MAPA 3011A Special topics in approximation theory.

1997: CF-Hankel-BMO(A) Rational approximation.

Alphonse Magnus,
Institut de Mathématique Pure et Appliquée,
Université Catholique de Louvain,
Chemin du Cyclotron,2,
B-1348 Louvain-la-Neuve
(Belgium)
(0)(10)473157 , magnus@anma.ucl.ac.be, http://www.math.ucl.ac.be/~magnus/

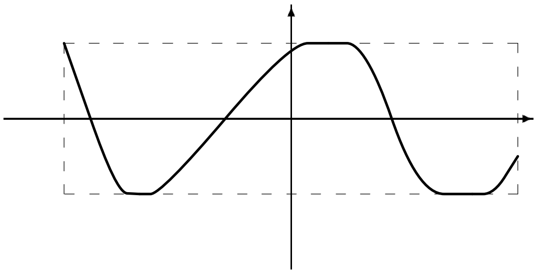
There has been quite a mood about 15 years ago about near-best rational approximation. The subject is strongly linked to Hankel approximation (system & control theory in engineering) and approximation in BMO(A) function spaces (highbrow mathematical analysis). While these two latter fields are alive and well, nobody (but me) speaks on near-best approximation anymore. What happened?

1. Approximation by one-step reductions.

1.1. Polynomial and rational approximations.

Polynomial approximation.

Best approximation p_n of degree $\leqslant n$ of a real function f on $[-1,1]$ is such that the graph of the error $f-p_n$ equioscillates $n+2$ times

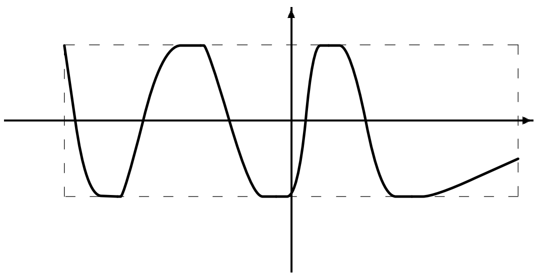


The determination of p_n happens to be quite easy when f happens to be a polynomial p_{n+1} of degree $\leqslant n+1$: then,

$$p_n(x) = p_{n+1}(x) - c_{n+1}T_{n+1}(x),$$

Rational approximation.

Best approximation r_n of degree $\leqslant n$ of a real function f on $[-1,1]$ is such that the graph of the error $f-r_n$ equioscillates $2n+2$ times (normally)



We certainly would like to know if something as simple as what is going on on the left side of this page holds for rational approximation ...

where T_{n+1} is the ***Chebyshev*** polynomial of degree $n+1$.

Near-best approximation is seen through a sequence of such one-step reductions (or *economizations*), starting with a high degree approximation $p_N \approx f$

$$\begin{array}{rl}
p_{N-1} = p_N - c_N T_N & \qquad f \approx p_N = p_{N-1} + c_N T_N \\
p_{N-2} = p_{N-1} - c_{N-1} T_{N-1} & \qquad f \approx p_N = p_{N-2} + c_{N-1} + c_N T_N \\
\cdots & \qquad \cdots \\
p_0 = p_1 - c_1 T_1 & \qquad f \approx p_N = p_0 + c_1 T_1 + \cdots + c_N T_N.
\end{array}$$

1.2. **Rational one-step reduction.**

$p_{n+1} - p_n$ = constant times $T_{n+1}(x) = \cos((n+1)\theta)$ = real part of

$$z^{n+1} = \exp(i(n+1)\theta)$$

, a function which describes a circle with winding number $n+1$ when $x = \cos\theta$ goes from -1 to 1.

Let us try to construct r_n such that $r_{n+1} - r_n$ is the real part of a function which describes a circle with winding number $2n+1$. Let $r_{n+1}(x) = \mathrm{Re}\,\dfrac{n_{n+1}(z)}{d_{n+1}(z)}$, where n_{n+1} and d_{n+1} are polynomials of degree $\leqslant n+1$. This can be done, for instance from the partial fractions expansion of r_{n+1}

$$r_{n+1}(x) = \pi_{n+1}(x) + \sum_{k=1}^{n+1} \frac{\rho_{n+1,k}}{x - x_{n+1,k}},$$

where π_{n+1} is a polynomial of degree $\leqslant n+1$ and the poles $x_{n+1,k} \notin [-1,1]$ (only simple poles are assumed; the number of poles is actually $n+1-$ degree π_{n+1}).

Let π_{n+1} be expanded in Chebyshev polynomials $\pi_{n+1}(x) = \pi_{n+1,0} + \pi_{n+1,1}T_1(x) + \cdots$ and $x_{n+1,k} = (\zeta_{n+1,k} + \zeta_{n+1,k}^{-1})/2$ with $|\zeta_{n+1,k}| > 1$, then

$$\begin{aligned} r_{n+1}(x) = \frac{N_{n+1}(x)}{D_{n+1}(x)} &= \frac{1}{2}\left[r_{n+1}^{(+)}(z) + r_{n+1}^{(+)}(z^{-1})\right] \\ &= \frac{1}{2}\left[\frac{n_{n+1}(z)}{d_{n+1}(z)} + \frac{n_{n+1}(z^{-1})}{d_{n+1}(z^{-1})}\right] \end{aligned} \tag{1}$$

with

$$r_{n+1}^{(+)}(z) = \sum_{k=1}^{n+1} \frac{2\rho_{n+1,k}}{\zeta_{n+1,k} - \zeta_{n+1,k}^{-1}} + \pi_{n+1,0} + \pi_{n+1,1}z + \cdots + \sum_{k=1}^{n+1} \frac{4\rho_{n+1,k}/(1-\zeta_{n+1,k}^{-2})}{z - \zeta_{n+1,k}} \tag{2}$$

which is indeed a rational function of degree $n+1$ in z [1].
Now, here is the ***trick***:

Proposition. *The best rational approximation $r_n^{(+)}$ of degree n to the given rational function $r_{n+1}^{(+)}$ of degree $n+1$ is such that the error curve $\left\{r_{n+1}^{(+)}(e^{i\theta}) - r_n^{(+)}(e^{i\theta})\right\}_{\theta=0}^{\theta=2\pi}$ is a circle of winding number[2] $2n+1$. Actually, the error function is the **Blaschke product***

$$r_{n+1}^{(+)}(z) - r_n^{(+)}(z) = \lambda_{n+1} \prod_{k=1}^{n+1} \frac{1 - \overline{\zeta_{n+1,k}}z}{z - \zeta_{n+1,k}} \prod_{k=1}^{n} \frac{1 - \overline{\zeta_{n,k}}z}{z - \zeta_{n,k}} = \lambda_{n+1}\frac{d_{n+1}^*(z)}{d_{n+1}(z)}\frac{d_n^*(z)}{d_n(z)} = \lambda_{n+1}B_{n+1}(z)B_n(z), \tag{3}$$

where $|\zeta_{n,k}| > 1, k = 1, \ldots, n$, $d^*(z) = \delta_p z^p + \delta_{p-1}z^{p-1} + \cdots + \delta_0$ if $d(z) = \delta_0 z^p + \delta_1 z^{p-1} + \cdots + \delta_p$. Only polynomials with ***real*** coefficients are considered, so that the zeros of d_n and d_{n+1} are real or come by complex conjugate pairs, and

$$r_{n+1}^{(+)}(z) - r_n^{(+)}(z) = \lambda_{n+1} \prod_{k=1}^{n+1} \frac{1 - \zeta_{n+1,k}z}{z - \zeta_{n+1,k}} \prod_{k=1}^{n} \frac{1 - \zeta_{n,k}z}{z - \zeta_{n,k}} = \lambda_{n+1}\frac{d_{n+1}^*(z)}{d_{n+1}(z)}\frac{d_n^*(z)}{d_n(z)} = \lambda_{n+1}B_{n+1}(z)B_n(z) \tag{4}$$

[1] Standard procedure [43] for extracting $f^{(+)}$ from a general real function f so to have $f(x) = [f^{(+)}(z) + f^{(+)}(z^{-1})]/2$, with $x = (z + z^{-1})/2$, is to manage to construct its [possibly approximate] Chebyshev expansion $f(x) = \sum_0^\infty c_k T_k(x)$, then, $f^{(+)}(z) = \sum_0^\infty c_k z^k$.

[2] ***Winding number*** $W(R)$ of curve $\{R(\exp(i\theta))\}_0^{2\pi} = (2\pi)^{-1}$ increase of argument of $R(\exp(i\theta)) = (2\pi)^{-1}$ increase of imaginary part of $\log R(\exp(i\theta))$, $W(RS) = W(R) + W(S)$, $W(z-a) = 1$ if $|a| < 1$, $W(z-a) = 0$ if $|a| > 1$, $W(R)$ = number of zeros − number of poles of R in $|z| < 1$ if R is rational.
Winding number extends easily to holomorphic functions in the open unit disk, continuous in the closed unit disk, see later remarks **??**, p. **??** for BMO ...

is as good as (3)

Before going further, remark that the form (3-4) holds only for contiguous degrees n and $n+1$: the $n+1$ known residues of $r_{n+1}^{(+)}$ are matched by the n unknown coefficients of d_n (say, $\delta_0 = 1$) and λ_{n+1}. Matching does not occur for other degrees, say $n+1$ and $n-1$.

Before exploring a proof of the proposition (I don't think that a complete proof will be given here), another remark: if (3) holds, then $r_n(x) = r_n^{(+)}(z) + r_n^{(+)}(z^{-1})$ certainly is the best approximant to $r_{n+1}(x) = (r_{n+1}^{(+)}(z) + r_{n+1}^{(+)}(z^{-1}))/2$, as the error function $r_{n+1}(\cos\theta) - r_n(\cos\theta) = \lambda_{n+1}[B_{n+1}(e^{i\theta})B_n(e^{i\theta}) + B_{n+1}(e^{-i\theta})B_n(e^{-i\theta})]/2$ shows the right number of oscillations between $x = \cos\theta = -1$ and $x = \cos\theta = 1$ (we shall have to show that λ_{n+1} is real).

Special best real approximants involving real parts of Blaschke products already appear in works of Chebyshev and Bernstein, thanks to J. Meinguet for pointing that, as well as the references [37, 38] to Talbot.

Optimality of $r_n^{(+)}$ as rational approximation to $r_{n+1}^{(+)}$ on the unit circle: suppose $\tilde r$ is a better approximant of degree n, then the curve

$$(\tilde r - r_n^{(+)})(e^{i\theta}) = [(r_{n+1}^{(+)} - r_n^{(+)})(e^{i\theta})]\left[1 - \frac{(r_{n+1}^{(+)} - \tilde r)(e^{i\theta})}{(r_{n+1}^{(+)} - r_n^{(+)})(e^{i\theta})}\right]$$

would still have a winding number of $2n+1$ (the big second factor remains in the right half plane, so has a winding number of 0), which is impossible, as $\tilde r - r_n^{(+)}$ has only numerator degree $\leqslant 2n$, so cannot have $2n+1$ zeros in the unit disk[3] (poles of $\tilde r$ within the unit disk would only make things worse).

As error functions (3-4) appear only in special cases, their study was considered as a field of limited interest, see for instance the historical introduction of [24]. L.N. Trefethen astonished the world (well, at least the world of the numerical rational approximators, who then only swore by modified Padé approximation[4]) by showing [39] that nearly circular error curves are far from being uncommon, and unveiling the new science of asymptotic circularity, more on this later on, I hope.

1.3. **Eigenvalue problem for solving the rational one-step reduction.**

Let us look at (3-4) as a problem for the coefficients $\delta_{n,0}, \ldots, \delta_{n,n}$ of $d_n(z) = \delta_{n,0}z^n + \cdots \delta_{n,n}$ and the coefficients $\nu_{n,0}, \ldots, \nu_{n,n}$ of the numerator $n_n(z) = \nu_{n,0}z^n + \cdots \nu_{n,n}$. The coefficients of z^0, z^1, ..., z^{2n+1} of

$$n_{n+1}d_n - n_n d_{n+1} = \lambda_{n+1} d_n^* d_{n+1}^* \qquad (5)$$

are

$$\left\{\lambda_{n+1}\begin{bmatrix} \delta_{n+1,0} & & & \\ \delta_{n+1,1} & \delta_{n+1,0} & & \\ \vdots & & \ddots & \\ \delta_{n+1,n} & \cdots & \delta_{n+1,1} & \delta_{n+1,0} \\ \delta_{n+1,n+1} & \cdots & \delta_{n+1,2} & \delta_{n+1,1} \\ & \ddots & & \vdots \\ & & & \delta_{n+1,n+1} \end{bmatrix} - \begin{bmatrix} & & & \nu_{n+1,n+1} \\ & & \nu_{n+1,n+1} & \nu_{n+1,n} \\ & \cdot^{\cdot^{\cdot}} & & \vdots \\ \nu_{n+1,n+1} & \nu_{n+1,n} & \cdots & \nu_{n+1,1} \\ \nu_{n+1,n} & \nu_{n+1,n-1} & \cdots & \nu_{n+1,0} \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ \nu_{n+1,0} & & & \end{bmatrix}\right\} \begin{bmatrix} \delta_{n,0} \\ \delta n, 1 \\ \vdots \\ \delta_{n,n-1} \\ \delta_{n,n} \end{bmatrix}$$

[3] ***Theorem of Rouché***: if rational functions f and g have same number of poles in a Jordan region, and if $|g| < |f|$ everywhere on the boundary, then f and $f-g$ have the same number of zeros in the region.

[4] By summing Padé approximants $r_{n,\text{Padé}}(z) = f^{(+)}(z) - K_n z^{2n+1} - \cdots$ and $r_{n,\text{Padé}}(z^{-1}) = f^{(+)}(z^{-1}) - K_n z^{-2n-1} - \cdots$, $r_{n,\text{Chebyshev-Padé}}(x) = f(x) - K_n T_{2n+1}(x) - \cdots$ seems convincingly close to equioscillation, but balance is often poor.

$$+\begin{bmatrix} & & & \delta_{n+1,n+1} \\ & & \delta_{n+1,n+1} & \delta_{n+1,n} \\ & \cdot^{\cdot^{\cdot}} & & \vdots \\ \delta_{n+1,n+1} & \delta_{n+1,n} & \cdots & \delta_{n+1,1} \\ \delta_{n+1,n} & \delta_{n+1,n-1} & \cdots & \delta_{n+1,0} \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ \delta_{n+1,0} & & & \end{bmatrix}\begin{bmatrix}\nu_{n,0} \\ \nu n, 1 \\ \vdots \\ \nu_{n,n-1} \\ \nu_{n,n}\end{bmatrix} = 0,$$

which can be rearranged as a generalized eigenvalue problem

$$(\lambda_{n+1}\boldsymbol{S}_{n+1} - \boldsymbol{N}_{n+1})\boldsymbol{x} = 0, \tag{6}$$

where $\boldsymbol{S}_{n+1}$ is the transposed Sylvester matrix

$$\boldsymbol{S}_{n+1} = \begin{bmatrix} \delta_{n+1,0} & & & & & \delta_{n+1,n+1} \\ \vdots & \ddots & & & \cdot^{\cdot^{\cdot}} & \vdots \\ \delta_{n+1,n} & \cdots & \delta_{n+1,0} & \delta_{n+1,n+1} & \cdots & \delta_{n+1,1} \\ \delta_{n+1,n+1} & \cdots & \delta_{n+1,1} & \delta_{n+1,n} & \cdots & \delta_{n+1,0} \\ & \ddots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \\ & & \delta_{n+1,n+1} & \delta_{n+1,0} & & \end{bmatrix},$$

$$\boldsymbol{N}_{n+1} = \begin{bmatrix} & & \nu_{n+1,n+1} & 0 & \cdots & \cdots & 0 \\ & \cdot^{\cdot^{\cdot}} & \vdots & \vdots & & & \vdots \\ \nu_{n+1,n+1} & \cdots & \nu_{n+1,1} & \vdots & & & \vdots \\ \nu_{n+1,n} & \cdots & \nu_{n+1,0} & \vdots & & & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots & & & \vdots \\ \nu_{n+1,0} & & & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

and where $\boldsymbol{x}$ is the vector $[\delta_{n,0}, \ldots, \delta_{n,n}, \nu_{n,0}/\lambda_{n+1}, \ldots, \nu_{n,n}/\lambda_{n+1}]^T$.

We immediately see that λ_{n+1} must be sought among the eigenvalues of the upper-left block $\boldsymbol{A}_{n+1} := (\boldsymbol{S}_{n+1}^{-1}\boldsymbol{N}_{n+1})_{1:n+1}^{1:n+1}$ of $\boldsymbol{S}_{n+1}^{-1}\boldsymbol{N}_{n+1}$. It would be nice to find a matrix proof that λ_{n+1} is real, probably by showing that $\boldsymbol{A}_{n+1}$ is similar to a real symmetric matrix, but I know no other proof that the one using (finite rank) Hankel operators (see later; for the equivalence between SVD of finite rank Hankel operators and (generalized) eigenvalue problems of matrices of finite order, see Meinguet [30, p.329]).

For other finite-dimensional (alas, unsymmetric) settings, such as the so called Kung equations, see [22, p.112]: multiply (4) by the unknown polynomial $d_n(z) = \delta_{n,0}z^n + \cdots \delta_{n,n}$ and look at the coefficients of $z^0, z^1, \ldots, z^n, z^{n+1}, \ldots, z^{2n+1}, \ldots$ of the Taylor-Maclaurin expansion

$$\begin{bmatrix} & & & c_0 \\ & & c_0 & c_1 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_{n+1} & c_{n+2} & \cdots & c_{2n+1} \\ \vdots & \vdots & & \vdots \end{bmatrix}\begin{bmatrix}\delta_{n,0} \\ \delta_{n,1} \\ \vdots \\ \delta_{n,n}\end{bmatrix} - \begin{bmatrix}\nu_{n,n} \\ \nu_{n,n-1} \\ \vdots \\ \nu_{n,0}\end{bmatrix} = \lambda\begin{bmatrix} \rho_0^* & & & \\ \rho_1^* & \rho_0^* & & \\ \vdots & & \ddots & \\ \rho_n^* & \rho_{n-1}^* & \cdots & \rho_0^* \\ \rho_{n+1}^* & \rho_n^* & \cdots & \rho_1^* \\ \vdots & \vdots & & \vdots \\ \rho_{2n+1}^* & \rho_{2n}^* & \cdots & \rho_{n+1}^* \\ \vdots & \vdots & & \vdots \end{bmatrix}\begin{bmatrix}\delta_{n,0} \\ \delta_{n,1} \\ \vdots \\ \delta_{n,n}\end{bmatrix},$$

where $\sum_0^\infty c_k z^k$ and $\sum_0^\infty \rho_k^* z^k$ are the expansions of $r_{n+1}^{(+)}(z)$ and $d_{n+1}^*(z)/d_{n+1}(z)$.

The $n+1$ first equations give the coefficients of the numerator; any slice of $n+1$ further equations are an eigensystem for the denominator's coefficients.

Remark however that, if $r^{(+)}_{n+1}$ is a ***polynomial*** (i.e., $d_{n+1} = 1 \Rightarrow d^*_{n+1}(z) = z^{n+1}$, we immediately have a Hankel eigenproblem for $[\delta_{n,0}, \ldots, \delta_{n,n}]^T$.

(Tedious) ***exercise.*** *For* $n = 1$, *if* $\delta_0, \delta_1, \delta_2, \nu_0, \nu_1$, *and* ν_2 *are real, and if* $d_2(z) = \delta_0 z^2 + \delta_1 z + \delta_2$ *has no zero in* $|z| \leqslant 1$, *show that* $\boldsymbol{X}(\boldsymbol{S}_2^{-1}\boldsymbol{N}_2)^{1:2}_{1:2}\boldsymbol{X}^{-1}$ *is real symmetric, with* $\boldsymbol{X} = \begin{bmatrix} \rho\cos\alpha & \rho\sin\beta \\ \rho\sin\alpha & \rho\cos\beta \end{bmatrix}$, *provided* $\sin(\alpha+\beta) = -\dfrac{\delta_1}{\delta_0+\delta_2}$.

But which eigenvalue?

Achilles: Obvious! as we are searching a ***best*** approximation, let us take the eigenvalue of ***smallest*** modulus.

Tortoise: But each eigenvalue leads to a valid realization of (5), and, from (1), to a real rational approximation to r_{n+1} with the right number of equioscillation points, so to several best rational approximations of same degree, a situation unheard of.

The result is right, but the reasoning is not. The point is that the $n+1$ first entries of an eigenvector of (6) make a denominator $d_n(z) = \delta_{n,0}z^n + \cdots + \delta_{n,n}$ which may have zeros in $|z| \leqslant 1$. Here again, it would be nice to have a matrix discussion of the localization of the zeros of the denominators constructed with the eigenvectors of (6), here is a more functional proof (oversimplification of [22, 24, 25, 40–42], etc.):

Theorem. *If the nonzero eigenvalues of* (6) *have distinct absolute values*

$$|\lambda_{n+1,1}| > |\lambda_{n+1,2}| > \cdots > |\lambda_{n+1,n+1}| > 0,$$

the $n+1$ *first elements of the eigenvector of* (6) *associated to* $\lambda_{n+1,k}$ *are the coefficients of a polynomial* $d_{n,k}(z) = \delta_{n,0}z^n + \cdots + \delta_{n,n}$ *which has exactly* $n+1-k$ *zeros in* $|z| < 1$ *and* $k-1$ *zeros in* $|z| > 1$.

We must therefore take $k = n+1$ for the best approximation.

Proof: let $d_{n,k}$ have p zeros in $|z| < 1$, and let us suppose that there is no zero on the unit circle $|z| = 1$ (to be settled later on). Then,

Lemma. $n_{n,k}/d_{n,k}$ *is the best approximation to* $r^{(+)}_{n+1}$ *on the unit circle, among all rational functions of degree* n *with* $\geqslant p$ *poles in the unit disk:*

suppose $\tilde r$ of degree n with $q \geqslant p$ poles in $|z| < 1$ is better than $n_{n,k}/d_{n,k}$, then, from (4),

$$\begin{aligned}\left(\tilde r - \frac{n_{n,k}}{d_{n,k}}\right)(e^{i\theta}) &= \left(r^{(+)}_{n+1} - \frac{n_{n,k}}{d_{n,k}}\right)(e^{i\theta})\left[1 - \frac{(r^{(+)}_{n+1} - \tilde r)(e^{i\theta})}{\left(r^{(+)}_{n+1} - \dfrac{n_{n,k}}{d_{n,k}}\right)(e^{i\theta})}\right] \\ &= \lambda_{n+1,k}\frac{d^*_{n+1}(e^{i\theta})}{d_{n+1}(e^{i\theta})}\frac{d^*_{n,k}(e^{i\theta})}{d_{n,k}(e^{i\theta})}\left[1 - \frac{(r^{(+)}_{n+1} - \tilde r)(e^{i\theta})}{\left(r^{(+)}_{n+1} - \dfrac{n_{n,k}}{d_{n,k}}\right)(e^{i\theta})}\right]\end{aligned}$$

has a winding number of $n+1$ (d^*_{n+1} has $n+1$ zeros in the disk) $+n-p$ ($d^*_{n,k}(z) = z^n d_{n,k}(1/z)$ has $n-p$ zeros in the disk) $-p$ ($d_{n,k}$ has p zeros, and d_{n+1} has no zero in the unit disk), making $2n+1-2p$. But the winding number of $(\tilde r - n_{n,k}/d_{n,k})(e^{i\theta})$ is at most $2n$ (the maximum number of zeros of the numerator, of degree $\leqslant 2n$) $-q-p = 2n-q-p < 2n+1-2p$. QED of this part.

So, starting with the smallest eigenvalue, $k = n+1$, if $d_{n,n+1}$ has p zeros in the unit disk, $d_{n,n}$ *must* have more than p zeros in the unit disk, etc., and as $d_{n,1}$ can not have more than n zeros[5] in the unit disk (or anywhere), $p = 0$: each $d_{n,j}$ has *exactly* $n+1-j$ zeros in the unit disk. □

Wait! If $d_{n,k}$ happens to have zeros on the unit circle, they disappear in the Blaschke product B_n of (3-4), so that the degree of the denominator AND the numerator of (3-4) is less than $2n+1$, say $2n+1-\partial$. The rational function $r^{(+)}_n$ has only $n-\partial$ true poles and can be written as a ratio

[5] Here is the advantage of dealing with matrices of finite $(n+1)$ order ...

of polynomials of degrees $\leqslant n - \partial$ ($\partial = defect$). The lemma above remains true, the winding number falls to $2n + 1 - 2p - \partial$, but $r_n^{(+)}$ is still optimal within the class of rational functions of degree $\leqslant n$ (and not $n - \partial$) with $\geqslant p$ poles in the unit disk. □ □Actually, it will be shown that $d_{n,k}$ has zeros on the unit circle only if (6) has more than one eigenvalue of modulus $|\lambda_{n+1,n+1-k}|$.

Remark that $|\lambda_{n+1,1}|$ is the least norm on the unit circle of $r_{n+1}^{(+)}$ minus a rational function without any pole in $|z| \geqslant 1$, an elementary setting of ***Nehari's theorem.***

```
m301197.m :

% Test of onestepr
diary
disp(' enter the coeff of numerator in square brackets, starting with power 0');
numnn=input(' '),
disp(' enter the coeff of denominator in square brackets, starting with power 0');
dennn=input(' '),
[aux,degn]=size(numnn);[aux,degd]=size(dennn);np1=max(degn,degd)-1;
for k=1:np1,
[numn,denn]=onestepr(numnn,dennn);
numnn=numn;dennn=denn;
end


onestepr.m  :

function [numn,denn]=onestepr(numnp1,dennp1);

% Best rational approximation of degree n of rational function of degree n+1
% numnp1 and dennp1 contain coeff. of x^0,1,2,... of given numerator and
% denominator
[aux,degn]=size(numnp1);[aux,degd]=size(dennp1);
np1=max(degn,degd)-1;n=np1-1;
disp(' degree ');disp(np1);
xres=[];xroots=[];zres=[];zroots=[];
if degd==1, xpolpart=numnp1(degn:-1:1)/dennp1(1);
else, [xres,xroots,xpolpart]=residue(numnp1(degn:-1:1),dennp1(degd:-1:1));end;
[pold1,pold2]=size(xpolpart);if pold1==0, pold1=1;pold2=1;xpolpart(1)=0;end;
zpolpart=zeros(1,2*pold2-1);
for k=1:pold2;zpolpart=([zpolpart(2:2*pold2-1),0]+[0,zpolpart(1:2*pold2-2)])/2;
     zpolpart(pold2)=zpolpart(pold2)+xpolpart(k);
end;
zpolpart=zpolpart(1:pold2);if pold2>1, zpolpart(1:pold2-1)=2*zpolpart(1:pold2-1);end;
if degd==1,  znumnp1=zpolpart;zdennp1=zeros(1,pold2);zdennp1(pold2)=1;
else,
  zroots=xroots+sqrt(xroots.*xroots-1);
  for k=1:degd-1;if abs(zroots(k))<1, zroots(k)=1/zroots(k);end;end;
  [sr1,sr2]=size(xres);onone=ones(sr1,sr2);
  zres=4*xres./(onone - onone./(zroots.*zroots));
  zpolpart(pold2)=zpolpart(pold2)+sum(zres./zroots)/2;
  [znumnp1,zdennp1]=residue(zres,zroots,zpolpart);
end;
  xres,xroots,xpolpart,
  zres,zroots,zpolpart,
  znumnp1,zdennp1,
[aux,degzn]=size(znumnp1);[aux,degzd]=size(zdennp1);
smatr=[toeplitz([zdennp1,zeros(1,2*n+2-degzd)],zeros(1,np1)), ...
        hankel([zeros(1,2*np1-degzd),zdennp1(degzd:-1:1)],zeros(1,np1))],
nmatr=[hankel([zeros(1,2*np1-degzn),znumnp1(degzn:-1:1)],zeros(1,np1)),zeros(2*np1,np1)],
smatr\nmatr,
nmatr/smatr,
```

```
[vecp,lambd]=eig(nmatr,smatr),
[lambs,ksort]=sort(-abs(diag(lambd)));
for k=1:np1,
  ks=ksort(k);
  disp(ks);format long;disp(lambd(ks,ks));format short;
  rv=roots(vecp(1:np1,ks));
  [rv,abs(rv)],
  if k==np1, kmin=ks; zroots2=rv;lamb2=lambd(ks,ks);end;
end;
numnz=lamb2*vecp(np1+1:2*np1,kmin),
dennz=vecp(1:np1,kmin),
% in x=(z+1/z)/2  :
denn=dennz(1);if n>0,denn=poly( (zroots2+ones(n,1)./zroots2)/2 );end;
% (n(z)/d(z)+n(1/z)/d(1/z))/2 :
numn=numnz(1);if n>0,
 chebp=exp(i*pi*(0:n)/n);chebp2=ones(1,np1)./chebp;
 zval=0.5*( polyval(numnz,chebp)./polyval(dennz,chebp)    ...
             +polyval(numnz,chebp2)./polyval(dennz,chebp2));
 numn=polyfit((chebp+chebp2)/2,polyval(denn,(chebp+chebp2)/2).*zval,n);
end;
denn=denn(np1:-1:1),
numn=numn(np1:-1:1),
```

Example [38] $r_2(x) = \dfrac{5x^2 - 9x + 10}{(3x-5)(4x-5)}$:

```
 enter the coeff of numerator in square brackets, starting with power 0
 [10 -9 5]
numnn = 10    -9     5
 enter the coeff of denominator in square brackets, starting with power 0
 [25 -35 12]
dennn = 25   -35    12

 degree 2

xres =                   xroots =                 xpolpart =
    1.7778                 1.6667                     0.4167
   -1.3125                 1.2500
```

so, $r_2(x) = \dfrac{5}{12} + \dfrac{16/9}{x - 5/3} - \dfrac{21/16}{x - 5/4}$.

```
zres =                      zroots =             zpolpart =
    8.0000                     3.0000              1.2768e-15
   -7.0000                     2.0000

znumnp1 = 0.0000    1.0000    5.0000

zdennp1 = 1.0000   -5.0000    6.0000
```

$$\begin{aligned}
\text{and } r_2(x) &= \frac{5}{12} + \frac{32/9}{z + z^{-1} - 10/3} - \frac{21/8}{z + z^{-1} - 5/2} = \frac{5}{12} - \frac{32/3}{(z-3)(z^{-1}-3)} + \frac{21/4}{(z-2)(z^{-1}-2)} \\
&= \frac{4}{z-3} + \frac{4}{z^{-1}-3} - \frac{7/2}{z-2} - \frac{7/2}{z^{-1}-2} \\
&= \frac{1}{2}\left(r_2^{(+)}(z) + r_2^{(+)}(z^{-1})\right),
\end{aligned}$$

where $r_2^{(+)}(z) = \dfrac{8}{z-3} - \dfrac{7}{z-2} = \dfrac{z+5}{z^2-5z+6}$.

```
Column wins diagonal conflict.
Column wins anti-diagonal conflict.

smatr =
    1.0000         0         0    6.0000
   -5.0000    1.0000    6.0000   -5.0000
    6.0000   -5.0000   -5.0000    1.0000
         0    6.0000    1.0000         0

Column wins anti-diagonal conflict.
nmatr =
         0    5.0000         0         0
    5.0000    1.0000         0         0
    1.0000    0.0000         0         0
    0.0000         0         0         0

ans =
    1.6000    1.0000         0         0
   -0.3333   -0.2667         0         0
    2.0000    1.6000         0         0
   -0.2667    0.6667         0         0

ans =

   -0.2083   -0.2917   -0.2083    0.7083
    0.7083    1.1917    1.7083    1.3917
    0.1500    0.2500    0.3500    0.2500
    0.0000    0.0000    0.0000    0.0000


vecp =
    0.6248   -0.0434    0.0000    0.0000
   -0.1250    0.0723    0.0000   -0.0000
    0.7498   -0.4340    0.0000    0.3969
   -0.1785   -0.8970   -1.0000    0.9178


lambd =
    1.4000         0         0         0
         0   -0.0667         0         0
         0         0    0.0000         0
         0         0         0   -0.0000

     1
   1.40000000000000

ans = 0.2000    0.2000

     2
  -0.06666666666667

ans = 1.6667    1.6667

numnz =
    0.0289
    0.0598
dennz =
   -0.0434
    0.0723
```

```
denn =
   -1.1333    1.0000


numn =
   -0.4889    0.0800
```

Non zero eigenvalue of smallest modulus of $\boldsymbol{S}_2^{-1}\boldsymbol{N}_2$ is $-1/15$, corresponding eigenvector yields $r_1^{(+)}(z) = \dfrac{-2z - 62/15}{3z-5}$ and $r_1(x) = \dfrac{1}{2}\left(r_1^{(+)}(z) + r_1^{(+)}(z^{-1})\right) = \dfrac{2x/25 - 22/45}{x - 17/15}$.

A further one-step reduction, from r_n to r_{n-1} disclose an incredibly interesting phenomenon: the eigenvalues are the same ... (excepting $\lambda_{n+1,n+1}$, which disappears). What happens?

Exercise. Use the `matlab` programs `m301197.m` and `onestepr.m` (available in `http://www.math.ucl.ac.be/~magnus/num3/`) on the function $r_3(x) = \dfrac{1+2x}{6+2x+x^3}$. Look at the zeros of $d_{2,2}$ and $d_{2,3}$! See also that the same eigenvalues appear in the further one-step reductions $r_2 \to r_1$ and $r_1 \to r_0$.

2. Laurent series and Hankel operators.

2.1. The Hankel infinite matrix.

Let us write (3-4) or (5) as

$$\frac{d_n}{d^*_{n+1}}\,\frac{n_{n+1}}{d_{n+1}} - \frac{n_n}{d^*_{n+1}} = \lambda_{n+1}\frac{d^*_n}{d_{n+1}}. \tag{7}$$

and we look at the Laurent expansions which converge on $|z|=1$. n_{n+1}/d_{n+1} and d^*_n/d_{n+1} are rational functions with all their poles in $|z|>1$, their Taylor-Maclaurin series $\sum_0^\infty c_k z^k$ and $\sum_0^\infty v_k z^k$ are therefore convergent in a disk containing $|z| \leqslant 1$; $d_n(z)/d^*_{n+1}(z) = (z^n d^*_n(z^{-1}))/(z^{n+1}d_{n+1}(z^{-1})) = \sum_0^\infty v_k z^{-k-1}$, n_n/d^*_{n+1} is similarly a series in z^{-1}, z^{-2}, ..., so (7) becomes

$$\left(\sum_0^\infty v_k z^{-k-1}\right)\left(\sum_0^\infty c_k z^k\right) - \sum_0^\infty w_k z^{-k-1} = \lambda_{n+1}\sum_0^\infty v_k z^k.$$

The coefficients of nonnegatives powers of z are

$$\sum_{\ell=0}^\infty c_{k+\ell+1} v_\ell = \lambda_{n+1} v_k,$$

for $k = 0, 1, \ldots$, i.e.,

$$\mathcal{H}_{n+1} = \mathcal{H}(r^{(+)}_{n+1})\boldsymbol{v} = \lambda_{n+1}\boldsymbol{v}, \tag{8}$$

with

$$\begin{bmatrix} c_1 & c_2 & c_3 & \cdots & \cdots \\ c_2 & c_3 & c_4 & \cdots & \cdots \\ c_3 & c_4 & c_5 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ \vdots \end{bmatrix} = \lambda_{n+1} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ \vdots \end{bmatrix},$$

the famous *Hankel* eigenvalue problem. Now we see that the eigenvalues of (6) are real: if λ is any eigenvalue of (6), (8) holds with exponentially decreasing series and the usual proof $\overline{\boldsymbol{v}}^T\mathcal{H}\boldsymbol{v} = \lambda\overline{\boldsymbol{v}}^T\boldsymbol{v}$ is valid.

2.2. **Equivalent finite eigensystems.**

$$\sum_0^\infty v_k z^k = \frac{d_n^*(z)}{d_{n+1}(z)} = \frac{\delta_{n,0}+\delta_{n,1}z+\cdots+\delta_{n,n}z^n}{\delta_{n+1,n+1}+\delta_{n+1,n}z+\cdots+\delta_{n+1,0}z^{n+1}} \Rightarrow$$

$$\begin{bmatrix} \delta_{n+1,n+1} & & & & & \\ \delta_{n+1,n} & \delta_{n+1,n+1} & & & & \\ \delta_{n+1,n-1} & \delta_{n+1,n} & \delta_{n+1,n+1} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \delta_{n+1,0} & \delta_{n+1,1} & \cdots & & \delta_{n+1,n+1} & \\ & \ddots & \ddots & & & \ddots \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta_{n,0} \\ \delta_{n,1} \\ \delta_{n,2} \\ \vdots \\ \delta_{n,n} \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{9}$$

$$\text{say:} \quad \boldsymbol{L}_{n+1}\boldsymbol{v} = \boldsymbol{\delta}_n$$

shows the connection between the Hankel (infinite) eigenvector and the eigenvector of (6). Note that the v_k's are solution of the recurrence relation $\delta_{n+1,0}v_k + \cdots + \delta_{n+1,n+1}v_{k+n+1} = 0$, $k \geqslant 0$, therefore behave like k^{th} powers of the zeros of $\delta_{n+1,0}+\cdots+\delta_{n+1,n+1}z^{n+1} = d_{n+1}^*(z)$ indeed inside the disk $|z| < 1$.

The same rate of decrease holds for the c_j's themselves, as

$$\delta_{n+1}(z)r_{n+1}^{(+)}(z) = (\delta_{n+1,n+1}+\delta_{n+1,n}z+\cdots+\delta_{n+1,0}z^{n+1})(c_0+c_1z+c_2z^2+\cdots) = \text{ polynomial},$$

so, $\delta_{n+1,0}c_j + \delta_{n+1,1}c_{j+1} + \cdots + \delta_{n+1,n+1}c_{j+n+1} = 0$, $j = 0, 1, \ldots$, and $\boldsymbol{L}_{n+1}\mathcal{H}_{n+1}$ has only $n+1$ nonzero rows. So,

$$\boldsymbol{L}_{n+1}\mathcal{H}_{n+1}\boldsymbol{L}_{n+1}^{-1}\boldsymbol{\delta}_n = \lambda_{n+1}\boldsymbol{\delta}_n$$

shows that $\boldsymbol{L}_{n+1}\mathcal{H}_{n+1}\boldsymbol{L}_{n+1}^{-1}$ is reduced to a nonzero upper-left corner which is the matrix of finite order once called $\boldsymbol{A}_{n+1} = (\boldsymbol{S}_{n+1}^{-1}\boldsymbol{N}_{n+1})_{1:n+1}^{1:n+1}$: the eigenvectors and eigenvalues are the same.

And now, still another view of the matrices $\boldsymbol{A}_n$ and $\boldsymbol{A}_{n+1}$:
in the generic case, $r_n^{(+)}$ and $r_{n+1}^{(+)}$ have distinct finite simple poles $\zeta_{n,j}$ and $\zeta_{n+1,j}$ in $|\zeta| > 1$, let

$$r_n^{(+)}(z) = \rho'_{n,0} + \sum_1^n \frac{\rho'_{n,j}}{z-\zeta_{n,j}}, \quad r_{n+1}^{(+)}(z) = \rho'_{n+1,0} + \sum_1^{n+1} \frac{\rho'_{n+1,j}}{z-\zeta_{n+1,j}},$$

with each $\rho' = 4\rho/(1-\zeta^{-2})$, from (2).

Let us consider again (4), *considering the residues ρ' as the unknowns*:

$$r_{n+1}^{(+)}(z) - r_n^{(+)}(z) = \rho'_{n+1,0} + \sum_1^{n+1} \frac{\rho'_{n+1,\ell}}{z-\zeta_{n+1,\ell}} - \rho'_{n,0} - \sum_1^n \frac{\rho'_{n,\ell}}{z-\zeta_{n,\ell}} = \lambda_{n+1,n+1}\frac{\prod_1^{n+1}(1-\zeta_{n+1,j}z)\prod_1^n(1-\zeta_{n,j}z)}{\prod_1^{n+1}(z-\zeta_{n+1,j})\prod_1^n(z-\zeta_{n,j})}.$$

So,

$$\rho'_{n+1,\ell} = \lambda_{n+1,n+1}\frac{\prod_1^{n+1}(1-\zeta_{n+1,j}\zeta_{n+1,\ell})\prod_1^n(1-\zeta_{n,j}\zeta_{n+1,\ell})}{\prod_{\substack{j=1\\ j\neq\ell}}^{n+1}(\zeta_{n+1,\ell}-\zeta_{n+1,j})\prod_1^n(\zeta_{n+1,\ell}-\zeta_{n,j})}, \tag{10}$$

$$\rho'_{n,\ell} = -\lambda_{n+1,n+1}\frac{\prod_1^{n+1}(1-\zeta_{n+1,j}\zeta_{n,\ell})\prod_1^n(1-\zeta_{n,j}\zeta_{n,\ell})}{\prod_1^{n+1}(\zeta_{n,\ell}-\zeta_{n+1,j})\prod_{\substack{j=1\\ j\neq\ell}}^n(\zeta_{n,\ell}-\zeta_{n,j})}, \tag{11}$$

telling what are the residues of two rational functions $r_{n+1}^{(+)}$ and $r_n^{(+)}$ such that $r_n^{(+)}$ is the best approximation to $r_{n+1}^{(+)}$ when the poles and the error amplitude are known.

Now, any other rational approximation $r_n^{(+)(k)}$ to $r_{n+1}^{(+)}$ with circular error curve has a denominator $d_n^{(k)}(z) = \delta_{n,0}^{(k)} z^n + \cdots + \delta_{n,n}^{(k)}$ such that

$$d_n^{(k)} r_{n+1}^{(k)}(z) - n_n^{(k)}(z) = (\delta_{n,0}^{(k)} z^n + \cdots + \delta_{n,n}^{(k)}) r_{n+1}^{(+)}(z) - n_n^{(k)}(z) = \lambda_{n+1,k} \frac{d_{n+1}^*(z) d_n^{*(k)}(z)}{d_{n+1}(z)}$$
$$= \lambda_{n+1,k} \frac{\prod_1^{n+1}(1-\zeta_{n+1,j} z)}{\prod_1^{n+1}(z-\zeta_{n+1,j})} (\delta_{n,n}^{(k)} z^n + \cdots + \delta_{n,0}^{(k)}).$$

One has, at each $\zeta_{n+1,\ell}$,

$$(\delta_{n,0}^{(k)} \zeta_{n+1,\ell}^n + \cdots + \delta_{n,n}^{(k)}) \rho'_{n+1,\ell} = \lambda_{n+1,k} \frac{\prod_1^{n+1}(1-\zeta_{n+1,j}\zeta_{n+1,\ell})}{\prod_{\substack{j=1\\ j\neq \ell}}^{n+1} (\zeta_{n+1,\ell}-\zeta_{n+1,j})} (\delta_{n,n}^{(k)} \zeta_{n+1,\ell}^n + \cdots + \delta_{n,0}^{(k)}).$$

amounting again to $\boldsymbol{A}_{n+1}\boldsymbol{\delta}_n = \lambda_{n+1,k}\boldsymbol{\delta}_n$, with

$$\boldsymbol{A}_{n+1} = \boldsymbol{J}_{n+1}\boldsymbol{V}_{n+1}^{-1}\boldsymbol{D}_{n+1}\boldsymbol{V}_{n+1}, \tag{12}$$

where $\boldsymbol{J}_{n+1} = \begin{bmatrix} & & 1 \\ & 1 & \\ \cdot^{\cdot^{\cdot}} & & \\ 1 & & \end{bmatrix}$, $\boldsymbol{V}_{n+1}$ is the Vandermonde matrix $\begin{bmatrix} \zeta_{n+1,1}^n & \cdots & \zeta_{n+1,1} & 1 \\ \zeta_{n+1,2}^n & \cdots & \zeta_{n+1,2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \zeta_{n+1,n+1}^n & \cdots & \zeta_{n+1,n+1} & 1 \end{bmatrix}$,
and $\boldsymbol{D}_{n+1}$ is the diagonal matrix of elements $\rho'_{n+1,\ell} \prod_{\substack{j=1\\ j\neq \ell}}^{n+1}(\zeta_{n+1,\ell}-\zeta_{n+1,j}) / \prod_1^{n+1}(1-\zeta_{n+1,j}\zeta_{n+1,\ell})$.

According to (10), $\boldsymbol{D}_{n+1}$ is *also* the diagonal matrix of elements

$$(\boldsymbol{D}_{n+1})_{\ell,\ell} = \lambda_{n+1,n+1} \prod_{j=1}^{n} (1-\zeta_{n,j}\zeta_{n+1,\ell}) / \prod_{j=1}^{n} (\zeta_{n+1,\ell}-\zeta_{n,j}). \tag{13}$$

Similarly, as we come to $\boldsymbol{A}_n$, $\boldsymbol{D}_n$ happens to be also the diagonal matrix

$$(\boldsymbol{D}_n)_{\ell,\ell} = -\lambda_{n+1,n+1} \prod_{j=1}^{n+1} (1-\zeta_{n+1,j}\zeta_{n,\ell}) / \prod_{j=1}^{n+1} (\zeta_{n,\ell}-\zeta_{n+1,j}), \tag{14}$$

from (11)

well, well

Experiments with $\boldsymbol{A}_n$ and $\boldsymbol{A}_{n+1}$:

```
%  onestepr2.m
%  test matrices built with zeros of d n+1   and  dn
diary;
disp('  enter the zeta n  vector ,  in square brackets ');
ze=input(' '),
[aux,n]=size(ze),
disp('  enter the zeta n+1  vector ,  in square brackets ');
ze1=input(' '),
[aux,np1]=size(ze1),
DD=(ones(n,np1)-kron(ze',ze1))./(kron(ze',ones(1,np1))-kron(ones(n,1),ze1)),
ii=eye(n);hh=ii(:,n);jn=hankel(hh),
vn=vander(ze),
an=-(jn/vn)*diag(prod(DD'))*vn,
[V,D]=eig(an),
ii=eye(np1);hh=ii(:,np1);jnp1=hankel(hh),
vnp1=vander(ze1),
anp1=(jnp1/vnp1)*diag(prod(-DD))*vnp1,
[V1,D1]=eig(anp1),
```

2.3. **Spectrum of $\mathcal{H}_{n+1}$ = spectrum of $\mathcal{H}_n \bigcup\{\lambda_{n+1,n+1}\}$.** (unfinished)

First, let $r_{n+1}^{(+)}(z) = \sum_0^\infty c_j z^j$ and $r_n^{(+)}(z) = \sum_0^\infty c'_j z^j$.

Then, trace $\mathcal{H}_{n+1}-$ trace $\mathcal{H}_n = \sum_0^\infty (c_{2j+1} - c'_{2j+1}) = \frac{1}{2}\left[r_{n+1}^{(+)}(1) - r_{n+1}^{(+)}(-1) - r_n^{(+)}(1) + r_n^{(+)}(-1)\right]$
$= \lambda_{n+1,n+1}$, from (4), as $d_n(\pm 1) = d_n^*(\pm 1)$. *Aha!*

And determinants, from (12), (14), and (13):

$$\det \boldsymbol{A}_n = (-1)^{(n-1)n/2} \prod_1^n (\boldsymbol{D}_n)_{\ell,\ell} = (-1)^{(n-1)n/2} (-\lambda_{n+1,n+1})^n \prod_{\ell=1}^{n} \prod_{j=1}^{n+1} \frac{1-\zeta_{n+1,j}\zeta_{n,\ell}}{\zeta_{n,\ell}-\zeta_{n+1,j}},$$

$$\det \boldsymbol{A}_{n+1} = (-1)^{n(n+1)/2} \prod_1^{n+1} (\boldsymbol{D}_{n+1})_{\ell,\ell} = (-1)^{n(n+1)/2} (\lambda_{n+1,n+1})^{n+1} \prod_{\ell=1}^{n+1} \prod_{j=1}^{n} \frac{1-\zeta_{n,j}\zeta_{n+1,\ell}}{\zeta_{n+1,\ell}-\zeta_{n,j}}$$

$= \lambda_{n+1,n+1} \det \boldsymbol{A}_n$.
!

3. Numerical CF approximation.

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