Rational interpolation to solutions of Riccati difference equations on elliptic lattices.

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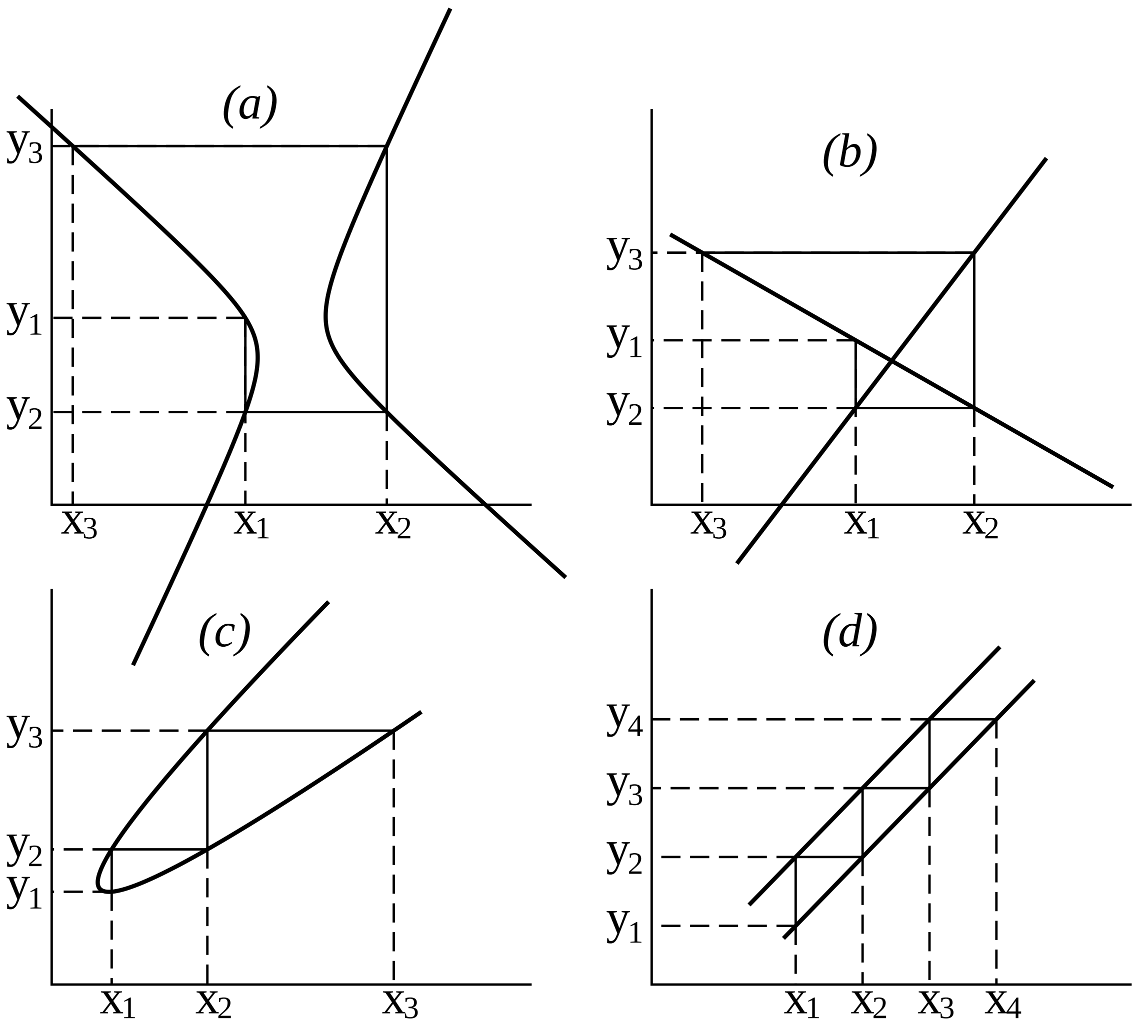
(a)
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Rational interpolation to solutions of Riccati difference equations on elliptic lattices.

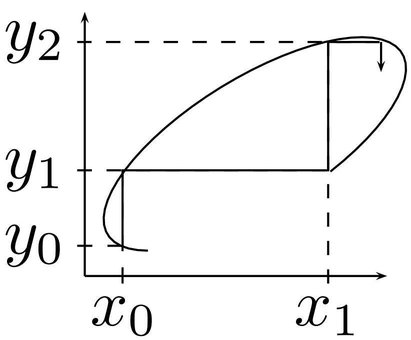
The new new new difference calculus.

Alphonse Magnus, Université Catholique de Louvain.

. . . opresivo y lento y plural.
J.L. Borges

Difference equations and lattices.

$$(\mathcal{D}f)(x) = \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)}, \tag{1}$$



The simplest choice for φ and ψ is to take the two determinations of an algebraic function of degree 2, i.e., the two $y-$roots of

$$F(x, y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \tag{2a}$$

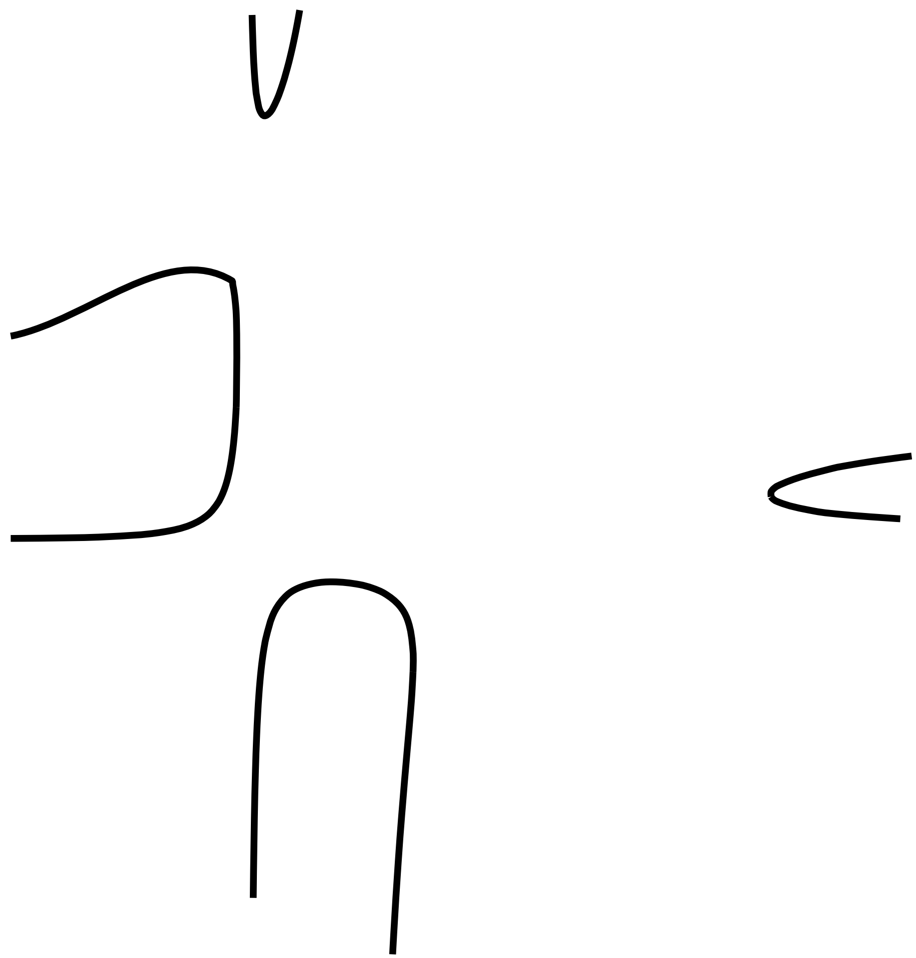
where X_0, X_1, and X_2 are rational functions.

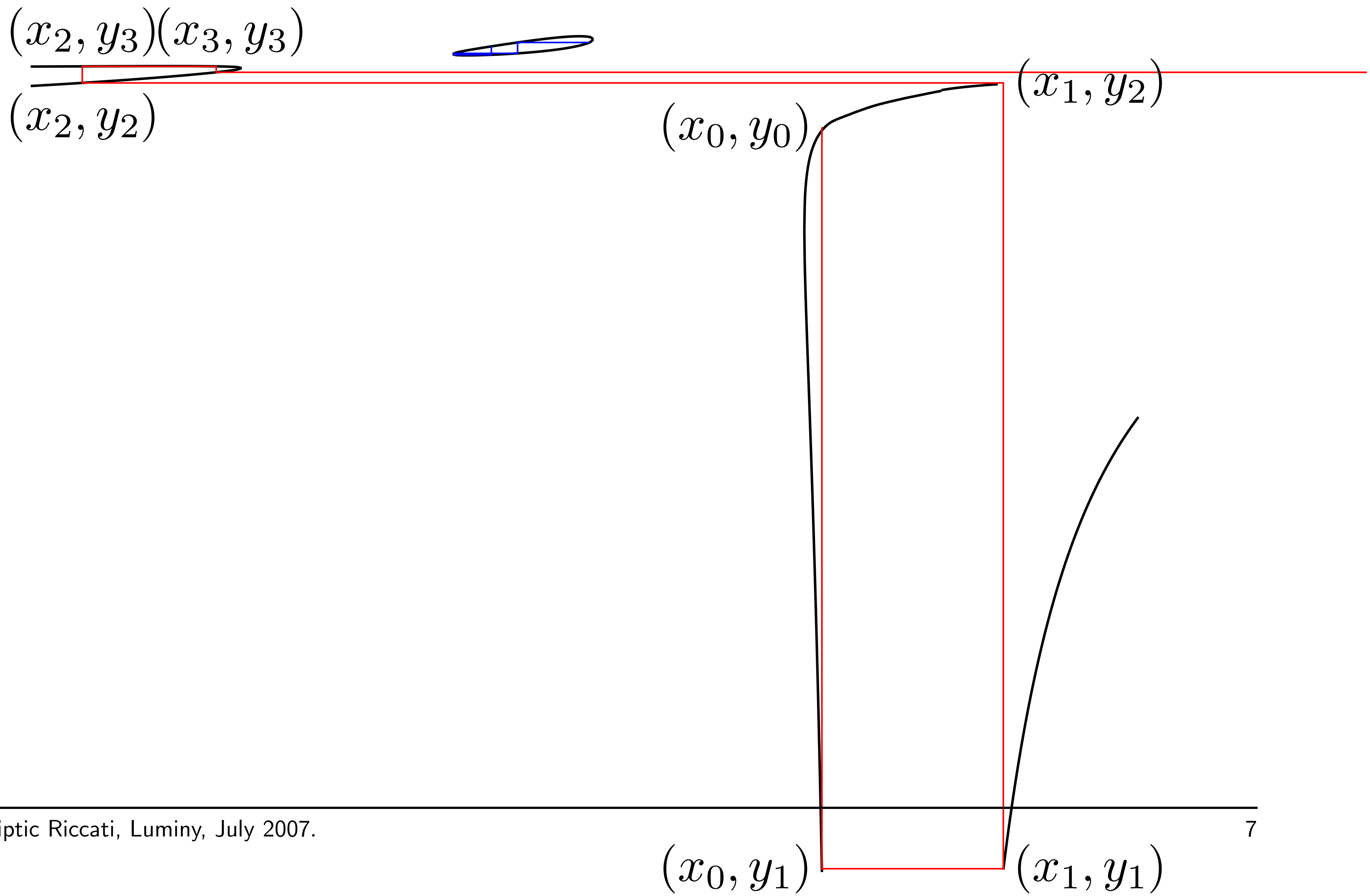
The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where y_1 is the second root of (2a) at x_0. We need x_1 such that y_1 is one of the two roots of (2a) at x_1, so for one of the roots of $F(x, y_1) = 0$ which is not x_0. Here again, the simplest case is when F is of degree 2 in x:

$$F(x,y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \tag{2b}$$

Both forms (2a) and (2b) hold simultaneously when F is ***biquadratic***:

$$F(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}x^i y^j. \tag{3}$$




(x_2, y_3)
(x_3, y_3)
(x_1, y_2)
(x_2, y_2)
(x_0, y_0)
(x_0, y_1)
(x_1, y_1)

Pons asinorum

Find f such that $\mathcal{D}f(x) = x$.

i.e., $$\frac{f\left(\dfrac{Q(x)+\sqrt{P(x)}}{R(x)}\right) - f\left(\dfrac{Q(x)-\sqrt{P(x)}}{R(x)}\right)}{2\sqrt{P(x)}/R(x)} = x,$$

where P, Q, R are given polynomials of degrees $4, 2$, and 2!

Why elliptic?

-Orthogonal polynomials on several intervals,
-contined fraction continuation,
-band structure in solid-state physics

Let S be a polynomial of degree 2 and we consider the root of

$$\zeta_0(z-z_0)(z-x_0)f^2(z)-2S(z)f(z)+\zeta_1(z-z_0)(z-x_1)=0 \quad (4)$$

which is regular at z_0. It is also

$$f(z) = \frac{S(z) - \sqrt{P(z)}}{\zeta_0(z - z_0)(z - x_0)}.$$

$$f(z) = \cfrac{z - z_0}{\alpha_0 z + \beta_0 - \cfrac{(z - z_0)^2}{\alpha_1 z + \beta_1 - \cfrac{(z - z_0)^2}{\alpha_2 z + \beta_2 - \cdots}}}, \qquad (5)$$

or $f_n(z) = \dfrac{z - z_0}{\alpha_n z + \beta_n - (z - z_0) f_{n+1}(z)}\,, \quad n = 0, 1, \ldots \qquad (6)$

with $f_0 = f$.

The form of f is kept in all the f_n's (basically from Perron):

$$\zeta_n(z-z_0)(z-x_n)f_n^2(z)-2S_n(z)f_n(x)+\zeta_{n+1}(z-z_0)(z-x_{n+1})=0, \tag{7}$$

and we have the

Proposition *The continued fraction expansion* (5) *of the quadratic function* f *defined by* (4) *involves a sequence of quadratic functions defined by* (7). *The related sequence* $\{x_n\}$ *is an elliptic sequence.*

One has

$$2n\int_{z_0}^{z_1}\frac{dt}{\sqrt{P(t)}}=\int_{z_1}^{x_0}\frac{dt}{\sqrt{P(t)}}\pm\int_{z_0}^{x_n}\frac{dt}{\sqrt{P(t)}}+2\sum_j N_j\int_{z_1}^{z_{j+1}}\frac{dt}{\sqrt{P(t)}}, \tag{8}$$

Elliptic functions, at last:

$$\theta(u;p)=\prod_{j=0}^{\infty}(1-p^j u)(1-p^{j+1}/u),$$

$$x_n=C\frac{\theta(q^n q_0\eta_0)\theta(q^n q_0/\eta_0)}{\theta(q^n q_0\eta_\infty)\theta(q^n q_0/\eta_\infty)}$$

What's going on here.

A special (bi-)orthogonal set of functions is known if one has

1. a formula for the scalar product, measure, weight, Stieltjes transform
2. a formula, or at least a recurrence relation formula, for the (bi-)orthogonal functions
3. difference relations and equations
4. hypergeometric expansions

Elliptic Pearson's equation.

A famous theorem by Pearson relates the classical orthogonal polynomials to the differential equation $w' = rw$ satisfied by the weight function, where r is a rational function of degree $\leqslant 2$.

Theorem. *Let* $\{(x(s_0+k), y(s_0+k))\}$ *be an elliptic lattice built on the biquadratic curve* (2a)-(2b)-(3). *If there are polynomials* a *and* c*, with*
$a(x(s_0)) + (y(s_0+1) - y(s_0))c(x(s_0)) = 0,$
$a(x(s_0+N)) - (y(s_0+N+1) - y(s_0+N))c(x(s_0+N)) = 0,$

such that

$$a(x'_k)\frac{\dfrac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)}-\dfrac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}}{y'_{k+1}-y'_k}$$

$$=c(x'_k)\left[\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)}+\frac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}\right],\quad (9)$$

$k = 0, 1, \ldots, N$, *where* (x'_k, y'_k) *is a shorthand for* $(x(s_0 +$

$k), y(s_0 + k))$, *and* $w_0 = w_{N+1} = 0$, *then,*

$$f(x) = \sum_{k=1}^{N} \frac{w_k}{x - y'_k} \tag{10}$$

satisfies

$$a(x)\mathcal{D}f(x) = a(x)\frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)} = c(x)[f(\varphi(x))+f(\psi(x))]+d(x) \tag{11}$$

where d *is a polynomial too.*

"Elliptic logarithm"

We extend $f(x) = \log \dfrac{x-a}{x-b}$ which satisfies $f'(x) = \dfrac{a-b}{(x-a)(x-b)}$ by looking for a function whose divided difference is a rational function of low degree.

Answer: $w_k = (x'_k - x'_{k-1}) Y_2(y'_k)$,

$$\mathcal{D}f(x) = \frac{(x'_N - x'_0) X_2(x)}{(x - x'_0)(x - x'_N)}.$$

Orthogonality and biorthogonality.

Padé expansion to f about z_0: denominators of

$$\frac{q_n(x)}{p_n(x)} = \alpha_0' + \cfrac{x - z_0}{\alpha_0 x + \beta_0 - \cfrac{(x-z_0)^2}{\cfrac{\ddots}{\alpha_{n-2}x + \beta_{n-2} + \cfrac{(x-z_0)^2}{\alpha_{n-1}x + \beta_{n-1}}}}}$$

satisfy $p_{n+1}(x) = (\alpha_n x + \beta_n)p_n(x) - (x - z_0)^2 p_{n-1}(x)$, asks for $f(z_0), f'(z_0), f''(z_0), \ldots$ available if f is solution to a differential equation.

Recurrences of biorthogonal rational functions.

$$\frac{q_n(x)}{p_n(x)} = \alpha_0' + \cfrac{x - y_0}{\alpha_0 x + \beta_0 - \cfrac{(x-y_1)(x-y_2)}{\cfrac{\ddots}{\alpha_{n-2}x + \beta_{n-2} + \cfrac{(x-y_{2n-3})(x-y_{2n-2})}{\alpha_{n-1}x + \beta_{n-1}}}}}$$

which agree with a given set up to $x = y_{2n}$.

The recurrence relations for p_n and q_n are

$$p_{n+1}(x) = (\alpha_n x + \beta_n)p_n(x) - (x - y_{2n-1})(x - y_{2n})p_{n-1}(x),$$
$$q_{n+1}(x) = (\alpha_n x + \beta_n)q_n(x) - (x - y_{2n-1})(x - y_{2n})q_{n-1}(x),$$
with $q_0 = \alpha_0'$, $p_0 = 1$, $q_1(x) = \alpha_0'(\alpha_0 x + \beta_0) + x - y_0$, $p_1(x) = \alpha_0 x + \beta_0$. We could as well start with $q_{-1}(x) = -1/(x - y_{-1})$ and $p_{-1} = 0$.

Consider now rational functions $R_n(x) = \dfrac{p_n(x)}{(x - y_2)(x - y_4)\cdots(x - y_{2n})}$

$$(x - y_{2n+2})R_{n+1}(x) = (\alpha_n x + \beta_n)R_n(x) - (x - y_{2n-1})R_{n-1}(x),$$

$$(x - y_{2n+1})T_{n+1}(x) = (\alpha_n x + \beta_n)T_n(x) - (x - y_{2n})T_{n-1}(x),$$

which is of the same structure that the recurrence of the R_n's, but with the odd x's interchanged with the even x's. Actually, $T_n(x)$ is a constant times the same $p_n(x)$ as before, divided by $(x - y_1)(x - y_3)\dots(x - y_{2n-1})$.

Let q_n/p_n interpolate a formal Stieltjes transform-like function

$$f(x) = \int_S \frac{d\mu(t)}{x - t},$$

then q_n interpolates $p_n f$ at the $2n+1$ points y_0, $y_1, \ldots, y_{2n}$. Also, for $k < n$, $\tilde{q}(x) = q_n(x)p_k(x)(x - y_{2k+3})(x - y_{2k+5}) \cdots (x - y_{2n-1})$, still of degree $< 2n$, interpolates $p_n(x)p_k(x))(x - y_{2k+3})(x - y_{2k+5}) \cdots (x - y_{2n-1})f(x)$, still has a vanishing divided difference at these $2n+1$ points:

$$[y_0, \ldots, y_{2n}] \text{ of } p_n(x)p_k(x))(x-y_{2k+3})(x-y_{2k+5}) \cdots (x-y_{2n-1})f(x)$$

$$= \int_S \frac{p_n(t)p_k(t))(t - y_{2k+3})(t - y_{2k+5}) \cdots (t - y_{2n-1})\ d\mu(t)}{(t-y_0)(t-y_1)\cdots(t-y_{2n})} = 0,$$

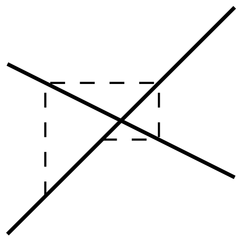
as the divided difference of a rational function $A(x)/(x-t)$

is $A(t)/\{(t-y_0)(t-y_1)\cdots(t-y_{2n})\}$ (Milne-Thomson § 1.7).

So, R_n is orthogonal to T_k with respect to the formal scalar product

$$\langle g_1, g_2\rangle = \int_S g_1(t)g_2(t)\, d\mu(t).$$

Where are the orthogonal polynomials on elliptic lattices?



Interpolation $=$ Padé if $z_0 = x_\infty$ is the limit of the x_n's.

A generic elliptic lattice has no convergence point.

(However. . .)

Elliptic Riccati equations.

Definition

An elliptic Riccati equation is

$$a(x)\frac{f(\psi(x))-f(\varphi(x))}{\psi(x)-\varphi(x)} = b(x)f(\varphi(x))f(\psi(x)) + \\ c(x)(f(\varphi(x))+f(\psi(x))) + d(x). \tag{12}$$

If $x = x_m$, some point of our $x-$lattice, then $\varphi(x) = y_m$ and $\psi(x) = y_{m+1}$.

A first-order difference equation of the kind (13) relates $f(y_0)$ to $f(y_1)$ when $x = x_0$; $f(y_1)$ to $f(y_2)$ when $x = x_1$, etc. The direct relation is

$$f(\psi) = \frac{\left[\dfrac{a}{\psi - \varphi} + c\right] f(\varphi) + d}{\dfrac{a}{\psi - \varphi} - c - bf(\varphi)}.$$

It is sometimes easier to write (13) as

$$e(x)f(\varphi(x))f(\psi(x)) + g(x)f(\varphi(x)) + h(x)f(\psi(x)) + k(x) = 0,$$

where $e = -b$, $g = -\dfrac{a}{\psi - \varphi} - c$, $h = \dfrac{a}{\psi - \varphi} - c$, and $k = -d$.

However, if a, b, c, and d are rational functions, g and h are conjugate algebraic functions: $h + g$ and hg are symmetric functions of φ and ψ, hence rational functions. This also happens with $2a = (h - g)(\psi - \varphi)$.

Now, if f_n satisfies the Riccati equation

$$a_n(x)\frac{f_n(\psi(x)) - f_n(\varphi(x))}{\psi(x) - \varphi(x)} = b_n(x) f(\varphi(x)) f_n(\psi(x)) \\ + c_n(x)(f(\varphi(x)) + f(\psi(x))) + d_n(x), \tag{13}$$

Theorem.

If f_n satisfies the Riccati equation (14) with rational coefficients a_n, b_n, c_n, and d_n, and if $f_n(x) = \dfrac{x - y_{2n}}{\alpha_n x + \beta_n - (x - y_{2n+1}) f_{n+1}(x)}$, then f_{n+1} satisfies an equation with coefficients a_{n+1} etc. of same complexity (degree of the rational functions).

$$a_{n+1} = \frac{2X_0 + (y_{2n} + y_{2n+1})X_1 + 2y_{2n}y_{2n+1}X_2}{2F(x, y_{2n})} a_n$$

$$+(X_1^2 - 4X_0X_2)\frac{[(y_{2n+1} - y_{2n})c_n + 2(\alpha_n y_{2n+1} + \beta_n)d_n]/X_2}{2F(x, y_{2n})}. \tag{14}$$

$$c_{n+1} = -(h_{n+1} + g_{n+1})/2 = -\frac{2X_0 + (y_{2n} + y_{2n+1})X_1 + 2y_{2n}y_{2n+1}X}{2F(x, y_{2n})}$$

$$+\frac{y_{2n+1} - y_{2n}}{2F(x, y_{2n})} X_2 a_n - \frac{\alpha_n(2X_0 + y_{2n+1}X_1) - \beta_n(X_1 + 2y_{2n+1}X_2)}{2F(x, y_{2n})} d_n$$

$$b_{n+1} = \frac{(\varphi - y_{2n+1})(\psi - y_{2n+1})}{(\varphi - y_{2n})(\psi - y_{2n})} d_n = \frac{F(x, y_{2n+1})}{F(x, y_{2n}} d_n,$$

$$d_{n+1} = -\frac{\alpha_n y_{2n} + \beta_n}{F(x, y_{2n})} X_2 a_n + b_n$$

$$+\frac{[\alpha_n(2X_0 + y_{2n}X_1) - \beta_n(X_1 + 2y_{2n}X_2)]c_n + (\alpha_n^2 X_0 - \alpha_n\beta_n X_1 + \beta}{F(x, y_{2n})}$$

$$\frac{c_n^2(x) - b_n(x)d_n(x)}{X_2^2(x)} P(x) - a_n^2(x) = C_n \frac{x - x_{2n-1}}{x - x_{-1}} \left[\frac{c_0^2(x) - b_0(x)d_0(x)}{X_2^2(x)} \right. \tag{15}$$

where $C_n = \dfrac{Y_2(y_{2n-1})Y_2(y_{2n-3})\cdots Y_2(y_1)}{Y_2(y_{2n-2})Y_2(y_{2n-4})\cdots Y_2(y_0)}$.

Classical case.

We keep the lowest possible degree, which is 3, considering that b_n and d_n must be $X_2(x)$ times a polynomial containing the factor $x - x_{2n-1}$.

Let $d_n(x) = \zeta_n(x - x_{2n-1})X_2(x)$, a_n of degree 3, and $c_n = X_2$ times a polynomial of degree 1.

$$b_{n+1}(x) = \frac{F(x, y_{2n+1})}{F(x, y_{2n})}\zeta_n(x - x_{2n-1})X_2(x) = \frac{Y_2(y_{2n+1})}{Y_2(y_{2n})}\zeta_n(x - x_{2n+1})$$

From the Riccati equation (14) at $x = x_{2n-1}$ and $f_n(y_{2n}) = 0$, we have

$$\frac{a_n(x_{2n-1})}{y_{2n} - y_{2n-1}} = c_n(x_{2n-1}),$$

allowing the divison of the left-hand side of (16), leaving

$$\frac{c_n^2(x) - b_n(x)d_n(x)}{X_2^2(x)}P(x) - a_n^2(x) = C_n(x - x_{2n-1})Q(x),$$

where Q is a fixed polynomial of degree 5.

At each of the four zeros $z_1, \ldots, z_4$ of P,

$$a_n(z_j) = \pm\sqrt{-C_n(z_j - x_{2n-1})Q(z_j)},$$

allowing to recover the third degree polynomial a_n from four values. . . should the square roots be determined! Square root-free relations come from (15) at z_j, knowing that $\varphi(z_j) = \psi(z_j)$, which we call φ_j:

$$a_{n+1}(z_j) = \frac{\varphi_j - y_{2n+1}}{\varphi_j - y_{2n}} a_n(z_j)$$

Remark that, from (16), $Q(z_j) = -a_0^2(z_j)/(z_j - x_{-1})$, so there is a subtle relation between the product of the $(\varphi_j - y_{2n+1})/(\varphi_j - y_{2n})$'s and a square root of $(z_j - x_{2n-1})/(z_j - x_{-1})$.

Linear difference relations and equations for the numerators and the denominators of the interpolants.

$$\frac{g_0(x_m)}{e_{n+1}(x_m)}\frac{p_n(y_{m+1})}{(y_{m+1}-y_0)(y_{m+1}-y_2)\cdots(y_{m+1}-y_{2n})}$$

$$-\frac{h_0(x_{m-1})}{e_{n+1}(x_{m-1})}\frac{p_n(y_{m-1})}{(y_{m-1}-y_0)(y_{m-1}-y_2)\cdots(y_{m-1}-y_{2n})}$$

$$=\left[\frac{h_{n+1}(x_m)}{e_{n+1}(x_m)}-\frac{g_{n+1}(x_{m-1})}{e_{n+1}(x_{m-1})}\right]\frac{p_n(y_m)}{(y_m-y_1)(y_m-y_3)\cdots(y_m-y_{2n-1})},$$

Hypergeometric expansions.

Building blocks:

$$\mathcal{D}\frac{(x-y_0)(x-y_1)\cdots(x-y_{n-1})}{(x-y_1')(x-y_2')\cdots(x-y_n')}$$

$$= C_n X_2(x)\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-2})}{(x-x_0')(x-x_1')\cdots(x-x_n')}.$$

(Zhedanov)

Recommended reading.

V.P. Spiridonov, Elliptic hypergeometric functions, Abstract: This is a brief overview of the status of the theory of elliptic hypergeometric functions to the end of 2006 written as a complement to a Russian edition (to be published by the Independent University press, Moscow, 2007) of the book by G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encycl. of Math. Appl. **71**, Cambridge Univ. Press, 1999. Report number: RIMS-1589 Cite as: `http://arxiv.org/abs/0704.3099` : `arXiv:0704.3099v1 [math.CA]`

Spiridonov, V.P.; Zhedanov, A.S., Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, *in* Bustoz, Joaquin (ed.) et al., *Special functions 2000: current perspective and future directions. Proceedings of the NATO Advanced Study Institute, Tempe, AZ, USA, May 29-June 9, 2000*, Dordrecht: Kluwer Academic Publishers. NATO Sci. Ser. II, Math. Phys. Chem. **30**, 365-388 (2001).

V. P. Spiridonov and A. S. Zhedanov: Elliptic grids, rational functions, and the Padé interpolation *The Ramanujan Journal* **13**, Numbers 1-3, June, 2007, p. 285–310.

A.S. Zhedanov, Padé interpolation table and biorthogonal rational functions. In: *Proceedings of RIMS Workshop on Elliptic Integrable Systems. Kyoto, November 8-11* (2004) to be published

`http://www.math.kobe-u.ac.jp/publications/rlm18/20.pdf`