by direct computation of the integral in the left-hand side with the help of formula (13). Appearance of the two-index orthogonality relations for functions of one variable is a new phenomenon in the theory of special functions. It should be remarked that only for $k = l = 0$ there exists the limit $p \to 0$ and functions $R_n(x;q,0)$, $T_n(x;q,0)$, appearing from that, coincide with Rahman's family of continuous ${}_{10}\varphi_9$-biorthogonal rational functions [26]. A special limit $\mathrm{Im}(\omega_3) \to \infty$ in the modular transformed R_{nm} and T_{nm} leads to the two-index biorthogonal functions which are expressed as products of two modular conjugated ${}_{10}\varphi_9$-series [44]. A special restriction for one of the parameters in $R_n(x;q,p)$ and $T_n(x;q,p)$ leads to the biorthognal rational functions of a discrete argument [47] generalizing Wilson's functions [53]. All these functions are natural generalizations of the Askey-Wilson polynomials [4]. Note that $R_{nm}(x)$ and $T_{nm}(x)$ are meromorphic functions of $x \in \mathbb{C}^*$ with essential singularities at $x = 0, \infty$ and only for $k = l = 0$ or $n = m = 0$ do they become rational functions of some argument depending on x. It is expected that there are more general systems of biorthogonal functions based on the V-function and connected to solutions of the generalized eigenvalue problem with continuous spectrum or recurrence relation (34) with the shifted index n.

Elliptic beta integrals on root systems. We introduce an analogue of the constant κ for the C_n (or BC_n) root system $\kappa_n = (p;p)_\infty^n (q;q)_\infty^n/(2\pi i)^n 2^n n!$ and describe a C_n-elliptic beta integral representing a multiparameter generalization of integral (13).

Theorem 3. *We take n variables $z_1, \dots, z_n \in \mathbb{T}$ and complex parameters $t_1, \dots, t_{2n+4}$ and p, q satisfying the constraints $|p|, |q|, |t_j| < 1$ and $\prod_{j=1}^{2n+4} t_j = pq$. Then*

$$\kappa_n \int_{\mathbb{T}^n} \prod_{1\le j<k\le n} \frac{1}{\Gamma(z_j^{\pm} z_k^{\pm}; p, q)} \prod_{j=1}^n \frac{\prod_{m=1}^{2n+4} \Gamma(t_m z_j^{\pm}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$
$$= \prod_{1\le m<s\le 2n+4} \Gamma(t_m t_s; p, q). \tag{37}$$

Formula (37) was suggested and partially confirmed in [11], and it was proved by different methods in [27, 43, 44]. It reduces to one of Gustafson's integration formulas [18] in a special $p \to 0$ limit.

Theorem 4. *We take complex parameters $t, t_m\,(m = 1, \dots, 6), p$ and q restricted by the conditions $|p|, |q|, |t|, |t_m| < 1$, and $t^{2n-2} \prod_{m=1}^6 t_m = pq$. Then*

$$\kappa_n \int_{\mathbb{T}^n} \prod_{1\le j<k\le n} \frac{\Gamma(t z_j^{\pm} z_k^{\pm}; p, q)}{\Gamma(z_j^{\pm} z_k^{\pm}; p, q)} \prod_{j=1}^n \frac{\prod_{m=1}^{6} \Gamma(t_m z_j^{\pm}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$
$$= \prod_{j=1}^n \left(\frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \prod_{1\le m<s\le 6} \Gamma(t^{j-1} t_m t_s; p, q) \right). \tag{38}$$

In order to prove (38), we consider the following $(2n-1)$-tuple integral

$$\kappa_n \kappa_{n-1} \int_{\mathbb{T}^{2n-1}} \prod_{1\le j<k\le n} \frac{1}{\Gamma(z_j^{\pm} z_k^{\pm}; p, q)} \prod_{j=1}^n \frac{\prod_{r=0}^5 \Gamma(t_r z_j^{\pm}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)}$$
$$\times \prod_{\substack{1\le j\le n \\ 1\le k\le n-1}} \Gamma(t^{1/2} z_j^{\pm} w_k^{\pm}; p, q) \prod_{1\le j<k\le n-1} \frac{1}{\Gamma(w_j^{\pm} w_k^{\pm}; p, q)}$$
$$\times \prod_{j=1}^{n-1} \frac{\Gamma(w_j^{\pm} t^{n-3/2} \prod_{s=1}^5 t_s; p, q)}{\Gamma(w_j^{\pm 2}, w_j^{\pm} t^{2n-3/2} \prod_{s=1}^5 t_s; p, q)} \frac{dw_1}{w_1} \cdots \frac{dw_{n-1}}{w_{n-1}} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}, \tag{39}$$

with p, q, t and t_r, $r = 0, \dots, 5$, lying inside the unit circle such that $t^{n-1} \prod_{r=0}^5 t_r = pq$. We denote the integral in the left-hand side of (38) as $I_n(t, t_1, \dots, t_5; p, q)$. Integration over the w-variables with the help of formula (37) brings expression (39) to the form $\Gamma^n(t) \Gamma^{-1}(t^n)\ I_n(t, t_1, \dots, t_5; p, q)$, where we have introduced the parameter t_6 via the relation $t^{2n-2}\prod_{j=1}^6 t_j = pq$. Because the integrand is bounded on the integration contour, we can change the order of integrations. As a result, integration over the z-variables with the help of formula (37) brings expression (39) to the form $\Gamma^{n-1}(t)\prod_{0\le r<s\le 5}\Gamma(t_rt_s)\ \ I_{n-1}(t,t^{1/2}t_1,\dots,t^{1/2}t_5;p,q)$, i.e. we obtain the following recurrence relation in the dimensionality of the integral of interest n:

$$I_n(t,t_1,\dots,t_5;p,q)=\frac{\Gamma(t^n;p,q)}{\Gamma(t;p,q)}\prod_{0\le r<s\le 5}\Gamma(t_rt_s;p,q)\ I_{n-1}(t,t^{1/2}t_1,\dots,t^{1/2}t_5;p,q).$$

Iterating it with known initial condition (13) for $n=1$, we obtain (38).

Integral (38) was suggested by van Diejen and the author in [10]. The given proof is taken from [11], it models Anderson's proof of the Selberg integral described in [2] (see Theorem 8.1.1 and Sect. 8.4). It represents also a direct generalization of Gustafson's method [18] of derivation of the multiple q-beta integral obtained from (38) after expressing t_6 via other parameters, removing the multipliers pq with the help of the reflection formula for $\Gamma(z;p,q)$, and taking the limit $p\to 0$. A number of further limiting relations in parameters leads to the Selberg integral – the fundamentally important integral because of many applications in mathematical physics. Therefore formula (38) represents an elliptic analogue of the Selberg integral (an analogous extension of Aomoto's integral described in Theorem 8.1.2 of [2] is derived in [27]). It can be interpreted also as an elliptic extension of the BC_n Macdonald-Morris constant term identities.

There are several other elliptic beta integrals on root systems. In particular, the author has suggested three different integrals for the A_n root system (two of them look different for even and odd values of n) in [41]. In [46], Warnaar and the author have found one more A_n-integral which appeared to be new even after degeneration to the q- and plain hypergeometric levels.

In analogy with the one dimensional case [41], it is natural to expect that the multiple elliptic beta integrals define measures in the biorthogonality relations for some functions of many variables generalizing relations (36). In [27, 28], Rains has constructed the first system of such functions on the basis of integral (38). These functions generalize also the Macdonald and Koornwinder orthogonal polynomials, as well as the interpolating polynomials of Okounkov. In this sense, the results of [27, 28] represent to the present moment the top level achievements of the theory of elliptic hypergeometric functions of many variables. The author especially likes the following BC_n-generalization of transformation (24) proven in [27]: $I(t_1,\dots,t_8;t;q,p)=I(s_1,\dots,s_8;t;q,p)$, where

$$\frac{I(t_1,\dots,t_8;t;q,p)}{\prod_{1\le j<k\le 8}\Gamma(t_jt_k;p,q,t)}=\kappa_n\int_{\mathbb{T}^n}\prod_{1\le j<k\le n}\frac{\Gamma(tz_j^{\pm}z_k^{\pm};p,q)}{\Gamma(z_j^{\pm}z_k^{\pm};p,q)}\prod_{j=1}^n\frac{\prod_{k=1}^8\Gamma(t_kz_j^{\pm};p,q)}{\Gamma(z_j^{\pm 2};p,q)}\frac{dz_j}{z_j},$$

$$\left\{\begin{array}{ll} s_j=\rho^{-1}t_j, & j=1,2,3,4\\ s_j=\rho t_j, & j=5,6,7,8\end{array}\right. ;\quad \rho=\sqrt{\frac{t_1t_2t_3t_4}{pqt^{1-n}}}=\sqrt{\frac{pqt^{1-n}}{t_5t_6t_7t_8}},\quad |t|,|t_j|,|s_j|<1,$$

and $\Gamma(z;p,q,t) = \prod_{j,k,l=0}^{\infty}(1-zt^jp^kq^l)(1-z^{-1}t^{j+1}p^{k+1}q^{l+1})$ is the elliptic gamma function of the higher level connected to the Barnes gamma function $\Gamma_4(u;\omega)$.

Integral analogues of the Bailey chains. The Bailey chains, discovered by Andrews, serve as a powerful tool for building constructive identities for hypergeometric series (see Ch. 12 in [2]). They describe mappings of given sequences of numbers to other sequences with the help of matrices admitting explicit inversions. So, the most general Bailey chain for the univariate q-hypergeometric series suggested in [1] is connected with the matrix attached to the ${}_8\varphi_7$ Jackson sum [7]. An elliptic generalization of this chain for the ${}_{s+1}V_s$-series was built in [40], but we do not consider here neither it nor its complement [52]. Instead of that, we present a generalization of the formalism of Bailey chains to the level of integrals discovered in [42].

We say that two functions $\alpha(z,t)$ and $\beta(z,t)$ form an elliptic integral Bailey pair with respect to the parameter t, if they are connected by the relation

$$\beta(w,t) = \kappa\int_{\mathbb{T}}\Gamma(tw^{\pm}z^{\pm};p,q)\,\alpha(z,t)\frac{dz}{z}. \tag{40}$$

Theorem 5. *For a given elliptic integral Bailey pair $\alpha(z,t)$ and $\beta(z,t)$ with respect to the parameter t, the functions*

$$\alpha'(w,st) = \frac{\Gamma(tuw^{\pm};p,q)}{\Gamma(ts^2uw^{\pm};p,q)}\,\alpha(w,t),$$
$$\beta'(w,st) = \kappa\frac{\Gamma(t^2s^2,t^2suw^{\pm};p,q)}{\Gamma(s^2,t^2,suw^{\pm};p,q)}\int_{\mathbb{T}}\frac{\Gamma(sw^{\pm}x^{\pm},ux^{\pm};p,q)}{\Gamma(x^{\pm2},t^2s^2ux^{\pm};p,q)}\,\beta(x,t)\frac{dx}{x},$$

where $w\in\mathbb{T}$, form a new Bailey pair with respect to the parameter st, and the functions

$$\alpha'(w,t) = \kappa\frac{\Gamma(s^2t^2,uw^{\pm};p,q)}{\Gamma(s^2,t^2,w^{\pm2},t^2s^2uw^{\pm};p,q)}\int_{\mathbb{T}}\frac{\Gamma(t^2sux^{\pm},sw^{\pm}x^{\pm}p,q)}{\Gamma(sux^{\pm};p,q)}\alpha(x,st)\frac{dx}{x},$$
$$\beta'(w,t) = \frac{\Gamma(tuw^{\pm};p,q)}{\Gamma(ts^2uw^{\pm};p,q)}\beta(w,st)$$

form a new Bailey pair with respect to the parameter t.

The proof is sufficiently simple. In the first case, it is necessary to substitute the key relation for $\beta(x,t)$ in the definition of $\beta'(w,st)$, to change the order of integrations and to take off one of the integrations with the help of the elliptic beta integral (we omit restrictions for the parameters necessary for validity of this procedure). The second relation is proved in a similar way. These chain substitution rules introduce two new parameters u and s at each step of their application. In fact, they are related to each other by the inversion of the integral operator entering the definition of integral Bailey pairs [46].

This theorem is used in a way analogous to the series case: it is necessary to take initial $\alpha(z,t)$ and $\beta(z,t)$, found, say, from formula (13), and to generate new pairs with the help of the described chain substitution rules. Equality (40) for these pairs leads to a binary tree of identities for elliptic hypergeometric integrals of different multiplicities. As an illustration, we would like to give one nontrivial relation. With the help of formula (13), one can easily verify validity of the following recurrence relation

$$I^{(m+1)}(t_1,\ldots,t_{2m+8}) = \frac{\prod_{2m+5\leq k<l\leq 2m+8}\Gamma(t_kt_l;p,q)}{\Gamma(\rho_m^2;p,q)} \times \kappa\int_{\mathbb{T}} \frac{\prod_{k=2m+5}^{2m+8}\Gamma(\rho_m^{-1}t_kw^{\pm};p,q)}{\Gamma(w^{\pm 2};p,q)} I^{(m)}(t_1,\ldots,t_{2m+4},\rho_m w,\rho_m w^{-1})\frac{dw}{w}, \tag{41}$$

where $\rho_m^2 = \prod_{k=2m+5}^{2m+8} t_k/pq$ and the integral $I^{(m)}$ was defined in (22). By an appropriate change of notation for parameters, we obtain a concrete realization of the Bailey pairs: $\alpha \propto I^{(m)}$ and $\beta \propto I^{(m+1)}$. For $m = 0$, substitution of the explicit expression for $I^{(0)}$ (13) in the right-hand side of (41) yields identity (24). Other interesting consequences of recursion (41) (an elliptic analogue of formula (2.2.2) in [2]) are considered in [44]. Various generalizations of the integral transformation (40) to root systems and their inversions are described in [46].

Conclusion. We would like to finish by listing some other achievements of the theory of elliptic hypergeometric functions. Multiple elliptic hypergeometric series were considered for the first time by Warnaar [51]. We described mostly properties of the elliptic hypergeometric integrals, since many results for the elliptic hypergeometric series represent their particular subcases and they can be derived via the residue calculus. A combinatorial derivation of the Frenkel-Turaev summation formula is given in [37]. Various generalizations of this sum to the root systems were found in [10, 31, 41, 51], and multivariable analogues of the Bailey transformation were described in [21, 27, 31, 51]. Expansions in partial fractions of the products of theta functions and the identities connected to them were discussed by Rosengren [31] (see also [11, 27]). The terminating continued fraction generated by three term recurrence relation (34) and the Racah type termination condition was computed in [48]. The raising and lowering operators connected with rational functions were discussed in [27, 49]. Connection to the Sklyanin algebra is considered in [24, 28, 32, 33]. In particular, in [33] Rosengren proved an old Sklyanin conjecture on the reproducing kernel. A systematic treatment of the elliptic determinant formulas connected to the root systems is given in the work of Rosengren and Schlosser [34]. In [22], it was shown that the ${}_{12}V_{11}$-series appears as a particular solution of the elliptic Painlevé equation discovered by Sakai [36]. An analogous role is played by the general solution of the elliptic hypergeometric equation [44, 45] and some multiple elliptic hypergeometric integrals [29]. The page limits of the present review did not allow the author to cite a number of other interesting results, an essentially more complete review of the literature is given in [44].

Derivation of the ${}_{12}V_{11}$-function from a similarity reduction of some discrete integrable system, given in [47], reflects the essence of a heuristic approach to all special functions of one variable. This approach is described in detail in [44] using a number of examples, including some other new special functions.

The main part of the plain hypergeometric constructions admits thus natural elliptic generalization, although such parallels emerge for a rather large number of free parameters and structural restrictions. Nevertheless there remain many open problems, in particular, analysis of the conditions of convergence of infinite elliptic hypergeometric series, investigation of specific properties of the points of finite order on the elliptic curve, computation of the nonterminating elliptic hypergeometric continued fraction, detailed analysis of the non-self-dual biorthogonal functions of [47] and construction of their multivariable analogues, and so on.

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Bogoliubov laboratory of theoretical physics, JINR, Dubna, Moscow reg., 141980, Russia; e-mail: spiridon@theor.jinr.ru