6. '1/9'.

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6.1. **Early history and CF link.**

Numerical solution of ordinary differential equations and evolution partial differential equations ask for approximations to the exponential function. The authors of [14] fell on a very remarkable phenomenon: e^{-x} can be approximated by an n^{th} degree rational function on the whole positive line with an exponentially (in n) decreasing error norm. Just try $r_n(x) = \dfrac{1}{1+x+x^2/2!+\cdots+x^n/n!}$: $e^{-x} - r_n(x)$ is negative everywhere and reaches its minimum ($n \geqslant 1$) at $\xi > 0$ such that $-e^{-\xi} + (1+\xi+\cdots+\xi^{n-1}/(n-1)!)/(1+\xi+\cdots+\xi^n/n!)^2 = 0 \Rightarrow |e^{-\xi} - r_n(\xi)| = (\xi^n/n!)/(1+\xi+\cdots+\xi^n/n!)^2 \leqslant (1/n!)/\text{coefficient of } \xi^n \text{ in } (1+\xi+\cdots+\xi^n/n!)^2 = 1/2^n$. Later on, Schönhage showed how the best approximations of the form 1/polynomial behave for various kinds of entire function. The rate of decrease for e^{-x} is $1/3^n$. General real n^{th} degree rational approximants to e^{-x} on $(0,\infty)$ were suspected to have error norms decreasing like $1/9^n$ by Saff and Varga (see [45] for more).

Trefethen and Gutknecht had the wonderful idea in [44] to test CF near-best approximation on THIS problem:

n	E_n	λ_n
0	5.000000000000000E-01	5.601715174207940E-01
1	6.683104216185045E-02	-6.680573308019967E-02
2	7.358670169580528E-03	7.355581867871742E-03
3	7.993806363356878E-04	-7.994517064498902E-04
4	8.652240695288851E-05	8.652095258749368E-05
5	9.345713153026646E-06	-9.345740936352446E-06
6	1.008454374899671E-06	1.008453857122026E-06
7	1.087497491375248E-07	-1.087497586430036E-07
8	1.172265211633491E-08	1.172265194363526E-08
9	1.263292483322314E-09	-1.263292486435714E-09
10	1.361120523345448E-10	1.361120522787731E-10
11	1.466311194937487E-11	-1.466311195036850E-11
12	1.579456837051239E-12	1.579456837033622E-12
13	1.701187076340353E-13	-1.701187076343463E-13
14	1.832174378254041E-14	1.832174378253495E-14
15	1.973138996612803E-15	-1.973138996612899E-15
16	2.124853710495224E-16	2.124853710495207E-16
17	2.288148563247892E-17	-2.288148563247895E-17
18	2.463915737765169E-18	2.463915737765169E-18
19	2.653114658063313E-19	-2.653114658063313E-19
20	2.856777383549094E-20	2.856777383549094E-20

(E_n values from [12]). The near-best quality appears to be better and better as n increases, a very favourable phenomenon, not yet well understood (some possible hints in [28, 1994, pp. 182-183] , see also § 6.3 a little further).

6.2. CF derivation through formal orthogonal polynomials.

Everybody (Cf as well as Remez) start by remarking that

$$e^{-x} - r_n(x), x \in [0,\infty) \iff \exp\left(c\frac{x-1}{x+1}\right) - r_n\left(c\frac{1-x}{x+1}\right),\ x \in [-1,1],$$

where c is any positive number. Through $x = (z+z^{-1})/2$, $\exp(c(x-1)/(x+1)) = \exp(c[(z-1)/(z+1)]^2) := f(z)$ whose Laurent expansion $(\sum_{-\infty}^{\infty} c_k z^k)/2$ yields $f^{(+)}(z)$ and $f^{(+)}(z^{-1})$.

According to § 5.4 of p. 20, we have to estimate the sup [not the sum] of $|(B_n(e^{i\theta})f^{(+)}(e^{i\theta}))^{(+)}|$ where we ca as well replace $f^{(+)}$ by f (as $B_n(z)$ has a convergent expansion in *negative* powers of $z \Rightarrow B_n f^{(-)}$ leaves no trace in the final $(\ )^{(+)}$ operation). So we have to manage to find B_n and C such that $\int_C B_n(z)f(z)\frac{dz}{z-e^{i\theta}}$ is small, the zeros ζ_j of B_n will of course be in C (which will depend on n: C_n).

Orthogonality: the integral of $B_n f(z-e^{i\theta})^{-1}$ will be much smaller if $B_n f$ is formally orthogonal (no complex conjugate) to various $(z - \text{something})^{-1}$ functions, for instance, to the $(1-\zeta_j z)^{-1}$ functions, where $1/\zeta_j$ are the poles of B_n. Then one may replace $(z-e^{i\theta})^{-1}$ in the integral by $B_n(z)(z-e^{i\theta})^{-1}/B_n(e^{i\theta})$, so the problem is to estimate

$$B_n^2(z)f(z) = \prod_{j=1}^{n}\left(\frac{z-\zeta_j}{1-\zeta_j z}\right)^2 \exp c\left(\frac{z-1}{z+1}\right)^2$$

on a valid contour C_n.

Obviously, take $u = \sqrt{c}(z-1)/(z+1)$. Problem becomes:

$$?\sigma_n \approx \min_{\{p_j\}}\max_{u\in D_n}\left|\prod_{j=1}^{n}\left(\frac{u+p_j}{u-p_j}\right)^2 e^{u^2}\right|.$$

Scaling and dequantifying: let $u = n^{1/2}v$ and suppose the $n^{-1/2}p_j$ tend to be distributed with weight w on a fixed locus D:

$$\sigma_n^{1/n} \approx \exp(\mathrm{Re}\,(2\phi(v)+v^2)) \text{ on } \widetilde{D},\ \ \phi(v) = \int_{\widetilde{D}} \log\left(\frac{v+p}{v-p}\right) w(p)\,dp,$$

where $\widetilde{D}$ is the part of D where $\mathrm{Re}\,(\phi + v^2/2)$ is maximal. $\mathrm{Re}\,(\phi+v^2/2)$ is also minimal when one crosses $\widetilde{D}$: if not, a better D could be found. Should $\mathrm{Re}\,\phi$ be smooth up to $\widetilde{D}$, its gradient, so ϕ', would vanish on all $\widetilde{D}$. Actually, what happens is $⨍_{\widetilde{D}} \frac{2pw(p)}{v^2-p^2}\,dp = v$ on $\widetilde{D} \Rightarrow \left(\frac{\phi'(v)}{v}\right)' = \text{constant}\ \frac{1}{v^2\sqrt{(v^2-\xi^2)(v^2-\overline{\xi}^2)}}$, (similar to [21, p. 323]), lots of elliptic integrals and functions, $\sigma^{1/n} \to' 1/9' = 1/9.28902549\ldots$,

$k = \sin\theta_1$	$=$	0.90890855754854147823611890874479350490101396934041
$k' = \cos\theta_1$	$=$	0.41699548440604205639041957807087776692610248051382
$1/q = \exp(\pi K'/K)$	$=$	9.28902549192081891875544943595174506103169486775012
K	$=$	2.32104973253042114734283739983633918849213061106173
$E = K/2$	$=$	1.16052486626521057367141869991816959424606530553086
K'	$=$	1.64669144431946837372958069030713103423036178930922
E'	$=$	1.50010688965199892576311071782207995131063998866470
$\xi_1, \xi_2 = \pi(k' \pm ik)/K$	$=$	0.564412701731271 $\pm$ 1.230228033100522 i

6.3. Relation with complex rational approximation.

Exactly the same formulas were found by Gončar & Rahmanov [21] in their study of '1/9'! A possible explanation:
one starts with the Hermite-Walsh remainder formula with n zeros and poles:

$$e^{-x} - R_n(x) = \frac{1}{2\pi i} \prod_1^n \frac{x - \text{zeros}}{x - \text{poles}} \int_{\Gamma_n} \frac{e^{-t}}{t-x} \prod_1^n \frac{t - \text{poles}}{t - \text{zeros}}\, dt,$$

we get squares by asking $\prod_1^n (t - \text{poles})$ to be orthogonal to $e^{-t}/\prod_1^{\mathbf{2}n}(t - \text{zeros})$:

$$e^{-x} - R_n(x) = \frac{1}{2\pi i} \prod_1^{2n} \frac{x - \text{zeros}}{x - \text{poles}} \int_{\Gamma_n} \frac{e^{-t}}{t-x} \prod_1^{2n} \frac{t - \text{poles}}{t - \text{zeros}}\, dt.$$

It begins to look to the $B_n^2 f$ above, up to the change of variable $x = -u^2$, $t = -p^2$. The Blaschke product was changed into $\mathcal{B}_n(u) = \prod_1^n \dfrac{u + p_j}{u - p_j}$. How to recover $\prod_1^{2n}(x - \text{zeros})/(x - \text{poles}) = \prod_1^{2n}(u^2 - \cdots)/(u^2 - p_j^2)$?

Possible answer: take $(\mathcal{B}_n(u))^2 + (\mathcal{B}_n(-u))^2 = (\mathcal{B}_n(u))^2 + \dfrac{1}{(\mathcal{B}_n(u))^2}$.

This could be a link between the CF auxiliary functions and the rational approximation.

6.4. About formal complex orthogonal polynomials and non formal ones.

The minimization of $\left| \displaystyle\int_S \frac{P_n(t)}{t-x} w(t)dt \right|$ through orthogonal polynomials is a quite sensible idea when the support S is real and $w > 0$: a simple interpolatory remainder formula is turned as a Gaussian integration remainder formula.

On a complex support, any sensible estimate should involve the orthogonal polynomials such that $\displaystyle\int_S P_n(z)\overline{P_m(z)}\, d\mu(z) = 0, m \neq n$, with $d\mu$ a positive measure. Not exactly the same thing than what we need: $\displaystyle\int_S P_n(z) P_m(z) g(z)\, dz = 0, m \neq n$, where everything is complex.

HOWEVER, near S, P_n behaves like $\exp(n\psi)$, with $\operatorname{Re}\psi =$ constant on the meaningful part of S, let ψ_+ and ψ_- be the limit values of ψ on the the two sides of S. The normal derivatives of $\operatorname{Re}\psi_+$ and $\operatorname{Re}\psi_-$ are opposite (Stahl's symmetry property [36, etc.]). So (Cauchy-Riemann), the *tangent* derivatives of $\operatorname{Im}\psi_+$ and $\operatorname{Im}\psi_-$ are opposite along S. So, P_n behaves on S like a mixture of constant times some $\exp(in\theta)$ and $\exp(-in\theta)$ ...

The two kinds of orthogonal polynomials essentially behave in the same way on these optimal supports S.

Unfortunately, these nice results only hold in a weak sense, in capacity and/or on subsequences, as far as we want them to hold for a wide class of function. But things may be much better for small function classes.

Morality: switch to speciaL functioN (no s please) theory.

6.5. The integral Hankel operator.

A strange feature of early CF '1/9' studies was that the eigenvalues of the Hankel matrix have a much simpler asymptotic behaviour that the matrix elements themselves. However, by playing with the arbitrary positive parameter $c \to \infty$ in $\exp[c(x-1)/(x+1)]$ (there must be a neater proof), it appears that we have to deal with the eigenvalues and eigenfunctions of $\mathcal{H}$:

$$(\mathcal{H}f)(x) := \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(x+y)^2} f(y)\, dy. \tag{18}$$

See some eigenvalues and eigenfunctions in the figures.

I suspect the following: n^{th} eigenfunction behaves like $A\exp(n\psi(2xn^{-1/2}))+B\exp(-n\psi(2xn^{-1/2}))$, when $0<x<n^{1/2}$, where ψ, with $\psi(0)=0$, is such that ψ' is inverse to ϕ' (which amounts to inverting elliptic integrals of the second kind, gasp).

6.6. **More (open) problems.**

Main riddle: n^{th} eigenvalue looks like

$$\lambda_n \sim 2(-1)^n q^{n+1/2}\ \exp\left(-\frac{1}{12(n+1/2)}+\frac{A}{(n+1/2)^5}+O(n^{-7}\right)$$

what is A (and further coefficients)?

$$192A=-0.40799445880\ldots$$

Is it useful to approach the Hankel eigenproblem by a ... *differential* eigenproblem???

Replace $\exp(-(x+y)^2)$ in (18) by $\left(1-\dfrac{(x+y)^2}{N}\right)^N$ on $0\leqslant y\leqslant\sqrt{N}-x$. Then, $\mathcal{H}_n f=\lambda f \Rightarrow$

$$\frac{2\sqrt{N}}{\sqrt{\pi}}\int_0^{1-x}F^{(N+1)}(y)\frac{d^N}{dy^N}((x+y)^2-1)^N\,dy=\frac{2\sqrt{N}}{\sqrt{\pi}}C_N\int_0^{1-x}F^{(N+1)}(y)P_N(x+y)\,dy=\lambda\,F^{(2N+1)}(x),$$

if $f(x\sqrt{N})=F^{(2N+1)}(x)$, with $F(0)=\cdots=F^{(2N)}(0)=0$.

More integrations by parts:

$$\lambda F^{(2N+1)}(x)=\frac{2\sqrt{N}}{\pi}C_N[P_N(1)F^{(N)}(1-x)-P_N'(1)F^{(N-1)}(1-x)+\cdots+(-1)^N F(1-x)P_N^{(N)}(1)],$$

Put $F(x)=\sum_\xi(X(\xi)e^{\xi x}+Y(\xi)e^{-\xi x}$:

$$\lambda X(\xi)\xi^{2N+1}=\frac{2\sqrt{N}}{\pi}e^{-\xi}Y(\xi)\Pi_N(\xi),$$
$$-\lambda Y(\xi)\xi^{2N+1}=\frac{2\sqrt{N}}{\pi}e^{\xi}X(\xi)\Pi_N(-\xi),$$

where $\Pi_N(\xi)=C_N(-1)^N[P_N(1)\xi^N+P_N'(1)\xi^{N-1}+\cdots+P_N^{(N)}]$.

$$\frac{4N}{\pi}\Pi_N(\xi)\Pi_N(-\xi)+\lambda^2\xi^{4N+2}=0.$$