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Elliptic grids, rational functions, and the Padé interpolation

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Dedicated to Richard Askey on the occasion of his seventieth birthday.
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Abstract We consider rational functions with n prescribed poles for which there exists a divided difference operator transforming them to rational functions with $n-1$ poles. The poles of such functions are shown to lie on the elliptic grids. There is a one-to-one correspondence between this problem of admissible grids and the Poncelet problem on two quadrics. Additionally, we outline an explicit scheme of the Padé interpolation with prescribed poles and zeros on the elliptic grids.

Keywords Divided difference operators · Elliptic functions · Poncele porism · Padé interpolation

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1 Introduction

We consider a meromorphic function $x(s),\ s \in \mathbb{C}$, which will be called the "grid", and associate with it the following divided difference operator

$$\mathcal{D}_{x(s)} F(x) = \frac{F(x(s+1)) - F(x(s))}{x(s+1) - x(s)}. \tag{1.1}$$

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The operator $\mathcal{D}_{x(s)}$ transforms a function $F(x)$ of the argument $x \in \mathbb{C}$ to a function of the argument s. We assume that the function $x(s)$ is invertible in some domain of the complex plane, so that it is possible to write $s = s(x)$ and $\mathcal{D}_{x(s)}F(x) = \tilde{F}(x)$, where the function $\tilde{F}(x)$ is defined by the right-hand side of (1.1) with s re-expressed in terms of x.

As a simplest example, we consider the case $x(s) = s$ (the linear grid). We have $\mathcal{D}_{x(s)}F(x) = \tilde{F}(x) = F(x+1) - F(x)$, so that the operator $\mathcal{D}_{x(s)}$ coincides with the ordinary finite-difference operator. For the quadratic grid $x(s) = s^2$, we have

$$\tilde{F}(x) = \frac{F(x+1+\sqrt{x}) - F(x)}{2\sqrt{x}+1}.$$

Orthogonal polynomials. We demand that the operator $\mathcal{D}_{x(s)}$ maps any polynomial of x of order n to a polynomial of order $n-1$ in $y(s)$, where $y(s)$ is another grid which will be called "adjacent" to $x(s)$. More precisely,

$$\mathcal{D}_{x(s)}P_n(x) = \frac{P_n(x(s+1)) - P_n(x(s))}{x(s+1) - x(s)} = P_{n-1}(y(s)), \tag{1.2}$$

where $P_n(x)$ is an arbitrary polynomial of the degree n and $P_{n-1}(x)$ is a polynomial of the degree $n-1$. The grids $x(s)$ for which such an operator does exist will be called *admissible*.

In order to verify existence of admissible grids it is sufficient to verify property (1.2) for monomials x^n:

$$\frac{x(s+1)^n - x(s)^n}{x(s+1) - x(s)} = q_{n-1}(y(s)), \tag{1.3}$$

where $q_{n-1}(x)$ are some polynomials of the degree $n-1$. We have step-by-step,

$$\mathcal{D}_{x(s)}1 = 0, \quad \mathcal{D}_{x(s)}x = 1, \quad \mathcal{D}_{x(s)}x^2 = x(s+1) + x(s) = q_1(y(s)) \tag{1.4}$$

and

$$\mathcal{D}_{x(s)}x^3 = x(s)^2 + x(s+1)^2 + x(s)x(s+1) = q_2(y(s)) \tag{1.5}$$

with arbitrary polynomials $q_{1,2}(x)$ of the degrees 1 and 2.

Theorem 1. *The relations*

$$x(s) + x(s+1) = \pi_1(y(s)), \quad x(s)x(s+1) = \pi_2(y(s)), \tag{1.6}$$

where $\pi_i(y)$ are polynomials of the degree $\leq i$, are necessary and sufficient for the operator $\mathcal{D}$ to satisfy the key demand (1.2).

The proof (by induction) is elementary and we omit it.

Lemma 1. *If the grid $y(s)$ is adjacent to $x(s)$ then the grid $\alpha y(s) + \beta$ with arbitrary parameters α, β is also adjacent to the same grid $x(s)$. Analogously, if the grid $x(s)$ is admissible then the grid $\alpha x(s) + \beta$ is also admissible. Moreover, for any fixed parameter s_0 the grid $x(s + s_0)$ is admissible with $y(s + s_0)$ being its adjacent grid.*

This lemma means that both admissible $x(s)$ and adjacent $y(s)$ grids are defined modulo the equivalence relations with respect to linear transformations of the argument s. Therefore it is sufficient to classify grids up to this equivalence class. Using this property we can set $q_1(x) = x$, so that

$$x(s) + x(s+1) = y(s). \tag{1.7}$$

Relation (1.7) can be considered as a definition of the grid $y(s)$. Then we have from Theorem 1

$$x(s)x(s+1) = \pi_2(y(s)), \tag{1.8}$$

where $\pi_2(x) = ax^2 + bx + c$ is an arbitrary polynomial of the second degree. The latter relation can be rewritten in the form

$$\mu_1(x(s)^2 + x(s+1)^2) + \mu_2 x(s)x(s+1) + \mu_3(x(s) + x(s+1)) + \mu_4 = 0 \tag{1.9}$$

with arbitrary parameters μ_i. It is well known that general analytical solution of Eq. (1.9) can be parametrized by elementary functions [13, 16]:

$$x(s) = a_1 q^s + a_2 q^{-s} + a_0, \quad \text{if} \quad 2\mu_1 + \mu_2 \neq 0, \tag{1.10}$$

or

$$x(s) = a_2 s^2 + a_1 s + a_0, \quad \text{if} \quad 2\mu_1 + \mu_2 = 0, \ \mu_3 \neq 0, \tag{1.11}$$

or

$$x(s) = a_1 s + a_0 \quad \text{if} \quad 2\mu_1 + \mu_2 = 0, \ \mu_3 = 0. \tag{1.12}$$

The admissible grids of types (1.10), (1.11), or (1.12) are called the Askey-Wilson grids [13, 14].

Theorem 2. *By an appropriate linear transformation, the adjacent grid $y(s)$ can be reduced to the form*

$$y(s) = x(s + 1/2). \tag{1.13}$$

The proof is elementary and is based on the explicit parametrization of the admissible grids.

Thus for the admissible grids $x(s)$ we have the fundamental property

$$\mathcal{D}_{x(s)} P_n(x) = P_{n-1}(x(s+1/2)), \tag{1.14}$$

where $P_n(x)$ is an arbitrary polynomial of the degree n. Notice that the adjacent grid $x(s+1/2)$ is again admissible and its adjacent grid will be $x(s+1)$.

We define the operator $\mathcal{M}_{x(s)}$ on the grid $x(s)$ by the relation

$$\mathcal{M}_{x(s)} F(x) = F(x(s)) + F(x(s+1)),$$

i.e. $\mathcal{M}_{x(s)}$ maps functions to their sums evaluated at $x(s)$ and $x(s+1)$.

Lemma 2. *For any polynomial $P_n(x)$ of the degree n the operator $\mathcal{M}_{x(s)}$ maps it to a polynomial $\tilde{P}_n(y(s))$ of the same degree n.*

Proof: It is sufficient to check this statement for the elementary monomials:

$$\begin{aligned} x(s+1)^n + x(s)^n &= (x(s+1)^{n-1} + x(s)^{n-1})(x(s+1) + x(s)) \\ &\quad - x(s)x(s+1)(x(s+1)^{n-2} + x(s)^{n-2}). \end{aligned}$$

Then the needed statement follows by induction (for $n=0$ and $n=1$ it is obvious).

Now we would like to construct an operator $\mathcal{R}_{x(s)}$ of the form

$$\mathcal{R}_{x(s)} F(x) = A(s) F(x(s+1)) + B(s) F(x(s)),$$

with some functions $A(s)$ and $B(s)$, mapping polynomials $P_n(x)$ of the degree n to polynomials of the degree $n+1$. Obviously,

$$\mathcal{R}_{x(s)} = E_2(s)\mathcal{D} + E_1(s)\mathcal{M}$$

for some functions $E_{1,2}(s)$. Applying $\mathcal{R}_{x(s)}$ to a constant, we have $\mathcal{R}_{x(s)}\{1\} = 2E_1(s)$, whence $E_1(s) = Q_1(y(s))$ is an arbitrary linear function in y. Applying $\mathcal{R}_{x(s)}$ to x, we obtain

$$\mathcal{R}_{x(s)} x = E_2(s) + E_1(s)(x(s+1) + x(s)) = Y_2(y(s)),$$

where $Y_2(y)$ is a polynomial of the second degree. Hence we have necessarily $E_2(s) = Q_2(y(s))$, where $Q_2(y)$ is a polynomial with $\deg(Q) \le 2$. By induction, we have the following statement. $\square$

Theorem 3. *The conditions*

$$E_i(s) = Q_i(y(s)), \quad i = 1, 2, \tag{1.15}$$

are necessary and sufficient for the operator $\mathcal{R}_{x(s)}$ to map an n-th degree polynomial $P_n(x(s))$ to a polynomial $\tilde{P}_{n+1}(y(s))$ of the degree $n+1$.

We thus have two operators $\mathcal{D}_{x(s)}$ and $\mathcal{R}_{x(s)} = Q_2(y(s))\mathcal{D}_{x(s)} + Q_1(y(s))\mathcal{M}_{x(s)}$ such that both map polynomials on the grid $x(s)$ to polynomials on the adjacent grid $y(s)$ but the operator $\mathcal{D}_{x(s)}$ lowers the degree by one while the operator $\mathcal{R}_{x(s)}$ raises this degree by one (i.e., we have the raising and lowering operators).

Obviously, the operator $\mathcal{R}_{x(s+1/2)}\mathcal{D}_{x(s)}$ maps polynomials of $x(s)$ to polynomials of the same degree but of the argument $x(s+1)$. Hence, the operator

$$\mathcal{L}_{x(s)} = \mathcal{R}_{x(s-1/2)}\mathcal{D}_{x(s-1)}$$

maps polynomials of $x(s)$ to polynomials on the same grid $x(s)$ and of the same degree. Therefore it is natural to look for polynomials $P_n(x)$ forming eigenfunctions of this operator:

$$\mathcal{L}_{x(s)}P_n(x(s)) = \lambda_n P_n(x(s)). \tag{1.16}$$

It appears that such polynomials coincide with the Askey-Wilson orthogonal polynomials [1] and $\mathcal{L}_{x(s)}$ is called the Askey-Wilson difference operator [13,16]. In order to find their explicit form, we need elementary polynomials transforming in a simple way under the action of $\mathcal{D}_{x(s)}$ and $\mathcal{M}_{x(s)}$:

$$\phi_n(x(s)) = \prod_{k=0}^{n-1}(x(s) - x(\alpha + k)), \tag{1.17}$$

where α is an arbitrary parameter. Function $\phi_n(x(s))$ vanishes at $s = \alpha + i,\ i = 0, 1, \ldots, n-1$. Since $x(s)$ is invertible in some domain, these points are in one-to-one correspondence with n (distinct) zeros of the polynomial $\phi_n(x)$. It is obvious that both functions of the argument s, $\mathcal{D}_{x(s)}\phi_n(x)$ and $\mathcal{M}_{x(s)}\phi_n(x)$, vanish at $s = \alpha + i,\ i = 0, 1, \ldots, n-2$. However, as we know already, $\mathcal{D}_{x(s)}\phi_n(x)$ is a polynomial in $x(s+1/2)$ of the degree $n-1$. Hence, we have necessarily

$$\mathcal{D}_{x(s)}\phi_n(x) = \kappa_n\tilde{\phi}_{n-1}(x(s+1/2)), \tag{1.18}$$

where $\tilde{\phi}_{n-1}(x(s+1/2)) = \prod_{k=0}^{n-1}(x(s+1/2) - x(\alpha+1/2+k))$. Analogously,

$$\begin{aligned}\mathcal{M}_{x(s)}\phi_n(x) &= (\mu_n x(s+1/2) + \nu_n)\tilde{\phi}_{n-1}(x(s+1/2))\\ &= \mu_n\tilde{\phi}_n(x(s+1/2)) + \rho_n\tilde{\phi}_{n-1}(x(s+1/2))\end{aligned} \tag{1.19}$$

with some constants κ_n, μ_n, ν_n, ρ_n independent on s.

Using these relations, we can represent polynomial solutions $P_n(x)$ of the eigenvalue problem (1.16) in the form

$$P_n(x) = \sum_{k=0}^{n} C_k\phi_k(x). \tag{1.20}$$

Then for the coefficients C_n we obtain two-term recurrence relation leading to the well-known expression of $P_n(x)$ in terms of the ${}_4\Phi_3$ basic hypergeometric function [1,13,16].

2 Rational functions. Formulation of the problem

We consider now rational functions $R_n(x)$ of the order $[n/n]$, which means that $R_n(x)$ are given by ratios of two n-th degree polynomials in x.

We take $\alpha_1, \alpha_2, \ldots, \alpha_n$ as n distinct prescribed positions of the poles of $R_n(x)$. Then $R_n(x)$ can be written as a sum of partial fractions

$$R_n(x) = t_0^{(n)} + \sum_{i=1}^{n} \frac{t_i^{(n)}}{x - \alpha_i} \tag{2.1}$$

with the coefficients $t_i^{(n)}$, $i = 1, 2, \ldots, n$, playing the role of residues of $R_n(x)$ at the poles α_i. The coefficient $t_0^{(n)}$ can be interpreted as $\lim_{x\to\infty} R_n(x)$.

Similar to the polynomial case, we would like to construct lowering and raising operators $\mathcal{D}_{x(s)}$, $\mathcal{R}_{x(s)}$ in the space of rational functions $R_n(x)$. We take as a definition of the lowering operator $\mathcal{D}_{x(s)}$ a divided difference operator in the parametrizing variable s, which obeys the following properties:

(i) the grid $x(s)$ is a meromorphic function of $s \in \mathbb{C}$ which is invertible in some domain of the complex plane;
(ii) for any function $F(x)$ one has

$$\mathcal{D}_{x(s)} F(x) = \chi_0(s)(F(x(s+1)) - F(x(s))),$$

where $\chi_0(s)$ is some function to be determined;
(iii) $\mathcal{D}_{x(s)} R_1(x) = const$, where $R_1(x)$ is an arbitrary rational function of the order $[1/1]$ with the only pole at $x = \alpha_1$;
(iv) the operator $\mathcal{D}_{x(s)}$ transforms any rational function $R_n(x)$ with the *prescribed* poles $\alpha_1, \ldots, \alpha_n$ to a rational function $\tilde{R}_{n-1}(y(s))$ of the adjacent grid $y(s)$ with some other poles $\beta_1, \ldots, \beta_{n-1}$;
(v) the operator $\mathcal{D}_{x(s)}$ is "transitive": for $j \in \mathbb{N}$ there exists a sequence of functions $\chi_j(s)$ such that the operators $\mathcal{D}_{x(s)}^{(j)} = \chi_j(s)\chi_0^{-1}(s)\mathcal{D}_{x(s)}$ transform rational functions $R_n(x)$ with the *prescribed* poles $\alpha_{j+1}, \ldots, \alpha_{j+n}$ to rational functions $\tilde{R}_{n-1}(y(s))$ with one and the same adjacent grid $y(s)$ and the poles $\beta_{j+1}, \ldots, \beta_{j+n-1}$;
(vi) the set of poles is nondegenerate: there are infinitely many distinct values of α_n and β_n and $\alpha_n \neq \alpha_{n+1}, \alpha_{n+2}$ and, similarly, $\beta_n \neq \beta_{n+1}, \beta_{n+2}$ for all n.

An important restriction, similar to the polynomials' case, is the condition of independence of $\mathcal{D}_{x(s)}$ on the order n of a rational function. We would like to deduce admissible forms of the functions $\chi_j(s)$, $x(s)$, $y(s)$ as well as of the poles' coordinates α_j, β_j from the given set of requirements.

Evidently, from (ii) we have $\mathcal{D}_{x(s)} R_0(x) = 0$ and from requirements (ii)–(iii) we easily find

$$\chi_0(s) = \left(\frac{1}{x(s+1) - \alpha_1} - \frac{1}{x(s) - \alpha_1} \right)^{-1} = \frac{(x(s) - \alpha_1)(x(s+1) - \alpha_1)}{x(s) - x(s+1)}. \tag{2.2}$$

This function is defined up to an inessential constant multiplier.

The most non-trivial problem consists in establishing properties (iv)–(v). For $n = 2$ we have

$$\mathcal{D}_{x(s)} R_2(x) = \mathcal{D}_{x(s)} \left\{ t_0^{(2)} + \frac{t_1^{(2)}}{x(s) - \alpha_1} + \frac{t_2^{(2)}}{x(s) - \alpha_2} \right\}.$$

Properties (ii), (iii) yield

$$\mathcal{D}_{x(s)} R_2(x) = t_1^{(2)} + t_2^{(2)} \frac{(x(s) - \alpha_1)(x(s+1) - \alpha_1)}{(x(s) - \alpha_2)(x(s+1) - \alpha_2)}. \tag{2.3}$$

Due to the demand (iv), we should have

$$\mathcal{D}_{x(s)} R_2(x) = r_0^{(1)} + \frac{r_1^{(1)}}{y(s) - \beta_1}, \tag{2.4}$$

where $r_i^{(1)}$ are some constants and β_1 is a new pole. Comparing (2.3) and (2.4), we come to the condition

$$\frac{(x(s) - \alpha_1)(x(s+1) - \alpha_1)}{(x(s) - \alpha_2)(x(s+1) - \alpha_2)} = \frac{A_1(y(s) - \gamma_1)}{y(s) - \beta_1}, \tag{2.5}$$

where A_1 and γ_1 are some constants.

In a similar way, considering arbitrary rational functions of order $[n/n]$, we derive the constraint

$$\frac{(x(s) - \alpha_1)(x(s+1) - \alpha_1)}{(x(s) - \alpha_n)(x(s+1) - \alpha_n)} = \frac{q_{n-1}(y(s))}{(y(s) - \beta_1)(y(s) - \beta_2) \cdots (y(s) - \beta_{n-1})}, \quad n = 2, 3, \ldots, \tag{2.6}$$

where $q_{n-1}(y)$ is a polynomial of the degree $\leq n - 1$. Now it is not difficult to see validity of the following statement.

Lemma 3. *Conditions (2.6) are necessary and sufficient for fulfilling property (iv).*

Now we should satisfy the transitivity condition (v). First of all, we find in an obvious way that

$$\chi_j(s) = \left(\frac{1}{x(s+1)-\alpha_{j+1}} - \frac{1}{x(s)-\alpha_{j+1}}\right)^{-1}$$
$$= \frac{(x(s)-\alpha_{j+1})(x(s+1)-\alpha_{j+1})}{x(s)-x(s+1)}. \tag{2.7}$$

Repeating the same considerations as for the case $j = 0$, we obtain

$$\frac{(x(s)-\alpha_{j+1})(x(s+1)-\alpha_{j+1})}{(x(s)-\alpha_{n+j})(x(s+1)-\alpha_{n+j})}$$
$$= \frac{q_{n-1}(y(s);j)}{(y(s)-\beta_{j+1})(y(s)-\beta_{j+2})\cdots(y(s)-\beta_{j+n-1})}, \tag{2.8}$$

where $n = 2, 3, \dots$, and $q_{n-1}(y; j)$ is a polynomial of the degree $\le n-1$ (depending on j as a discrete parameter). From Lemma 3 we find that conditions (2.8) are necessary and sufficient for validity of (v).

For $n = 2$ we have from (2.8)

$$\frac{(x(s)-\alpha_j)(x(s+1)-\alpha_j)}{(x(s)-\alpha_{j+1})(x(s+1)-\alpha_{j+1})} = \frac{A_j(y(s)-\gamma_j)}{y(s)-\beta_j}, \quad j = 1, 2, \dots, \tag{2.9}$$

where A_j and γ_j are some constants, $\gamma_j \neq \beta_j$. This relation can be considered as a direct generalization of the property (2.5) (which corresponds to $j = 1$).

Lemma 4. *Relations (2.9) are necessary and sufficient for fulfilling the demand (v).*

Indeed, starting from (2.9) we obtain for $m > n$

$$\frac{(x(s)-\alpha_n)(x(s+1)-\alpha_n)}{(x(s)-\alpha_m)(x(s+1)-\alpha_m)}$$
$$= A_n A_{n+1}\cdots A_{m-1}\,\frac{(y(s)-\gamma_n)\cdots(y(s)-\gamma_{m-1})}{(y(s)-\beta_n)\cdots(y(s)-\beta_{m-1})}, \tag{2.10}$$

which coincides with condition (2.8).

Thus it is sufficient to investigate relations (2.9). Our further program consists in the determination of functional equations for the grids $x(s)$, $y(s)$ and positions of the poles α_i, β_i.

3 Functional equations for the grids

We start from relations (2.9) which should be valid for all $j = 1, 2, \dots$. Unknowns are the grids $x(s)$ and $y(s)$, poles α_j and β_j and parameters γ_j and A_j. First of all we establish covariance of (2.9) with respect to linear fractional transformations.

Theorem 4. *If the grid $x(s)$ and poles α_n are admissible then the grid $\tilde{x}(s) = (ax(s) + b)/(cx(s) + d)$ and poles $\tilde{\alpha}_n = (a\alpha_n + b)/(c\alpha_n + d)$ are also admissible for all $a, b, c, d \in \mathbb{C}$, $ad \neq bc$, with the same adjacent grid $y(s)$, parameters β_n, γ_n and rescaled parameters A_n. If the grid $x(s)$ is admissible with the poles α_n and $y(s)$ is the adjacent grid with the poles β_n and parameters A_n, γ_n, then the grid $\tilde{y}(s) = (ay(s) + b)/(cy(s) + d)$ is also adjacent with respect to the grid $x(s)$ and poles α_n. Corresponding adjacent poles are $\tilde{\beta}_n = (a\beta_n + b)/(c\beta_n + d)$ and $\tilde{\gamma}_n = (a\gamma_n + b)/(c\gamma_n + d)$ with the rescaled parameters A_n.*

The proof of this theorem is elementary. It simply means that linear fractional transformation of the argument $x \to (ax + b)/(cx + d)$ maps any rational function of order $[n/n]$ to a rational function of the same order (with the poles transformed in an appropriate way). Thus if the grid $x(s)$ is admissible for rational functions with the prescribed poles, then the transformed grid $\tilde{x}(s)$ is also admissible for the transformed rational function.

This theorem allows us to consider equivalence classes of the grids $x(s)$ and $y(s)$. We say that two grids $x_1(s)$ and $x_2(s)$ belong to the same class if they can be obtained from each other by a nondegenerate linear fractional transformation $x_1(s) = (ax_2(s) + b)/(cx_2(s) + d)$. The same is applicable to adjacent grids $y(s)$ (these grids can be transformed independently on $x(s)$). Note that the class of admissible equivalence transformations for rational functions is wider than for polynomials, where only linear transformations were allowed. We consider now relations (2.9) for $j = 1, 2$. They yield two compatible linear equations for two unknowns $z_1(s) = x(s)x(s + 1)$ and $z_2(s) = x(s) + x(s + 1)$. We assume that these two equations are independent. Indeed, if these equations were reducible to each other, then either $\alpha_n = const$ or $y(s) = const$, which is forbidden. Thus we can solve them uniquely and obtain

$$z_1(s) = \frac{Q_0(y(s))}{Q_2(y(s))}, \qquad z_2(s) = -\frac{Q_1(y(s))}{Q_2(y(s))}, \tag{3.1}$$

where $Q_{0,1,2}(y)$ are polynomials in y of the degrees not exceeding 2. It is well known [4] that formulas (3.1) give rational parametrization for an arbitrary quadric in the coordinates z_1 and z_2. This means that variables z_1, z_2 (3.1) satisfy the equation

$$\mu_1 z_1^2 + \mu_2 z_2^2 + \mu_3 z_1 z_2 + \mu_4 z_1 + \mu_5 z_2 + \mu_6 = 0, \tag{3.2}$$

where $\mu_i,\ i = 1, \dots, 6$, are some unconstrained parameters. In the variables $x(s)$ and $x(s+1)$ we obtain the equation

$$\mu_1 x(s)^2 x(s+1)^2 + \mu_2(x(s)^2 + x(s+1)^2) + \mu_3(x(s)x(s+1)^2 + x(s+1)x(s)^2) + (\mu_4 + 2\mu_2)x(s)x(s+1) + \mu_5(x(s) + x(s+1)) + \mu_6 = 0. \tag{3.3}$$

Theorem 5. *Variables $u = x(s)$ and $v = x(s+1)$ satisfy the equation*

$$G(u, v) \equiv E_1 u^2 v^2 + E_2 uv(u+v) + E_3 uv + E_4(u^2 + v^2) + E_5(u+v) + E_6 = 0 \tag{3.4}$$

with some arbitrary constants E_i.

This is one of the main results of the present paper and it was announced briefly in our earlier work [25]. The polynomial $G(u, v)$ can be characterized as the general symmetric polynomial in two variables u and v having degree 2 with respect to each of them. In what follows we call polynomial (3.4) the Euler polynomial, because Euler was perhaps the first who considered it in the context of addition theorems for elliptic functions. The Euler polynomial describes some elliptic curve [28] which will be called the *Euler curve*.

From (3.1) we find that $x(s)$ and $x(s+1)$ are two roots of the quadratic equation

$$Q_2(y(s))x^2 + Q_1(y(s))x + Q_0(y(s)) = 0. \tag{3.5}$$

We introduce the polynomial

$$Z(x, y) = Q_2(y)x^2 + Q_1(y)x + Q_0(y) = S_2(x)y^2 + S_1(x)y + S_0(x) \tag{3.6}$$

of two variables x and y, where $S_{1,2}(x)$ are some polynomials of the degrees ≤ 2. This $Z(x, y)$ plays a fundamental role in the following considerations. We have by definition the elliptic curve equations $Z(x(s), y(s)) = Z(x(s+1), y(s)) = 0$. Excluding $y(s)$ from these two equations by the resultant technique and assuming that $x(s) \neq x(s+1)$, we return to the Euler Eq. (3.4) for variables $x(s)$ and $x(s+1)$. Now we can repeat our procedure by shifting $s \to s+1$ and obtain relations $Z(x(s+1), y(s)) = Z(x(s+1), y(s+1)) = 0$. Excluding $x(s+1)$ from them (assuming that $y(s) \neq y(s+1)$), we find that variables $y(s)$ and $y(s+1)$ also satisfy the Euler equation

$$\tilde{G}(y(s), y(s+1)) = 0, \tag{3.7}$$

where $\tilde{G}(u, v)$ is another symmetric polynomial which has the degree at most 2 with respect to each variable u and v. Equivalently, we can write down a pair of relations similar to (3.1)

$$y(s)y(s+1) = \frac{S_0(x(s))}{S_2(x(s))}, \qquad y(s) + y(s+1) = -\frac{S_1(x(s))}{S_2(x(s))}, \tag{3.8}$$

where polynomials $S_i(x)$ are defined in (3.6).

Now we define a procedure which we call the T-algorithm. We denote I_1 and I_2 two involutions of the curve $Z(x, y) = 0$ defined as follows. We take an arbitrary point $P_1 = (x_1, y_1)$ of the algebraic curve $Z(x, y) = 0$. Then $I_1(x_1, y_1) = (x_2, y_1)$, where x_1, x_2 are two roots of the quadratic equation $Z(x, y_1) = 0$. Similarly, $I_2(x_2, y_1) = (x_2, y_2)$, where y_1, y_2 are two roots of the quadratic equation $Z(x_2, y) = 0$. Clearly $I_1^2 = I_2^2 = 1$, that is both transformations $I_{1,2}$ are indeed involutions: $I_1(x_2, y_1) = (x_1, y_1)$ and $I_2(x_2, y_2) = (x_2, y_1)$. This allows us to construct a set of points $P_n = (x_n, y_n)$ on the curve $Z(x, y) = 0$ by the algorithm $P_{n+1} = T P_n = I_2 I_1 P_n$. The involutions I_1 and I_2 in general do not commute and we have $P_{n-1} = T^{-1} P_n = I_1 I_2 P_n$. The transformation $T = I_2 I_1$ can be thus interpreted as a shift along the curve $Z(x, y) = 0$. The grid points $x(s)$ and $y(s)$ are then obtained as natural projections of the points P_n to the coordinate axes (x, y). There are exceptional cases, when the involution I_1 is applied to the fixed points: $I_1(x_n, y_n) = (x_n, y_n)$. This can occur only if $Q_1(y_n)^2 = 4Q_2(y_n)Q_0(y_n)$. Analogously, the involution I_2 is degenerate only if $S_1(x_n)^2 = 4S_2(x_n)S_0(x_n)$. Clearly, there are no more than 8 points of such kind on the curve $Z(x, y) = 0$. In what follows we assume that the polynomial $Z(x, y)$ has degree 2 in both variables x, y and the set of points (x_n, y_n) is not degenerate (this is the generic case). In this way our T-algorithm generates both grids $x(s)$ and $y(s)$.

We can rewrite relations (2.9) in the form

$$\frac{Z(\alpha_n, y(s))}{Z(\alpha_{n+1}, y(s))} = A_n \frac{y(s) - \gamma_n}{y(s) - \beta_n} \tag{3.9}$$

for all $s = 1, 2, \dots$. We assume that $y(s)$ takes infinitely many distinct values so that we can continue analytically relation (3.9) to

$$\frac{Z(\alpha_n, y)}{Z(\alpha_{n+1}, y)} = A_n \frac{y - \gamma_n}{y - \beta_n}. \tag{3.10}$$

By our assumptions, $\gamma_n \neq \beta_n$ and we find from (3.10) that the polynomial $Z(\alpha_n, y)$ should have the root $y = \gamma_n$, whereas the polynomial $Z(\alpha_{n+1}, y)$ should have the root $y = \beta_n$. There are two possibilities:

(1) $\gamma_n = \beta_{n-1}$. In this case we can write $Z(\alpha_n, y) = \nu_n(y - \beta_{n-1})(y - \epsilon)$, where ϵ is some unknown parameter. Since $Z(\alpha_n, \epsilon) = 0$, the poles α_n can take only two distinct values for all $n = 1, 2, \dots$, which is forbidden.

(2) $\gamma_n \neq \beta_{n-1}$. In this case the polynomial $Z(\alpha_n, y)$ has two distinct roots:

$$Z(\alpha_n, y) = \nu_n(y - \gamma_n)(y - \beta_{n-1}).$$

Then from (3.10) we find

$$\gamma_n = \beta_{n-2} \tag{3.11}$$

and from (3.6) we deduce

$$Z(\alpha_n, y) = S_2(\alpha_n)(y - \beta_{n-2})(y - \beta_{n-1}). \tag{3.12}$$

Therefore,

$$\beta_{n-1} + \beta_{n-2} = -\frac{S_1(\alpha_n)}{S_2(\alpha_n)}, \qquad \beta_{n-1}\beta_{n-2} = \frac{S_0(\alpha_n)}{S_2(\alpha_n)}. \tag{3.13}$$

Equivalently, this means that

$$Z(\alpha_n, \beta_{n-1}) = Z(\alpha_n, \beta_{n-2}) = 0 \tag{3.14}$$

for all n. Hence, the grid of poles α_n satisfies the same Euler equation as the grid $x(s)$: $G(\alpha_n, \alpha_{n+1}) = 0$. Similarly, the adjacent poles' coordinates β_n satisfy the Euler equation coinciding with that for the grid y_s: $\tilde{G}(\beta_n, \beta_{n+1}) = 0$.

Now we can rewrite the basic relation (2.9) in the form

$$\frac{(x(s)-\alpha_n)(x(s+1)-\alpha_n)}{(x(s)-\alpha_{n+1})(x(s+1)-\alpha_{n+1})} = \frac{S_2(\alpha_n)}{S_2(\alpha_{n+1})}\frac{y(s)-\beta_{n-2}}{y(s)-\beta_n}, \ n = 1, 2, \dots, \tag{3.15}$$

which assumes that

$$\frac{(x(s)-\alpha_n)(x(s+1)-\alpha_n)}{(x(s)-\alpha_m)(x(s+1)-\alpha_m)} = \frac{S_2(\alpha_n)}{S_2(\alpha_m)}\frac{(y(s)-\beta_{n-2})(y(s)-\beta_{n-1})}{(y(s)-\beta_{m-2})(y(s)-\beta_{m-1})}, \tag{3.16}$$

where $S_2(x)$ is defined in (3.6). Relation (3.16) is valid for all pairs of integers n, m. It should be stressed that the grids α_n and β_{n-2} can be constructed using the same T-algorithm as for the grids $x(s)$ and $y(s)$. This algorithm can be defined for negative n (the backward shift algorithm) in the same way as for $n > 0$. So we can assume that the grids α_n and β_n are defined for all integers $-\infty < n < \infty$.

Admissible grids $x(s)$ and $y(s)$ lead to a biquadratic polynomial $Z(x, y)$. Conversely, for a given biquadratic polynomial $Z(x, y)$ the T-algorithm generates admissible grids $x(s)$ and $y(s)$ as well as the corresponding sequences of poles α_n and β_{n-2}. Therefore,

Theorem 6. *The T-algorithm provides the necessary and sufficient conditions for an admissibility of the grids $x(s)$ and $y(s)$.*

It is interesting to note that in the case of polynomials A. Magnus found [13, 14] a similar T-algorithm but for conics of the form

$$Z(x, y) = a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0. \tag{3.17}$$

More precisely, he showed that for arbitrary nondegenerate conics (3.17) an application of the T-algorithm generates two Askey-Wilson grids $x(s)$ and $y(s)$. In his approach, for a given point y the points x_1 and x_2 should be found such that for any polynomial $P(x)$ of n-th degree the expression $(P(x_2(y)) - P(x_1(y)))/(x_2(y) - x_1(y))$ is a polynomial in y of the degree $n - 1$. Magnus showed that the T-algorithm generates the grids $y(s)$ and $x_{1,2}(y(s))$ for $s = 1, 2, \dots$. Our approach is equivalent to the Magnus one after the obvious identifications $y \to y(s)$, $x_1 \to x(s)$, $x_2 \to x(s+1)$. The generalization from the polynomials' case to rational functions is quite natural: we replace conics by an arbitrary algebraic curve $Z(x, y) = 0$ with the property that for fixed y equation $Z(x, y) = 0$ has exactly two roots for x (and, similarly, for fixed x we have two roots for y). This property is obvious from the geometric point of view: any horizontal or vertical line intersects our curve exactly at two points (which are in general complex). This defines the needed curve (3.6).

4 Parametrization by elliptic functions

There is natural parametrization of admissible grids $x(s)$, $y(s)$ and poles' coordinates α_n, β_n by elliptic functions of the second order. Indeed, for arbitrary non-degenerate biquadratic polynomial $Z(x, y)$, the curve $Z(x, y) = 0$ admits a uniformization in terms of elliptic functions:

$$x = \Phi(s + s_1), \qquad y = \Phi(s + s_0), \tag{4.1}$$

where

$$\Phi(s) = C\, \frac{\theta_1(h(s-e))\theta_1(h(s+e))}{\theta_1(h(s-d))\theta_1(h(s+d))} \tag{4.2}$$

is the general elliptic function of the second order (i.e., it has two poles in the fundamental parallelogram of periods). Here $\theta_1(z)$ is the standard Jacobi θ_1-function [29] with quasiperiods 1 and τ (dependence on τ is not indicated for brevity). The variables $e, d, C, h, s_1 - s_0, \tau$ are determined from the parameters of $Z(x, y)$ (the variable h can be removed by rescalings of s and other parameters).

Conversely, for any two elliptic functions $x(s)$ and $y(s)$ of the second order (4.1) there exists a polynomial $Z(x, y)$ quadratic in both x and y such that $Z(x(s), y(s)) = 0$ (see, e.g. [29]).

The grids $x(s)$ and $y(s)$ admit linear fractional transformations

$$\tilde{x}(s) = \frac{a_1 x(s) + b_1}{c_1 x(s) + d_1}, \qquad \tilde{y}(s) = \frac{a_2 y(s) + b_2}{c_2 y(s) + d_2}.$$

If parameters $a_1, \ldots, d_2$ are arbitrary (so that 6 of them form independent variables), then we get a curve determined by a generic biquadratic polynomial equation $\tilde{Z}(\tilde{x}, \tilde{y}) = 0$ which is birationally equivalent to the original $Z(x, y) = 0$. Such a polynomial contains $8 = 9 - 1$ independent parameters (with one parameter being a common multiplier). We can choose $a_1, \ldots, d_2$ in such a way that the curve $\tilde{Z}(x, y) = 0$ will contain only 2 independent parameters. There are different possibilities for such a choice. One of them is the canonical Euler-Baxter form [3]:

$$\tilde{Z}(x, y) = x^2 y^2 + 1 + c(x^2 + y^2) + 2dxy, \tag{4.3}$$

where c and d are two independent parameters. In this case we obtain very simple expressions for $x(s)$ and $y(s)$:

$$x(s) = \sqrt{k}\,\mathrm{sn}(h(s+s_0);k), \qquad y(s) = \sqrt{k}\,\mathrm{sn}(h(s+s_0+1/2);k), \tag{4.4}$$

where s_0 is an arbitrary parameter and the elliptic modulus k and h can be determined from the equations [3]

$$k + 1/k = (d^2 - 1 - c)/c, \qquad k\,c\,\mathrm{sn}^2(h/2) + 1 = 0.$$

Using an appropriate projective transformation applied simultaneously to the variables x and y it is possible to give to $Z(x, y)$ other forms corresponding to the parametrization

$$x(s) = \Phi(s+s_0), \qquad y(s) = \Phi(s+s_0+1/2) \tag{4.5}$$

for different elliptic functions of the second order $\Phi(s)$. For the poles α_n and β_n we obtain expressions

$$\alpha_n = \Phi(n+n_0), \qquad \beta_n = \Phi(n+n_0+3/2), \tag{4.6}$$

where n_0 is an arbitrary parameter.

By an appropriate Möbius transformation of the variables x, y we can make $Z(x, y)$ symmetric in x, y and force the polynomial $Q_2(y) = S_2(y)$ in expression (3.6) to have a double root y_0, $Q_2(y) = (y - y_0)^2$. Using further linear transformations $x \to \alpha x + \beta$, $y \to \alpha y + \beta$, it is possible to reduce $Z(x, y)$ to the form

$$Z = (xy + (x+y)y_0 + g_2/4)^2 - (x+y+y_0)(4xyy_0 - g_3), \tag{4.7}$$

where g_2, g_3 are two remaining independent parameters of the polynomial Z. Due to the addition theorem for Weierstrass function $\wp(z)$ [29], we obtain parametrization of (4.7) of the form

$$x(s) = \wp(h(s+s_0)), \qquad y(s) = \wp(h(s+s_0+1/2)), \tag{4.8}$$

where g_2 and g_3 are fundamental invariants of the Weierstrass function appearing in the differential equation for $\wp$:

$$\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3.$$

5 Further properties of the basic operators

In the previous sections we have constructed the operator $D_{x(s)}$ defined on rational functions of the order $[n/n]$ with the poles $\alpha_1, \ldots, \alpha_n$ and mapping them to rational functions of the order $[n-1/n-1]$ with the argument $y(s)$ and the poles $\beta_1, \ldots, \beta_{n-1}$.

Now we introduce the operator $\mathcal{M}_{x(s)}$ by the same formula as for polynomials, that is for any rational function $F(x)$ with poles $\alpha_1, \alpha_2, \dots,$ we define

$$\mathcal{M}_{x(s)}F(x) = F(x(s+1)) + F(x(s)). \tag{5.1}$$

It is sufficient to study how the operator $\mathcal{M}_{x(s)}$ acts on the simplest fractions, i.e.,

$$g_n(s) \equiv \mathcal{M}_{x(s)}\{(x-\alpha_n)^{-1}\}$$

for arbitrary n. We have, by definition,

$$g_n(s) = (x(s+1)-\alpha_n)^{-1} + (x(s)-\alpha_n)^{-1} = \frac{z_2 - 2\alpha_n}{z_1 - \alpha_n z_2 + \alpha_n^2},$$

where $z_1 = x(s)x(s+1)$ and $z_2 = x(s) + x(s+1)$. From relations (3.1) and (3.12) we deduce

$$\begin{aligned} g_n(s) &= -\frac{2\alpha_n + Q_1(y(s))}{\alpha_n^2 Q_2(y(s)) + \alpha_n Q_1(y(s)) + Q_0(y(s))} \\ &= \frac{\xi_n}{y(s)-\beta_{n-2}} + \frac{\eta_n}{y(s)-\beta_{n-1}} \end{aligned}$$

with some coefficients ξ_n, η_n independent on s. By induction, we come to the following statement.

Lemma 5. *For the admissible grids $x(s)$ and $y(s)$ fixed in (4.1), the operator $\mathcal{M}_{x(s)}$ maps any rational function $F(x)$ of the order $[n/n]$ with the poles $\alpha_1, \alpha_2, \dots, \alpha_n$ to a rational function $\tilde{F}(y(s))$ of the order $[n+1/n+1]$ with the poles $\beta_{-1}, \beta_0, \dots, \beta_{n-1}$. The only exception is $F(x) = F = const$ which is transformed into $2F$.*

In what follows we stick to the choice $y(s) = x(s+1/2)$ with the function $x(s)$ parametrized by an elliptic function of the second order as fixed in (4.5). Such a choice is always achievable by an appropriate Möbius transformation of the grid. In this case the explicit expressions for the poles is fixed in (4.6), that is $\alpha_n = x(n+n_1)$, $\beta_n = \alpha_{n+3/2} = x(n+n_1+3/2)$, where $n_1 = n_0 - s_0$.

Now we introduce some elementary rational functions

$$\phi_n(x) = \prod_{i=1}^{n} \frac{x-\mu_i}{x-\alpha_i}, \tag{5.2}$$

and

$$\psi_n(y) = \prod_{i=1}^{n} \frac{y-\nu_i}{y-\beta_i}, \tag{5.3}$$

where $\mu_i = \alpha_{i+n_2}$, $\nu_i = \alpha_{i+1/2+n_2}$ with some parameter n_2. Then it is easily verified that

$$\mathcal{D}_{x(s)}\phi_n(x) = \kappa_n \psi_{n-1}(y(s)) \tag{5.4}$$

with some coefficients κ_n independent on s. Similarly,

$$\mathcal{M}_{x(s)}\phi_n(x) = \frac{\pi_n(y(s))}{(y(s)-\beta_{-1})(y(s)-\beta_0)}\,\psi_{n-1}(y(s)), \tag{5.5}$$

where $\pi_n(y)$ is a polynomial of the second degree in y with the coefficients depending on n. The elementary rational functions $\phi_n(x)$ can be considered as a natural basis for construction of concrete examples of rational functions obeying nice transformation properties under the action of the operators $\mathcal{D}_{x(s)}$ and $\mathcal{M}_{x(s)}$. In particular, the latter operators can be used for building the second order difference equation (a generalized eigenvalue problem) for discrete biorthogonal rational functions introduced by the authors in [24] and their continuous measure extension [23]. Such a construction should match with the scheme considered in [18] (for the biorthogonal functions of [17] and [30]) and its generalization proposed recently in [19, 20]. We hope to address this correspondence elsewhere.

6 Admissible grids and the Poncelet-Chasles correspondence

In this section we demonstrate a nice correspondence between the T-algorithm for the grids $x(s)$ and $y(s)$ and the famous Poncelet problem for two conics. We start by recalling the Poncelet problem [4]. We take two arbitrary nondegenerate conics C and D and assume that C circumscribes D. We pick up an arbitrary point x_1 on the conic D and pass a tangent to C at this point. This tangent crosses the conic D at two points, say y_1 and y_2. Then we take the point y_2 on C and pass the second tangent to the conic D. We denote as x_2 the point on D where this tangent crosses D. This tangent meets the conic C in two points y_2 and y_3. Take the point y_3 and repeat this procedure. We then obtain two sets of points x_n and y_n on the conics D and C, respectively.

The first interesting problem is to describe these crossing points explicitly. It was solved by Jacobi and Chasles who showed that the sequences x_n and y_n can be parametrized by the elliptic functions. The second problem, the so-called Poncelet porism, consists in showing that if a particular trajectory on conics is closed (e.g., if $y_N = y_1$ for some $N > 2$) then this property does not depend on the choice of the initial point y_1 on the conic C. The modern treatment of this problem from the algebro-geometric point of view can be found, e.g., in [9].

In order to relate our grids $x(s)$ and $y(s)$ to the Poncelet problem, we introduce the standard rational parametrization of an arbitrary conic [4]. We assume that the conic D is described by the coordinates ξ, η of an affine plane, that is

$$A_1\xi^2 + A_2\eta^2 + A_3\xi\eta + A_4\xi + A_5\eta + A_6 = 0.$$

Then it is possible to find polynomials $E_0(x)$, $E_1(x)$, $E_2(x)$ with $\deg(E_i(x)) \le 2$ such that

$$\xi = \frac{E_1(x)}{E_0(x)}, \qquad \eta = \frac{E_2(x)}{E_0(x)}. \tag{6.1}$$

Quite analogously, the conic C can be parametrized as

$$\xi = \frac{F_1(y)}{F_0(y)}, \qquad \eta = \frac{F_2(y)}{F_0(y)}, \tag{6.2}$$

where $F_i(y)$ are some other polynomials of the degrees not exceeding 2. Thus values of the parameters x and y completely characterize points on the conics D and C, respectively. The tangent line L_1 to conics D at a point x_1 with the affine coordinates $\xi_1(x_1)$, $\eta_1(x_1)$ is described by the equation

$$(\xi - \xi_1)\frac{d\eta}{dx}\Big|_{x=x_1} = (\eta - \eta_1)\frac{d\xi}{dx}\Big|_{x=x_1}. \tag{6.3}$$

In order to find intersection points of the tangent line L_1 with C we should replace ξ, η in (6.3) by their expressions (6.2). Omitting elementary calculations, we present the final result: for a given point x_1 on the conic D, the points y_1 and y_2 of the intersection of the tangent L_1 with C are determined as two roots of the quadratic equation:

$$Z(x_1, y) = 0, \tag{6.4}$$

where $Z(x, y)$ has the form

$$Z(x, y) = F_0(y)M_0(x) + F_1(y)M_1(x) + F_2(y)M_2(x) \tag{6.5}$$

with the polynomials $M_i(y)$ defined as

$$M_i(x) = \epsilon_{ikl}(E_k'(x)E_l(x) - E_k(x)E_l'(x)), \quad i, k, l = 0, 1, 2. \tag{6.6}$$

Here the primes denote the derivative with respect to x and ϵ_{ikl} is the completely antisymmetric tensor. It is easily seen that $\deg(M_i(x)) \le 2$.

In the generic situation, $Z(x, y)$ is a polynomial of the exact degree 2 with respect to each variable. Indeed, for a nondegenerate situation a tangent to the conic D at a point x should intersect C in two distinct points, and, moreover, from a point y on C there are two distinct tangents to D. These arguments are sufficient to conclude that $Z(x, y)$ should be a polynomial of the degree 2 with respect to each variable. Conversely, for any polynomial $Z(x, y)$ of the degree 2 in x and y it is possible to find polynomials $F_i(y)$ and $E_i(x)$ such that relation (6.5) holds, and, hence, we can relate with any such curve $Z(x, y) = 0$ two conics C and D parametrized as in (6.2) and (6.1). Note that similar considerations were exploited in [9] in order to show that the tangent and intersection points belong to an elliptic curve. The authors of [9] also introduced two involutions I_1 and I_2, which are equivalent to our ones in the T-algorithm for the Euler curve. In our approach we derive the curve $Z(x, y) = 0$ explicitly, whereas in [9] all considerations are rather abstract.

Thus we can apply the T-algorithm in order to construct two sequences of points x_n and y_n on conics D and C. It is seen that these two sequences correspond to the admissible grids $x(s)$ and $y(s)$. So, we have an interesting geometric interpretation: the initial grid $x(s)$ corresponds to the tangent points x_n on the conic D whereas the adjacent grid $y(s)$ corresponds to the intersection points of the tangent lines with the conic C.

Both x_n and y_n have an elliptic parametrization of the form (4.1) (with the replacement $s \to n$). The parameter s_0 depends on a choice of the initial point on D. If the trajectory is closed after N steps, then $h = (k + l\tau)/N$ for some integers k and l (remind that 1 and τ are quasiperiods of the $\theta_1(s)$ function). This condition does not depend on the choice of the initial point s_0 and we obtain a very elementary explanation of the Poncelet's porism.

Periodic trajectories in the Poncelet problem correspond to finite grids $x(s)$ in our problem when only N points $x(1), x(2), \dots, x(N)$ of the grid $x(s)$ are distinct and $x(s + N) = x(s)$ (the same is valid for the grid $y(s)$). This means, in particular, that we should restrict ourselves to rational functions of the order $[n/n]$, $n \le N$, having no more than N distinct poles $\alpha_1, \alpha_2, \dots, \alpha_N$.

Since the Poncelet problem is invariant under the arbitrary projective transformation of the plane (ξ, η) we can reduce the conics C and D to some simple shapes. Some options are listed below.

(i) C and D are reduced to concentric quadrics determined by the equations

$$\xi^2/a_1 + \eta^2/b_1 = 1, \quad \xi^2/a_2 + \eta^2/b_2 = 1.$$

In this case we have the parametrization

$$\xi = 2a_1^{1/2} y/(1 + y^2), \quad \eta = b_1^{1/2}(1 - y^2)/(1 + y^2)$$

for the conic C and

$$\xi = 2a_2^{1/2} x/(1 + x^2), \quad \eta = b_2^{1/2}(1 - x^2)/(1 + x^2)$$

for the conic D. The polynomial $Z(x, y)$ defined by (6.5) is reduced in this case to the simplest Euler-Baxter form (4.3). Hence, we have a simple form solution of the Poncelet problem in this case:

$$x_n = \sqrt{k}\, \mathrm{sn}(h(n + s_0); k), \quad y_n = \sqrt{k}\, \mathrm{sn}(h(n + s_0 + 1/2); k). \tag{6.7}$$

Parameters a_i and b_i can still be specialized further. For instance, it is possible to choose $a_1 = b_1 = 1$ reducing C to the unit circle. In this case the points x_n are isomorphic to the godograph distribution of spins in the classical XY-chain [8]. Another possible choice $a_2 - a_1 = b_2 - b_1$ corresponds to the confocal quadrics. In this case the Poncelet problem is equivalent to the elliptic billiard [28].

(ii) It is possible to reduce the conics D and C to two parabolas in the euclidean plane (ξ, η). One of them, say D, can be fixed by the choice $\xi = x, \eta = x^2$, whereas another parabola remains arbitrary: $\xi = F_1(y)/F_0(y),\ \ \eta = F_2(y)/F_0(y)$, where $F_0(y) = (ay + b)^2$ is a square of the linear function (this is a characteristic property of any parabola). Then we get

$$Z(x, y) = x^2 F_0(y) - 2x F_1(y) + F_2(y). \tag{6.8}$$

Performing additional projective transformation of the variable y we can fix polynomials $F_1(y)$, $F_2(y)$, and $F_0(y)$ in such a way that the polynomial $Z(x, y)$ becomes symmetric in x, y. Then we can reduce $Z(x, y)$ to the form (4.7). In this case we obtain the grids parametrized by the Weierstrass function

$$x_n = A_1\wp(h(n + s_0)) + A_0, \quad y_n = A_1\wp(h(n + s_0 + 1/2)) + A_0.$$

(iii) Finally, we note that when $F_0 = const$ and $F_1(y)$ is a linear function in y then the parabola C has an asymptotic axis parallel to that of the parabola D. But then the curve $Z(x, y) = 0$ describes arbitrary conics in the coordinates x and y without x^2y^2, x^2y, and xy^2 terms. Hence in this special case of two parabolas with the parallel asymptotic axes we return to the Magnus treatment of the Askey-Wilson grids. This gives a geometric interpretation of the origin of orthogonal polynomials' admissible grids from the Poncelet problem point of view.

7 Padé interpolation on the elliptic grids

In this section we describe a practical application of the elliptic grids and corresponding biorthogonal rational functions in the approximation theory. A preliminary discussion of this subject was given in [26]. Here we outline a procedure allowing construction of explicit schemes for the Padé interpolation. We restrict ourselves to the announcement of results the detailed derivation of which will be given separately [34].

The Cauchy-Jacobi interpolation problem (CJIP) for a sequence Y_j of complex numbers can be formulated as follows [2, 15]. Given two nonnegative integers n and m, we fix a system of (distinct) points $x_j,\ j = 0, 1, \dots, n + m$, on the complex plane. Then we seek for polynomials $Q_m(x; n)$ and $P_n(x; m)$ of the degrees m and n, respectively, such that

$$Y_j = \frac{Q_m(x_j; n)}{P_n(x_j; m)}, \quad j = 0, 1, \dots, n + m. \tag{7.1}$$

In this notation the polynomial $Q_m(x; n)$, being of the degree m in x, depends on n as a parameter (and similarly for $P_n(x_j; m)$). The CJIP is called *normal* if the polynomials $Q_m(x; n)$ and $P_n(x; m)$ exist for all values of $m, n = 0, 1, \dots$, and they have no common zeros. This means, in particular, that the polynomials $Q_m(x; n)$ and $P_n(x; m)$ have no zeros coinciding with the interpolatory points, that is

$$Q_m(x_j; n) \neq 0, \quad P_n(x_j; m) \neq 0, \quad j = 0, 1, \dots, n + m. \tag{7.2}$$

In a special case when there exists an analytic function $f(z)$, $z \in \mathbb{C}$, such that $f(x_j) = Y_j$, the corresponding CJIP is called the multipoint Padé approximation problem [2].

There is a modification of CJIP with the prescribed poles and zeros [27]. We take two sequences a_i and b_i and assume that $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$ and, moreover, $a_i \neq b_j$ for all i and j. For any integer $n = 0, 1, \dots$, we introduce the n-th degree polynomials $A_n(x) = (x - a_1)\cdots(x - a_n)$ and $B_n(x) = (x - b_1)\cdots(x - b_n)$ (it is assumed that $A_0 = B_0 = 1$). Then we again seek for polynomials $Q_m(x;n)$ and $P_n(x;m)$ such that

$$Y_j = \frac{A_n(x_j)}{B_m(x_j)} \frac{Q_m(x_j;n)}{P_n(x_j;m)}, \quad j = 0, 1, \dots, n+m. \tag{7.3}$$

In this case we extract explicitly the part $A_n(x)$ with the prescribed zeros $a_1, a_2, \dots, a_n$ and the part $B_m(x)$ with the prescribed poles $b_1, b_2, \dots, b_m$. Equivalently, conditions (7.3) can be rewritten in the form

$$Y_j = \frac{V_m(x_j;n)}{U_n(x_j;m)}, \quad j = 0, 1, \dots, n+m, \tag{7.4}$$

where

$$V_m(x) = \frac{Q_m(x;n)}{B_m(x)}, \qquad U_n(x) = \frac{P_n(x)}{A_n(x)} \tag{7.5}$$

are rational functions with the prescribed poles. Thus, the scheme with prescribed poles and zeros is obtained from the ordinary CJIP by changing polynomials $Q_m(x;n)$ and $P_n(x;m)$ to rational functions $V_m(x;n)$ and $U_n(x;m)$.

We recall that the standard *Padé approximation* [2] problem consists in finding polynomials $Q_m(x;n)$ and $P_n(x;m)$ such that for a given function $f(x)$ (which is assumed to be analytical near the point $x = 0$) we have the condition

$$f(x) - \frac{Q_m(x;n)}{P_n(x;m)} = O(x^{n+m+1}). \tag{7.6}$$

It is clear that the ordinary Padé problem can be obtained by the limiting process $x_j \to 0$ for all j. In this respect, CJIP can be considered as a generalization of the ordinary Padé problem.

As far as we know, no explicit Padé interpolation tables (for both ordinary and CJIP with prescribed poles and zeros) were suggested in the literature except of the case of exponential function on the uniform grid described by Iserles [11]. We present new examples of such Padé interpolation tables. These examples are connected to the discrete Askey-Wilson (or q-Racah) polynomials [1] for the ordinary CJIP and to the elliptic biorthogonal rational functions for the scheme with prescribed poles and zeros.

First we consider the ordinary scheme. As shown in [5], for $n + m = N$, where N is a positive integer, the denominator polynomials $P_n(x; N - n)$ satisfy the three term recurrence relation

$$P_{n+1}(x, N - n - 1) + (\xi_n x + \eta_n) P_n(x; N - n) + \zeta_n P_{n-1}(x; N - n + 1) = 0 \tag{7.7}$$

and, hence, the set $P_0(x;N)$, $P_1(x;N-1),\ldots,P_N(x;0)$ can be considered as a finite set of orthogonal polynomials satisfying a discrete orthogonality relation on the interpolation grid x_s:

$$\sum_{s=0}^{N} w_s P_n(x_s;N-n)P_{n'}(x_s;N-n') = h_n\delta_{nn'}, \tag{7.8}$$

where the weights are

$$w_s = \frac{Y_s}{\omega'_N(x_s)} \tag{7.9}$$

with $\omega_N(x) = (x-x_0)(x-x_1)\cdots(x-x_N)$ and $\omega'_N(x) = d\omega(x)/dx$.

It is thus natural to expect that the families of explicitly known finite orthogonal polynomials give rise to some Padé interpolation schemes. The most general explicit discrete polynomials (the q-Racah polynomials) were discovered by Askey and Wilson [1].

We adopt the standard notation [7, 12]

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$$

for the q-shifted factorial and for the basic hypergeometric function ${}_4\Phi_3$. We fix three arbitrary parameters α, e, f and introduce the interpolation sequence

$$Y_j = \left(\frac{ef}{\alpha q}\right)^j \frac{(q\alpha/e;q)_j(q\alpha/f;q)_j}{(e;q)_j(f;q)_j} \tag{7.10}$$

and the interpolation grid $x_j = q^{-j} + \alpha q^j,\ j = 0, 1, \ldots$. Then the CJIP is fulfilled for all m and n with the polynomials

$$Q_m(x;n) = {}_4\Phi_3\left(\begin{matrix} q^{-m}, q^{-t}, \alpha q^t, efq^{-n-1}/\alpha \\ q^{-n-m}, e, f \end{matrix}; q, q\right) \tag{7.11}$$

and

$$P_n(x;m) = {}_4\Phi_3\left(\begin{matrix} q^{-n}, q^{-t}, \alpha q^t, \alpha q^{1-m}/ef \\ q^{-n-m}, q\alpha/e, q\alpha/f \end{matrix}; q, q\right), \tag{7.12}$$

where the argument x depends on the parameter t as $x(t) = q^{-t} + \alpha q^t$. Note that for $t = j$ the argument $x(t)$ takes the value from the interpolation grid: $x(j) = x_j$.

By specialization and limiting processes one can obtain all possible cases of the finite orthogonal polynomials from the Askey table [12], in particular, the Racah, Hahn, Krawtchouk, etc polynomials. This gives the corresponding Padé interpolating tables. For example, the Krawtchouk polynomials generate the Padé interpolation table for the exponential function on the uniform grid whereas the q-Krawtchouk polynomials generate the Padé interpolation table for the power function on the exponential grid (these cases were considered in [32]).

We describe now the scheme with the prescribed poles and zeros. As shown in [26], for $n+m=N$, where N is a positive integer, the denominator rational functions $U_n(x;m)$ satisfy the recurrence relation

$$(x-a_{n+1})U_{n+1}(x;m)+(\xi_n x+\eta_n)U_n(x;m)+\zeta_n(x-b_{m+1})U_{n-1}(x;m)=0, \tag{7.13}$$

which can be represented in the form of a generalized eigenvalue problem

$$J_1\vec{U}=xJ_2\vec{U}, \tag{7.14}$$

where J_1 and J_2 are two Jacobi (i.e., tridiagonal) matrices and $\vec{U}$ is an $(N+1)$-dimensional vector with the components $(U_0(x;N), U_1(x;N-1), \dots, U_N(x;0))$. A similar recurrence relation holds for the numerator functions $V_m(x;n)$.

It was shown in [31] that the generalized eigenvalue problem (7.14) with two arbitrary Jacobi matrices J_1 and J_2 generates two families of biorthogonal rational functions $U_n(x)$ and $T_n(x)$ with the property

$$\langle\sigma, U_n(x)T_{n'}(x)\rangle=h_n\delta_{nm'} \tag{7.15}$$

for some functional σ defined in the space of rational functions with poles a_i and b_k, $i,k=0,1,\dots,N$. In our case this biorthogonality relation can be represented explicitly as

$$\sum_{s=0}^{N} w_s U_n(x_s)T_{n'}(x_s)=h_n\delta_{nn'}, \tag{7.16}$$

where the discrete weights w_s have the same form (7.9) as in the ordinary polynomials case. The rational functions $T_n(x)$ are defined as

$$T_n(x)=\frac{P_n^*(x)}{(x-b_{N-n})(x-b_{N-n+1})\cdots(x-b_N)}$$

and the polynomials $P_n^*(x)$ are expressed as

$$P_n^*(x)=\frac{P_{n+1}(x;N-n-1)+\epsilon_n(x-b_{N-n})P_n(x;N-n)}{x-a_n}, \tag{7.17}$$

where the coefficients ϵ_n are uniquely defined from the fact that the right-hand side of (7.17) has no pole at $x=a_n$.

Thus it is natural to expect that the discrete elliptic biorthogonal rational functions discovered in [24, 25] give rise to some Padé interpolation scheme with the prescribed poles and zeros.

We need the very-well-poised elliptic hypergeometric series [6] which, in the notation of [22, 26], have the form

$$ {}_{r+1}E_r = \sum_{n=0}^{\infty} \frac{[u_1+2n][u_1]_n}{[u_1][n]!} \prod_{m=1}^{r-4} \frac{[u_{3+m}]_n}{[u_1+1-u_{3+m}]_n}, \tag{7.18} $$

where $\sum_{m=1}^{r-4} u_{3+m} = u_1(r-5)/2 + (r-7)/2$. The elliptic shifted factorials are defined as follows

$$ [u]_n \equiv [u][u+1]\cdots[u+n-1], \quad [n]! = [1]_n, $$

and $[u] \equiv \theta_1(hu)/\theta_1(h)$, where $\theta_1(x)$ is the standard Jacobi θ_1-function with the quasiperiods 1, τ and h is an arbitrary parameter.

We introduce the interpolation sequence

$$ Y_s = \frac{[1+\alpha-\beta]_s[1-\epsilon+\beta]_s[\delta-\alpha]_s[\epsilon+\delta]_s}{[1+\alpha]_s[1-\epsilon]_s[\beta+\delta-\alpha]_s[\epsilon+\delta-\beta]_s}, \tag{7.19} $$

where $\alpha, \beta, \epsilon, \delta$ are arbitrary parameters. Then we have the solution of the Padé interpolation problem (7.5), where both rational functions $V_m(x;n)$ and $U_n(x;m)$ are expressed as elliptic hypergeometric functions ${}_{12}E_{11}$ with the parameters:

$$ u_1 = \alpha - \beta, \quad u_4 = -m, \quad u_5 = -\beta - n, \quad u_6 = \alpha - \beta + 1 + m + n, $$
$$ u_7 = \epsilon + \alpha - \beta, \quad u_8 = 1 + \alpha - \epsilon - \delta, \quad u_9 = -t, \quad u_{10} = \delta + t $$

for the numerator functions $V_m(x(t);n)$ and

$$ u_1 = \alpha, \quad u_4 = -n, \quad u_5 = \beta - m, \quad u_6 = \alpha + 1 + m + n, \quad u_7 = \epsilon + \alpha - \beta, $$
$$ u_8 = 1 + \alpha - \epsilon - \delta, \quad u_9 = -t, \quad u_{10} = \delta + t $$

for the denominator functions $U_n(x(t);m)$.

The argument $x(t)$ is parametrized as

$$ x(t) = \frac{[t-\xi_1][t-\xi_2]}{[t-\eta_1][t-\eta_2]}, $$

where ξ_i, η_i are arbitrary parameters satisfying restrictions $\xi_1 + \xi_2 = \eta_1 + \eta_2 = -\delta$. The interpolation grid x_s is found to be

$$ x_s = x(s), \quad s = 0, 1, \ldots. \tag{7.20} $$

The prescribed zeros are $a_j = x(-\alpha - j)$ and the prescribed poles are $b_j = x(\beta - \alpha - j)$.

As expected, for any fixed positive integer $N > 1$ both $U_n(x; N-n)$ and $V_{N-n}(x;n)$ coincide with the biorthogonal functions on the elliptic grids of [24–26] (see also [21] for a different approach to building these functions). For further details on construction of different kinds of biorthogonal rational functions on elliptic grids from the Padé interpolation scheme see [34].

By limiting procedures it is possible to obtain simpler examples of the Padé interpolation tables with the prescribed poles and zeros. We mention here only one particular example: the interpolation of a ratio of two gamma functions. We take

$$f(x) = \frac{\Gamma(\alpha - \beta - x)}{\Gamma(\alpha - x)}$$

as an interpolated function of the complex variable x with arbitrary parameters α and β. We introduce the rational functions

$$V_m(x;n) = {}_3F_2\left(\begin{matrix} -m, -x, -\beta - n \\ -n - m, \alpha - \beta - x \end{matrix}; 1 \right)$$

and

$$U_n(x;m) = {}_3F_2\left(\begin{matrix} -n, -x, \beta - m \\ -n - m, \alpha - x \end{matrix}; 1 \right).$$

Then we have the solution of the Padé interpolation problem

$$f(s) = \kappa \frac{V_m(s;n)}{U_n(s;m)}, \quad s = 0, 1, \ldots, n + m,$$

for all possible entries $n, m = 0, 1, \ldots,$ of the Padé table, where $\kappa = \Gamma(\alpha - \beta)/\Gamma(\alpha)$. In this special case the grid is uniform: $x_s = s = 0, 1, \ldots$. The prescribed poles in this case are $b_j = \alpha - \beta + j - 1,\ j = 1, 2, \ldots, n,$ and the prescribed zeroes are $a_j = \alpha + j - 1,\ j = 1, 2, \ldots, m.$

We note finally that the Padé interpolation scheme with the prescribed poles and zeros leads to the generalized Frobenius relations between adjacent interpolants $P_n(x;m)$ (i.e., polynomials with $n \to n \pm 1, m \to m \pm 1$):

$$A_{nm} P_{n+1}(x;m) + B_{nm}(x - b_{m+1}) P_n(x;m+1) = (x - x_{n+m+1}) P_n(x;m) \quad (7.21)$$

and

$$C_{nm} P_{n+1}(x;m) + D_{nm}(x - a_{n+1}) P_n(x;m+1) = P_{n+1}(x;m+1) \quad (7.22)$$

with some coefficients $A_{nm}, B_{nm}, C_{nm}, D_{nm}$. The compatibility conditions between relations (7.21) and (7.22) result in the discrete time chain (the R_{II} chain) for these coefficients which was described in [24]. Thus the R_{II} chain arises naturally from the Padé interpolation scheme with the prescribed poles and zeros.

As a final remark, we mention that already for the simplest case of the Lagrange polynomial interpolation scheme there is a non-trivial biorthogonality relation for the corresponding interpolation polynomials [33].

8 Conclusion

We have derived explicit forms of the admissible grids $x(s)$ and $y(s)$ for which there exists a divided difference operator mapping rational functions of $x(s)$ of the order $[n/n]$ with the poles $\alpha_1, \dots, \alpha_n$ to rational functions of $y(s)$ of the order $[n-1/n-1]$ with the poles $\beta_1, \dots, \beta_{n-1}$. We have shown that these grids are in one-to-one correspondence with sequences of the Poncelet problem points on two conics. The admissible poles α_n and β_n also correspond to the solution of the Poncelet problem for the same conics but with the shifted initial point. The points $x(s)$ and α_n lie on one conic and generate tangents to this conic. The points $y(s)$ and β_n lie on another conic and appear from intersections of the mentioned tangents with it. Using projective transformations the grids $x(s)$ and $y(s)$ can be made equivalent to the trajectories of an elliptic billiard or to the godograph of the classical XY-chain. The Askey-Wilson grids emerge from a particular degeneration of these conics. We outlined also some explicit schemes of the Padé interpolation with the prescribed poles and zeros on the elliptic grids.

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