3. About Constantin Carathéodory and Leopold Fejér.

At the request of a kind member of the audience, a word on Carathéodory and Fejér's own work [11]: they proceeded to give constructive solutions to the following problems:

1. *Given the complex numbers* $\alpha_0, \alpha_1, \ldots$, *how to know if* $\mathrm{Re}\left(\Psi(z) = \sum_0^\infty \alpha_k z^k\right)$ *is bounded in* $\mathbb{D} = \{z : |z| < 1\}$?

Answer: for each $n = 1, 2, \ldots$, construct $\tilde{\alpha}_{n+1}, \tilde{\alpha}_{n+2}, \ldots$, such that $\mathrm{Re}\left(\Psi_n(z) = \sum_0^n \alpha_k z^k + \sum_{n+1}^\infty \tilde{\alpha}_k z^k\right)$ has the ***least*** possible supremum on the unit disk.

They show that $\sup \mathrm{Re}\,\Psi_n$ on the unit disk is $\Lambda_{n,1}$, the largest eigenvalue of the Hermitian Toeplitz matrix of order $n+1$ $\boldsymbol{T}_n = \begin{bmatrix} (\alpha_0 + \overline{\alpha_0})/2 & \alpha_1/2 & \cdots & \alpha_n/2 \\ \overline{\alpha_1}/2 & (\alpha_0 + \overline{\alpha_0})/2 & \cdots & \alpha_{n-1}/2 \\ \vdots & \vdots & \cdots & \vdots \\ \overline{\alpha_n}/2 & \cdots & \cdots & (\alpha_0 + \overline{\alpha_0})/2 \end{bmatrix}$, that the limit of the increasing sequence $\{\Lambda_{n,1}\}_n$ is the answer (for the infimum, look at the decrasing sequence $\{\Lambda_{n,n}\}_n$ of the lowest eigenvalues), that Ψ_n is a rational function with constant real part on the unit circle

$$\Psi_n(z) = \frac{w_n + w_{n-1}z + \cdots + w_0 z^n}{v_n + v_{n-1}z + \cdots + v_0 z^n},$$

where $[v_0, v_1, \ldots, v_n]^T$ is the eigenvector of $\boldsymbol{T}_n$, and where $w_0, \ldots, w_n$ care for the match of the Taylor coefficients of Ψ_n. One has $\Psi_n(1/\overline{z}) = 2\Lambda_{n,1} - \overline{\Psi_n(z)}$.

2. *Given* $M > \mathrm{Re}\,\alpha_0$, *what is the largest disk where* Ψ *is regular and has a real part* $\leqslant M$?

   Answer: construct the Toeplitz matrix $\boldsymbol{T}_n(r)$ with the $\alpha_k r^k$ and increase r until $\Lambda_{n,1}(r) = M$, then increase n.

3. *Given the complex numbers* $\alpha_0, \alpha_1, \ldots$, *how to know if* $\Phi(z) = \sum_0^\infty \alpha_k z^k$ *is bounded in* $|z| < 1$?

   Answer: for each $n = 1, 2, \ldots$, construct $\tilde{\alpha}_{n+1}, \tilde{\alpha}_{n+2}, \ldots$, such that $\Phi_n(z) = \sum_0^n \alpha_k z^k + \sum_{n+1}^\infty \tilde{\alpha}_k z^k$ has the ***least*** possible L^∞ norm on the unit disk.

   This is the classical CF problem. Their construction of Φ_n is rather clumsy and will be superseded by Schur and Takagi: the harmonic function $\mathrm{Re}\left(\Psi(z) = \dfrac{\Phi(z) - \mu}{\Phi(z) + \mu}\right)$ will be negative in the open unit disk with zero as supremum only if $\mu = \|\Phi_n\|_\infty$, therefore this μ leads to $\Lambda_{n,1} = 0$ with the Toeplitz matrix built with the Taylor expansion of Ψ ... Then, the sequence $\{\|\Phi_n\|_\infty\}_n$ is increasing and its limit is the essential sup of $|\Phi|$.

4. *Given the complex numbers* $\alpha_0, \alpha_1, \ldots$, ($\alpha_0 \neq 0$ *and* $\alpha_0 \neq 1$), *what is the largest disk where* Φ *is regular and has* **never value 0 nor 1**? (Picard-Landau-type problem).

   Answer: consider the famous ***modular function*** $\mathcal{M}$ which 1) is analytic in any domain which does not contain 0 nor 1; 2) has a negative real part in any such domain. Therefore, consider the expansions of $\Psi(z) := \mathcal{M}(\Phi(rz)) = \beta_0 + \beta_1 z + \cdots$ for increasing values of r: while $\Phi(r\mathbb{D})$ is still in an admissible domain for $\mathcal{M}$, $\Lambda_{n,1}$ is negative. The requested radius is reached when $\Lambda_{n,1} = 0$.

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Of course, direct solution to problem 3 is given by Schur's construction: one has

$$\Phi_n(z) = \mu\frac{1 + \Psi_n(z)}{1 - \Psi_n(z)} = \mu\frac{v_n + w_n + \cdots + (v_0 + w_0)z^n}{v_n - w_n + \cdots + (v_0 - w_0)z^n} = \mu\gamma\frac{\overline{u_n} + \cdots + \overline{u_0}z^n}{u_0 + \cdots + u_n z^n},$$

with $|\gamma| = 1$, leading to

$$\begin{bmatrix} \alpha_n & \alpha_{n-1} & \cdots & \alpha_1 & \alpha_0 \\ \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_0 & \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & & \\ \alpha_0 & & & & \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix} = \mu\gamma \begin{bmatrix} \overline{u_0} \\ \overline{u_1} \\ \vdots \\ \overline{u_n} \end{bmatrix} \qquad (15)$$

$$\text{i.e.} \quad \boldsymbol{H}_n \boldsymbol{u} = \mu\gamma\overline{\boldsymbol{u}}.$$

4. Numerical CF approximation.

First construct a large number of Chebyshev coefficients of $f(x) = \sum_0^{\infty}{}' c_k T_k(x)$, for instance through FFT sampling [43].

```
% cftref.m
%  Approximation Theory V
% RCF -- REAL RATIONAL CF APPROXIMATION ON THE UNIT INTERVAL
%
% Loyd N. Trefethen, Dept. of Math., M.I.T., March 1986
% Reference: L.N.T. and M.H. Gutknecht,
%            SIAM J. Numer. Anal. 20 (1983), pp. 420-436
%
%     Fx(x) - function to be approximated by R(x)=P(x)/Q(x)
%      m,n  - degree of P,Q
%     nfft  - number of points in FFT (power of 2)
%        K  - degree at which Chebyshev series is truncated
%   F,P,Q,R - functions evaluated on FFT mesh (Chebyshev points)
%    Pc,Qc  - Chebyshev coefficients of P and Q
%
%  If Fx is even, take (m,n)=( odd, even).
%  If Fx is odd , take (m,n)=(even,even ).
%
%  CONTROL PARAMETERS
    m=input('m? ') ; n=input('n? '); np=n+1;
    nfft=input('nfft? ');nfft2=nfft/2;
    K=input('K? '); dim=K+n-m;
%
%  CHEBYSHEV COEFFICIENTS OF Fx
    z = exp(2*pi*sqrt(-1)*(0:nfft-1)/nfft);
    x = real(z); F = Fx(x); Fc = real(fft(F))/nfft2;
%
%  SVD OF HANKEL MATRIX H
    H = toeplitz(Fc(1+rem((dim:-1:1)+nfft+m-n,nfft)));
       H = triu(H); H=H(:,(dim:-1:1));
    [u,s,v] = svd(H);
    s = s(np,np); u = u((dim:-1:1),np)';v = v(:,np)';
%
%  DENOMINATOR POLYNOMIAL Q
    zr = roots(v); qout = []; for i=1:dim-1;
        if abs(zr(i))>1 qout = [qout, zr(i)];end; end;
    qc = real(poly(qout)); qc = qc/qc(np); q = polyval(qc,z);
    Q = q.*conj(q); Qc = real(fft(Q))/nfft2;
    Qc(1)= Qc(1)/2; Q=Q/Qc(1); Qc = Qc(1:np)/Qc(1);
%
%  NUMERATOR POLYNOMIAL P
    b = fft([u zeros(1,nfft-dim)])./fft([v zeros(1,nfft-dim)]);
```

```
      Rt = F-real(s*z.^K.*b); Rtc = real(fft(Rt))/nfft2;
      gam = real(fft((1)./Q))/nfft2; gam = toeplitz(gam(1:2*m+1));
      if m==0 Pc = 2*Rtc(1)/gam;
        else Pc = 2*[Rtc(m+1:-1:2) Rtc(1:m+1)]/gam; end;
      Pc = Pc(m+1:2*m+1); Pc(1) = Pc(1)/2;
      P = real(polyval(Pc(m+1:-1:1),z)); R = P./Q;
%
% RESULTS
    plot(x,F-R,'-',x,[s;0;-s]*ones(1,nfft),':'); pause;
    s, err = norm(F-R,'inf'), Pc, Qc
```

Then, $f(x) = (f^{(+)}(z) + f^{(-)}(z))/2$, $f^{(+)}(z) = \sum_0^{\prime\infty} c_k z^k$; $f^{(-)}(z) = f^{(+)}(1/z)$. Proceed with a big truncated Hankel operator, either $\mathcal{H}_N = \begin{bmatrix} c_1 & c_2 & \cdots & \cdots & c_N \\ c_2 & c_3 & \cdots & c_N & \\ \vdots & \vdots & \iddots & & \\ c_N & & & & \end{bmatrix}$ (L.N. Trefethen's choice, Hankel consistent), or $\mathcal{H}_N = \begin{bmatrix} c_1 & c_2 & \cdots & \cdots & c_N \\ c_2 & c_3 & \cdots & c_N & c_{N+1} \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ c_N & c_{N+1} & \cdots & \cdots & c_{2N-1} \end{bmatrix}$ (A.P. Magnus' choice, operator projection consistent).

Compute the eigenvalues $\lambda_1, \ldots, \lambda_N$ of $\mathcal{H}_N$ (with $\sigma_1 = |\lambda_1| \geqslant \sigma_2 = |\lambda_2| \geqslant \cdots \geqslant \sigma_N = |\lambda_N|$). No need to keep 2 subscripts, as the reductions $N \to N-1$, $N-1 \to N-2 \cdots n+1 \to n$ keep $\lambda_1, \ldots, \lambda_n$.

Nor shall we reach $r_n^{(+)}$ through exhausting multiple one step reductions from N to n: as far as near-best approximations are considered, we may as well work directly on the eigenvectors of $\mathcal{H}_N$. Take the lemma p. 5, with N instead of $n+1$, and $n+1$ instead of k $(n \geqslant 0)$:
$\tilde{r}_{N,n} := n_{N-1,n+1}/d_{N-1,n+1}$ *is the best approximation to* $f^{(+)}$ *on the unit circle, among all rational functions of degree* $N-1$ *with* $\geqslant N-n-1$ *poles in the unit disk,*
or: $\tilde{r}_{N,n} := n_{N-1,n+1}/d_{N-1,n+1}$ *is the best approximation to* $f^{(+)}$ *on the unit circle, among all rational functions of degree* $N-1$ *with* $\leqslant n$ *poles* ***outside*** *the unit disk,*
the number of these poles is the right one ($r_n^{(+)}$ must have its n poles outside the unit disk), we simply keep them to make a reasonable denominator $\tilde{d}_n$ of $r_n^{(+)}$.

$\dfrac{\tilde{d}_n(z)}{z^n}\tilde{r}_{N,n}(z)$ is therefore a rational function of high degree without pole in $|z| > 1$, and so has a Laurent expansion with nonpositive powers:

$$\tilde{r}_{N,n}(z) = \frac{\tilde{\nu}_0 z^n + \tilde{\nu}_1 z^{n-1} + \cdots + \tilde{\nu}_n + \tilde{\nu}_{n+1} z^{-1} + \cdots}{\tilde{d}_n(z)},$$

and we check again (p. 9)

$$f^{(+)}(z) - \tilde{r}_{N,n}(z) = \sum_{k=0}^{\infty} c_k z^k - \tilde{r}_{N,n}(z) = \lambda_{n+1} \frac{v_0 + v_1 z + v_2 z^2 + \cdots}{v_0 z^{-1} + v_1 z^{-2} + v_2 z^{-3} + \cdots}.$$

Finally, we construct the CF numerator by keeping only the nonnegative powers of z in

$$\tilde{d}_n(z)\tilde{r}_{N,n}(z) = \tilde{d}_n(z)\left(f^{(+)}(z) - \lambda_{n+1}\frac{v_0 + v_1 z + v_2 z^2 + \cdots}{v_0 z^{-1} + v_1 z^{-2} + v_2 z^{-3} + \cdots}\right).$$

We compare the two techniques (CF in [24-25, 39-44] sense, and successive one-step degree reductions) for $f(x) = (1+2x)/(6+2x+x^3)$, looking for r_1.

We split $f(x) = (f^{(+)}(z) + f^{(+)}(z^{-1}))/2$:

$$f^{(+)}(z) = \frac{-0.0952z^3 - 0.2871z^2 + 15.1044z + 4.3088}{z^3 - 0.2444z^2 + 11.0651z + 45.2721},$$

with poles at $z = 1.3795 \pm 4.0125i$ and -2.5147.

$\mathcal{H}_3$ has the three eigenvalues

$$\lambda_1 = 0.333359\,, \qquad \lambda_2 = -0.007973\,, \qquad \lambda_3 = 0.007947\,.$$

CFGT

Take $\lambda_{n+1} = \lambda_2$,
poles are -3.8188 and -0.9985.

Keep only the pole -3.8188, so:
$\tilde{d}_1(z) = z + 3.8188$.
Final $z-$fraction is

$$\frac{1.282z + 0.363}{z + 3.8188},$$

and final $x-$fraction is

$$\frac{0.3493 + 0.6884x}{2.0404 + x}.$$

One-step

Take $\lambda_3 = 0.007947$, poles are -3.821 and 1.0016, one finds

$$r_2^{(+)}(z) = \frac{1.2818z^2 - 0.9200z - 0.3644}{z^2 + 2.8194z - 3.8273}.$$

Restart with $r_2^{(+)}$, now, λ_2 is the least modulus (non vanishing) eigenvalue, pole is -3.5227, final $z-$fraction is

$$r_1^{(+)}(z) = \frac{1.1743z + 0.3334}{z + 3.5227},$$

and final $x-$fraction is

$$r_1(x) = \frac{0.3334 + 0.6345x}{1.9033 + x}.$$

CFGT fraction is found to have 0.0083 as error norm, one-step fraction shows 0.0147, almost twice as bad for this particular experiment. There must be some hidden tricks in L.N.T.'s program.

5. BMO, ETC.

A fascinating topic of function theory is the connection between function classes and approximation properties.

Best known examples are the Jackson-Bernstein theorems on polynomial errors and Lipschitz classes:

$$f \in \mathscr{C}^r(\mathbb{T}) \Rightarrow E_n(f) \leqslant C_r n^{-r}\omega\left(f^{(r)}, n^{-1}\right)\ , \quad E_n(f) \leqslant C_r n^{-r-\alpha}, 0<\alpha<1 \Rightarrow f^{(r)} \in \mathrm{Lip}_\alpha.$$

The subject has been thoroughly amplified ...

5.1. **BMO, BMOA, and VMO.**

Ref: [7, p. 40-42]

You probably did not suspect any problem in dealing with $f^{(+)}$ in

$$f(x) = f(\cos\theta) = f\left(\frac{z+z^{-1}}{2}\right) = {\sum_0^\infty}' c_k \cos(k\theta) = \frac{1}{2}\sum_{-\infty}^{\infty} c_k z^k \Rightarrow f^{(+)}(z) = {\sum_0^\infty}' c_k z^k.$$

Well, there could be one: f could be bounded on the unit circle $\mathbb{T}$, and $f^{(+)}$ could be unbounded on the unit disk $\mathbb{D}$.

For instance, $f(\cos\theta) = \mathrm{sign}(\pi/2 - |\theta|) = \dfrac{4}{\pi}\displaystyle\sum_{\substack{k\geqslant 1\\ k \text{ odd}}} (-1)^{(k-1)/2}\frac{\cos(k\theta)}{k}$ in $(-\pi,\pi) \Rightarrow f^{(+)}(z) = \dfrac{2}{\pi i}\log\dfrac{1+iz}{1-iz}$.

BMO is the space where the $f^{(+)}$'s and the $f^{(-)}$'s of bounded functions live:

$$\text{BMO } = \{f^{(+)} + g^{(-)}, f \text{ and } g \in L^\infty(\mathbb{T})\}. \tag{16}$$

BMO is a Banach space containing L^∞, but contained in any other L^p. A valuable norm is

$$h \in \text{ BMO } \Rightarrow \|h\| = \inf_{\substack{f,g\in L^\infty \\ f^{(+)}+g^{(-)}=h}} \|f\|_\infty + \|g\|_\infty.$$

BMOA is the analytic part of BMO, just ignore g above:
BMOA $= \{f^{(+)}, f \in L^\infty(\mathbb{T})\}$.

$$h \in \text{ BMOA } \Rightarrow \|h\| = \inf_{\substack{f\in L^\infty \\ f^{(+)}=h}} \|f\|_\infty, \tag{17}$$

which is of course the Hankel norm, i.e., $\sigma_1 = |\lambda_1|$. For the BMO meaning of λ_n, see a later section on AAK.

BMO means "Bounded Mean Oscillation", referring to the initial definition
BMO$= \{h \in L^1(\mathbb{T}) : \sup_{0<\delta<2\pi} M_\delta(h) < \infty$, where $M_\delta(h) = \sup_{|I|\leqslant\delta} \int_I |h(z) - h_I|\,|dz|$, with $h_I = \dfrac{1}{|I|}\displaystyle\int_I h(z)|dz|$,
mean value of h on the arc $I \subset \mathbb{T}$.

Equivalence with the definition (16) above: Fefferman.

VMO, set of functions of vanishing mean oscillation, is the subset of BMO with $M_0 = 0$. VMO is exactly to the continuous functions what BMO is for L^∞ (Sarason). Winding numbers can still be computed with functions of VMO.

5.2. **Approximation spaces.**

Ref: [32-34]

Let f be in a space X of functions defined on $\mathbb{T}$, and consider various norms of best rational approximation $\rho_n(f, X)$:

$$\rho_n(f,p) = \inf \|f-r\|_p, \quad \rho_n(f) = \inf \|f-r\|_\infty, \quad \rho_n(f,*) = \inf \|f-r\|_{\text{BMO}},$$

The problem is: what is the smoothness of the functions with a given rate of decrease of $\rho_n(f, X)$?

Some results [32]:
$f^{(r)}$ of bounded variation $\Rightarrow \rho_n = O(n^{-r-1})$ (Popov)
$f \in$ Lip$_1 \Rightarrow \rho_n = o(1/n)$ (small o!) (Popov: Newman's conjecture)
f piecewise analytic $\Rightarrow \rho_n = O(e^{-c\sqrt{n}})$ (Szűsz & Turan (after Newman))
Let $\kappa(f;n) = \sup \sum_1^{n-1} |f(z_{i+1}) - f(z_i)|$ be the modulus of variation of f. Then,
$\kappa(f;n) = O(n^\beta), f \in$ Lip$_\alpha, \alpha > 0, \Rightarrow, \rho_n(f) = O(n^{\beta+\varepsilon-1})$ (Petrushev)
$\rho_n(f) = O(n^{\beta-1}), 0 < \beta < 1 \Rightarrow \kappa(f;n) = O(n^\beta)$ (Popov)

f is in the ***approximation space*** $R^\alpha_{X,q}$ if

$$\left[\sum_\nu \left(2^{\nu\alpha}\rho_{2^\nu}(f,X)\right)^q\right]^{1/q} < \infty, \qquad \alpha > 0, 0 < q \leqslant \infty.$$

$q = \infty$: $f \in R^\alpha_{X,\infty} \iff \rho_n(f,X) = O(n^{-\alpha})$.

How do the functions in $R^\alpha_{X,q}$ behave?

Answer: $R^\alpha_{p,\sigma} = B^\alpha_\sigma$, where $1/\sigma = 1/p + \alpha$, $1 < p < \infty$, $\alpha > 0$ (***Pekarskii's main theorem***).

For $p = \infty$, use the, yes, BMO metric: $R^\alpha_{*,1/\alpha} = B^\alpha_{1/\alpha}$ (Peller, Semmes, Pekarskii),

where B^α_p is the *Besov* space $\left\{f \in L^p(\mathbb{T}) : \displaystyle\int_{-\pi}^{\pi} |t|^{-1-\alpha p}\|\Delta^n_t f\|^p_p\,dt < \infty\right\}$, with any integer $n > \alpha$, and where Δ^n_t is the n^{th} power of $\Delta_t : (\Delta_t f)(e^{i\theta}) = f(e^{i(\theta+t)}) - f(e^{i\theta})$.

Analytic Besov: $A^\alpha_p = \left\{f^{(+)} \in H^p : \displaystyle\iint_{\mathbb{D}} (1-|z|)^{(n-\alpha)p-1}\left|\left(f^{(+)}\right)^{(n)(z)}\right|^p dxdy < \infty\right\}.$

5.3. **Hankel: AAK and Peller.**

Main result of AAK [2-5], from [29]:

Let $f^{(+)}(z) = \sum_0^\infty c_k z^k$ such that $\mathcal{H}(f^{(+)} = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots \\ c_2 & c_3 & c_4 & \cdots \\ c_3 & c_4 & c_5 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$ is compact on ℓ^2, true iff there is a $g^{(-)}(z) = \sum_1^\infty d_k z^{-k}$ such $f^{(+)} + g^{(-)}$ continuous on $\mathbb{T}$.

Then,

1. $\sigma_{n+1} = \min \|\mathcal{H} - C\|_{\ell^2}$ on all the bounded C's of rank $\leqslant n$ (a general property of singular values of compact operators).
2. $\sigma_{n+1} = \min \|\mathcal{H} - C\|_{\ell^2}$ on all the bounded ***Hankel operators*** C's of rank $\leqslant n$ (first big AAK result).

An excerpt from [29], pp. 241-242:

For the applications we have in mind (which essentially involve finite rank Hankel matrices), the most profitable point of view is usually *sequential* rather than *functional*; that is why we prefer here, unlike the authors of [4], to speak of *singular value decomposition of matrices* (whose exceptionally fruitful idea apparently dates back to E. Beltrami and C. Jordan, 1873-1874, see [C.C. MacDuffee: *The Theory of Matrices*, Chelsea, New York, 1946, p.78], rather than of *Schmidt expansion of operators* (which was introduced by E. Schmidt in the quite different setting of integral equations with non-Hermitian kernels). Now, unlike the best approximation property 1. and the reminder complementing its proof, the following is conditioned by the property of $\mathcal{H}$ to be a Hankel matrix.

3. $\sigma_{n+1} = \min \|\mathcal{H}(B_n f^{(+)})\|_{\ell^2}$ on the Blaschke products $B_n(z) = \prod_{j=1}^n \dfrac{z-\zeta_j}{1-\overline{\zeta_j}\, z}$ with $|\zeta_j| > 1$.

   ($\mathcal{H}(B_n f^{(+)})$ means the Hankel operator built with the coefficients of z, $z^2 \ldots$ of the Laurent expansion on $\mathbb{T}$ of $B_n f^{(+)}$).
4. $\sigma_{n+1} = \min \sup_\theta \left| f^{(+)}(e^{i\theta}) - g^{(-)}(e^{i\theta}) - \sum_{j=1}^n \dfrac{\rho_j}{e^{i\theta} - \zeta_j} \right|$ on $g^{(-)}$, the ρ's and the $|\zeta_j| > 1$'s.

Rate of decrease of the σ's:

$\mathcal{H}(f^{(+)})$ is in the Schatten-von Neumann class $\mathfrak{S}_p$, i.e., $\sum_1^\infty \sigma_n^p < \infty \iff f^{(+)} \in A_p^{1/p}$ $(1 \leqslant p < \infty)$ (Peller [34]).

Exercise: check when $p = 2$: $n = 1$ in the definition of $A_p^{1/p}$ above, and the trace of $\mathcal{H}^2$ is recovered.

Other: Chui, Glover

5.4. **Exponential decrease of the σ's for analytic functions.**

Let $f^{(+)}$ be analytic in a domaining containing $\overline{\mathbb{D}}$, so
$\sigma_{n+1} =$ norm of $\mathcal{H}\left(\left(B_n f^{(+)}\right)^{(+)}\right) \leqslant \sum_\theta \left| \left(B_n(e^{i\theta}) f^{(+)}(e^{i\theta})\right)^{(+)} \right|$.

$\left(B_n(e^{i\theta}) f^{(+)}(e^{i\theta})\right)^{(+)} = \dfrac{1}{2\pi i} \displaystyle\int_C B_n(z) f^{(+)}(z) \, \dfrac{dz}{z - e^{i\theta}}$, where C is any contour outside the unit disk.

We build C and B_n such that: 1. $f^{(+)}$ is still analytic in the interior of C, 2. $\max_z |B_n(z) f^{(+)}(z)|$ is as small as possible on C: weighted Zolotarev problem.

Example: $C = \{z : |z| = R > 1\}$, $\zeta_j = \infty \Rightarrow B_n(z) = z^{-n}$: $\sigma_{n+1} \leqslant \text{const.}/(R - \varepsilon)^n$.

And: 3. numerator of B_n orthogonal with respect to complex weight function $f^{(+)}/$ square of denominator of B_n: connection with Stahl's theory.