Rational interpolation to exponential-like functions on elliptic lattices

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The present file is http://www.math.ucl.ac.be/membres/magnus/num3/ellexp.pdf

Abstract. A function is called exponential-like with respect to a difference operator if it satisfies D f(x) = a[f(psi(x))+f(phi(x))], where the (divided) difference operator is D f(x) =[f(psi(x))-f(phi(x))]/[psi(x)-phi(x)]. The functions phi and psi define the setting of the theory, from the most elementary choice (x,x+h) to forms R(x) plus or minus square root of S(x), where R and S are rational functions of degrees up to 2 and 4. Remark that the difference equation is a symmetric combination of the two conjugate algebraic functions phi and psi. The difference equation is also a recurrence relation on a lattice built from y(n)= phi(x(n)), y(n+1)=psi(x(n)), from which x(n+1) is found through y(n+1)=phi(x(n+1)). When the degrees of R and S are 2 and 4, we get a so-called elliptic lattice, or grid, as x(n) and y(n) appear to be elliptic functions of n (Baxter, Spiridonov, Zhedanov). The exponential-like function of above is interpolated on such a lattice y(0), y(1),... by rational functions with poles on a well-chosen sequence y'(0), y'(1),... of the same family of lattices. The relevant continued fraction expansion will be investigated.

Contents

1. Difference equations and lattices: the elliptic lattice.

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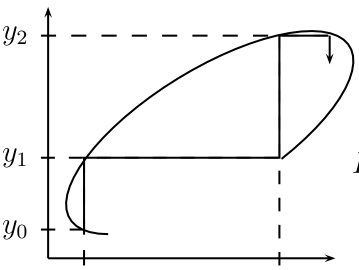
1.1. The divided difference operator.

$$(\mathcal{D}f)(x) = \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)}, \tag{1}$$



The simplest choice for φ and ψ is to take the two determinations of an algebraic function of degree 2, i.e., the two $y-$roots of

$$F(x,y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \tag{2a}$$

where X_0, X_1, and X_2 are rational functions.

The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where y_1 is the second root of (2a) at x_0. We need x_1 such that y_1 is one of the two roots of (2a) at x_1, so for one of the roots of $F(x, y_1) = 0$ which is not x_0. Here again, the simplest case is when F is of degree 2 in x:

$$F(x,y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \tag{2b}$$

Both forms (2a) and (2b) hold simultaneously when F is ***biquadratic***:

$$F(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}x^i y^j. \tag{3}$$

Then,

$$\varphi(x) \text{ and } \psi(x) = \frac{-X_1(x)/2 \pm \sqrt{P(x)}}{X_2(x)} \tag{4}$$

where P is the fourth degree polynomial $X_1^2/4 - X_0X_2$.

1.2. Relations for elliptic sequences.

From (3), the coefficients in (2b) and (2a) are polynomials of degrees $\leqslant 2$. From $F(x_n, y_n) = F(x_n, y_{n+1}) = 0$, $y_n + y_{n+1}$ and $y_n y_{n+1}$ are the rational functions

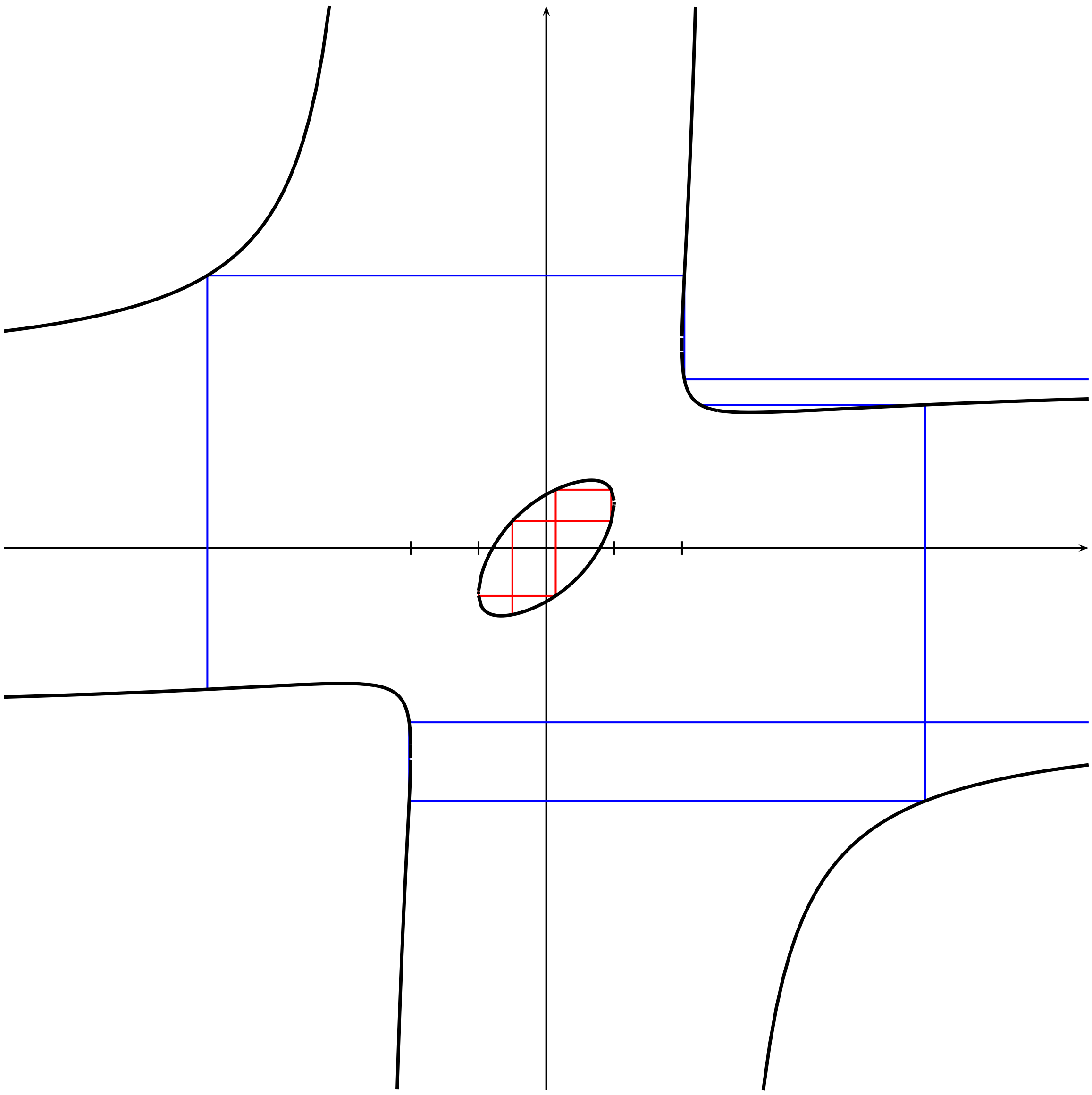
$$y_n + y_{n+1} = -\frac{X_1(x_n)}{X_2(x_n)} = -\frac{c_{2,1}x_n^2 + c_{1,1}x_n + c_{0,1}}{c_{2,2}x_n^2 + c_{1,2}x_n + c_{0,2}}, y_n y_{n+1} = \frac{X_0(x_n)}{X_2(x_n)} = \frac{c_{2,0}x_n^2 + c_{1,0}x_n + c_{0,0}}{c_{2,2}x_n^2 + c_{1,2}x_n + c_{0,2}}. \tag{5}$$

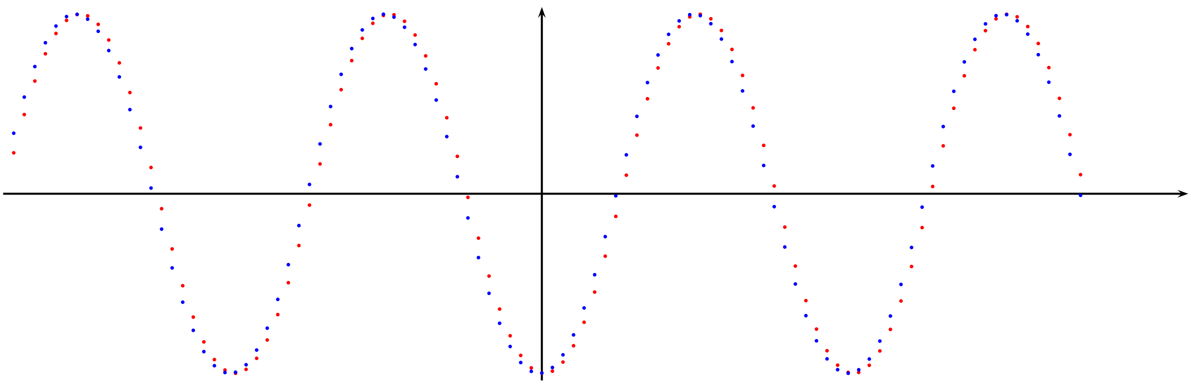
Also, from $F(x_{n-1}, y_n) = F(x_n, y_n) = 0$, $x_n + y_{n-1}$ and $x_n x_{n-1}$ are the rational functions

$$x_n + x_{n-1} = -\frac{Y_1(y_n)}{Y_2(y_n)} = -\frac{c_{1,2}y_n^2 + c_{1,1}y_n + c_{1,0}}{c_{2,2}y_n^2 + c_{2,1}y_n + c_{2,0}}, x_n x_{n-1} = \frac{Y_0(y_n)}{Y_2(y_n)} = \frac{c_{0,2}y_n^2 + c_{0,1}y_n + c_{0,0}}{c_{2,2}y_n^2 + c_{2,1}y_n + c_{2,0}}. \tag{6}$$

Sums and products of successive instances of (5):

$$y_{n-1} + 2y_n + y_{n+1} =$$





1.3. **Hypergeometric expansion.**

Difference of a simple rational function:

$$\mathcal{D}\frac{1}{x-a} = \frac{\dfrac{1}{\psi(x)-a} - \dfrac{1}{\varphi(x)-a}}{\psi(x)-\varphi(x)} = \frac{-1}{(\psi(x)-a)(\varphi(x)-a)} = \frac{-X_2(x)}{X_0(x)+aX_1(x)+a^2X_2(x)=F(x,a)}.$$

Let a be some y_α, then, $\mathcal{D}\dfrac{1}{x-a} = \dfrac{-X_2(x)}{Y_2(a)(x-x_\alpha)(x-x_{\alpha-1})}$.

Residues at x_α and $x_{\alpha-1}$

$$\frac{1}{(y_{\alpha+1}-y_\alpha)\varphi'(x_\alpha)} = \frac{X_2(x_\alpha)}{Y_2(y_\alpha)(x_\alpha-x_{\alpha-1})}, \frac{1}{(y_\alpha-y_{\alpha-1})\psi'(x_{\alpha-1})} = \frac{-X_2(x_{\alpha-1})}{Y_2(y_\alpha)(x_{\alpha-1}-x_\alpha)}. \quad (7)$$

Remark that $\varphi'(x_\alpha) = \dfrac{(x_\alpha-x_{\alpha-1})Y_2(y_\alpha)}{(y_{\alpha+1}-y_\alpha)X_2(x_\alpha)}, \psi'(x_\alpha) = \dfrac{(x_{\alpha+1}-x_\alpha)Y_2(y_{\alpha+1})}{(y_{\alpha+1}-y_\alpha)X_2(x_\alpha)}$.

The difference operator acting on a rational function of degree k is normally a rational function of degree $2k$.

Consider now the rational functions

$$\Phi_k(x) = \frac{(x-y_0)\cdots(x-y_{k-1})}{(x-y'_0)\cdots(x-y'_{k-1})}, \quad \Psi_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})}{(x-x'_0)\cdots(x-x'_{k-1})} \quad (8)$$

$$(\mathcal{D}\Phi_k)(x) = \frac{1}{\psi(x)-\varphi(x)}\left[\frac{(\psi(x)-y_0)\cdots(\psi(x)-y_{k-1})}{(\psi(x)-y'_0)\cdots(\psi(x)-y'_{k-1})} - \frac{(\varphi(x)-y_0)\cdots(\varphi(x)-y_{k-1})}{(\varphi(x)-y'_0)\cdots(\varphi(x)-y'_{k-1})}\right],$$

is a rational function of denominator $(x-x'_{-1})(x-x'_0)\cdots(x-x'_{k-1})$ and of numerator of same degree with factor X_2.

Furthermore, $(\psi(x)-y_0)\cdots(\psi(x)-y_{k-1})$ vanishes when $x = x_{-1},\dots,x_{k-2}$, and $(\varphi(x)-y_0)\cdots(\varphi(x)-y_{k-1})$ vanishes when $x = x_0,\dots,x_{k-1}$.

This leaves

$$(\mathcal{D}\Phi_k)(x) = C_kX_2(x)\frac{(x-x_0)\cdots(x-x_{k-2})}{(x-x'_{-1})(x-x'_0)\cdots(x-x'_{k-1})} = \frac{C_kX_2(x)\Psi_{k-1}(x)}{(x-x'_{-1})(x-x'_{k-1})}, k=1,2,\dots \quad (9)$$

where C_k can be found from a particular value, or from a particular residue.

So, at $x = x_{k-1}$, where $\Phi_k(\varphi(x) = y_{k-1}) = 0$:

$$\frac{\Phi_k(y_k)}{y_k-y_{k-1}} = \frac{C_kX_2(x_{k-1})\Psi_{k-1}(x_{k-1})}{(x_{k-1}-x'_{-1})(x_{k-1}-x'_{k-1})}, \quad (10a)$$

and at $x = x_{-1}$, where $\Phi_k(\psi(x) = y_0) = 0$:

$$\frac{\Phi_k(y_{-1})}{(y_0-y_{-1})} = -\frac{C_kX_2(x_{-1})\Psi_{k-1}(x_{-1})}{(x_{-1}-x'_{-1})(x_{-1}-x'_{k-1})}. \quad (10b)$$

Also, residues at $x = x'_{-1}$ are

$$\frac{1}{(y'_0-y'_{-1})}\frac{(y'_0-y_0)\cdots(y'_0-y_{k-1})}{\psi'(x'_{-1})(y'_0-y'_1)\cdots(y'_0-y'_{k-1})} = C_kX_2(x'_{-1})\frac{\Psi_{k-1}(x'_{-1})}{x'_{-1}-x'_{k-1}}, \quad (10c)$$

where ψ' means the derivative of ψ. At x'_{k-1}:

$$\frac{-1}{(y'_k - y'_{k-1})}\frac{(y'_{k-1} - y_{k-1})\Phi_{k-1}(y'_{k-1})}{\varphi'(x'_{k-1})} = C_k X_2(x'_{k-1})\frac{\Psi_{k-1}(x'_{k-1})}{x'_{k-1} - x'_{-1}}. \tag{10d}$$

Four equations for the same C_k! An interesting identity from (10a) and (10d) is

$$\frac{-(y'_{k-1} - y_{k-1})\Phi_{k-1}(y'_{k-1})}{(y'_k - y'_{k-1})\varphi'(x'_{k-1})}\frac{x'_{k-1} - x'_{-1}}{X_2(x'_{k-1})\Psi_{k-1}(x'_{k-1})} = \frac{\Phi_k(y_k)}{y_k - y_{k-1}}\frac{(x_{k-1} - x'_{-1})(x_{k-1} - x'_{k-1})}{X_2(x_{k-1})\Psi_{k-1}(x_{k-1})} \tag{11}$$

Check: from

$(y'_k - y'_{k-1})\varphi'(x'_{k-1})X_2(x'_{k-1}) = (x'_{k-1} - x'_{k-2})Y_2(y'_{k-1}) = (x'_{k-1} - x'_{k-2})\dfrac{(y'_{k-1} - y_{k-1})(y'_{k-1} - y_k)X_2(x_{k-1})}{(x_{k-1} - x'_{k-1})(x_{k-1} - x'_{k-2})}$

the left-hand side is $\dfrac{-(y'_{k-1} - y_{k-1})(x'_{k-1} - x'_{-1})(x_{k-1} - x'_{k-1})(x_{k-1} - x'_{k-2})\Phi_{k-1}(y'_{k-1})}{(x'_{k-1} - x'_{k-2})(y'_{k-1} - y_{k-1})(y'_{k-1} - y_k)X_2(x_{k-1})\Psi_{k-1}(x'_{k-1})}$.

Is it possible that

$\dfrac{-(x'_{k-1} - x'_{-1})(x_{k-1} - x'_{k-2})\Phi_{k-1}(y'_{k-1})}{(x'_{k-1} - x'_{k-2})(y'_{k-1} - y_k)\Psi_{k-1}(x'_{k-1})} = \dfrac{\Phi_k(y_k)}{y_k - y_{k-1}}\dfrac{(x_{k-1} - x'_{-1})}{\Psi_{k-1}(x_{k-1})}$

or $\dfrac{(x'_{k-1} - x'_{-1})\Phi_{k-1}(y'_{k-1})}{(x'_{k-1} - x'_{k-2})\Psi_{k-1}(x'_{k-1})} = \Phi_{k-1}(y_k)\dfrac{(x_{k-1} - x'_{-1})}{(x_{k-1} - x'_{k-2})\Psi_{k-1}(x_{k-1})}$?

$k = 1$: OK. $k = 2$: $\dfrac{(x'_1 - x'_{-1})(y'_1 - y_0)}{(y'_1 - y'_0)(x'_1 - x'_0)} = \dfrac{y_2 - y_0}{y_2 - y'_0}\dfrac{x_1 - x'_{-1}}{x_1 - x_0}$

or $\dfrac{(y'_1 - y_0)(y_2 - y'_0)}{(y'_1 - y'_0)(y_2 - y_0)} = \dfrac{(x'_1 - x_0)(x_1 - x'_{-1})}{(x'_1 - x'_{-1})(x_1 - x_0)}$.

$k = 3$: $\dfrac{(x'_2 - x'_{-1})(y'_2 - y_0)(y'_2 - y_1)(x'_2 - x'_0)}{(y'_2 - y'_0)(y'_2 - y'_1)(x'_2 - x_0)(x'_2 - x_1)} = \dfrac{(y_3 - y_0)(y_3 - y_1)}{(y_3 - y'_0)(y_3 - y'_1)}\dfrac{(x_2 - x'_{-1})(x_2 - x'_0)}{(x_2 - x_0)(x_2 - x_1)}$

$\dfrac{(x'_2 - x'_{-1})(y'_2 - y_0)(y'_2 - y_1)(x'_2 - x'_0)}{(y'_2 - y'_0)(y'_2 - y'_1)(x'_2 - x_0)(x'_2 - x_1)} = \dfrac{(y_3 - y_0)(y_3 - y_1)}{(y_3 - y'_0)(y_3 - y'_1)}\dfrac{(x_2 - x'_{-1})(x_2 - x'_0)}{(x_2 - x_0)(x_2 - x_1)}$

For the sum

$$\Phi_k(\varphi(x)) + \Phi_k(\psi(x)) = Z_k(x)\frac{(x - x_0)\cdots(x - x_{k-2})}{(x - x'_{-1})(x - x'_0)\cdots(x - x'_{k-1})}, \tag{12}$$

where Z_k is a second degree polynomial. Similarly to (10a)-(10d),

$$\frac{(y_k - y_0)\cdots(y_k - y_{k-1})}{(y_k - y'_0)\cdots(y_k - y'_{k-1})} = \frac{Z_k(x_{k-1})(x_{k-1} - x_0)\cdots(x_{k-1} - x_{k-2})}{(x_{k-1} - x'_{-1})(x_{k-1} - x'_0)\cdots(x_{k-1} - x'_{k-1})}, \tag{13a}$$

$$\frac{(y_{-1} - y_0)\cdots(y_{-1} - y_{k-1})}{(y_{-1} - y'_0)\cdots(y_{-1} - y'_{k-1})} = \frac{Z_k(x_{-1})(x_{-1} - x_0)\cdots(x_{-1} - x_{k-2})}{(x_{-1} - x'_{-1})(x_{-1} - x'_0)\cdots(x_{-1} - x'_{k-1})}. \tag{13b}$$

$$\frac{(y'_0 - y_0)\cdots(y'_0 - y_{k-1})}{\psi'(x'_{-1})(y'_0 - y'_1)\cdots(y'_0 - y'_{k-1})} = Z_k(x'_{-1})\frac{(x'_{-1} - x_0)\cdots(x'_{-1} - x_{k-2})}{(x'_{-1} - x'_0)\cdots(x'_{-1} - x'_{k-1})}, \tag{13c}$$

$$\frac{(y'_{k-1} - y_0)\cdots(y'_{k-1} - y_{k-1})}{(y'_{k-1} - y'_0)\cdots(y'_{k-1} - y'_{k-2})\varphi'(x'_{k-1})} = Z_k(x'_{k-1})\frac{(x'_{k-1} - x_0)\cdots(x'_{k-1} - x_{k-2})}{(x'_{k-1} - x'_{-1})\cdots(x'_{k-1} - x'_{k-2})}. \tag{13d}$$

2. Exponential-like functions.

Solution of

$$(\mathcal{D}f)(x) = a[f(\varphi(x)) + f(\psi(x))] \tag{14}$$

or

$$f(\psi(x)) = \frac{1 + a[\psi(x) - \varphi(x)]}{1 - a[\psi(x) - \varphi(x)]}\, f(\varphi(x))$$

on a lattice: $f(y_n) = \dfrac{[1 + a(y_n - y_{n-1})][1 + a(y_{n-1} - y_{n-2})]\cdots[1 + a(y_1 - y_0)]}{[1 - a(y_n - y_{n-1})][1 - a(y_{n-1} - y_{n-2})]\cdots[1 - a(y_1 - y_0)]}\, f(y_0)$

e^{2ax}

on the arithmetic lattice, still an exponential function $f(x) = \left[\dfrac{1 + ah}{1 - ah}\right]^{(x-y_0)/h}$

on the $q-$lattice, $\varphi(x) = x$, $\psi(x) = qx$, $f(x) = \dfrac{e_q(ax)}{e_q(-ax)}$ [qbook, § 2]

3. Known rational approximations.

3.1. Padé.

for e^z,

$$[m/n] = \frac{1 + \dfrac{m}{m+n}\dfrac{z}{1!} + \dfrac{m(m-1)}{(m+n)(m+n-1)}\dfrac{z^2}{2!} + \cdots + \dfrac{m(m-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(n+1)}\dfrac{z^m}{m!}}{1 - \dfrac{n}{m+n}\dfrac{z}{1!} + \dfrac{n(n-1)}{(m+n)(m+n-1)}\dfrac{z^2}{2!} - \cdots + (-1)^n\dfrac{n(n-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(m+1)}\dfrac{z^n}{n!}} \tag{15}$$

$$\begin{aligned} e^z \text{ den.} - \text{num.} &= \frac{(-1)^n}{(m+1)\cdots(m+n)} \sum_{k=m+n+1}^{\infty} \frac{(k-m-1)\cdots(k-m-n)}{k!} z^k \\ &= \frac{(-1)^n}{(m+n)!}\left[\int_{-\infty}^{z} - \int_{-\infty}^{0} = \int_0^z e^t (z-t)^m t^n\, dt\right] \end{aligned} \tag{16}$$

[25]

$$e^{ax} = 1 + \cfrac{ax}{1 - ax/2 + \cfrac{a^2x^2}{12 + \cfrac{a^2x^2}{5 + \cfrac{a^2x^2}{\ddots + \cfrac{a^2x^2}{a_k + \ddots}}}}} \tag{17}$$

with $a_k = 8k - 4$ if k is even, $2k - 1$ if k is odd.

3.2. Equidistant points. [10]

As far as we only need e^{Az} at $z = z_0, z_0 + h, \ldots, z_0 + (m+n)h$,

$$
\begin{aligned}
e^{Az} &= (\boldsymbol{I}+\boldsymbol{\Delta})^{(z-z_0)/h} e^{Az_0} \\
&= \sum_{k=0}^{m+n} \binom{(z-z_0)/h}{k} \boldsymbol{\Delta}^k e^{Az_0} \\
&= \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{1}{k!}(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h)e^{Az_0},
\end{aligned}
$$

which we multiply by the denominator $Q(z)=\sum_{j=0}^{n} q_j(z-z_0)\cdots(z-z_0-(j-1)h)$, using

$$
\begin{aligned}
&(z-z_0)(z-z_0-h)\cdots(z-z_0-(j-1)h)e^{Az} = \\
&\qquad e^{A(z_0+jh)} \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^{k-j} \frac{1}{(k-j)!}(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),
\end{aligned}
$$

$$
Q(z)e^{Az} = e^{Az_0} \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{C(k)}{k!}(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),
$$

where $C(k)=\sum_{j=0}^{n} q_j e^{Ajh}\left(\frac{e^{Ah}-1}{h}\right)^{-j}\frac{1}{(k-j)!}$ is a polynomial of degree n in k, which must vanish at $k=m+1, m+2, \ldots, m+n$,

$$
P(z) = e^{Az_0} \sum_{k=0}^{m} \left(\frac{e^{Ah}-1}{h}\right)^k \binom{m}{k}(m+n-k)!(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),
$$

$$
Q(z) = \sum_{k=0}^{n} \left(\frac{e^{-Ah}-1}{h}\right)^k \binom{n}{k}(m+n-k)!(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),
$$

and, formally:

$$
\begin{aligned}
&Q(z)e^{Az}-P(z) = \\
&e^{Az_0} m!(-1)^n \sum_{k=m+n+1}^{\infty} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{(k-m-1)(k-m-2)\cdots(k-m-n)}{k!}(z-z_0)(z-z_0-h)\cdots(z-z_0-(k
\end{aligned}
\qquad (18)
$$

4. The elliptic exponential-like function.

4.1. Zeros and poles.

At some $x=x_\alpha$, (14) is

$$
f(y_{\alpha+1}) = \frac{1+a[y_{\alpha+1}-y_\alpha]}{1-a[y_{\alpha+1}-y_\alpha]}\, f(y_\alpha)
$$

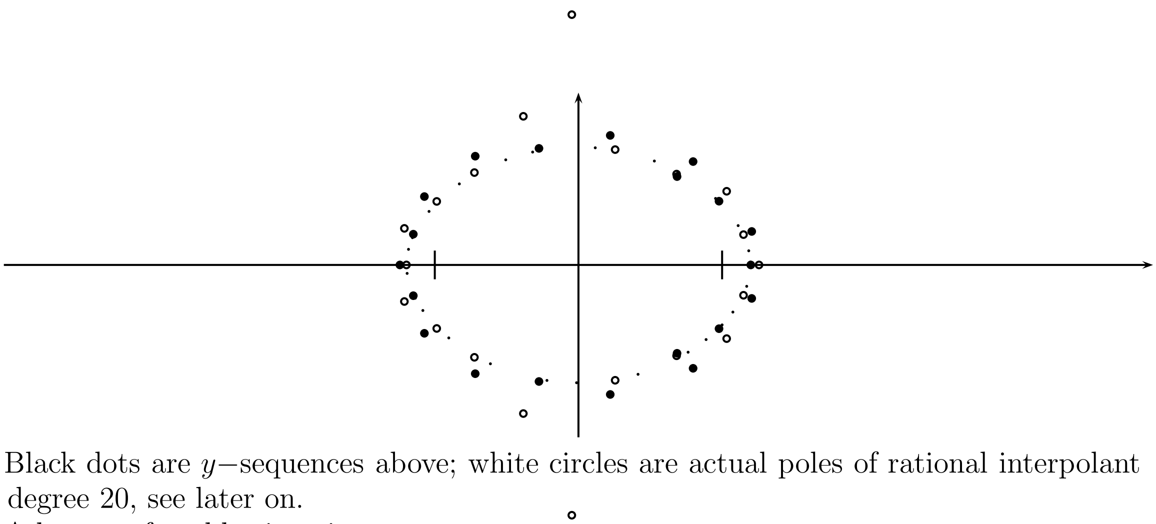
so that we expect poles at all the y_αs where $y_{\alpha+1-n}-y_{\alpha-n}=1/a$, for some integer $n>0$ and zeros when $y_{\alpha+1-n}-y_{\alpha-n}=-1/a$.

From (4), the x−roots of $\psi(x)-\varphi(x)=\pm 1/a$ are the 4 roots of $4a^2P(x)-X_2^2(x)=0$. Let x_β be such a root, then, the sequence $\{\dots,y_{\beta-1},y_\beta,y_{\beta+1},\dots\}$ is liable to contain zeros and poles.

For instance, with $a=1/2$,
k=1/2, z0=3 $X_0(x)=2x^2-1.25, X_1(x)=-2.25x, X_2(x)=-0.3125x^2+2$, so that $P(x)=X_1^2(x)/4-X_0(x)X_2(x)=2.5(1-x^2)(1-x^2/4)$, the roots are $x=\pm 2.0563 \Rightarrow \varphi$ and $\psi=\pm(4.4086, 2.4086)$; $x=\pm 0.82019i \Rightarrow \varphi$ and $\psi=\pm(1+0.41748i, -1+0.41748i)$, and the y−sequences are $\pm\{\dots,-4.4086,-2.4086,2.6785,3.4686,-2.1483,-8.6905,2.0010,-12.691,\dots\}$ and $\pm\{\dots,-1+0.41748i,1+0.41748i,0.69127-0.64922i,-1.1556-0.18970i,-0.21929+0.80408i,1.1962-0.019606i,-0.31895-0.78586i,-1.1358+0.23132i,0.76339+0.60765i,\dots\}$

Some values of the rational interpolants:

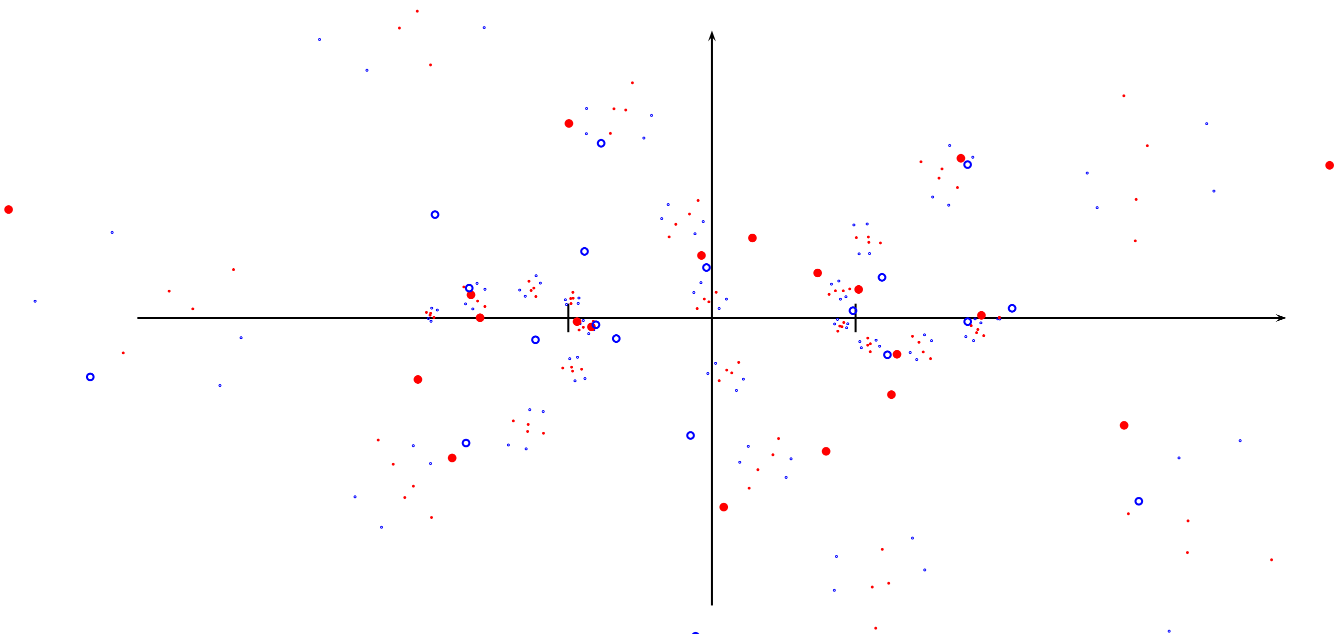
	-1.5	-1	-0.5	0	0.5	1	1.5
1	-0.3186707	0.0106012	0.4343382	1.000000	1.793233	2.985950	4.981495
2	0.2590631	0.3097855	0.4770833	1.000000	2.257913	2.935437	2.253881
3	-0.7447123	0.3600602	0.4684872	1.000000	2.183663	2.918343	1.124377
4	0.6246931	0.3494623	0.4623021	1.000000	2.107666	2.947874	3.279223
5	0.0971869	0.3404011	0.4607317	1.000000	2.146329	2.948048	4.050295
6	0.3508917	0.3411494	0.5006247	1.000000	2.166102	2.949342	1.083328
7	-1.800737	0.3387593	0.4615957	1.000000	2.166283	2.949189	0.2481340
8	0.4737387	0.3403256	0.4615740	1.000000	2.166444	2.948994	-99.08147
9	-0.1343806	0.3390977	0.4615842	1.000000	2.166454	2.948999	-6.033262
10	0.3570745	0.3390971	0.4615828	1.000000	2.166457	2.949005	-0.9676665
...							
20	0.3049597	0.3390975	0.4615828	1.000000	2.166458	2.949005	1.769250
50	1.925918	0.3390975	0.4615828	1.000000	2.166458	2.949005	-0.3163812



Black dots are y−sequences above; white circles are actual poles of rational interpolant of degree 20, see later on.

A less comfortable situation:

Censored



4.2. **Hypergeometric expansion.**

As we want to put a Φ_k expansion in the difference equation (14),

$$(\mathcal{D}\Phi_k)(x) - a[\Phi_k(\varphi(x)) + \Phi_k(\psi(x))] = W_k(x)\frac{(x-x_0)\cdots(x-x_{k-2})}{(x-x'_{-1})(x-x'_0)\cdots(x-x'_{k-1})}, \tag{19}$$

with $W_k(x) = C_k X_2(x) - aZ_k(x)$ for $k>0$.

Check: $(\mathcal{D}\Phi_0)(x) - a[\Phi_0(\varphi(x)) + \Phi_0(\psi(x))] = -2a$, $(\mathcal{D}\Phi_1)(x) - a[\Phi_1(\varphi(x)) + \Phi_1(\psi(x))] = -2a - \dfrac{(y'_0-y_0)[1+a(\varphi(x)+\psi(x)-2y'_0)]}{(\varphi(x)-y'_0)(\psi(x)-y'_0) = F(x,y'_0)/X_2(x)} = -2a - \dfrac{(y'_0-y_0)[X_2(x) - a(X_1(x)+2y'_0X_2(x))]}{Y_2(y'_0)(x-x'_{-1})(x-x'_0)}$, so, $W_0(x) = -2a(x-x'_{-1})(x-x_{-1})$, $W_1(x) = -2a(x-x'_{-1})(x-x'_0) - (y'_0-y_0)[X_2(x) - a(X_1(x)+2y'_0X_2(x))]/Y_2(y'_0)$.

An important simplification occurs when $W_k(x'_{-1})=0$, i.e., from (10c) and (13c), when $1 = a(y'_0 - y'_{-1})$, which is precisely the condition ensuring that $y'_0, y'_1, \dots$ will be actual poles of f!

Then, $W_k(x)/(x-x'_{-1})$ is a first degree polynomial

$\dfrac{W_0(x)}{x-x'_{-1}} = -2a(x-x_{-1})$, $\dfrac{W_1(x)}{x-x'_{-1}}$ is found from its values $(x_0-x'_0)\left[-2a + \dfrac{1+a(y_0+y_1-2y'_0)}{y_1-y'_0}\right] = (x_0-x'_0)\dfrac{1+a(y_0-y_1)}{y_1-y'_0}$ at $x=x_0$, and $-(y'_0-y_0)X_2(x'_0)[1+a(y'_1-y'_0)]/Y_2(y'_0) = -(y'_0-y_0)\dfrac{1+a(y'_1-y'_0)}{\varphi'(x'_0)(y'_1-y'_0)} = -(y'_0-y_0)X_2(x'_0)[1+a(y'_1-y'_0)]\dfrac{(x_0-x'_0)(x_0-x'_{-1})}{(y'_0-y_0)(y'_0-y_1)X_2(x_0)}$ at $x=x'_0$,

The values at x_{k-1} and x'_{k-1} are

$$\frac{W_k(x_{k-1})}{x_{k-1}-x'_{-1}} = \frac{(y_k-y_0)\cdots(y_k-y_{k-2})}{(y_k-y'_0)\cdots(y_k-y'_{k-1})}\frac{(x_{k-1}-x'_0)\cdots(x_{k-1}-x'_{k-1})}{(x_{k-1}-x_0)\cdots(x_{k-1}-x_{k-2})}[1-a(y_k-y_{k-1})]$$
$$= \frac{(x_{k-1}-x'_{k-1})\Phi_{k-1}(y_k)}{(y_k-y'_{k-1})\Psi_{k-1}(x_{k-1})}[1-a(y_k-y_{k-1})],$$

and
$$\frac{W_k(x'_{k-1})}{x'_{k-1}-x'_{-1}}=\frac{(y'_{k-1}-y_0)\cdots(y'_{k-1}-y_{k-1})}{(y'_{k-1}-y'_0)\cdots(y'_{k-1}-y'_{k-2})\varphi'(x'_{k-1})}\frac{(x'_{k-1}-x'_0)\cdots(x'_{k-1}-x'_{k-2})}{(x'_{k-1}-x_0)\cdots(x'_{k-1}-x_{k-2})}\left[-\frac{1}{y'_k-y'_{k-1}}-a\right],$$
which is also, thanks to (11), $\dfrac{(x_{k-1}-x'_{k-1})(x_{k-1}-x'_{-1})X_2(x'_{k-1})\Phi_{k-1}(y_k)}{(y_k-y'_{k-1})(x'_{k-1}-x'_{-1})X_2(x_{k-1})\Psi_{k-1}(x_{k-1})}[1+a(y'_k-y'_{k-1})]$.

So, $\dfrac{W_k(x)}{x-x'_{-1}}=C'_k(x-x_{k-1})+C''_k(x-x'_{k-1})$, with

$C'_0=-2a, C''_0=0; C'_1=X_2(x'_0)[1+a(y'_1-y'_0)]\dfrac{x_0-x'_{-1}}{(y'_0-y_1)X_2(x_0)}, C''_1=\dfrac{1-a(y_1-y_0)}{y_1-y'_0}$.

$$C'_k=-\frac{(x_{k-1}-x'_{-1})X_2(x'_{k-1})\Phi_{k-1}(y_k)}{(y_k-y'_{k-1})(x'_{k-1}-x'_{-1})X_2(x_{k-1})\Psi_{k-1}(x_{k-1})}[1+a(y'_k-y'_{k-1})], C''_k=\frac{\Phi_{k-1}(y_k)}{(y_k-y'_{k-1})\Psi_{k-1}(x_{k-1})}[1-a(y_k-y_{k-1}) \tag{20}$$

and (9) becomes

$$(\mathcal{D}\Phi_k)(x)-a[\Phi_k(\varphi(x))+\Phi_k(\psi(x))]=C'_k\Psi_k(x)+C''_k\Psi_{k-1}(x). \tag{21}$$

Finally, we come to the formal hypergeometric expansion of f satisfying (14):

$$f(x)=\sum_0^\infty\gamma_k\Phi_k(x)\Rightarrow(\mathcal{D}f)(x)-a[f(\varphi(x))+f(\psi(x))]=-2a\gamma_0+\sum_1^\infty\gamma_k[C'_k\Psi_k(x)+C''_k\Psi_{k-1}(x)]=0,$$

from (21). So, $-2a\gamma_0+C''_1\gamma_1=0, \gamma_kC'_k+\gamma_{k+1}C''_{k+1}=0, k=1,2,\ldots$, or

$$\frac{\gamma_{k+1}\Phi_k(y_{k+1})/[(y_{k+1}-y'_k)\Psi_k(x_k)]}{\gamma_k\Phi_{k-1}(y_k)/[(y_k-y'_{k-1})\Psi_{k-1}(x_{k-1})]}=\frac{(x_{k-1}-x'_{-1})X_2(x'_{k-1})[1+a(y'_k-y'_{k-1})]}{(x'_{k-1}-x'_{-1})X_2(x_{k-1})[1-a(y_{k+1}-y_k)]},$$

$$\gamma_k=\frac{(y_k-y'_{k-1})\Psi_{k-1}(x_{k-1})}{\Phi_{k-1}(y_k)}\Psi_{k-1}(x'_{-1})\frac{X_2(x'_0)\cdots X_2(x'_{k-2})}{X_2(x_0)\cdots X_2(x_{k-2})}\frac{[1+a(y'_1-y'_0)]\cdots[1+a(y'_{k-1}-y'_{k-2})]}{[1-a(y_2-y_1)]\cdots[1-a(y_k-y_{k-1})]}\frac{\gamma_1}{y_1-y'_0} \tag{22}$$

the finite sum of N terms interpolates at N points $y_0,\cdots,y_{N-1}$. Check: $\gamma_0=f(y_0)$, $f(y_1)=\dfrac{1+a(y_1-y_0)}{1-a(y_1-y_0)}f(y_0)=\gamma_0+\gamma_1\dfrac{y_1-y_0}{y_1-y'_0}\Rightarrow\gamma_1=\dfrac{2a(y_1-y'_0)}{1-a(y_1-y_0)}\gamma_0$.

$f(y_2)=\dfrac{[1+a(y_1-y_0)][1+a(y_2-y_1)]}{[1-a(y_1-y_0)][1-a(y_2-y_1)]}f(y_0)=\gamma_0+\gamma_1\dfrac{y_2-y_0}{y_2-y'_0}+\gamma_2\dfrac{(y_2-y_0)(y_2-y_1)}{(y_2-y'_0)(y_2-y'_1)}\Rightarrow$
$\gamma_2=\dfrac{2a(y_2-y'_1)[1+a(y_1-y'_0)]}{[1-a(y_1-y_0)][1-a(y_2-y_1)]}\gamma_0$.

4.3. **Continued fraction.**

Let $f_0(x)=f(x)-f(y_0)$ be expanded in an interpolatory continued fraction ($R_{II}-$fraction [11, 12, 33, 36], or contracted Thiele's continued fraction [22, Chap. 5])

$$f_0(x)=\cfrac{x-y_0}{\alpha_0x+\beta_0-\cfrac{(x-y_1)(x-y_2)}{\ddots\over\alpha_{n-2}x+\beta_{n-2}+\cfrac{(x-y_{2n-3})(x-y_{2n-2})}{\alpha_{n-1}x+\beta_{n-1}+\cdots}}}$$

making clear that the n^{th} approximant (stopped at, and including the $\alpha_{n-1}x+\beta_{n-1}$ term) is the rational function of degree n interpolating f_0 at $x = y_0, y_1, \dots, y_{2n}$.

Let p_n be the denominator of this n^{th} approximant. If f is a Stieltjes tranform, it is known [12, 33] [36, § 5] [37] that $\{p_n(x)/((x-y_0)(x-y_2)\dots(x-y_{2n})\}$ and $\{p_m(x)/((x-y_1)(x-y_3)\dots(x-y_{2m+1})\}$ are biorthogonal sequences of rational functions.

From $f_n(x) = \dfrac{x-y_{2n}}{\alpha_n x+\beta_n-(x-y_{2n+1})f_{n+1}(x)}$, $\alpha_n x+\beta_n$ is the polynomial interpolant of degree 1 to $(x-y_{2n})/f_n(x)$ at y_{2n+1} and y_{2n+2}, so we need $f_n(y_{2n+1})$ and $f_n(y_{2n+2})$ in order to find α_n and β_n.

First instances

$$f_0(y_0)=0, f_0(y_1)=\frac{1+a(y_1-y_0)}{1-a(y_1-y_0)}-1=\frac{2a(y_1-y_0)}{1-a(y_1-y_0)},$$

$$f_0(y_2)=\frac{(1+a(y_1-y_0))(1+a(y_2-y_1))}{(1-a(y_1-y_0))(1-a(y_2-y_1))}-1=\frac{2a(y_2-y_1)}{(1-a(y_1-y_0))(1-a(y_2-y_1))}$$

$$\alpha_0y_1+\beta_0=\frac{y_1-y_0}{f_0(y_1)}=\frac{1-a(y_1-y_0)}{2a}, \alpha_0y_2+\beta_0=\frac{y_2-y_0}{f_0(y_2)}=\frac{(1-a(y_1-y_0))(1-a(y_2-y_1))}{2a}:$$

$$\alpha_0=\frac{a(y_1-y_0)-1}{2}, \beta_0=\frac{(1-a(y_1-y_0))(1+ay_1)}{2a}$$

$$f_0(y_3)=\cfrac{y_3-y_0}{\alpha_0y_3+\beta_0+\cfrac{(y_3-y_1)(y_3-y_2)(y_2-y_1)a(1-a^2(y_1-y_0)^2)}{2[y_3-y_0+a^2(y_1-y_0)(y_2-y_1)(y_3-y_2)]}}$$

$$f_0(y_4)=\cfrac{y_4-y_0}{\alpha_0y_4+\beta_0+\cfrac{(y_4-y_2)(y_4-y_3+y_2-y_1)(y_3-y_1)a(1-a^2(y_1-y_0)^2)}{2[y_4-y_0+a^2((y_2-y_1)(y_3-y_2)(y_4-y_3)+\cdots+(y_1-y_0)(y_2-y_1)(y_3-y_2))]}}$$

$$\alpha_1y_3+\beta_1=\frac{y_3-y_2}{f_1(y_3)}=-2\frac{y_3-y_0+a^2(y_1-y_0)(y_2-y_1)(y_3-y_2)}{a(y_2-y_1)(1-a^2(y_1-y_0)^2)},$$

$$\alpha_1y_4+\beta_1=\frac{y_4-y_2}{f_1(y_4)}=-2(y_4-y_1)\frac{y_4-y_0+a^2((y_2-y_1)(y_3-y_2)(y_4-y_3)+\cdots+(y_1-y_0)(y_2-y_1)(y_3-y_2))}{a(y_3-y_1)(y_4-y_3+y_2-y_1)(1-a^2(y_1-y_0)^2)}$$

5. **Extension**

The form of the equation

$$(\mathcal{D}f)(x)=\frac{X_2(x)}{Z(x)}[f(\varphi(x))+f(\psi(x))], \tag{23}$$

where Z is a polynomial of degree $\leqslant 2$, is invariant under a transformation $Y=(\alpha y+\beta)/(1+\gamma y) \Leftrightarrow y=(Y-\beta)/(\alpha-\gamma Y)$. Indeed, if $g(Y)=f(y)$, the equation for g is

$$\cfrac{g(\Psi(x))-g(\Phi(x))}{\psi(x)-\varphi(x)=\cfrac{\Psi(x)-\beta}{\alpha-\gamma\Psi(x)}-\cfrac{\Phi(x)-\beta}{\alpha-\gamma\Phi(x)}=\cfrac{(\alpha-\beta\gamma)(\Psi(x)-\Phi(x))}{(\alpha-\gamma\Phi(x))(\alpha-\gamma\Psi(x))}}=\frac{X_2(x)}{Z(x)}[g(\Phi(x))+g(\Psi(x))],$$

so that X_2/Z is multiplied by $(\alpha-\beta\gamma)/[(\alpha-\gamma\Phi(x))(\alpha-\gamma\Psi(x))]$. Now, the new biquadratic polynomial is $G(x,Y)=(\alpha-\gamma Y)^2F(x,(Y-\beta)/(\alpha-\gamma Y))=X_2(x)(Y-\beta)^2+X_1(x)(Y-\beta)(\alpha-\gamma Y)+X_0(x)(\alpha-\gamma Y)^2$, so that $\boldsymbol{X}_2(x)=X_2(x)-\gamma X_1(x)+\gamma^2X_0(x)$, $\boldsymbol{X}_1(x)=-2\beta X_2(x)+(\alpha-\beta\gamma)X_1(x)-2\alpha\gamma X_0(x)$, $\boldsymbol{X}_0(x)=\beta^2X_2(x)-\alpha\beta X_1(x)+$

$\alpha^2 X_0(x)$, and $(\alpha-\gamma\Phi(x))(\alpha-\gamma\Psi(x)) = (\alpha^2\boldsymbol{X}_2(x)+\alpha\gamma\boldsymbol{X}_1(x)+\gamma^2\boldsymbol{X}_0(x))/\boldsymbol{X}_2(x) = \gamma^2\lim_{Y\to\alpha/\gamma} G(x,Y)/\boldsymbol{X}_2(x) = (\alpha-\beta\gamma)^2 X_2(x)/\boldsymbol{X}_2(x)$. The new X_2/Z is therefore $(\alpha-\beta\gamma)^{-1}\boldsymbol{X}_2/Z$.

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