MAPA 3011A Special topics in approximation theory.

1997-1998: New difference calculus & orthogonal polynomials.

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We investigate a difference operator which extends the usual derivative and the classical difference operators, and appears as a kind of ultimate possibility within a certain class, especially as far as mappings of polynomials are considered.

The main bulk of the study is on special orthogonal polynomials satisfying difference equations. Although the new orthogonal families do not seem (yet) to have many applications in approximation problems[1], a unified theory is always liable to shed more light on a given field.

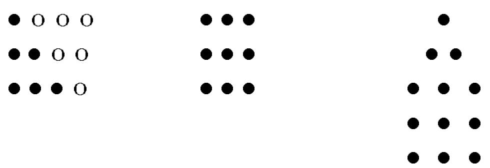
1. Difference calculus through the ages.

1.1. Arithmetic progressions.

Formula for, say, $1+2+\cdots+n$ can be constructed by looking at $f(n)$ such that $f(n+1)-f(n) = n+1$. $f(n) = n(n+1)/2$ is readily found by trial and error, or by amusing (or deep??) links with geometry (***polygonal numbers***[2]):

triangular square pentagonal

see [31, pp. 24-27, 499]



1.2. Summation formulas.

Not much more was found before the 17$^{\text{th}}$ century, when Jakob Bernoulli[3] built a unified theory of the sums

$$S_k(n) = 1 + 2^k + \cdots + n^k.$$

Of course, we see that $S_k(n+1) - S_k(n) = (n+1)^k$, so that we look for a discrete integral of x^k through a function whose discrete derivative (***difference***) is given. The fundamental theorem of discrete calculus is utterly trivial. The development of such an elementary difference calculus had to wait for the birth of the plain calculus, probably from competition.

Moreover, there is an extremely interesting interplay between difference and differential calculus: write
$S_k(x+1) - S_k(x) = (x+1)^k$ and ... differentiate in x!
$S'_k(x+1) - S'_k(x) = k(x+1)^{k-1}$, showing that
$S'_k = kS_{k-1}$, plus a constant. One finds, from $S_k(0) = 0$ and $S_k(1) = 1$[4]:

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[1] One of the first applications deals with a new kind of discrete spherical harmonics [27]. Interesting, as plain spherical harmonics involve the first orthogonal polynomials encountered in history, the Legendre polynomials!

[2] See also the appendix.

[3] (1654-1705) in his posthumus *Ars Conjectandi* (1713), see [20]

[4] The actual Bernoulli polynomials are not the S_k polynomials but $B_{k+1}(x) = (k+1)S_k(x-1)+$ a further constant such that $\int_0^1 B_{k+1}(x)\,dx = 0$: 1, $x - 1/2$, $x^2 - x + 1/6$, $x^3 - 3x^2/2 + x/2$, $x^4 - 2x^3 + x^2 - 1/30$ ...

$S_0(x)$	$S_1(x)$	$S_2(x)$	$S_3(x)$
x	$x(x+1)/2$	$x^3/3+x^2/2+x/6$	$x^4/4+x^3/2+x^2/4 = S_1^2(x)$ (incidentally (but see [4]).)

A much simpler set of polynomials with respect to difference calculus is the set of ***factorial polynomials***, for instance the lower[5] factorial polynomial
$p_n(x-a) = (x-a)(x-a-1)\cdots(x-a-n+1)$. We immediately find

$$\Delta p_n(x-a) = p_n(x-a+1) - p_n(x-a) = np_{n-1}(x-a). \qquad (1)$$

The simplest way to find the sum of a given polynomial is therefore to expand it in the p_n's (Newton's interpolation expansion), and to sum each of the p_n's from (1).

Of course, (1) is the discrete extension of the derivative of a power. ***Hypergeometric expansions*** contain as well factorial polynomials of some arguments and plain powers of other arguments, allowing for a rich set of difference and differential formulas.

1.3. Classical difference operators.

[24]:

Δ	∇	δ	E	$\boldsymbol{\mu}$
$f(x+h)-f(x)$	$f(x)-f(x-h)$	$f(x+h/2)-f(x-h/2)$	$f(x+h)$	(f(x+h/2)+f(x-h/2))/2

1.4. Geometric progressions and the first $q-$difference operator.

Solving simple instances of $f(qx)-f(x) = g(x)$ amounts to a sum of $g(x)$, $g(qx)$, $g(q^2x)$,etc.:

$$\begin{aligned}
f(qx) - f(x) &= g(x)\\
f(q^2x) - f(qx) &= g(qx)\\
f(q^3x) - f(q^2x) &= g(q^2x)\\
\cdots \quad &\quad \cdots\\
f(q^Nx) - f(q^{N-1}x) &= g(q^{N-1}x)\\
\hline
f(q^Nx) - f(x) &= g(x) + g(qx) + \cdots + g(q^{N-1}x)
\end{aligned}$$

Classical geometric progression, $g(x) = x$, asks for $f(x) = \dfrac{x}{q-1}$.

Remark that the difference of a polynomial of degree n is no more a polynomial of degree $n-1$. This is settled by using the ***divided difference*** $D_q f(x) = \dfrac{f(qx)-f(x)}{(q-1)x}$.

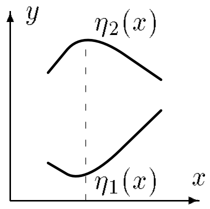
Here again, shall we find simple factorial-like polynomials recovered by the action of D_q? $p_n(x)$ should be a product such that $p_n(qx)$ contains almost all the factors of $p_n(x)$. Le me try $p_n(x) = (x-a)(x-qa)\ldots(x-q^{n-1}a)$, then $p_n(qx) = (qx-a)(qx-qa)\ldots(qx-q^{n-1}a) =$, hmm, well, let's go further, $p_n(qx) = q^n(x-a/q)(x-a)\ldots(x-q^{n-2}a)$, so,
$p_n(qx)-p_n(x) = (x-a)\ldots(x-q^{n-2}a)[q^nx - aq^{n-1} - x + q^{n-1}a]$,

$$D_q(x-a)(x-qa)\ldots(x-q^{n-1}a) = \frac{q^n-1}{q-1}(x-a)\ldots(x-q^{n-2}a).$$

[5] written $(x-a)_n$: the subscript is the *umbra* (shadow) of the exponent in the usual power [10].

2. *The* difference operator.

2.1. Introducing $\mathscr{D}$.



We look at the most general (divided) difference operator of the form

$$(\mathscr{D}f)(x) = \frac{f(\eta_2(x)) - f(\eta_1(x))}{\eta_2(x) - \eta_1(x)} \tag{2}$$

leaving a polynomial of degree $n-1$ when f is a polynomial of degree n. The first non trivial conditions are

$$\eta_1(x) + \eta_2(x) = \text{ a polynomial of degree 1;}$$
$$\eta_1^2(x) + \eta_1(x)\eta_2(x) + \eta_2^2(x) = \text{ a polynomial of degree 2.}$$

This is enough to show that $\eta_1(x)$ and $\eta_2(x)$ MUST be the two roots y of some quadratic equation

$$F(x,y) := \mathfrak{a}y^2 + 2\mathfrak{b}xy + \mathfrak{c}x^2 + 2\mathfrak{d}y + 2\mathfrak{e}x + \mathfrak{f} = 0, \tag{3}$$

and that

$$\eta_1(x) = \mathfrak{p}(x) + \sqrt{\mathfrak{q}(x)}, \quad \eta_2(x) = \mathfrak{p}(x) - \sqrt{\mathfrak{q}(x)},$$

with $\mathfrak{p}$ and $\mathfrak{q}$ polynomials of degrees $\leqslant 1$ and 2,

$$\mathfrak{p}(x) = p_0x + p_1 = -\frac{\mathfrak{b}x+\mathfrak{d}}{\mathfrak{a}}, \quad \mathfrak{q}(x) = q_0x^2 + q_1x + q_2 = \frac{(\mathfrak{b}^2 - \mathfrak{a}\mathfrak{c})x^2 + (\mathfrak{b}\mathfrak{d} - \mathfrak{a}\mathfrak{e})x + \mathfrak{d}^2 - \mathfrak{a}\mathfrak{f}}{\mathfrak{a}^2}.$$

So, we have, up to now,

$f(x)$	$\mathscr{D}f(x)$
1	0
x	1
x^2	$2\mathfrak{p}(x)$
x^3	$3\mathfrak{p}^2(x) + \mathfrak{q}(x)$

the action of $\mathscr{D}$ on polynomials up to degree 3. Great! We can check that $\mathscr{D}$ applied to a polynomial of any degree yields another polynomial by expanding powers of $\mathfrak{p} \pm \sqrt{\mathfrak{q}}$, a not very elegant procedure. We can try to begin a list of algebraic properties of $\mathscr{D}$. We need the companion operator

$$\mathscr{M}f(x) = \frac{f(\eta_1(x)) + f(\eta_2(x))}{2}$$

then

$$\begin{aligned}(\mathscr{D}fg)(x) &= \frac{f(\eta_2(x))g(\eta_2(x)) - f(\eta_1(x))g(\eta_1(x))}{\eta_2(x) - \eta_1(x)} \\ &= \frac{\left[\mathscr{M}f + \sqrt{\mathfrak{q}}\mathscr{D}f\right]\left[\mathscr{M}g + \sqrt{\mathfrak{q}}\mathscr{D}g\right] - \left[\mathscr{M}f - \sqrt{\mathfrak{q}}\mathscr{D}f\right]\left[\mathscr{M}g - \sqrt{\mathfrak{q}}\mathscr{D}g\right]}{2\sqrt{q}} \\ &= \mathscr{M}f\ \mathscr{D}g + \mathscr{M}g\ \mathscr{D}f.\end{aligned} \tag{4}$$

$$\begin{aligned}(\mathscr{M}fg)(x) &= \frac{f(\eta_2(x))g(\eta_2(x)) + f(\eta_1(x))g(\eta_1(x))}{2} \\ &= \frac{\left[\mathscr{M}f + \sqrt{\mathfrak{q}}\mathscr{D}f\right]\left[\mathscr{M}g + \sqrt{\mathfrak{q}}\mathscr{D}g\right] + \left[\mathscr{M}f - \sqrt{\mathfrak{q}}\mathscr{D}f\right]\left[\mathscr{M}g - \sqrt{\mathfrak{q}}\mathscr{D}g\right]}{2} \\ &= \mathscr{M}f\ \mathscr{M}g + \mathfrak{q}\mathscr{D}f\ \mathscr{D}g.\end{aligned} \tag{5}$$

So, the operators $\mathscr{D}$ and $\mathscr{M}$ map polynomials of degree n to polynomials of degrees $\leqslant n-1$ and n:

$$\begin{bmatrix}\mathscr{D}xf \\ \mathscr{M}xf\end{bmatrix} = \begin{bmatrix}\mathfrak{p} & 1 \\ \mathfrak{q} & \mathfrak{p}\end{bmatrix}\begin{bmatrix}\mathscr{D}f \\ \mathscr{M}f\end{bmatrix},$$

and also $\mathscr{D}x^2f = 2\mathfrak{p}\mathscr{D}xf + (\mathfrak{q}-\mathfrak{p}^2)\mathscr{D}f$. Is this $(\mathscr{D},\mathscr{M})$ treatment so efficient? Painful experiences lead me to prefer often the plain $f(\eta_1(x))$, $f(\eta_2(x))$ form. For instance, what is the coefficient of x^{n-1} of $\mathscr{D}x^n$? A little bit of brute force does no harm:

$$\mathscr{D}x^n = \frac{\eta_2^n(x)-\eta_1^n(x)}{\eta_2(x)-\eta_1(x)} = \frac{(\mathfrak{p}+\sqrt{\mathfrak{q}})^n-(\mathfrak{p}-\sqrt{\mathfrak{q}})^n}{2\sqrt{\mathfrak{q}}} = \frac{(p_0+\sqrt{q_0})^n-(p_0-\sqrt{q_0})^n}{2\sqrt{q_0}}x^{n-1}+\cdots$$

Remember that we had $D_q x^n = \dfrac{q^n-1}{q-1}x^{n-1}+\cdots$. Will $\dfrac{p_0+\sqrt{q_0}}{p_0-\sqrt{q_0}}$ act as q? Suspense ...

Difference of a ratio:

$$\begin{aligned}\left(\mathscr{D}\frac{f}{g}\right)(x) &= \frac{f(\eta_2(x))/g(\eta_2(x))-f(\eta_1(x))/g(\eta_1(x))}{\eta_2(x)-\eta_1(x)}\\ &= \frac{\left[\mathscr{M}f+\sqrt{\mathfrak{q}}\mathscr{D}f\right]\left[\mathscr{M}g-\sqrt{\mathfrak{q}}\mathscr{D}g\right]-\left[\mathscr{M}f-\sqrt{\mathfrak{q}}\mathscr{D}f\right]\left[\mathscr{M}g+\sqrt{\mathfrak{q}}\mathscr{D}g\right]}{2\sqrt{q}g(\eta_1(x))g(\eta_2(x))}\\ &= \frac{\mathscr{D}f\ \mathscr{M}g-\mathscr{D}g\ \mathscr{M}f}{g(\eta_1(x))g(\eta_2(x))=(\mathscr{M}g)^2-\mathfrak{q}(\mathscr{D}g)^2}.\end{aligned}$$

2.2. $\mathscr{D}$–factorial polynomials, lattices.

Trying to find a basis of polynomials behaving nicely under $\mathscr{D}$, we look for a product

$$\mathsf{p}_n(x) = (x-y_1)(x-y_2)\cdots(x-y_n) \tag{6}$$

such that $\mathsf{p}_n(\eta_1(x))$ and $\mathsf{p}_n(\eta_2(x))$ have $n-1$ common factors:

$$\begin{aligned}\mathsf{p}_n(\eta_1(x)) &= (\eta_1(x)-y_1)(\eta_1(x)-y_2)\cdots(\eta_1(x)-y_{n-1})(\eta_1(x)-y_n)\\ \mathsf{p}_n(\eta_2(x)) &= (\eta_2(x)-y_1)(\eta_2(x)-y_2)\cdots(\eta_2(x)-y_{n-1})(\eta_2(x)-y_n)\end{aligned}$$

Well, well, we can't ask for some $\eta_1(x)-y_j$ to be the same as some $\eta_2(x)-y_k$ *for all* x! This worked only with Δ, as $\eta_2(x)-\eta_1(x) = $ constant only then.
With D_q, we found the way out by requesting $\eta_2(x)-y_{j+1} = $ constant $(\eta_1(x)-y_j)$, $j=1,\ldots,n-1$. And now?

We shall still manage to make $\mathsf{p}_n(\eta_1(x))$ and $\mathsf{p}_n(\eta_2(x))$ to vanish *simultaneously* for $n-1$ values of x by having

$$y_1 = \eta_1(x_1), y_2 = \eta_1(x_2) = \eta_2(x_1), \ldots, y_n = \eta_1(x_n) = \eta_2(x_{n-1}).$$

So, $\mathsf{p}_n(\eta_1(x)) = 0$ at $x = x_1, x_2, \ldots, x_n$,
$\mathsf{p}_n(\eta_2(x)) = 0$ at $x = x_1, \ldots, x_{n-1}$, (and $x = x_0$), and
$\mathscr{D}\mathsf{p}_n$

1. is a polynomial of degree $n-1$,
2. vanishes when $x = x_1, x_2, \ldots, x_{n-1}$.

So,

$$(\mathscr{D}\mathsf{p}_n)(x) = \mathscr{D}(x-y_1)(x-y_2)\cdots(x-y_n) = C_n(x-x_1)(x-x_2)\cdots(x-x_{n-1}) = C_n\tilde{\mathsf{p}}_{n-1}(x), \tag{7}$$

where $y_j = \eta_1(x_j), y_{j+1} = \eta_2(x_j)$:

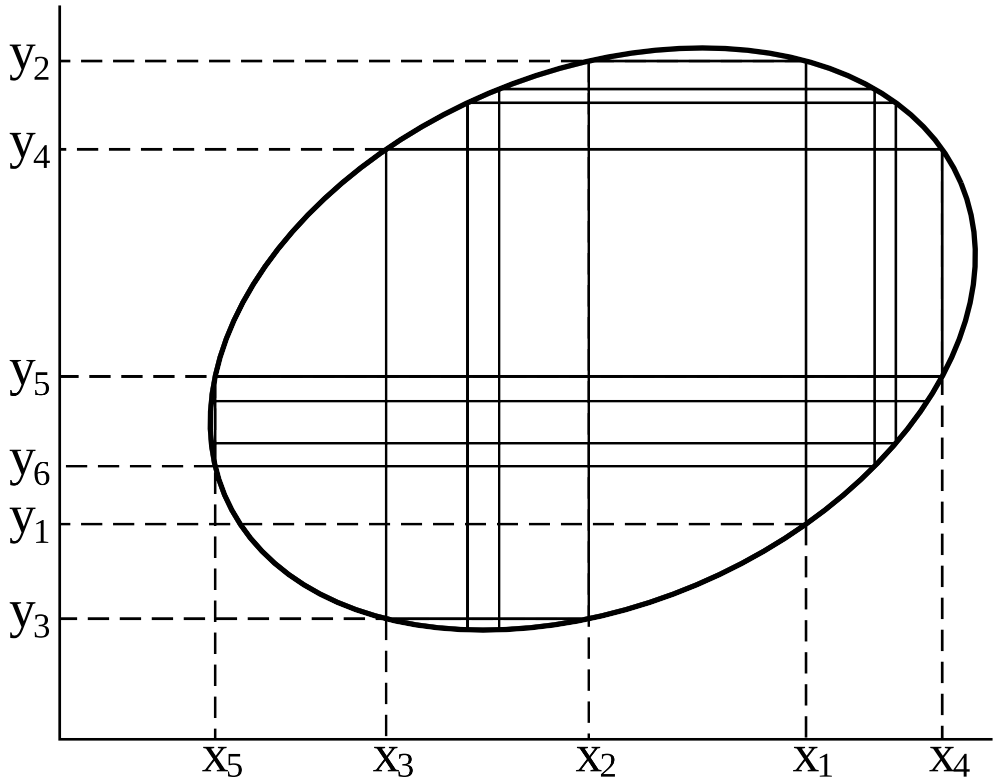


Figure 1. $x-$ and $y-$ lattices.

and where we already encountered the constant $C_n = \dfrac{(p_0+\sqrt{q_0})^n - (p_0-\sqrt{q_0})^n}{2\sqrt{q_0}}$.

Extension to negative indexes:

$$\text{if } n<0,\, \mathsf{p}_n(x) = \frac{1}{(x-y_{n+1})(x-y_{n+2})\cdots(x-y_0)}\,.$$

Problem: we can discuss the effect of $\mathscr{D}$ on factorial or inverse factorial series [Nörlund]. What are the convergence regions of $\mathscr{D}-$factorial series??

Exercise. Show that

$$\begin{aligned}\mathscr{M}(x-y_1)(x-y_2)\cdots(x-y_n) &= [D_n(x-x_n)+E_n](x-x_1)(x-x_2)\cdots(x-x_{n-1}),\\ &= D_n\tilde{\mathsf{p}}_n(x)+E_n\tilde{\mathsf{p}}_{n-1}(x)\end{aligned} \tag{8}$$

and look at some values of D_n and E_n, $n=0,1,\ldots$

The points $\{x_i\}$ and $\{y_i\}$ make the ***lattices*** which are our main subject of study. The relevant discrete operators act on functions defined on such lattices.

A set of $x-$ and $y-$ lattices is completely determined by a starting point (x_1, y_1) on the conic (3). The, y_2 is the second ordinate corresponding to x_1, x_2 is the second abscissa corresponding to y_2, etc.

Exercise (already stated by people trying to embarrass the author): what if there is *no* second ordinate, i.e., if x_1 is the abscissa of a vertical tangent to the conic?

Note that we should not define η_1 as the lower part of the conic as, after some steps, y_k may very well be the upper ordinate corresponding to x_k. The geometry of the points (x_k, y_k) supersedes clumsy algebraic definitions. Relevant items are symmetric functions of η_1 and η_2 anyhow.

Action of $\mathscr{D}$ and $\mathscr{M}$ on factorial polynomials.

Let us work (7) and (8) more explicitly: as $\mathsf{p}_{n+1}(x) = (x - y_{n+1})\mathsf{p}_n(x)$, use (4)-(5):

$$\mathscr{D}\mathsf{p}_{n+1}(x) = \mathscr{M}\mathsf{p}_n(x) + (\mathfrak{p}(x) - y_{n+1})\mathscr{D}\mathsf{p}_n(x),$$
$$\mathscr{M}\mathsf{p}_{n+1}(x) = (\mathfrak{p}(x) - y_{n+1})\mathscr{M}\mathsf{p}_n(x) + \mathfrak{q}(x)\mathscr{D}\mathsf{p}_n(x),$$

and put (7)-(8):

$$C_{n+1}\tilde{\mathsf{p}}_n(x) = D_n\tilde{\mathsf{p}}_n(x) + E_n\tilde{\mathsf{p}}_{n-1}(x) + (\mathfrak{p}(x) - y_{n+1})C_n\tilde{\mathsf{p}}_{n-1}(x),$$
$$D_{n+1}\tilde{\mathsf{p}}_{n+1}(x) + E_{n+1}\tilde{\mathsf{p}}_n(x) = (\mathfrak{p}(x) - y_{n+1})[D_n\tilde{\mathsf{p}}_n(x) + E_n\tilde{\mathsf{p}}_{n-1}(x)] + \mathfrak{q}(x)C_n\tilde{\mathsf{p}}_{n-1}(x),$$

or, after division by $\tilde{\mathsf{p}}_{n-1}(x)$,

$$C_{n+1}(x - x_n) = D_n(x - x_n) + E_n + (\mathfrak{p}(x) - y_{n+1})C_n,$$
$$D_{n+1}(x - x_n)(x - x_{n+1}) + E_{n+1}(x - x_n) = (\mathfrak{p}(x) - y_{n+1})[D_n(x - x_n) + E_n] + \mathfrak{q}(x)C_n,$$

we already know C_n, check $D_n = C_{n+1} - p_0C_n$, and find $E_n = -(\mathfrak{p}(x_n) - y_{n+1})C_n = -C_n\mathfrak{q}(x_n)/(\mathfrak{p}(x_n) - y_{n+1})$:

$$C_n = \frac{(p_0 + \sqrt{q_0})^n - (p_0 - \sqrt{q_0})^n}{2\sqrt{q_0}},\ D_n = \frac{(p_0 + \sqrt{q_0})^n + (p_0 - \sqrt{q_0})^n}{2},\ E_n = C_n\frac{y_{n+1} - y_n}{2}. \tag{9}$$

So, $\mathscr{D}$ and $\mathscr{M}$ transform expansions of p's as expansions of $\tilde{\mathsf{p}}$'s.
We will of course need operators making the reverse action.
Let

$$(\widetilde{\mathscr{D}}f)(y) := \frac{f(\xi_2(y)) - f(\xi_1(y))}{\xi_2(y) - \xi_1(y)}, \tag{10}$$

where $\xi_1(y)$ and $\xi_2(y)$ are the roots in x of (3) for a given y. Then,
$\tilde{\mathsf{p}}_n(\xi_1(y)) = (\xi_1(y) - x_1)\cdots(\xi_1(y) - x_n)$ vanishes at $y = y_1, \ldots, y_n$, and
$\tilde{\mathsf{p}}_n(\xi_2(y)) = (\xi_2(y) - x_1)\cdots(\xi_2(y) - x_n)$ vanishes at $y = y_2, \ldots, y_n$, (and $y = y_{n+1}$):

$$\widetilde{\mathscr{D}}(y - x_1)\cdots(y - x_n) = \widetilde{C}_n(y - y_2)\cdots(y - y_n) \tag{11}$$

2.3. q again.

Let us try to get the inside secrets of these lattices. Remember that the two ordinates y_k and y_{k+1} corresponding to x_k are the two roots of (3) when $x = x_k$, so, their sum is

$$y_k + y_{k+1} = 2\mathfrak{p}(x_k) = -2(\mathfrak{b}x_k + \mathfrak{d})/\mathfrak{a}.$$

Write the same at $k - 1$ and sum:

$$y_{k-1} + 2y_k + y_{k+1} = -2(\mathfrak{b}(x_{k-1} + x_k) + 2\mathfrak{d})/\mathfrak{a}.$$

But x_{k-1} and x_k are the two abscissae corresponding to y_k: look now at x as a root of (3) for given y:

$$x_{k-1} + x_k = -2(\mathfrak{b}y_k + \mathfrak{e})/\mathfrak{c}.$$

And voila!

$$\begin{aligned} y_{k-1} + (2 - 4\mathfrak{b}^2/\mathfrak{a}\mathfrak{c})y_k + y_{k+1} &= 4\frac{\mathfrak{b}\mathfrak{e} - \mathfrak{c}\mathfrak{d}}{\mathfrak{a}\mathfrak{c}}, \\ x_{k-1} + (2 - 4\mathfrak{b}^2/\mathfrak{a}\mathfrak{c})x_k + x_{k+1} &= 4\frac{\mathfrak{b}\mathfrak{d} - \mathfrak{a}\mathfrak{e}}{\mathfrak{a}\mathfrak{c}}. \end{aligned} \tag{12}$$

Remark that $\mathfrak{a}\mathfrak{c} \neq 0$ if we want the two lattices to exist.
The common exponential part of the solutions of the recurrences (12) above will of course be called q^k and q^{-k}:

$$\begin{aligned} y_k &= \text{ const. } q^k + \text{const. } q^{-k} + \frac{\mathfrak{b}\mathfrak{e}-\mathfrak{c}\mathfrak{d}}{\mathfrak{a}\mathfrak{c}-\mathfrak{b}^2}, \\ x_k &= \text{ const. } q^k + \text{const. } q^{-k} + \frac{\mathfrak{b}\mathfrak{d}-\mathfrak{a}\mathfrak{e}}{\mathfrak{a}\mathfrak{c}-\mathfrak{b}^2}, \end{aligned} \tag{13}$$

where

$$q+q^{-1} = 4\frac{\mathfrak{b}^2}{\mathfrak{a}\mathfrak{c}} - 2. \tag{14}$$

Remark that the last terms are the coordinates of the *center* of the conic (3). (13) is no more valid if $\mathfrak{b}^2 = \mathfrak{a}\mathfrak{c}$, i.e., $q=1$. However, (12) is then easily solved as

$$\begin{aligned} y_k &= \text{ const. } + \text{const. } k + 2\frac{\mathfrak{b}\mathfrak{e}-\mathfrak{c}\mathfrak{d}}{\mathfrak{a}\mathfrak{c}}k^2, \\ x_k &= \text{ const. } + \text{const. } k + 2\frac{\mathfrak{b}\mathfrak{d}-\mathfrak{a}\mathfrak{e}}{\mathfrak{a}\mathfrak{c}}k^2. \end{aligned} \tag{15}$$

The shape of real conics is linked to q:

q	conic	Fig.
$\lvert q\rvert = 1$	Ellipse	1.
q real $\neq \pm 1$	Hyperbola	$2a, 2b$
$q=1$	Parabola	$2c, 2d$

Actually, the four constants in (13) or (15) are not independent: choose a square root of $\mathfrak{a}\mathfrak{c}$ and take $q^{1/2}$ from

$$q^{1/2}+q^{-1/2} = 2\frac{\mathfrak{b}}{\sqrt{\mathfrak{a}\mathfrak{c}}} \tag{14'}$$

then, if

$$x_k = Aq^k + Bq^{-k} + \frac{\mathfrak{b}\mathfrak{d}-\mathfrak{a}\mathfrak{e}}{\mathfrak{a}\mathfrak{c}-\mathfrak{b}^2},$$

$x_{k-1}+x_k = -2(\mathfrak{b}y_k+\mathfrak{e})/\mathfrak{c}$ shows that

$$y_k = -\sqrt{\frac{\mathfrak{c}}{\mathfrak{a}}}\left[Aq^{k-1/2} + Bq^{-k+1/2}\right] + \frac{\mathfrak{b}\mathfrak{e}-\mathfrak{c}\mathfrak{d}}{\mathfrak{a}\mathfrak{c}-\mathfrak{b}^2}.$$

Similarly, if $q = q^{1/2} = 1$,i.e., $\mathfrak{b} = \sqrt{\mathfrak{a}\mathfrak{c}}$,

$$\begin{aligned} x_k &= A + Bk + 2\mathfrak{a}\frac{\mathfrak{d}\sqrt{\mathfrak{c}}-\mathfrak{e}\sqrt{\mathfrak{a}}}{\mathfrak{a}\mathfrak{c}}k^2 \Rightarrow \\ &\qquad y_k = -\left(\frac{\mathfrak{d}}{2\mathfrak{a}} + \frac{\mathfrak{e}}{2\sqrt{\mathfrak{a}\mathfrak{c}}}\right) - \sqrt{\frac{\mathfrak{c}}{\mathfrak{a}}}\left[A + B(k-1/2) + 2\mathfrak{a}\frac{\mathfrak{d}\sqrt{\mathfrak{c}}-\mathfrak{e}\sqrt{\mathfrak{a}}}{\mathfrak{a}\mathfrak{c}}(k-1/2)^2\right] \end{aligned}$$

Funny problem: So, $q+q^{-1} \in (-1,1)$ is for ellipses, $q+q^{-1} \notin (-1,1)$ for hyperbolas, and $q+q^{-1} = 2$, i.e., $q=1$ for parabolas, as expected. But what happens at $q=-1$??

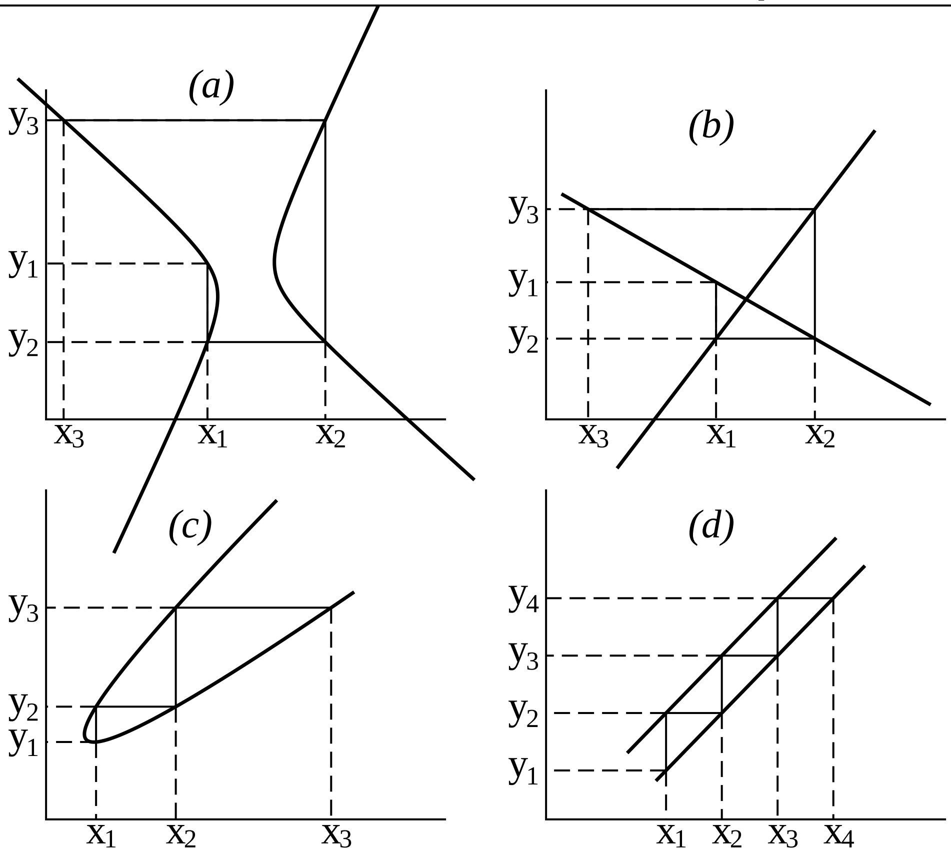


Figure 2. Other kinds of lattices.

The figures are picked from [22].

A single pair of (x, y) lattices determine the conic (3) if there are at least 5 distinct points. That's why $q = -1$ is so special: each lattice pair has only 4 points, belonging to ellipses as well as to hyperbolas (solution to the funny problem).

2.4. About factorial series as (basic or not) hypergeometric series.

"Old" plain difference calculus needs factorial series containing products of the form $\prod_k (x - ak - b)$, i.e., plain hypergeometric series. With "old" q–difference calculus, $\prod_k (x - aq^k - b)$, so the so-called ***basic*** hypergeometric expansions.

(Remember [18] that the ratio of the $(k+1)^{\text{th}}$ term to the k^{th} one of ${}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)$ is

$$\frac{\text{term}_{k+1}}{\text{term}_k} = \frac{(1 - a_1 q^k) \cdots (1 - a_r q^k)(-q^k)^{1+s-r} z}{(1 - b_1 q^k) \cdots (1 - b_s q^k)(1 - q^{k+1})}$$

[one starts with the 0^{th} term]).

And now, new expansions?

No.

If $q = 1$, factorize $x - ak^2 - bk - c$ in $\prod_k (x - ak^2 - bk - c)$ as const. $(k - \xi_1(x))(k - \xi_2(x))$! Result is again a plain hypergeometric expansion, ***with twice as many parameters***, involving the algebraic functions $\xi_1(x)$ and $\xi_2(x)$ in the parameters.

Similarly, if $q \neq 1$, one factorizes $x - Aq^k - Bq^{-k} - C$ as, say, const. $q^{-k}(q^k - \xi_1(x))(q^k - \xi_2(x))$ to get a basic hypergeometric series.

So, the familiar ${}_2F$ and ${}_2\phi$ expansions of the classical orthogonal polynomials will be succeeded by ${}_4F$ and ${}_4\phi$ expansions, very likely. See also [2, § 5, p. 33].

2.5. **Parametric representation, the Eastern way.**

Just replace k above by a continuous parameter s:

$$\begin{aligned} x = x(s) &= Aq^s + Bq^{-s} + \text{ const.},\\ y = y(s) &= -\sqrt{\frac{\mathfrak{c}}{\mathfrak{a}}}\left(Aq^{s-1/2} + Bq^{-s+1/2}\right) + \text{ const.} \end{aligned} \tag{16}$$

So, the $x-$ and $y-$lattices are basically the same: $y(s) =$ const. $x(s-1/2)+$ const. The $\mathscr{D}$ operator can be written as a combination of ***elementary*** difference operators in terms of this s variable: at $x = x(s)$,

$$(\mathscr{D}f)(x(s)) = \frac{(\Delta f)(x(s-1/2))}{\Delta x(s-1/2)} = \frac{(\delta f)(x(s))}{\delta x(s)} \tag{17}$$

This is the "Eastern way" of Nikiforov *et al.* [25–27]

Considering the limit case $q = 1$ and the comfort of real formulas, 6 kinds of lattices are distinguished:

"Eastern" lattice number	formula
I	$as + b$
II	$as^2 + bs + c$
III	(16), with $AB = 0$
IV	(16), with $AB < 0$
V	(16), with $AB > 0$
VI	(16), with $\lvert q\rvert = 1$

2.6. **Canonical choices.**

$x-$ and $y-$ lattices are easily transformed under separate translations and dilations of x and y:

$$\begin{aligned} x = \alpha x' + \beta, y = \gamma y' + \delta \Rightarrow G(x', y') &:= F(\alpha x' + \beta, \gamma y' + \delta) = 0,\\ x_k = \alpha x'_k + \beta, y_k &= \gamma y'_k + \delta. \end{aligned} \tag{18}$$

The points (x'_k, y'_k) now stay on the conic $G(x', y') = 0$ and their dance in the plane is still homothetic to the (x_k, y_k)'s one.

A conic has 5 degrees of freedom, the conics G's associated to F move in a fourdimensional manifold. So, ONE characteristic should be common to all the G's. Which one? Who did not guess?

Simplest way to get it is to look at large x (and y): from (3), the slopes of the asymptotes of the conic $F = 0$ are the two roots of $\mathfrak{a}\sigma^2 + 2\mathfrak{b}\sigma + \mathfrak{c} = 0$. Now, from (18), the same for G is given by $\mathfrak{a}\beta^2\sigma'^2 + 2\mathfrak{b}\alpha\beta\sigma' + \mathfrak{c}\alpha^2 = 0$. Of course, we find $\sigma' = \dfrac{\alpha}{\beta}\sigma$ (obvious: $\sigma = \lim \dfrac{y}{x} = \dfrac{\beta}{\alpha}\lim\dfrac{y'}{x'} = \dfrac{\beta}{\alpha}\sigma'$).

So, the *ratio of the slopes* of the asymptotes is kept, and this ratio is (this is your last chance) q: let ρ be this ratio, then the slopes are σ and $\rho\sigma$, with sum $\sigma(\rho+1) = -2\mathfrak{b}/\mathfrak{a}$ and product $\sigma^2\rho = \mathfrak{c}/\mathfrak{a}$, eliminate σ:

$$\frac{(\rho+1)^2}{\rho} = \rho + 2 + \rho^{-1} = \frac{4\mathfrak{b}^2/\mathfrak{a}^2}{\mathfrak{c}/\mathfrak{a}} = 4\frac{\mathfrak{b}^2}{\mathfrak{a}\mathfrak{c}},$$

compare with (14):

q ***is the ratio of the slopes of the asymptotes of*** (3)***.***

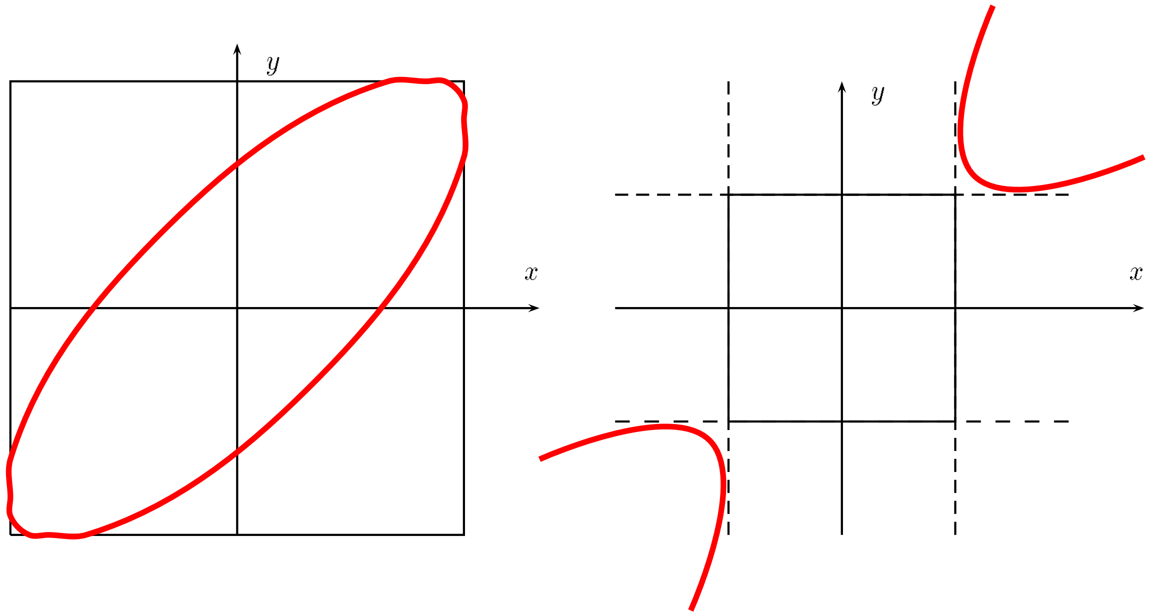
(Of course, we may choose beteween a ratio and its inverse. Final formulas are symmetric in q and q^{-1}).

As the root of the matter is basically left unchanged by translations and dilations, one often make special choices for the conic, such as

$$y^2 - (q^{1/2} + q^{-1/2})xy + x^2 + \frac{q + q^{-1} - 2}{4} = 0, \tag{19}$$

which is:

1. centered at origin,
2. symmetric in x and y,
3. inscribed, or circumscribed, to square $[-1, 1] \times [-1, 1]$:



The related difference operator is known as the ***Askey-Wilson*** difference operator $\mathcal{D}_q$ in the Western world.

$\mathcal{D}_q$ is often presented as an avatar of the elementary $q-$difference operator δ_q through a change of variable:

$$(\mathcal{D}_q f)(\cos\theta) = \frac{(\delta_q g)(e^{i\theta})}{\delta_q e^{i\theta}} = \frac{g(q^{1/2}e^{i\theta}) - g(q^{-1/2}e^{i\theta})}{i(q^{1/2} - q^{-1/2})\sin\theta}, \tag{20}$$

where g is defined by $g(e^{i\theta}) = f(\cos\theta)$, i.e., $g(z) = f((z + z^{-1}/2)$. So,

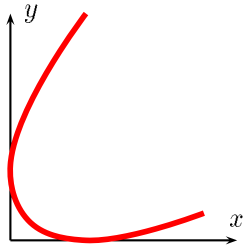
$$(\mathcal{D}_q f)(\cos\theta) = \frac{f((q^{1/2}e^{i\theta} + q^{-1/2}e^{-i\theta})/2) - f((q^{-1/2}e^{i\theta} + q^{1/2}e^{-i\theta})/2)}{i(q^{1/2} - q^{-1/2})\sin\theta},$$

and one recognizes that $y = (q^{\pm 1/2}e^{i\theta} + q^{\mp 1/2}e^{-i\theta})/2$ is indeed related to $x = \cos\theta$ by $(y - q^{1/2}x)(y - q^{-1/2}x) = (y - q^{\pm 1/2}x)(y - q^{\mp 1/2}x) = -(q^{\pm 1/2} - q^{\mp 1/2})^2/4$ which is indeed (19).

Being symmetric has some drawbacks, such as forbidding η_1 and η_2 to be real in the hyperbolic case when $x \in (-1, 1)$ ... [6]. Also, honest real hyperbolas like $y^2 - x^2 = 1$ don't appear real (because $q < 0$). And symmetric forms of elementary difference operators, as $\delta f(x) = (f(x +$

[6]However, this strange situation has triggered interesting investigations by the authors of [7], see next section on Chebyshev polynomials ...

$h/2) - f(x - h/2))/h$ do not supersede the unsymmetric forms Δ and ∇. That's why I don't devote my life to symmetric forms.



$q = 1$: The ***Wilson-Racah*** difference operator is usually written

$$\frac{f((\sqrt{x} + \alpha/2)^2) - f((\sqrt{x} - \alpha/2)^2)}{2\alpha\sqrt{x}}$$

(Racah: $\alpha = 1$, Wilson: $\alpha = i$).
The corresponding parabola is $(y - x)^2 - \alpha^2(x + y)/2 + \alpha^4/16 = 0$. Its axis is $y = x$ and the parabola touches the x and y axes at $(0, \alpha^2/4)$ and $(\alpha^2/4, 0)$.

"Western" name	symbol	conic	"Eastern" lattice number	canonical lattice [27, §3.4]
AW	$\mathcal{D}_q$	ell. hyp.	VI IV, V	$\cos(2\omega s)$ $\sinh, \cosh(2\omega s)$
Wilson		par.	II	$s(s+1)$
	D_q	2 lines	III	$e^{2\omega s}$
	Δ	2 parallel lines	I	s

2.7. **Chebyshev polynomials.**

Looking for polynomial families with remarkable behaviour under differencing, one encounters a surprising stability of the identity

$$\frac{dT_n(x)}{dx} = nU_{n-1}(x), \text{ where } T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}$$
$$\text{and } U_{n-1}(x) = \frac{(x + \sqrt{x^2 - 1})^n - (x - \sqrt{x^2 - 1})^n}{\sqrt{x^2 - 1}}.$$

Apply (20): $T_n(\cos\theta) = \cos n\theta$, so $g(z) = (z^n + z^{-n})/2$,

$$\begin{aligned}(\mathcal{D}_q T_n)(\cos\theta) &= \frac{q^{n/2}e^{in\theta} + q^{-n/2}e^{-in\theta} - q^{-n/2}e^{in\theta} - q^{n/2}e^{-in\theta}}{2i(q^{1/2} - q^{-1/2})\sin\theta} \\ &= \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \frac{\sin n\theta}{\sin\theta} \\ &= \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} U_{n-1}(\cos\theta).\end{aligned}$$

The formula is used in [7] in order to study $\mathscr{D}$ as operator on Hilbert spaces.

In a noncanonical setting, what is the driving force of such formulas? Here it is: let the functions ξ_1 and ξ_2 give the two abscissas for a given ordinate of the conic (3):

$$F(\xi, y) = 0 \Rightarrow \xi = \xi_1(y) \text{ and } \xi = \xi_2(y).$$

ξ_1 and ξ_2 are the inverse functions to η_1 and η_2.
Remark that $\xi_1(x) + \xi_2(x) = -2(\mathfrak{b}x + \mathfrak{e})/\mathfrak{c}$.

Let $f(x) = G(\xi_1(x)) + G(\xi_2(x))$, then

$$
\begin{aligned}
(\mathscr{D}f)(x) &= \frac{G(\xi_1(\eta_2(x))) + G(\xi_2(\eta_2(x))) - G(\xi_1(\eta_1(x))) - G(\xi_2(\eta_1(x)))}{\eta_2(x) - \eta_1(x)} \\
&= \frac{G(-2(\mathfrak{b}\eta_2(x) + \mathfrak{e})/\mathfrak{c} - \eta_2(x)) - G(-2(\mathfrak{b}\eta_1(x) + \mathfrak{e})/\mathfrak{c} - \eta_1(x))}{\eta_2(x) - \eta_1(x)} \\
&\quad \text{as } \xi_1(\eta_1(x)) = \xi_2(\eta_2(x)) = x, \text{ and from what we just saw about } \xi_1 + \xi_2 \\
&= (\mathscr{D}G)(-2(\mathfrak{b}x + \mathfrak{e})/\mathfrak{c} - x).
\end{aligned}
$$

Hmmm, what if $\mathfrak{b}/\mathfrak{c} = 1/2$?

3. First order difference equations.

3.1. Summation under $\mathscr{D}$, first difference equations.

If $\mathscr{D}f = g$, what kind of summation of g shall we encounter? At $x = x_1$,

$$f(y_2) - f(y_1) = (y_2 - y_1)g(x_1),$$

and we must go further with something $-f(y_2)$:

$$
\begin{aligned}
f(y_3) - f(y_2) &= (y_3 - y_2)g(x_2) \\
f(y_4) - f(y_3) &= (y_4 - y_3)g(x_3) \\
\cdots \quad &\quad \cdots \\
f(y_N) - f(y_{N-1}) &= (y_N - y_{N-1}g(x_{N-1}) \\
\hline
f(y_N) - f(y_1) &= \sum_1^{N-1} (y_{k+1} - y_k)g(x_k)
\end{aligned}
$$

which we may consider as a plain summation of, say, $G(x_k) = (y_{k-1} - y_k)g(x_k)$ as, after all, $y_{k+1} - y_k$ is a function of x_k.

3.2. The adjoint of $\mathscr{D}$.

Adjoint of $\mathscr{D}$ for the bilinear form $\langle f, g\rangle = \sum_k w_k f(y_k)g(y_k)$:

$$\langle f, \mathscr{D}g\rangle = \sum_k w_k f(y_k)\frac{g(y_{k+1}) - g(y_k)}{y_{k+1} - y_k} = \sum_k w_k g(y_k)\left[\frac{f(y_{k-1})w_{k-1}/w_k}{y_k - y_{k-1}} - \frac{f(y_k)}{y_{k+1} - y_k}\right]$$

So,

$$(\mathscr{D}^\dagger f)(x_k) = \frac{f(y_{k-1})w_{k-1}/w_k}{y_k - y_{k-1}} - \frac{f(y_k)}{y_{k+1} - y_k}$$

3.3. First order difference equations.

Just as $(\mathscr{D}f)(x) = g(x)$ involves $f(\eta_1(x))$ and $f(\eta_2(x))$, a first order $\mathscr{D}-$difference equation is

$$F(f(\eta_1(x)), f(\eta_2(x)), x) = 0,$$

or

$$f(\eta_2(x)) = H(f(\eta_1(x)), x),$$

allows the ***recurrence***

$$f(y_{k+1}) = H(f(y_k), x_k), \qquad k = 0, 1, \ldots$$

In $(\mathscr{D},\mathscr{M})$ form:

$$G((\mathscr{D}f)(x),(\mathscr{M}f)(x),x)=0.$$

3.4. **Linear first order difference equations.**

In $(\mathscr{D},\mathscr{M})$ form:

$$A(x)(\mathscr{D}f)(x)+B(x)(\mathscr{M}f)(x)+C(x)=0.$$

Becomes

$$\left(\frac{A(x)}{\eta_2(x)-\eta_1(x)}+\frac{B(x)}{2}\right)f(\eta_2(x))-\left(\frac{A(x)}{\eta_2(x)-\eta_1(x)}-\frac{B(x)}{2}\right)f(\eta_1(x))+C(x)=0.$$

Solution in factorial series. For instance, the $\mathscr{D}-$exponential function, solution of

$$\mathscr{D}f=\omega\mathscr{M}f$$

can be searched as $\sum_0^\infty c_n(x-y_1)\cdots(x-y_n)$. From (7) and (8) (p.6),

$$c_nC_n=\omega(c_{n-1}D_{n-1}+c_nE_n),\ \text{i.e. },c_n=\frac{\omega D_{n-1}}{C_n-\omega E_n}c_{n-1},n=1,2,\dots$$

For $A\mathscr{D}f=\omega\mathscr{M}f$, where $A(x)=A_0x+A_1$, one finds

$$c_n=\frac{\omega D_{n-1}-A_0C_{n-1}}{C_nA(x_n)-\omega E_n}c_{n-1},n=1,2,\dots$$

For $A\mathscr{D}f+B\mathscr{M}f+C=0$ with polynomial A,B,C:

$$A\sum c_n\mathscr{D}\mathsf{p}_n+B\sum c_n\mathscr{M}\mathsf{p}_n+C=0,$$

$$A\sum c_nC_n\tilde{\mathsf{p}}_{n-1}+B\sum c_n[D_n\tilde{\mathsf{p}}_n+E_n\tilde{\mathsf{p}}_{n-1}]+C=0,$$

expand A and B as Newton interpolants about x_n, x_{n+1}, $\dots$

$$\begin{aligned}&\sum_n c_nC_n\{A(x_n)+[x_n,x_{n+1}]_A(x-x_n)+\cdots\}\tilde{\mathsf{p}}_{n-1}\\&+\sum_n c_nD_n\{B(x_{n+1})+[x_{n+1},x_{n+2}]_B(x-x_{n+1})+\cdots\}\tilde{\mathsf{p}}_n\\&+\sum_n c_nE_n\{B(x_n)+[x_n,x_{n+1}]_B(x-x_n)+\cdots\}\tilde{\mathsf{p}}_{n-1}+C=0,\end{aligned}$$

Finally, coefficient of $\tilde{\mathsf{p}}_N$:

$$\sum_{n=N-\max((\text{degree }A)-1,\text{ degree }B)}^{N+1}c_n\{C_n[x_n,\dots,x_{N+1}]_A+D_n[x_{n+1},\dots,x_{N+1}]_B+E_n[x_n,\dots,x_{N+1}]_B\}=0,\quad(21)$$

$N\notin[0,\text{ degree }C]$, which is normally a linear recurrence relation of order max(degree $A,1+$ degree B).

3.5. **Riccati difference equations.**

$$A(x)(\mathscr{D}f)(x) = B(x)f(\eta_1(x))f(\eta_2(x)) + C(x)(\mathscr{M}f)(x) + D(x). \tag{22}$$

$$f(\eta_2(x)) = \frac{\left(\dfrac{A(x)}{\eta_2(x)-\eta_1(x)} + \dfrac{C(x)}{2}\right) f(\eta_1(x)) + D(x)}{\dfrac{A(x)}{\eta_2(x)-\eta_1(x)} - \dfrac{C(x)}{2} - B(x)f(\eta_1(x))} \tag{23}$$

Continued fraction:

$$\begin{aligned} f(y_{k+1}) &= \frac{\left(\dfrac{A(x_k)}{\Delta y_k} + \dfrac{C(x_k)}{2}\right) f(y_k) + D(x_k)}{\dfrac{A(x_k)}{\Delta y_k} - \dfrac{C(x_k)}{2} - B(x_k)f(y_k)} \\ &= -\frac{A(x_k)}{B(x_k)\Delta y_k} - \frac{C(x_k)}{2B(x_k)} + \frac{1}{B(x_k)} \frac{\left(\dfrac{A(x_k)}{\Delta y_k}\right)^2 - \dfrac{(C(x_k))^2}{4} + B(x_k)D(x_k)}{\dfrac{A(x_k)}{\Delta y_k} - \dfrac{C(x_k)}{2} - B(x_k)f(y_k)}. \end{aligned}$$

Remark the importance of $\Omega := A^2 - \mathfrak{q}(C^2 - 4BD)$: if f is finite at a zero of Ω, f can be computed in finite numbers of steps at each subsequent point of the lattice of this zero.

Relation with linear second order equation:
let $\dfrac{A(x_k)}{y_{k+1}-y_k} - \dfrac{C(x_k)}{2} - B(x_k)f(y_k) = \dfrac{F_k}{F_{k-1}}$, then

$$f(y_{k+1}) = \frac{-\dfrac{F_{k+1}}{F_k} + \dfrac{A(x_{k+1})}{y_{k+2}-y_{k+1}} - \dfrac{C(x_{k+1})}{2}}{B(x_{k+1})},$$

$$\begin{aligned} &B(x_{k+1})F_{k+1} = \\ &\quad \left\{ B(x_k)\left[\frac{A(x_{k+1})}{y_{k+2}-y_{k+1}} - \frac{C(x_{k+1})}{2}\right] - B(x_{k+1})\left[\frac{A(x_k)}{y_{k+1}-y_k} - \frac{C(x_k)}{2}\right]\right\} F_k \\ &\quad - B(x_{k+1})\left\{\left(\frac{A(x_k)}{y_{k+1}-y_k}\right)^2 - \frac{(C(x_k))^2}{4} + B(x_k)D(x_k)\right\} F_{k-1} \end{aligned}$$

4. Second order difference equations.

4.1.

$$(\mathscr{L}f)(x_k) = \alpha(x_k)f(y_{k+1}) + \beta(x_k)f(y_k) + \gamma(x_k)f(y_{k-1}) \tag{24}$$

If $\mathscr{L}$ must send polynomials to polynomials,

$$\begin{aligned} \alpha_k + \beta_k + \gamma_k &= \text{ polynomial } = p_0(x_k) \\ \alpha_k y_{k+1} + \beta_k y_k + \gamma_k y_{k-1} &= \text{ polynomial } = p_1(x_k) \\ \alpha_k y_{k+1}^2 + \beta_k y_k^2 + \gamma_k y_{k-1}^2 &= \text{ polynomial } = p_2(x_k) \end{aligned} \tag{25}$$

Use (3):

$$\alpha(x_k)F(x,y_{k+1}) + \beta(x_k)F(x,y_k) + \gamma(x_k)F(x,y_{k-1}) = \mathfrak{a}p_2(x_k) + 2(\mathfrak{b}x+\mathfrak{d})p_1(x_k) + (\mathfrak{c}x^2 + 2\mathfrak{e}x + \mathfrak{f})p_0(x_k)$$

$x = x_k \Rightarrow \gamma =$
$x = x_{k-1} \Rightarrow \alpha =$
Hmm, drop that subsection

4.2. Selfadjointness.

5. Orthogonal polynomials satisfying $\mathscr{D}-$difference equations.

5.1. Instant Padé-continued fractions- orthogonal polynomials theory.

Given a real or complex sequence $\{\mu_m\}_0^\infty$, the three following problems are basically the same ([Chebyshev, Frobenius, Padé, Perron, Stieltjes, Szegő, etc., etc.]):

1. ***Continued fraction:*** expand the (formal) series $\mathcal{S}(z) = \sum_{j=0}^{\infty} \frac{\mu_j}{z^{j+1}}$ in a Jacobi continued fraction
$$\cfrac{\mu_0}{z - b_0 - \cfrac{a_1^2}{z - b_1 - \cfrac{a_2^2}{z - b_2 - \cdots}}}$$
2. ***(Formal) orthogonal polynomials:*** look for polynomials $p_0, p_1, \ldots$, with p_n of degree n: $p_n(z) = \gamma_n z^n + \gamma_n' z^{n-1} + \cdots$, orthonormal with respect to the form of moments μ_m:
$$\langle p_n, p_m \rangle = \delta_{n,m}\,, n, m = 0, 1, \ldots, \qquad \text{where } \langle x^i, x^j \rangle = \mu_{i+j}.$$
3. ***Padé approximation:*** look for rational functions $\tilde{p}_{n-1}/p_n$ (yes, the same p_n than just before) such that
$$\mathcal{S}(z) - \frac{\tilde{p}_{n-1}(z)}{p_n(z)} = O\left(\frac{1}{z^{2n+1}}\right)\ , n = 0, 1, \ldots \tag{26}$$

The problems so stated have each a unique solution[7] if all the Hankel determinants $H_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}$ do not vanish.

The relations between the solutions of the three problems are seen from the recurrence relations
$$\begin{aligned} a_{n+1} p_{n+1}(z) &= (z - b_n) p_n(z) - a_n p_{n-1}(z), \\ a_{n+1} \tilde{p}_n(z) &= (z - b_n) \tilde{p}_{n-1}(z) - a_n \tilde{p}_{n-2}(z), \end{aligned} \tag{27}$$
$n = 1, 2, \ldots,$
$$\begin{gathered} p_0 = \gamma_0 = 1/\sqrt{\mu_0}, \qquad \tilde{p}_{-1} = 0, \\ p_1(z) = \gamma_0 (z - b_0)/a_1, \qquad \tilde{p}_0 = \mu_0 \gamma_1 = \sqrt{\mu_0}/a_1, \\ \text{i.e., extended by (27) to } n = 0 \text{ by} \\ a_0 p_{-1}(z) \equiv 0, \qquad a_0 \tilde{p}_{-2} = -\sqrt{\mu_0} \end{gathered} \tag{28}$$

Interesting identities are, among other ones,
$$\gamma_0^2 \gamma_1^2 \cdots \gamma_n^2 = H_n,$$
$$a_n = \frac{\gamma_{n-1}}{\gamma_n}, \quad n = 1, 2, \ldots \tag{29}$$

[7] Unique up to the sign of γ_n.

$$p_n\tilde{p}_n - p_{n+1}\tilde{p}_{n-1} = \frac{1}{a_{n+1}}, \quad n = 0, 1, \ldots \tag{30}$$

$$\left.\begin{aligned} p_n(z) &= \frac{\gamma_0}{a_1\cdots a_n}[z^n - (b_0 + \cdots + b_{n-1})z^{n-1} + \cdots] \\ \tilde{p}_n(z) &= \frac{1}{\gamma_0 a_1 \cdots a_{n+1}}[z^n - (b_1 + \cdots + b_n)z^{n-1} + \cdots] \\ \mathcal{S}(z)p_n(z) - \tilde{p}_{n-1}(z) &= \frac{a_1\cdots a_n}{\gamma_0 z^{n+1}}\left[1 + \frac{b_0 + \cdots + b_n}{z} + \cdots\right] \end{aligned}\right\} \tag{31}$$

A special family of orthogonal polynomials is considered essentially known if we have a compact formula for $\mathcal{S}$ (or the moments μ_j) ***and*** the recurrence (or continued fraction) coefficients $\{a_n\}$ and $\{b_n\}$.

For instance, $\mathcal{S}(z) = \int_{-1}^{1} \frac{dx}{z-x} = \frac{2}{z} + \frac{2}{3z^3} + \frac{2}{5z^5} + \cdots = \log\left(\frac{z+1}{z-1}\right)$ gives $b_n = 0$, $a_n^2 = \frac{n^2}{4n^2-1}$ (Legendre). How on earth is it being done?

5.2. **Semi-classical and Laguerre-Hahn orthogonal polynomials.**

Extensions of Gauss and Heine theory of orthogonal polynomials-continued fractions, emphasizing the role of differential (or difference [in x]) equations and relations received the name "semi-classical". The choice of this name [13, 23] is perhaps not so fortunate, as a body of calculations in quantum mechanics bears the same name.

I coined the name "Laguerre-Hahn" for what could be the ultimate big family of orthogonal polynomials satisfying differential or difference relations. Laguerre [19] started to investigate continued fractions satisfying differential equations, and showed how to construct the differential equations satisfied by each denominator (orthogonal polynomial). Hahn ([11,12] and earlier works quoted therein) worked in the converse direction: what happens when orthogonal polynomials satisfy linear differential equations of fixed order, with coefficients of fixed degree? The answer is that they are basically the family that can be reached by the Laguerre theory. Well, I suppose, because not many people, and certainly not me, went very far in the study of the most difficult Hahn's theorems.

Other smart people involved: Bonan-Lubinsky-Nevai [5, 6], the Chudnovsky brothers, Freud, Gammel-Nuttall, Shohat, ...

Here is the Laguerre-like theory, in $\mathscr{D}$–language: we consider Stieltjes functions satisfying a ***Riccati*** equation (22)

$$A(x)(\mathscr{D}\mathcal{S})(x) = B(x)\mathcal{S}(\eta_1(x))\mathcal{S}(\eta_2(x)) + C(x)(\mathscr{M}\mathcal{S})(x) + D(x)$$

with polynomial A, B, C, and D. The ***semi-classical*** special case is when $B(x) \equiv 0$.

Let $\mathcal{S}_m(z) = \cfrac{1}{z - b_m - \cfrac{a_{m+1}^2}{z - b_{m+1} - \cfrac{a_{m+2}^2}{\cdots}}}$.

Then, each $\mathcal{S}_m$ satisfies a Riccati equation

$$A_m(x)(\mathscr{D}\mathcal{S}_m)(x) = B_m(x)\mathcal{S}_m(\eta_1(x))\mathcal{S}_m(\eta_2(x)) + C_m(x)(\mathscr{M}\mathcal{S})(x) + D_m(x). \tag{32}$$

similar to (22) with polynomial coefficients with ***same degrees***: 1.) it's true for $m = 0$, as $\mathcal{S} = \mu_0\mathcal{S}_0$, 2.) enter $\mathcal{S}_m(x) = \frac{1}{x - b_m - a_{m+1}^2\mathcal{S}_{m+1}(x)}$ in (32). Things are easier with the recurrent form (23):

$$\frac{1}{\eta_2 - b_m - a_{m+1}^2\mathcal{S}_{m+1}(\eta_2)} = \frac{\left(\dfrac{A_m(x)}{\eta_2(x)-\eta_1(x)} + \dfrac{C_m(x)}{2}\right)\dfrac{1}{\eta_1 - b_m - a_{m+1}^2\mathcal{S}_{m+1}(\eta_1)} + D_m(x)}{\dfrac{A_m(x)}{\eta_2(x)-\eta_1(x)} - \dfrac{C_m(x)}{2} - \dfrac{B_m(x)}{\eta_1 - b_m - a_{m+1}^2\mathcal{S}_{m+1}(\eta_1)}}$$
$$= \frac{\dfrac{A_m(x)}{\eta_2(x)-\eta_1(x)} + \dfrac{C_m(x)}{2} + D_m(x)[\eta_1 - b_m - a_{m+1}^2\mathcal{S}_{m+1}(\eta_1)]}{-B_m(x) + \left(\dfrac{A_m(x)}{\eta_2(x)-\eta_1(x)} - \dfrac{C_m(x)}{2}\right)[\eta_1 - b_m - a_{m+1}^2\mathcal{S}_{m+1}(\eta_1)]},$$

or $-a_{m+1}^2\mathcal{S}_{m+1}(\eta_2) =$

$$\frac{-a_{m+1}^2\left(\dfrac{A_m(x)}{\eta_2(x)-\eta_1(x)} - \dfrac{C_m(x)}{2} - (\eta_2 - b_m)D_m(x)\right)\mathcal{S}_{m+1}(\eta_1) + X_m}{\dfrac{A_m(x)}{\eta_2(x)-\eta_1(x)} + \dfrac{C_m(x)}{2} + (\eta_1 - b_m)D_m(x) - a_{m+1}^2\mathcal{S}_{m+1}(\eta_1)},$$

where $X_m =$
$$\left(\frac{A_m(x)}{\eta_2(x)-\eta_1(x)} - \frac{C_m(x)}{2}\right)(\eta_1 - b_m) - B_m(x) - \left(\frac{A_m(x)}{\eta_2(x)-\eta_1(x)} + \frac{C_m(x)}{2}\right)(\eta_2 - b_m) - (\eta_1 - b_m)(\eta_2 - b_m)D_m(x),$$
so that

$$\frac{A_{m+1}(x)}{\eta_2(x)-\eta_1(x)} + \frac{C_{m+1}(x)}{2} = \frac{A_m(x)}{\eta_2(x)-\eta_1(x)} - \frac{C_m(x)}{2} - (\eta_2 - b_m)D_m(x)$$
$$\frac{A_{m+1}(x)}{\eta_2(x)-\eta_1(x)} - \frac{C_{m+1}(x)}{2} = \frac{A_m(x)}{\eta_2(x)-\eta_1(x)} + \frac{C_m(x)}{2} + (\eta_1 - b_m)D_m(x)$$
$$B_{m+1}(x) = a_{m+1}^2 D_m(x)$$
$$a_{m+1}^2 D_{m+1}(x) = -X_m = A_m(x) + (\eta_1 + \eta_2 - 2b_m)C_m(x)/2 + B_m(x) + (\eta_1 - b_m)(\eta_2 - b_m)D_m(x)$$

The two first equations above simplify as

$$A_{m+1}(x) = A_m(x) - \frac{(\eta_2(x) - \eta_1(x))^2}{2}D_m(x) = A_m(x) - 2\mathfrak{q}(x)D_m(x), \tag{33}$$

$$C_{m+1}(x) = -C_m(x) - (\eta_1(x) + \eta_2(x) - 2b_m)D_m(x) = -C_m(x) - 2(\mathfrak{p}(x) - b_m)D_m(x) \tag{34}$$

showing, together with the already encountered

$$B_{m+1}(x) = a_{m+1}^2 D_m(x), \tag{35}$$

and

$$\begin{aligned} a_{m+1}^2 D_{m+1}(x) &= A_m(x) + (\eta_1 + \eta_2 - 2b_m)C_m(x)/2 + B_m(x) + (\eta_1 - b_m)(\eta_2 - b_m)D_m(x) \\ &= A_m(x) - \frac{(\eta_2 - \eta_1)^2}{4}D_m(x) + \frac{(\eta_1 + \eta_2 - 2b_m)C_m(x)}{2} + B_m(x) + \frac{(\eta_1 + \eta_2 - 2b_m)^2 D_m(x)}{4} \\ &= A_m(x) - \frac{(\eta_2 - \eta_1)^2}{4}D_m(x) + \frac{(\eta_1 + \eta_2 - 2b_m)(C_m(x) - C_{m+1}(x))}{2} + B_m(x), \end{aligned} \tag{36}$$

that $A_{m+1}, B_{m+1}, C_{m+1}$, and D_{m+1} are polynomials if A_m, B_m, C_m, and D_m are polynomials ($\eta_1 + \eta_2$ and $\eta_1\eta_2$ are polynomials of degrees $\leqslant 1$ and 2 [see p. 3]). What about the degrees? Let

$$s = \max[(\text{degree } A_m) - 2, (\text{degree } B_m) - 2, (\text{degree } C_m) - 1]$$

(the notation was launched by Maroni [23]). Then, degree $D_m \leqslant s$, from (32), as $\mathcal{S}_m = O(z^{-1}) \Rightarrow \mathscr{D}\mathcal{S}_m = O(z^{-2})$ and $\mathscr{M}\mathcal{S}_m = O(z^{-1})$, so the degrees of A_{m+1}, B_{m+1}, and C_{m+1} are bounded by $s+2$, s, and $s+1$, from the equations above. And the degree of D_{m+1} ***must*** be bounded by s, as $\mathcal{S}_{m+1} = O(z^{-1})$ still holds.

(33)-(36) can be arranged in a matrix-vector recurrence with polynomial coefficients. However, painful experiments lead to prefer (again) the intermediate form in η_1 and η_2

$$\begin{bmatrix} -\dfrac{A_{m+1}(x)}{\eta_2(x)-\eta_1(x)} - \dfrac{C_{m+1}(x)}{2} \\ \dfrac{A_{m+1}(x)}{\eta_2(x)-\eta_1(x)} - \dfrac{C_{m+1}(x)}{2} \\ B_{m+1}(x) \\ D_{m+1}(x) \end{bmatrix} =$$
$$\begin{bmatrix} & -1 & & \eta_2 - b_m \\ -1 & & & \eta_1 - b_m \\ & & & a_{m+1}^2 \\ -\dfrac{\eta_2 - b_m}{a_{m+1}^2} & -\dfrac{\eta_1 - b_m}{a_{m+1}^2} & \dfrac{1}{a_{m+1}^2} & \dfrac{(\eta_1 - b_m)(\eta_2 - b_m)}{a_{m+1}^2} \end{bmatrix} \begin{bmatrix} -\dfrac{A_m(x)}{\eta_2(x)-\eta_1(x)} - \dfrac{C_m(x)}{2} \\ \dfrac{A_m(x)}{\eta_2(x)-\eta_1(x)} - \dfrac{C_m(x)}{2} \\ B_m(x) \\ D_m(x) \end{bmatrix} \tag{37}$$

That the degree of D_m stays $\leqslant s$ does not seem obvious from (36). This non-obviousness is a clue to the calculation of the a_m's and the b_m's!!!

Often useful, the sum of (36):

$$a_{m+1}^2 D_{m+1}(x) D_m(x) - a_m^2 D_m(x) D_{m-1}(x) = \frac{A_m^2(x) - A_{m+1}^2(x)}{(\eta_2 - \eta_1)^2} + \frac{C_{m+1}^2(x) - C_m^2(x)}{4},$$

so

$$\begin{aligned} a_m^2 D_m(x) D_{m-1}(x) - B_0(x) D_0(x) &= \frac{A_0^2(x) - A_m^2(x)}{(\eta_2 - \eta_1)^2} + \frac{C_m^2(x) - C_0^2(x)}{4}, \\ &= A(x) \sum_{j=0}^{m-1} D_j(x) \; + \frac{C_m^2(x) - C_0^2(x)}{4}, \text{ if } \eta_1(x) = \eta_2(x), \end{aligned} \tag{38}$$

which shows that the polynomial of degree $\leqslant 2s+4$

$$\Omega = A_m^2 - (\eta_2 - \eta_1)^2 \left[\frac{C_m^2}{4} - a_m^2 D_m D_{m-1} \right] = A_m^2 - (\eta_2 - \eta_1)^2 \left[\frac{C_m^2}{4} - B_m D_m \right] \tag{39}$$

does not depend on m ...

Summary:

	Semi-classical $B_0(x) \equiv 0$	Laguerre-Hahn
$s = 0$	classical	associate of classical if B_0 is a constant
$s > 0$	General semi-classical	includes transformed classical cases [37]

5.3. **Classical orthogonal polynomials from the semi-classical-Laguerre-Hahn point of view.**

Let us take again the ***Legendre*** example. Here, $\eta_1(x) \equiv \eta_2(x) \equiv x$; $\mathcal{S}' = -2/(z^2-1)$: $A(z) = A_0(z) = z^2-1$, $B(z) \equiv C(z) \equiv 0$, and $D(z) = -2 \Rightarrow D_0(z) = -1$, as $\mu_0 = 2$.

(33) $\Rightarrow A_m(x) = x^2 - 1$ for all m,
let $C_m(x) = \Gamma_m x + \Gamma'_m$, $D_m = \Delta_m$, (35) $\Rightarrow B_m = a_m^2 \Delta_m$. Equations (34) and (36) are
$\Gamma_{m+1}x + \Gamma'_{m+1} = -\Gamma_m x - \Gamma'_m - 2(x-b_m)\Delta_m$,
$a_{m+1}^2 \Delta_{m+1} = x^2 - 1 + (x-b_m)(\Gamma_m x + \Gamma'_m) + a_m^2 \Delta_{m-1} + (x-b_m)^2 \Delta_m$

x^2 : $\Gamma_m + \Gamma_{m+1} = -2\Delta_m$ and $0 = 1 + \Gamma_m + \Delta_m \Rightarrow \Delta_m = -(2m+1), \Gamma_m = 2m$,
x : $\Gamma'_m + \Gamma'_{m+1} = -2(2m+1)b_m$ and $\Gamma'_m = -2(m+1)b_m \Rightarrow b_m = 0$,
x^0 : $(2m+3)a_{m+1}^2 - (2m-1)a_m^2 = 1 \Rightarrow a_m^2 = m^2/(4m^2-1)$. I told you!

Hahn: $\eta_1(x) = x$, $\eta_2(x) = x+h$. Again, A_m has degree 2, $C_m(x) = \Gamma_m x + \Gamma'_m$. As D_m has degree 0, (33) shows that only the constant coefficient of A_m, say α_m, depends on m: let $A_m(x) = \mathsf{a}x^2 + \mathsf{b}x + \alpha_m$. Then, $\alpha_{m+1} - \alpha_m = -h^2 D_m/2$ from (33); constant $= 2\mathsf{a}\alpha_m - h^2\Gamma_m^2/4$, i.e., $\alpha_m = h^2\Gamma_m^2/(8\mathsf{a})+$ constant, from the x^2 term of (39), so, $\alpha_{m+1} - \alpha_m = -h^2 D_m/2 = h^2(\Gamma_{m+1}^2 - \Gamma_m^2)/(8\mathsf{a})$, but, as (34) $\Rightarrow \Gamma_m + \Gamma_{m+1} = -2D_m$, we find $\Gamma_{m+1} - \Gamma_m = 2\mathsf{a}$, so
$\Gamma_m = 2m\mathsf{a} + \Gamma_0$, and $D_m = -(2m+1)\mathsf{a} - \Gamma_0$, $\alpha_m = \alpha_0 + \Gamma_0 h^2 m/2 + \mathsf{a}h^2m^2/2$.

Remark that $\Gamma_m = -D_{m-1/2}$.

Coefficient of x in (39): $\Omega'(0) = 2\mathsf{b}\alpha_m - h^2\Gamma_m\Gamma' m/2 \Rightarrow \Gamma' m = 2(\Omega'(0) - 2\mathsf{b}\alpha_m)/(h^2 D_{m-1/2})$, and (34) at $x=0 \Rightarrow b_m = (\Gamma'_m + \Gamma'_{m+1})/(2D_m) + h/2$.

Finally, $x = 0$ in (39) $\Rightarrow a_m^2 = \dfrac{(\alpha_m^2 - \Omega(0))/h^2 - (\Gamma'_m)^2/4}{D_m D_{m-1}}$.

The full $\mathscr{D}-$classical orthogonal polynomials (***Askey-Wilson orthogonal polynomials***) will deserve the full next subsection.

A non classical semi-classical example: Freud. $\mathcal{S}_0(z) = \mu_0^{-1}\displaystyle\int_{-\infty}^{\infty}(z-x)^{-1}e^{-x^4/4}\,dx \Rightarrow$ $\mathcal{S}'(z) = -z^3\mathcal{S}(z) + z^2 + a_1^2$. So, $A_0(z) \equiv 1$, $B_0(z) \equiv 0$, $C_0(z) = -z^3$, and $D_0(z) = z^2 + a_1^2$. Repeated application of (33)-(36) leads to $A_m(z) \equiv 1$, $B_m(z) = a_m^2(z^2 + a_{m-1}^2 + a_m^2)$, $C_m(z) = -z^3 - 2a_m^2 z$, and $D_m(z) = z^2 + a_m^2 + a_{m+1}^2$, ***and*** $a_m^2(a_{m-1}^2 + a_m^2 + a_{m+1}^2) = m$, $m = 1, 2, \ldots$ $(a_0 = 0)$.

5.4. **Recurrence relations of the Askey-Wilson polynomials.**

From (3), $\eta_1(x) + \eta_2(x) = 2\mathfrak{p}(x) = -2(\mathfrak{b}x + \mathfrak{d})/\mathfrak{a}$, $\eta_1(x)\eta_2(x) = (\mathfrak{c}x^2 + 2\mathfrak{e}x + \mathfrak{f})/\mathfrak{a}$, $(\eta_2(x) - \eta_1(x))^2 = 4\mathfrak{q}(x) = 4[(\mathfrak{b}x + \mathfrak{d})^2 - \mathfrak{a}(\mathfrak{c}x^2 + 2\mathfrak{e}x + \mathfrak{f})]/\mathfrak{a}^2$.

Let us consider first the main coefficients $A_m(x) = \hat{A}_m x^{s+2} + \cdots$, $C_m(x) = \hat{C}_m x^{s+1} + \cdots$, $D_m(x) = \hat{D}_m x^s + \cdots$ (this part is valid for $s > 0$ as well). From (33)-(36),
$\hat{A}_{m+1} = \hat{A}_m - 2(\mathfrak{b}^2 - \mathfrak{a}\mathfrak{c})\hat{D}_m/\mathfrak{a}^2$,
$\hat{C}_{m+1} = -\hat{C}_m + 2\mathfrak{b}\hat{D}_m/\mathfrak{a}$,
$0 = \hat{A}_m - \mathfrak{b}\hat{C}_m/\mathfrak{a} + \mathfrak{c}\hat{D}_m/\mathfrak{a}$.

It yields $2\mathfrak{b}\hat{C}_{m+1}/\mathfrak{a} = \mathfrak{c}(\hat{D}_{m+1} + \hat{D}_m)/\mathfrak{a}$ and $2\mathfrak{b}\hat{C}_m/\mathfrak{a} = -\mathfrak{c}\hat{D}_{m+1}/\mathfrak{a} + (4\mathfrak{b}^2 - \mathfrak{a}\mathfrak{c})\hat{D}_m/\mathfrak{a}^2$, and so

$$\mathfrak{a}\mathfrak{c}\hat{D}_{m+1} - (4\mathfrak{b}^2 - 2\mathfrak{a}\mathfrak{c})\hat{D}_m + \mathfrak{a}\mathfrak{c}\hat{D}_{m-1} = 0,$$

again solved by a combination of q^m and q^{-m} if $q \neq 1$, by a combination of 1 and m if $q = 1$:

	$q \neq 1$	$q = 1$
$\hat{A}_m$	$\frac{\mathfrak{c}}{\mathfrak{a}}\frac{q^{-1/2}-q^{1/2}}{2}[\alpha q^{m-1/2}-\beta q^{1/2-m}]$	$\hat{A}_0 = -\frac{\mathfrak{c}\alpha}{2\mathfrak{a}}$
$\hat{C}_m$	$\sqrt{\frac{\mathfrak{c}}{\mathfrak{a}}}\hat{D}_{m-1/2}$	$\sqrt{\frac{\mathfrak{c}}{\mathfrak{a}}}\Big(\alpha(m-1/2)+\beta\Big)$
$\hat{D}_m$	$\alpha q^m + \beta q^{-m}$	$\alpha m + \beta$

For classical polynomials ($s = 0$), we "only" have yet to find formulas for $C_m(0) = \Gamma'_m$, a_m^2 and b_m.

From (33), as $D_m = \hat{D}_m$ is a known constant,

$$A_m(x) = A_0(x) - 2\mathfrak{q}(x)[D_0 + \cdots + D_{m-1}] = A_0(x) - 2\mathfrak{q}(x)\left[\alpha\frac{1-q^m}{1-q} + \beta\frac{1-q^{-m}}{1-q^{-1}}\right].$$

We now need to express $D_{m-1/2}$ as an odd function of $m+\mu$:

$$D_{m-1/2} = \mathfrak{D}(m+\mu).$$

This is obvious when $q = 1$, $\mathfrak{D}(x) =$ constant x, with $\mu = \beta/\alpha - 1/2$. When $q \neq 1$, write

$$D_{m-1/2} = \alpha q^{-\mu-1/2}q^{m+\mu} + \beta q^{\mu+1/2}q^{-m-\mu} = \text{ constant } (q^{m+\mu} - q^{-m-\mu}),$$

provided $q^{2\mu+1} = -\alpha/\beta$.

We then find that $D_0 + \cdots + D_{m-1}$ is an *even* function of $m+\mu$:

$$D_0 + \cdots + D_{m-1} = \frac{\alpha}{2}[(m+\mu)^2 - \mu^2] \text{ if } q = 1;\ \alpha q^{-\mu}\frac{q^\mu + q^{-\mu} - (q^{m+\mu} + q^{-m-\mu})}{1-q} \text{ if } q \neq 1.$$

We now look at the coefficient of x in the expansion of $\Omega/\mathfrak{q}$ about $x = \infty$, from (39):

$$\left(\frac{\Omega}{\mathfrak{q}}\right)_x = \text{ constant } = \left(\frac{A_m^2}{4\mathfrak{q}}\right)_x - 2\hat{C}_m\Gamma'_m,$$

and, as $A_m^2 = A_0^2 - 4\mathfrak{q}(D_0 + \cdots + D_{m-1}) + 4\mathfrak{q}^2(D_0 + \cdots + D_{m-1})^2$,

whence $\Gamma'_m = C_m(0) = \dfrac{\text{constant } - q_1(D_0 + \cdots + D_{m-1})^2}{2\hat{C}_m} = \dfrac{\text{a Laurent polynomial of degree 2 in } q^m}{\hat{C}_m = \text{ constant } D_{m-1/2}}$,

and, from (34),

$$b_m = \mathfrak{p}(0) + \frac{C_{m+1}(0) + C_m(0)}{D_m} = \frac{\text{a Laurent polynomial of degree 3 in } q^m}{D_{m-1/2}D_m D_{m+1/2}}.$$

Exercise. Show that D_m is a factor of the numerator of b_m and can therefore be simplified.

Indeed, $C_m(0) + C_{m+1}(0) =$ odd function of $(m+\mu)+$ odd function of $(m+\mu+1) =$ odd function of $(m+\mu+1/2)$.

Next, looking at (39) at a zero ω of Ω:

$$a_m^2 = \frac{C_m^2(\omega) - \dfrac{A_m^2(\omega)}{\mathfrak{q}(\omega)}}{4D_m D_{m-1}} = \frac{\text{Laurent pol. of degree 4 in } q^m}{D_m D_{m-1/2}^2 D_{m-1}}.$$

Remark that a_m^2 is an even function of $m+\mu$.

Let $F_m = [C_m(\omega) + A_m(\omega)/\sqrt{\mathfrak{q}(\omega)}]/(2D_{m-1})$ and $G_m = [C_m(\omega) - A_m(\omega)/\sqrt{\mathfrak{q}(\omega)}]/(2D_m)$, where we manage to have, say, $G_0 = 0$ (from (39), as $\Omega(\omega) = 0$ and $B_0(x) \equiv 0$).

Then,

$$a_m^2 = F_m G_m, \tag{40}$$

and

$$F_{m+1} + G_m = \frac{C_{m+1}(\omega) + C_m(\omega) + [A_{m+1}(\omega) - A_m(\omega)]/\sqrt{\mathfrak{q}(\omega)}}{2D_m} = b_m - \mathfrak{p}(\omega) - \sqrt{\mathfrak{q}(\omega)} = b_m - \eta(\omega), \tag{41}$$

which make four ways (the four zeros of Ω) to write a_m^2 and b_m as a product $F_m G_m$ and a sum $F_{m+1} + G_m +$ constant.

These four ways are no doubt linked to the four ways to express the Askey-Wilson polynomials as ${}_4\phi_3$ functions (Whipple-Sears identities, [2, p. 6]).

From [2, 17, 18, etc.], $p_n(x; a, b, c, d|q)$ is symmetric in a, b, c, and d, in

$$\frac{a^n p_n(x; a, b, c, d|q)}{(ab, ac, ad; q)_n} = r_n(x) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q; q \right), \quad x = \cos\theta.$$

$$2x r_n(x) = \mathsf{A}_n r_{n+1}(x) + [a + a^{-1} - (\mathsf{A}_n + \mathsf{C}_n)] r_n(x) + \mathsf{C}_n r_{n-1}(x),$$

where

$$\mathsf{A}_n = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^n)}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})},$$

$$\mathsf{C}_n = \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}.$$

Here, $b_n = [a + a^{-1} - (\mathsf{A}_n + \mathsf{C}_n)]/2$, and $a_n^2 = \mathsf{A}_{n-1}\mathsf{C}_n/4$.

These hypergeometric expansions aspects will be considered agaoin in section 5.7.

5.5. **Difference relations and equations for semi-classical and Laguerre-Hahn orthogonal polynomials.**

Here, we follow Laguerre [19] rather closely:

from (22) and (26), let

$$\mathcal{S}(z) = \frac{\tilde{p}_{n-1}(z)}{p_n(z)} + \varepsilon_n(z),$$

then

$$A(x)\left(\mathscr{D}\frac{\tilde{p}_{n-1}}{p_n}\right)(x) + A(x)(\mathscr{D}\varepsilon_n)(x) = B(x)\frac{\tilde{p}_{n-1}(\eta_1(x))}{p_n(\eta_1(x))}\frac{\tilde{p}_{n-1}(\eta_2(x))}{p_n(\eta_2(x))} + B(x)\frac{\tilde{p}_{n-1}(\eta_1(x))}{p_n(\eta_1(x))}\varepsilon(\eta_2(x))$$
$$+ B(x)\frac{\tilde{p}_{n-1}(\eta_2(x))}{p_n(\eta_2(x))}\varepsilon(\eta_1(x)) + B(x)\varepsilon(\eta_1(x))\varepsilon(\eta_2(x)) + C(x)\left(\mathscr{M}\frac{\tilde{p}_{n-1}}{p_n}\right)(x) + C(x)(\mathscr{M}\varepsilon_n)(x) + D(x).$$

So,

$$\Theta_n(x) = A(x)\frac{\tilde{p}_{n-1}(\eta_2(x))p_n(\eta_1(x)) - \tilde{p}_{n-1}(\eta_1(x))p_n(\eta_2(x))}{\eta_2(x) - \eta_1(x)} - B(x)\tilde{p}_{n-1}(\eta_1(x))\tilde{p}_{n-1}(\eta_2(x))$$
$$- C(x)\frac{\tilde{p}_{n-1}(\eta_2(x))p_n(\eta_1(x)) + \tilde{p}_{n-1}(\eta_1(x))p_n(\eta_2(x))}{2} - D(x)p_n(\eta_1(x))p_n(\eta_2(x)) \tag{42}$$

is 1.) a polynomial (symmetry in η_1 and η_2) 2.) of degree $\leqslant s$ (as $\varepsilon_n(x) \sim x^{-2n-1}$).

This Θ_n is not a completely new actor. To see things further, remark that (42) is a combination of products of solutions of the recurrence (27) at η_1 and at η_2.

We now consider the ***linear*** *recurrence satisfied by combinations of such products*, i.e., by combinations of

$$p_n(\eta_1)p_n(\eta_2), p_n(\eta_1)\tilde{p}_{n-1}(\eta_2), \tilde{p}_{n-1}(\eta_1)p_n(\eta_2), \text{ and } \tilde{p}_{n-1}(\eta_1)\tilde{p}_{n-1}(\eta_2).$$

We just have to consider a product $r_n s_n$, kowing that $a_{n+1}r_{n+1} = (\eta_1 - b_n)r_n - a_n r_{n-1}$, and $a_{n+1}s_{n+1} = (\eta_2 - b_n)s_n - a_n s_{n-1}$. Simplest is to write it as matrix-vector recurrence (*doesn't it ring a bell?*)

$$\begin{bmatrix} r_n s_{n+1} \\ r_{n+1}s_n \\ r_n s_n \\ r_{n+1}s_{n+1} \end{bmatrix} = \begin{bmatrix} & -\dfrac{a_n}{a_{n+1}} & & \dfrac{\eta_2 - b_n}{a_{n+1}} \\ -\dfrac{a_n}{a_{n+1}} & & & \dfrac{\eta_1 - b_n}{a_{n+1}} \\ & & & 1 \\ -\dfrac{a_n(\eta_2 - b_n)}{a_{n+1}^2} & -\dfrac{a_n(\eta_1 - b_n)}{a_{n+1}^2} & \dfrac{a_n^2}{a_{n+1}^2} & \dfrac{(\eta_1 - b_n)(\eta_2 - b_n)}{a_{n+1}^2} \end{bmatrix} \begin{bmatrix} r_{n-1}s_n \\ r_n s_{n-1} \\ r_{n-1}s_{n-1} \\ r_n s_n \end{bmatrix}$$

Looks damned close to (37), needs just a little bit of scaling:

$$\begin{bmatrix} a_{n+1}r_n s_{n+1} \\ a_{n+1}r_{n+1}s_n \\ a_{n+1}^2 r_n s_n \\ r_{n+1}s_{n+1} \end{bmatrix} = \begin{bmatrix} & -1 & & \eta_2 - b_n \\ -1 & & & \eta_1 - b_n \\ & & & a_{n+1}^2 \\ -\dfrac{\eta_2 - b_n}{a_{n+1}^2} & -\dfrac{\eta_1 - b_n}{a_{n+1}^2} & \dfrac{1}{a_{n+1}^2} & \dfrac{(\eta_1 - b_n)(\eta_2 - b_n)}{a_{n+1}^2} \end{bmatrix} \begin{bmatrix} a_n r_{n-1}s_n \\ a_n r_n s_{n-1} \\ a_n^2 r_{n-1}s_{n-1} \\ r_n s_n \end{bmatrix}$$

So, each vector $\begin{bmatrix} -\dfrac{A_n(x)}{\eta_2(x) - \eta_1(x)} - \dfrac{C_n(x)}{2} \\ \dfrac{A_n(x)}{\eta_2(x) - \eta_1(x)} - \dfrac{C_n(x)}{2} \\ B_n(x) \\ D_n(x) \end{bmatrix}$ will be a fixed (independent of n) combination of the four vectors

$$\begin{bmatrix} a_n p_{n-1}(\eta_1)\tilde{p}_{n-1}(\eta_2) \\ a_n p_n(\eta_1)\tilde{p}_{n-2}(\eta_2) \\ a_n^2 p_{n-1}(\eta_1)\tilde{p}_{n-2}(\eta_2) \\ p_n(\eta_1)\tilde{p}_{n-1}(\eta_2) \end{bmatrix}, \begin{bmatrix} a_n\tilde{p}_{n-2}(\eta_1)p_n(\eta_2) \\ a_n\tilde{p}_{n-1}(\eta_1)p_{n-1}(\eta_2) \\ a_n^2\tilde{p}_{n-2}(\eta_1)p_{n-1}(\eta_2) \\ \tilde{p}_{n-1}(\eta_1)p_n(\eta_2) \end{bmatrix}, \begin{bmatrix} a_n\tilde{p}_{n-2}(\eta_1)\tilde{p}_{n-1}(\eta_2) \\ a_n\tilde{p}_{n-1}(\eta_1)\tilde{p}_{n-2}(\eta_2) \\ a_n^2\tilde{p}_{n-2}(\eta_1)\tilde{p}_{n-2}(\eta_2) \\ \tilde{p}_{n-1}(\eta_1)\tilde{p}_{n-1}(\eta_2) \end{bmatrix}, \text{ and } \begin{bmatrix} a_n p_{n-1}(\eta_1)p_n(\eta_2) \\ a_n p_n(\eta_1)p_{n-1}(\eta_2) \\ a_n^2 p_{n-1}(\eta_1)p_{n-1}(\eta_2) \\ p_n(\eta_1)p_n(\eta_2) \end{bmatrix},$$

which are $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \mu_0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\mu_0 \end{bmatrix}$ at $n = 0$ (from (28)), so that

$$\begin{aligned} -\frac{A_n(x)}{\eta_2(x) - \eta_1(x)} - \frac{C_n(x)}{2} &= \alpha a_n p_{n-1}(\eta_1)\tilde{p}_{n-1}(\eta_2) + \beta a_n\tilde{p}_{n-2}(\eta_1)p_n(\eta_2) + \gamma a_n\tilde{p}_{n-2}(\eta_1)\tilde{p}_{n-1}(\eta_2) + \delta a_n p_{n-1}(\eta_1)p_n(\eta_2) \\ \frac{A_n(x)}{\eta_2(x) - \eta_1(x)} - \frac{C_n(x)}{2} &= \alpha a_n p_n(\eta_1)\tilde{p}_{n-2}(\eta_2) + \beta a_n\tilde{p}_{n-1}(\eta_1)p_{n-1}(\eta_2) + \gamma a_n\tilde{p}_{n-1}(\eta_1)\tilde{p}_{n-2}(\eta_2) + \delta a_n p_n(\eta_1)p_{n-1}(\eta_2) \\ B_n(x) &= \alpha a_n^2 p_{n-1}(\eta_1)\tilde{p}_{n-2}(\eta_2) + \beta a_n^2\tilde{p}_{n-2}(\eta_1)p_{n-1}(\eta_2) + \gamma a_n^2\tilde{p}_{n-2}(\eta_1)\tilde{p}_{n-2}(\eta_2) + \delta a_n^2 p_{n-1}(\eta_1)p_{n-1}(\eta_2) \\ D_n(x) &= \alpha p_n(\eta_1)\tilde{p}_{n-1}(\eta_2) + \beta\tilde{p}_{n-1}(\eta_1)p_n(\eta_2) + \gamma\tilde{p}_{n-1}(\eta_1)\tilde{p}_{n-1}(\eta_2) + \delta p_n(\eta_1)p_n(\eta_2) \end{aligned} \tag{43}$$

with $\alpha = -\dfrac{A_0(x)}{\eta_2-\eta_1} + \dfrac{C_0(x)}{2}$, $\beta = \dfrac{A_0(x)}{\eta_2-\eta_1} + \dfrac{C_0(x)}{2}$, $\gamma = \dfrac{B_0(x)}{\mu_0}$, $\delta = \mu_0 D_0(x)$.

One may now check that Laguerre's $\Theta_n = -D_n$, ***but the truly useful outcome of*** (43) ***is the production of relations linking*** p_n ***and*** p_{n-1}.

Trying to eliminate as many $\tilde{p}$'s as possible, let us take the first equation of (43) times $p_n(\eta_1)$ minus the last equation times $a_n p_{n-1}(\eta_1)$: the contributions of α and δ disappear, and

$$\left(-\frac{A_n(x)}{\eta_2(x)-\eta_1(x)} - \frac{C_n(x)}{2}\right) p_n(\eta_1) - a_n D_n(x) p_{n-1}(\eta_1) = \beta a_n X_n p_n(\eta_2) + \gamma a_n X_n \tilde{p}_{n-1}(\eta_2),$$

with $X_n = \tilde{p}_{n-2}(\eta_1)p_n(\eta_1) - \tilde{p}_{n-1}(\eta_1)p_{n-1}(\eta_1)$, whence the formidable simplification $a_n X_n = -1$, thanks to (30), and

$$\left(-\frac{A_n(x)}{\eta_2(x)-\eta_1(x)} - \frac{C_n(x)}{2}\right) p_n(\eta_1) - a_n D_n(x) p_{n-1}(\eta_1) = -\left(\frac{A_0(x)}{\eta_2-\eta_1} + \frac{C_0(x)}{2}\right) p_n(\eta_2) - \frac{B_0(x)}{\mu_0}\tilde{p}_{n-1}(\eta_2). \tag{44}$$

Similarly with the second and last equations of (43), and with η_2 instead of η_1:

$$\left(\frac{A_n(x)}{\eta_2(x)-\eta_1(x)} - \frac{C_n(x)}{2}\right) p_n(\eta_2) - a_n D_n(x) p_{n-1}(\eta_2) = \left(\frac{A_0(x)}{\eta_2-\eta_1} - \frac{C_0(x)}{2}\right) p_n(\eta_1) - \frac{B_0(x)}{\mu_0}\tilde{p}_{n-1}(\eta_1). \tag{45}$$

5.6. **Difference relations & equations: the importance of being semi-classical.**

In the semi-classical case, $B_0 = 0$, (44)-(45) do not contain $\tilde{p}_{n-1}$ any more. Summing (44) and (45) yields a good looking ***difference relation***

$$(A_n(x) + A_0(x))(\mathscr{D}p_n)(x) = (C_n(x) - C_0(x))(\mathscr{M}p_n)(x) + 2a_n D_n(x)(\mathscr{M}p_{n-1})(x), \tag{46}$$

but the true useful meaning of (44)-(45) is to show how $[p_n(\eta_1), p_n(\eta_2)]$ can be determined from $[p_{n-1}(\eta_1), p_{n-1}(\eta_2)]$, er, oops, it seems that it is more obvious to get $[p_{n-1}(\eta_1), p_{n-1}(\eta_2)]$ from $[p_n(\eta_1), p_n(\eta_2)]$.
Ah, this is the subject of a little ***exercise***: from the recurrence relation (27), $-a_n p_{n-1}(\eta) = -(\eta - b_n)p_n(\eta) + a_{n+1}p_{n+1}(\eta)$, and one can establish, using (37)

$$\left(-\frac{A_{n+1}(x)}{\eta_2(x)-\eta_1(x)} + \frac{C_{n+1}(x)}{2}\right) p_n(\eta_1) + a_{n+1} D_n(x) p_{n+1}(\eta_1) = -\left(\frac{A_0(x)}{\eta_2-\eta_1} + \frac{C_0(x)}{2}\right) p_n(\eta_2),$$
$$\left(\frac{A_{n+1}(x)}{\eta_2(x)-\eta_1(x)} + \frac{C_{n+1}(x)}{2}\right) p_n(\eta_2) + a_{n+1} D_n(x) p_{n+1}(\eta_2) = \left(\frac{A_0(x)}{\eta_2-\eta_1} - \frac{C_0(x)}{2}\right) p_n(\eta_1),$$

so, $a_n D_{n-1}(x) \begin{bmatrix} p_n(\eta_1) \\ p_n(\eta_2) \end{bmatrix} = \begin{bmatrix} A_n/\Delta\eta - C_n/2 & -A_0/\Delta\eta - C_0/2 \\ A_0/\Delta\eta - C_0/2 & -A_n/\Delta\eta - C_n/2 \end{bmatrix} \begin{bmatrix} p_{n-1}(\eta_1) \\ p_{n-1}(\eta_2) \end{bmatrix}$.

But we also have, from (44)-(45) (remember that $B_0 = 0$)

$$\begin{bmatrix} p_n(\eta_1) \\ p_n(\eta_2) \end{bmatrix} = \begin{bmatrix} -A_n/\Delta\eta - C_n/2 & A_0/\Delta\eta + C_0/2 \\ -A_0/\Delta\eta + C_0/2 & A_n/\Delta\eta - C_n/2 \end{bmatrix}^{-1} a_n D_n(x) \begin{bmatrix} p_{n-1}(\eta_1) \\ p_{n-1}(\eta_2) \end{bmatrix}$$

the two matrices are found to be the same (use (39)).

Most dramatic consequence of (44)-(45):
The second order difference equation.
Take (44) at $x = x_k$ and (45) at $x = x_{k-1}$, and eliminate $p_{n-1}(y_k)$ (as $\eta_1(x_k) = \eta_2(x_{k-1}) = x_k$):

$$D_n(x_{k-1})\left(\frac{A_0(x_k)}{y_{k+1}-y_k}+\frac{C_0(x_k)}{2}\right)p_n(y_{k+1})$$
$$-\left[D_n(x_{k-1})\left(\frac{A_n(x_k)}{y_{k+1}-y_k}+\frac{C_n(x_k)}{2}\right)+D_n(x_k)\left(\frac{A_n(x_{k-1})}{y_k-y_{k-1}}-\frac{C_n(x_{k-1})}{2}\right)\right]p_n(y_k)$$
$$+D_n(x_k)\left(\frac{A_0(x_{k-1})}{y_k-y_{k-1}}-\frac{C_0(x_{k-1})}{2}\right)p_n(y_{k-1})=0. \quad (47)$$

5.7. **Difference relations, difference equations, and hypergeometric expansions of the Askey-Wilson polynomials.**

We are at last able now to recover the difference equation of the $\mathscr{D}-$classical orthogonal polynomials (Nikiforov & *al.* starting point [25–27]), as well as the ${}_4\phi_3$ expansion (Askey & Wilson starting point [2]).

First, the difference equation: when $s=$ (degree D_n) $=0$ drop the same constant D_n from the three terms of (47). The dependence in n is quite interesting. Remark already that n appears only in the middle term. As C_n is only of degree 1, $C_n(x_k)-C_n(x_{k-1})=\hat{C}_n(x_k-x_{k-1})$ with the $\hat{C}_n$ of § 5.4. And $A_n(x_p)=A_0(x_p)+((y_{p+1}-y_p)^2/2)(D_0+\cdots+D_{n-1})$ from (33), whence the form

$$\mathscr{L}p_n=\lambda_n p_n, \quad (48)$$

with

$$\mathscr{L}p_n(y_k)=\frac{1}{x_k-x_{k-1}}\left(A_0(x_k)\frac{p_n(y_{k+1})-p_n(y_k)}{y_{k+1}-y_k}-A_0(x_{k-1})\frac{p_n(y_k)-p_n(y_{k-1})}{y_k-y_{k-1}}\right.$$
$$\left.+C_0(x_k)\frac{p_n(y_{k+1})+p_n(y_k)}{2}-C_0(x_{k-1})\frac{p_n(y_k)+p_n(y_{k-1})}{2}\right),$$

i.e.,

$$\mathscr{L}=\widetilde{\mathscr{D}}A_0\mathscr{D}+\widetilde{\mathscr{D}}C_0\mathscr{M}, \quad (49)$$

and

$$\lambda_n=-\frac{\mathfrak{b}}{\mathfrak{a}}(D_0+\cdots+D_{n-1})+\frac{\hat{C}_n+\hat{C}_0}{2}$$

(use $y_{k+1}-y_k+(y_k-y_{k-1})=y_{k+1}+y_k-(y_k+y_{k-1})=(-2\mathfrak{b}/\mathfrak{a})(x_k-x_{k-1})$).

To explore the ***hypergeometric expansion*** of p_n, look at the recurrence relation (27) (for the monic polynomials p_n/γ_n), when (40)-(41) hold:

$$p_{n+1}(x)/\gamma_{n+1}=(x-y_1-F_{n+1}-G_n)p_n(x)/\gamma_n-F_nG_np_{n-1}(x)/\gamma_{n-1},$$

where $x_0=\omega$, $y_1=\eta(\omega)=\eta_2(\omega)$, or

$$p_{n+1}(x)/\gamma_{n+1}+F_{n+1}p_n(x)/\gamma_n+G_n[p_n(x)/\gamma_n+F_np_{n-1}(x)/\gamma_{n-1}]=(x-y_1)p_n(x)/\gamma_n,$$

showing that $p_n(y_1)/\gamma_n=(-1)^nF_1\cdots F_n$, as $G_n=0$.

Now, (48)-(49) at y_1, i.e., (47) at $k=1$:

$$\left(\frac{A_0(x_1)}{y_2-y_1}+\frac{C_0(x_1)}{2}\right)p_n(y_2)=\left(\frac{A_0(x_1)}{y_2-y_1}+\frac{A_0(x_0)}{y_1-y_0}-\frac{C_0(x_1)-C_0(x_0)}{2}+(x_1-x_0)\lambda_n\right)p_n(y_1)$$
$$=\left(\frac{A_0(x_1)}{y_2-y_1}+C_0(x_0)-\frac{C_0(x_1)}{2}+(x_1-x_0)\lambda_n\right)p_n(y_1)$$
$$=\left(\frac{A_0(x_1)}{y_2-y_1}+\frac{C_0(x_1)}{2}+(x_1-x_0)(\lambda_n-\hat{C}_0)\right)p_n(y_1),$$

having used $2A_0(x_0) = (y_1 - y_0)C_0(x_0)$, so,

$$p_n(y_2) = \left[1 + \frac{(x_1 - x_0)(\lambda_n - \hat{C}_0)(y_2 - y_1)}{A_0(x_1) + (y_2 - y_1)C_0(x_1)/2}\right] p_n(y_1)$$

preparing an expansion $p_n(x) = p_n(y_1)[1 + c_1(x - y_1) + c_2(x - y_1)(x - y_2) + \cdots]$.

5.8. **Difference equation for Laguerre-Hahn orthogonal polynomials.**

When $B_0(x) \not\equiv 0$, there is no more a second order difference equation for p_n, but one can still do something with the solutions of (47) (Hahn).

Let φ_n and ψ_n be two independent solutions of (47), i.e.,

$$\begin{aligned} D_n(x_{k-1}) \left(\frac{A_0(x_k)}{y_{k+1} - y_k} + \frac{C_0(x_k)}{2}\right) \varphi_n(y_{k+1}) \\ - \left[D_n(x_{k-1}) \left(\frac{A_n(x_k)}{y_{k+1} - y_k} + \frac{C_n(x_k)}{2}\right) + D_n(x_k) \left(\frac{A_n(x_{k-1})}{y_k - y_{k-1}} - \frac{C_n(x_{k-1})}{2}\right)\right] \varphi_n(y_k) \\ + D_n(x_k) \left(\frac{A_0(x_{k-1})}{y_k - y_{k-1}} - \frac{C_0(x_{k-1})}{2}\right) \varphi_n(y_{k-1}) = 0, \end{aligned} \quad (50)$$

and the same for ψ_n. Now, nor φ_n nor ψ_n need to be a polynomial. The index n is no more related to a degree of any kind.

To be continued ...

6. New directions?

6.1. **Chasles-Euler-Poncelet.**

"**Poncelet Transverse:** Let a Circle C_1 lie inside another Circle $C2$. From any point on C_2, draw a tangent to C_1 and extend it to C_2. From the point, draw another tangent, etc. For n tangents, the result is called an $n-$sided Poncelet Transform.

References Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 192, 1965."

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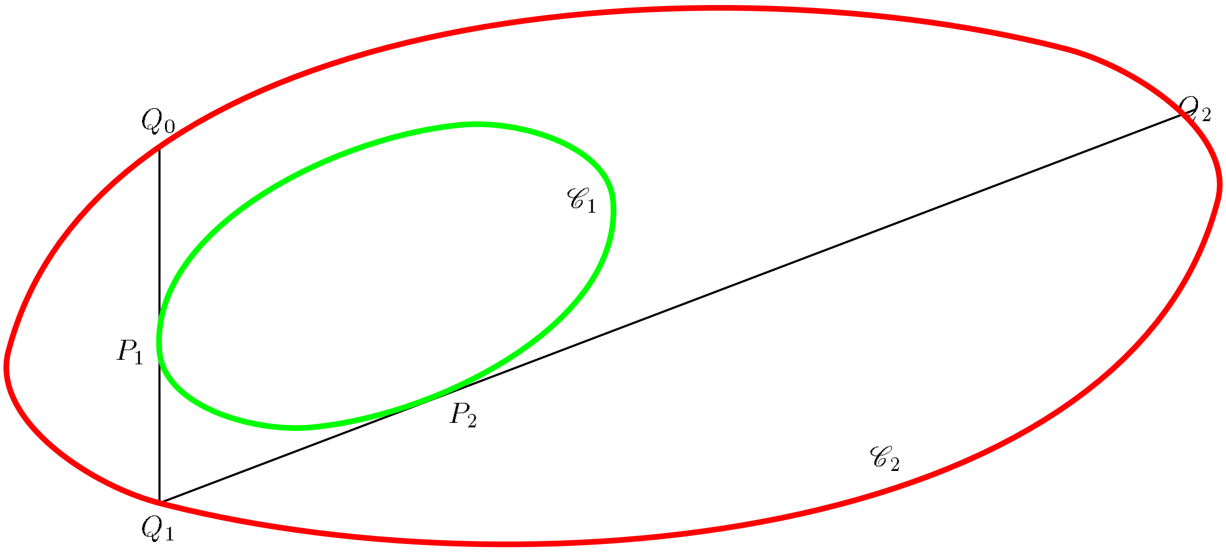
For two general conics $\mathscr{C}_1$ and $\mathscr{C}_2$, take a point Q_1 on $\mathscr{C}_2$, call Q_0 and Q_2 its two Poncelet transforms defining also P_1 and P_2 on $\mathscr{C}_1$, etc. We so have two lattices on the two conics. There is a theorem by Poncelet stating that if the polygonal line closes on itself after a finite number of steps for some Q_1, the it does so for any starting point on $\mathscr{C}_2$. There must be some $q-$geometric progression



somewhere ...

As one can always project one of the conics on a circle, we could have a clue about $\mathscr{D}-$orthogonal polynomials on the circle. Askey-Wilson polynomials on the circle have been considered in [36].

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See also the part "q-Special functions, a tutorial", in `ftp://unvie6.un.or.at/siam/opsf_new/koornwinder/koornwinder3/koornwinder3.tex` and *ibid.*`/wilsonqhahn.ps`

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Appendix. About polygonal, or figurate, numbers.

I just encountered this enthusiastic account and can't resist to include it:

sci.math #225968 (24 + 4158 more)
From: TechMaster Pinyan <jefpin@bergen.org>
[1]$n-$gonal numbers
Date: Sat Jul 19 09:07:43 EDT 1997

Ok... I am happy.

I have made a great mathematical discovery.

Actually, I'm hyping this up a bit.

I was looking a "Five Hundred Mathematical Challenges" by Edward J. Barbeau, Murray S. Klamkin, and William O. J. Moser. In the back, they have a "tool chest" of mathematical tidbits that can help you solve the 500 questions. One of them had to do with figurate numbers: triangular, square, pentagonal, hexagonal... And it got me thinking...

"I know the formula for triangular numbers... $(x/2)(x+1)$... square numbers are obviously x^2, and pentagonal numbers are $(x/2)(3x-1)$... there has to be a pattern, so you can find the formula for the xth $n-$gonal number."

There IS, but x^2 is a red herring. Instead, look at $(x/2)(x+1)$ and $(x/2)(3x-1)$:

$$1x + 1$$
$$3x - 1$$

Would it not follow that for a square number, you would use:

$$2x + 0$$

I was amazed. I was thrilled. I am pathetic :(. But I did it! I figured out the formula.

For the xth $n-$gonal number, use this:

$$(x/2)[(n-2)x + (4-n)]$$

Hooray... this will make my calc teacher happy. It may not be calculus, but it's math...

For now, I'm gonna calculate the 453rd 32-gonal number...

————-

— "I don't contemplate it, I just sit and think about it."
— - Sonia Balsky
————- Jeff "TechMaster" Pinyan — http://www.bergen.org/˜jefpin
HTML/CGI Designer and Consultant and JavaScripter
jefpin@bergen.org — TechMasterJeff@juno.com
Got a JavaScript/CGI/Perl question or problem? Let me know!
webXS - the new eZine for WebProgrammers! webXS@juno.com
Visit us @ http://www.bergen.org/ jefpin/webXS

End of article 225968 (of 226703) – what next? [npq]

sci.math #226021 (23 + 4158 more)
From: ptwahl@aol.com (PTWahl)
[1]Re : $n-$gonal numbers
Date: Sat Jul 19 16:16:59 EDT 1997

Good work. None of this is new, but there is value in finding things out for oneself; new perspectives sometimes result in new shoots off an old root.

For your amusement (I don't need to know the answer!) I offer these little tasks:

1. Note that "x" in your formulas "for the xth $n-$gonal number" can be a positive or negative integer. Investigate what happens when you do that.

2. We all know that $(x)^2 = (-x)^2$, for the square numbers ($n = 4$). How about for the triangular numbers ($n = 3$)? Something similar (not quite the same) is going on. Discover what it is, and write a formula.

3. List some hexagonal numbers ($n = 6$), say from $x = -5$ to $x = +5$. Compare these to the triangular numbers. What is going on here? Can that happen for other sets of n-gonal numbers?

You write, "It may not be calculus, but it's math ..." But you're just scratching the surface, so far. There's calculus around the corner (formal power series, modular forms, etc.) when you get farther along. Figurate numbers provide a set of neat examples, all the way along.

There is a lot of fascination, even today, with the figurate numbers. I enjoy the topic partly because of the connection between "elementary" techniques and much deeper results. For example, there turns out to be a connection between these two problems:

(A) The number of ways, counting sums in different order as different, to express a positive integer N as the sum of 24 triangular numbers. This turns out to be expressible in terms of the sum of the 12th powers of the divisors of N and $(N/2)$, together with Ramanujan's intriguing "Tau" function (essentially the 24th power of the "eta" function).

(B) The optimal packing of hyper-spheres in 24 dimensions, which among other applications resulted in an efficient code for correcting errors in data transmissions such as the Internet.

Good luck!

Patrick T. Wahl
(no institutional affiliation)

End of article 226021 (of 226703) – what next? [npq]

Afternote: adjoint in Wolfgang Hahn's *Lineare Geometrische Differenzengleichungen*, Bericht Nr. 169 der Mathematisch-Statistischen Sektion im Forschungszentrum Graz (1981):
the general linear homogeneous system is

$$\boldsymbol{f}(qx) = \boldsymbol{P}(x)\boldsymbol{f}(x), \tag{51}$$

where $\boldsymbol{f}$ is a column vector and $\boldsymbol{P}$ is a square matrix. If one comes from a scalar equation of order n

$$p_0(x)f(x) + p_1(x)f(qx) + \cdots + p_n(x)f(q^n x) = 0,$$

i.e.,

$$\mathcal{L}f := \sum_{k=0}^{n} \frac{p_k}{p_n} \mathsf{J}^k f = 0, \tag{52}$$

(with $\mathsf{J}f(x) := f(qx)$), one then has

$$\boldsymbol{f}(x) = \begin{bmatrix} f(x) \\ \mathsf{J}f(x) \\ \vdots \\ \mathsf{J}^{n-1}f(x) \\ \mathsf{J}^n f(x) \end{bmatrix}, \qquad \boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{p_0}{p_n} & -\dfrac{p_1}{p_n} & -\dfrac{p_2}{p_n} & \cdots & -\dfrac{p_{n-1}}{p_n} \end{bmatrix}$$

One defines the adjoint to (51) to be the equation

$$\boldsymbol{g}(qx) = \boldsymbol{Q}(x)\boldsymbol{g}(x), \tag{53}$$

in such a way that the scalar product $\boldsymbol{g}^T\boldsymbol{f}$ is constant by multiplication by q, i.e., $q-$periodic.
As $\boldsymbol{g}^T(qx)\boldsymbol{f}(qx) = \boldsymbol{g}^T(x)\boldsymbol{Q}^T(x)\boldsymbol{P}(x)\boldsymbol{f}(x)$, it follows $\boldsymbol{Q}^T\boldsymbol{P} = \boldsymbol{E}$, and the adjoint to (51) is

$$\boldsymbol{g}(x) = \boldsymbol{P}^T(x)\boldsymbol{g}(qx), \qquad \text{or} \qquad \boldsymbol{g}(q^{-1}x) = \boldsymbol{P}^T(q^{-1}x)\boldsymbol{g}(x). \tag{54}$$

If the initial equation is scalar, as (52), the adjoint (54) has not the same form, as $\boldsymbol{g}$ is not compelled to have the form $[g(x), g(qx), \ldots, g(q^{n-1}x)]^T$, but we simply have to look for the recurrence satisfied by the n^{th} component of $\boldsymbol{g}$. We get this by eliminating the other components: from (54),

$$g_{i+1}(q^{-1}x) = g_i(x) - \frac{p_i(q^{-1}x)}{p_n(q^{-1}x)} g_n(x), \qquad i = 0, 1, \ldots, n-1 \qquad (g_0 = 0) \tag{55}$$

or

$$\begin{aligned} g_i(x) &= g_{i+1}(q^{-1}x) + \frac{p_i(q^{-1}x)}{p_n(q^{-1}x)} g_n(x), \\ &= g_{i+2}(q^{-2}x) + \frac{p_{i+1}(q^{-2}x)}{p_n(q^{-2}x)} g_n(q^{-1}x) + \frac{p_i(q^{-1}x)}{p_n(q^{-1}x)} g_n(x), \\ &\cdots \\ &= g_n(q^{i-n}x) + \frac{p_{n-1}(q^{i-n}x)}{p_n(q^{i-n}x)} g_n(q^{i-n+1}x) + \cdots + \frac{p_i(q^{-1}x)}{p_n(q^{-1}x)} g_n(x) \end{aligned}$$

and look at $i = 0$:

$$(\mathcal{M}g)(x) := g_0(x) = \sum_{k=0}^{n} \frac{p_k(q^{-k-1}x)}{p_n(q^{-k-1}x)} (\mathsf{J}^{-k}g)(x) = 0, \tag{56}$$

(where $g(x) = g_n(x)$).

Finally, consider now, for any f and g, so that $\mathcal{L}f$ and $g_0 = \mathcal{M}g$ need not to vanish,

$$R(x) = \boldsymbol{g}^T\boldsymbol{f} = \sum_{i=0}^{n-1} g_{i+1}(x)f(q^i x) = \sum_{i=0}^{n-1} f(q^i x) \sum_{k=i+1}^{n} \frac{p_k(q^{i-k}x)}{p_n(q^{i-k}x)} g(q^{i-k+1}x),$$

which should be constant ($q-$periodic) when $\mathcal{L}f = 0$ and $g_0 = \mathcal{M}g = 0$. Check:

$$\begin{aligned}
R(x) - R(qx) &= \sum_{i=0}^{n-1} g_{i+1}(x)f(q^i x) - \sum_{j=0}^{n-1} g_{j+1}(qx)f(q^{j+1}x) \\
&= \sum_{i=0}^{n-1} g_{i+1}(x)f(q^i x) - \sum_{i=1}^{n} g_i(qx)f(q^i x) \\
&= g_0(qx)f(x) + \sum_{i=0}^{n-1}[g_{i+1}(x) - g_i(qx)]f(q^i x) \quad - g(qx)f(q^n x) \\
&= f(x)(\mathcal{M}g)(qx) - \sum_{i=0}^{n-1} \frac{p_i(x)}{p_n(x)} g(qx)f(q^i x) \quad - g(qx)f(q^n x) \qquad \text{(from (55))} \\
&= f(x)(\mathcal{M}g)(qx) - g(qx)(\mathcal{L}f)(x).
\end{aligned}$$

Sum this exact difference on a lattice $\{x_0, x_0 q, \ldots, x_0 q^{N-1}\}$:

$$R(x_0) - R(x_0 q^N) = \sum_{r=0}^{N-1} f(x_r)(\mathcal{M}g)(qx_r) - g(qx_r)(\mathcal{L}f)(x_r).$$

And the sensible duality form is

$$\langle F, G\rangle = \sum_{r=0}^{N-1} F(x_r)G(qx_r) = \sum_{r=0}^{N-1} F(x_r)G(x_{r+1}).$$

cf. N.E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, 1924 = Chelsea, N.Y., 1954 (p. 289); W. Hahn, Über die höheren Heineschen Reihen und eine einheitliche Theorie der sogenannten speziellen Funktionen, *Math. Nachr.* **3** (1950) 257-294.

(Of course, Nörlund's book is about *arithmetic* difference equations, he finds $\mu(x) + \sum_{i=1}^{n} q_{n-i}(x)\mu(x+i) = 0$ to be the adjoint of $u(x+n) + \sum_{i=0}^{n-1} q_i(x)u(x+i) = 0$, where $q_i(x) = p_i(x)/p_n(x)$).

Hilbertian theory: according to Brown, Evans, and Ismail [8], the adjoint of $\mathcal{D}_q$ (see (20), p. 10) with respect to $\langle F, G\rangle = \int_0^\pi F(\cos\theta)\overline{G(\cos\theta)}\, d\theta = \int_{-1}^1 F(x)\overline{G(x)}\,(1-x^2)^{-1/2}\,dx$ is $\mathcal{D}_q^*$ defined by $(\mathcal{D}_q^*)(f(x)) = -(1-x^2)^{1/2}\mathcal{D}_q[(1-x^2)^{-1/2}f(x)]$.

The proof is rather hard.

Other afternote: a (slightly) more complete version of the section

5.8. Difference equation for Laguerre-Hahn orthogonal polynomials.

We work again the general identities of § 5.5, introducing first other, hopefully more convenient, notations:

$$\alpha_n(x) = -\frac{A_n(x)n}{\eta_2(x)-\eta_1(x)}+\frac{C_n(x)}{2}, \beta_n(x) = \frac{A_n(x)n}{\eta_2(x)-\eta_1(x)}+\frac{C_n(x)}{2}, \gamma_n(x) = \frac{B_n(x)}{\mu_0}, \delta_n(x) = \mu_0 D_n(x),$$

so that (43) becomes

$$\begin{aligned}
-\beta_n(x) &= \alpha_0 a_n p_{n-1}(\eta_1)\tilde{p}_{n-1}(\eta_2) + \beta_0 a_n \tilde{p}_{n-2}(\eta_1) p_n(\eta_2) + \gamma_0 a_n \tilde{p}_{n-2}(\eta_1)\tilde{p}_{n-1}(\eta_2) + \delta_0 a_n p_{n-1}(\eta_1) p_n(\eta_2)\\
-\alpha_n(x) &= \alpha_0 a_n p_n(\eta_1)\tilde{p}_{n-2}(\eta_2) + \beta_0 a_n \tilde{p}_{n-1}(\eta_1) p_{n-1}(\eta_2) + \gamma_0 a_n \tilde{p}_{n-1}(\eta_1)\tilde{p}_{n-2}(\eta_2) + \delta_0 a_n p_n(\eta_1) p_{n-1}(\eta_2)\\
B_n(x) &= \alpha_0 a_n^2 p_{n-1}(\eta_1)\tilde{p}_{n-2}(\eta_2) + \beta_0 a_n^2 \tilde{p}_{n-2}(\eta_1) p_{n-1}(\eta_2) + \gamma_0 a_n^2 \tilde{p}_{n-2}(\eta_1)\tilde{p}_{n-2}(\eta_2) + \delta_0 a_n^2 p_{n-1}(\eta_1) p_{n-1}(\eta_2)\\
D_n(x) &= \alpha_0 p_n(\eta_1)\tilde{p}_{n-1}(\eta_2) + \beta_0 \tilde{p}_{n-1}(\eta_1) p_n(\eta_2) + \gamma_0 \tilde{p}_{n-1}(\eta_1)\tilde{p}_{n-1}(\eta_2) + \delta_0 p_n(\eta_1) p_n(\eta_2)
\end{aligned}$$

We already got (44)-(45) through combinations of the lines 1 and 4 above, then of lines 2 and 4. We go further, with now lines 1 and 3, then lines 2 and 3, always managing to use (30):

$$\begin{aligned}
-\beta_n p_n(\eta_1) - a_n D_n p_{n-1}(\eta_1) &= -\beta_0 p_n(\eta_2) - \gamma_0 \tilde{p}_{n-1}(\eta_2),\\
-\alpha_n p_n(\eta_2) - a_n D_n p_{n-1}(\eta_2) &= -\alpha_0 p_n(\eta_1) - \gamma_0 \tilde{p}_{n-1}(\eta_1),\\
-a_n\beta_n \tilde{p}_{n-2}(\eta_2) - B_n \tilde{p}_{n-1}(\eta_2) &= -a_n\beta_0 \tilde{p}_{n-2}(\eta_1) - a_n\delta_0 p_{n-1}(\eta_1),\\
-a_n\alpha_n \tilde{p}_{n-2}(\eta_1) - B_n \tilde{p}_{n-1}(\eta_1) &= -a_n\alpha_0 \tilde{p}_{n-2}(\eta_2) - a_n\delta_0 p_{n-1}(\eta_2),
\end{aligned}$$

which shows how the p's and the $\tilde{p}$'s at η_1 and η_2 are linked together, in other words:
the general Laguerre-Hahn orthogonal polynomials enter a first order difference system of 4 equations

$$\boldsymbol{M}_n \begin{bmatrix} p_n(\eta_2)\\ p_{n-1}(\eta_2)\\ \tilde{p}_{n-1}(\eta_2)\\ \tilde{p}_{n-2}(\eta_2)\end{bmatrix} = \boldsymbol{N}_n \begin{bmatrix} p_n(\eta_1)\\ p_{n-1}(\eta_1)\\ \tilde{p}_{n-1}(\eta_1)\\ \tilde{p}_{n-2}(\eta_1)\end{bmatrix},$$

$$\boldsymbol{M}_n = \begin{bmatrix} -\beta_0 & & -\gamma_0 & \\ -\alpha_n & -a_n D_n & & \\ & & -B_n & -a_n\beta_n \end{bmatrix}, \boldsymbol{N}_n = \begin{bmatrix} -\beta_n & -a_n D_n & & \\ -\alpha_0 & & -\gamma_0 & \\ & -a_n\delta_0 & & -a_n\beta_0 \\ & & -B_n & -a_n\alpha_n \end{bmatrix}. \quad (57)$$

It should be possible to extract from (57) a (probably ugly) ***scalar difference equation of order 4*** for p_n, but it is much more interesting to investigate the structure of the set of solutions of (57).

What about the matrix in $\boldsymbol{p}(\eta_2) = \boldsymbol{M}_n^{-1}\boldsymbol{N}_n\boldsymbol{p}(\eta_1)$? Remark first that $\boldsymbol{M_n}$ and $\boldsymbol{N}_n$ have the same determinant $a_n^2[\alpha_0\beta_0 B_n D_n - \gamma_0\delta_0\alpha_n\beta_n] = a_n^2[\alpha_0\beta_0 B_n D_n - \alpha_n\beta_n B_0 D_0]$.
This asks for a bit more investigation of earlier identities, the two first equations above (33) are

$$\beta_{m+1} = -\alpha_m - (\eta_2 - b_m)D_m, \qquad \alpha_{m+1} = -\beta_m - (\eta_1 - b_m)D_m,$$

and (36) times $\dfrac{B_{m+1}}{a_{m+1}^2} = D_m$ is rearranged as

$$\begin{aligned}
D_{m+1}B_{m+1} &= \frac{(\eta_2-\eta_1)(\beta_m-\alpha_m)}{2}D_m + \frac{(\eta_1+\eta_2-2b_m)(\alpha_m+\beta_m)}{2}D_m + B_mD_m + (\eta_1-b_m)(\eta_2-b_m)D_m^2 \\
&= B_mD_m + [(\eta_1-b_m)\alpha_m + (\eta_2-b_m)\beta_m]D_m + (\eta_1-b_m)(\eta_2-b_m)D_m^2 \\
&= B_mD_m + [(\eta_1-b_m)D_m+\beta_m][(\eta_2-b_m)D_m+\alpha_m] - \alpha_m\beta_m \\
&= B_mD_m + \alpha_{m+1}\beta_{m+1} - \alpha_m\beta_m
\end{aligned}$$

so, we recover (39) as

$$B_mD_m - \alpha_m\beta_m = \frac{\Omega}{(\eta_2-\eta_1)^2}.$$

The common determinant of $\boldsymbol{M}_n$ and $\boldsymbol{N}_n$ is therefore

$$a_n^2\left[\alpha_0\beta_0\left(\alpha_n\beta_n + \frac{\Omega}{(\eta_2-\eta_1)^2}\right) - \alpha_n\beta_n\left(\alpha_0\beta_0 + \frac{\Omega}{(\eta_2-\eta_1)^2}\right)\right] = a_n^2(\alpha_0\beta_0-\alpha_n\beta_n)\frac{\Omega}{(\eta_2-\eta_1)^2}.$$

Now, we come to $\boldsymbol{M}_n^{-1}\boldsymbol{N}_n$. I don't try to look for clever shortcuts, just follow (57):

$$\begin{aligned}
\boldsymbol{M}_n^{-1}\boldsymbol{N}_n &= \frac{(\eta_2-\eta_1)^2}{a_n^2(\alpha_0\beta_0-\alpha_n\beta_n)\Omega}\begin{bmatrix} -a_n^2\alpha_0B_nD_n & a_n^2\gamma_0\delta_0\beta_n & a_n^2\alpha_0\gamma_0D_n & -a_n^2\gamma_0\beta_nD_n \\ a_n\alpha_0\alpha_nB_n & -a_n\alpha_0\beta_0B_n & -a_n\alpha_0\gamma_0\alpha_n & a_n\gamma_0\alpha_n\beta_n \\ a_n^2\delta_0\alpha_n\beta_n & -a_n^2\beta_0\delta_0\beta_n & -a_n^2\alpha_0\beta_0D_n & a_n^2\beta_0\beta_nD_n \\ -a_n\delta_0\alpha_nB_n & a_n\beta_0\delta_0B_n & a_n\gamma_0\delta_0\alpha_n & -a_n\beta_0B_nD_n \end{bmatrix} \\
&\qquad\qquad \times \begin{bmatrix} -\beta_n & -a_nD_n & & \\ -\alpha_0 & & -\gamma_0 & \\ & -a_n\delta_0 & & -a_n\beta_0 \\ & & -B_n & -a_n\alpha_n \end{bmatrix} \qquad (58) \\
&= \frac{(\eta_2-\eta_1)^2(\alpha_0\beta_0-\alpha_n\beta_n)}{a_n^2(\alpha_0\beta_0-\alpha_n\beta_n)\Omega}\begin{bmatrix} -a_n^2\alpha_0\beta_n & -a_n^3\alpha_0D_n & -a_n^2\gamma_0\beta_n & -a_n^3\gamma_0D_n \\ a_n\alpha_0B_n & a_n^2\alpha_0\alpha_n & a_n\gamma_0B_n & a_n^2\gamma_0\alpha_n \\ a_n^2\delta_0\beta_n & a_n^3\delta_0D_n & a_n^2\beta_0\beta_n & a_n^3\beta_0D_n \\ -a_n\delta_0B_n & -a_n^2\delta_0\alpha_n & -a_n\beta_0B_n & -a_n^2\beta_0\alpha_n \end{bmatrix}.
\end{aligned}$$

Hmmm, an array mixing in a very remarkable way (Kronecker product) things with index n and things with index 0. Let us separate the two parts of the mixture: consider the system

$$\begin{bmatrix}\varphi_n(\eta_2)\\ \psi_n(\eta_2)\end{bmatrix} = \begin{bmatrix}\beta_n & a_nD_n \\ -a_nD_{n-1} & -\alpha_n\end{bmatrix}\begin{bmatrix}\varphi_n(\eta_1)\\ \psi_n(\eta_1)\end{bmatrix}. \qquad (59)$$

Remark that he determinant of the matrix is $a_n^2D_nD_{n-1} - \alpha_n\beta_n = B_nD_n - \alpha_n\beta_n = \dfrac{\Omega}{(\eta_2-\eta_1)^2}$.

Consider also

$$\begin{bmatrix}\phi_m(\eta_2)\\ \rho_m(\eta_2)\end{bmatrix} = \frac{(\eta_2-\eta_1)^2}{\Omega}\begin{bmatrix}-\alpha_m & -\gamma_m \\ \delta_m & \beta_m\end{bmatrix}\begin{bmatrix}\phi_m(\eta_1)\\ \rho_m(\eta_1)\end{bmatrix}. \qquad (60)$$

Then, one finds (using $a_nD_{n-1} = B_n/a_n$),

$$\begin{bmatrix}\varphi_n(\eta_2)\phi_0(\eta_2)\\ \psi_n(\eta_2)\phi_0(\eta_2)\\ \varphi_n(\eta_2)\rho_0(\eta_2)\\ \psi_n(\eta_2)\rho_0(\eta_2)\end{bmatrix} = \boldsymbol{M}_n^{-1}\boldsymbol{N}_n\begin{bmatrix}\varphi_n(\eta_1)\phi_0(\eta_1)\\ \psi_n(\eta_1)\phi_0(\eta_1)\\ \varphi_n(\eta_1)\rho_0(\eta_1)\\ \psi_n(\eta_1)\rho_0(\eta_1)\end{bmatrix}$$

So, if $\begin{bmatrix}\varphi_n^{(1)}\\ \psi_n^{(1)}\end{bmatrix}$, $\begin{bmatrix}\varphi_n^{(2)}\\ \psi_n^{(2)}\end{bmatrix}$; $\begin{bmatrix}\phi_n^{(1)}\\ \rho_n^{(1)}\end{bmatrix}$, $\begin{bmatrix}\phi_n^{(2)}\\ \rho_n^{(2)}\end{bmatrix}$ are fundamental systems of solutions of (59)-(60), any solution of (57), and $\begin{bmatrix}p_n(x)\\ p_{n-1}(x)\\ \tilde p_{n-1}(x)\\ \tilde p_{n-2}(x)\end{bmatrix}$ in particular, is a combination

$$\begin{bmatrix}\mathsf{A}\varphi_n^{(1)}(x)\phi_0^{(1)}(x)+\mathsf{B}\varphi_n^{(1)}(x)\phi_0^{(2)}(x)+\mathsf{C}\varphi_n^{(2)}(x)\phi_0^{(1)}(x)+\mathsf{D}\varphi_n^{(2)}(x)\phi_0^{(2)}(x)\\ \mathsf{A}\psi_n^{(1)}(x)\phi_0^{(1)}(x)+\mathsf{B}\psi_n^{(1)}(x)\phi_0^{(2)}(x)+\mathsf{C}\psi_n^{(2)}(x)\phi_0^{(1)}(x)+\mathsf{D}\psi_n^{(2)}(x)\phi_0^{(2)}(x)\\ \mathsf{A}\varphi_n^{(1)}(x)\rho_0^{(1)}(x)+\mathsf{B}\varphi_n^{(1)}(x)\rho_0^{(2)}(x)+\mathsf{C}\varphi_n^{(2)}(x)\rho_0^{(1)}(x)+\mathsf{D}\varphi_n^{(2)}(x)\rho_0^{(2)}(x)\\ \mathsf{A}\psi_n^{(1)}(x)\rho_0^{(1)}(x)+\mathsf{B}\psi_n^{(1)}(x)\rho_0^{(2)}(x)+\mathsf{C}\psi_n^{(2)}(x)\rho_0^{(1)}(x)+\mathsf{D}\psi_n^{(2)}(x)\rho_0^{(2)}(x)\end{bmatrix}$$

As we are interested in the special combinations leading to $\begin{bmatrix}p_n(x) & p_{n-1}(x) & \tilde p_{n-1}(x) & \tilde p_{n-2}(x)\end{bmatrix}^T$, let us rename the appropriated special combinations $\mathsf{A}\phi_0^{(1)}+\mathsf{B}\phi_0^{(2)}$, $\mathsf{C}\phi_0^{(1)}+\mathsf{D}\phi_0^{(2)}$, $\mathsf{A}\rho_0^{(1)}+\mathsf{B}\rho_0^{(2)}$, $\mathsf{C}\rho_0^{(1)}+\mathsf{D}\rho_0^{(2)}$: $\phi_0^{(1)}, -\phi_0^{(2)}, \rho_0^{(1)}, -\rho_0^{(2)}$ respectively, so that

$$\begin{aligned}p_n(x) &= \varphi_n^{(1)}(x)\phi_0^{(1)}(x)-\varphi_n^{(2)}(x)\phi_0^{(2)}(x),\\ p_{n-1}(x) &= \psi_n^{(1)}(x)\phi_0^{(1)}(x)-\psi_n^{(2)}(x)\phi_0^{(2)}(x),\\ \tilde p_{n-1}(x) &= \varphi_n^{(1)}(x)\rho_0^{(1)}(x)-\varphi_n^{(2)}(x)\rho_0^{(2)}(x),\\ \tilde p_{n-2}(x) &= \psi_n^{(1)}(x)\rho_0^{(1)}(x)-\psi_n^{(2)}(x)\rho_0^{(2)}(x).\end{aligned} \tag{61}$$

The minus sign will make things more comfortable at $n=0$ (and we also slowly approach important formulas by Hahn, such as [11, formula (16), p.4]), where the left-hand side is made of constants. That combinations of products of solutions of (59) and (60) may turn to constants requires a closer look. Indeed, inverting the matrix of (60) yields

$$\begin{bmatrix}\phi_m(\eta_1)\\ \rho_m(\eta_1)\end{bmatrix}=\begin{bmatrix}\beta_m & \gamma_m\\ -\delta_m & \alpha_m\end{bmatrix}\begin{bmatrix}\phi_m(\eta_2)\\ \rho_m(\eta_2)\end{bmatrix}\ ,\quad\text{or}\quad \begin{bmatrix}\phi_m(\eta_1)\\ -\dfrac{a_m\rho_m(\eta_1)}{\mu_0}\end{bmatrix}=\begin{bmatrix}\beta_m & -B_m/a_m\\ a_mD_m & \alpha_m\end{bmatrix}\begin{bmatrix}\phi_m(\eta_2)\\ -\dfrac{a_m\rho_m(\eta_2)}{\mu_0}\end{bmatrix}\ ,$$

the adjoint of (59) (when $m=n$). So, for any solutions of (59) and (60),

$$\varphi_m\phi_m - a_m\psi_m\rho_m/\mu_0$$

is constant (on any lattice), or:

$$\begin{bmatrix}\phi_m^{(1)} & -a_m\dfrac{\rho_m^{(1)}}{\mu_0}\\ \phi_m^{(2)} & -a_m\dfrac{\rho_m^{(2)}}{\mu_0}\end{bmatrix} = \text{ a constant matrix times } \begin{bmatrix}\psi_m^{(2)} & -\varphi_m^{(2)}\\ -\psi_m^{(1)} & \varphi_m^{(1)}\end{bmatrix}\frac{1}{\varphi_m^{(1)}\psi_m^{(2)}-\psi_m^{(1)}\varphi_m^{(2)}} \tag{62}$$

Let us look more closely at the solutions of (59). The scalar second order equations for φ_n and ψ_n are obtained by performing various eliminations in

$$\begin{aligned}\varphi_n(y_{k+1}) &= \beta_n(x_k)\varphi_n(y_k)+a_nD_n(x_k)\psi_n(y_k),\\ \varphi_n(y_k) &= \beta_n(x_{k-1})\varphi_n(y_{k-1})+a_nD_n(x_{k-1})\psi_n(y_{k-1}),\\ \psi_n(y_{k+1}) &= -a_nD_{n-1}(x_k)\varphi_n(y_k)-\alpha_n(x_k)\psi_n(y_k),\\ \psi_n(y_k) &= -a_nD_{n-1}(x_{k-1})\varphi_n(y_{k-1})-\alpha_n(x_{k-1})\psi_n(y_{k-1}),\end{aligned}$$

and are

$$D_n(x_{k-1})\varphi_n(y_{k+1})+[\alpha_n(x_{k-1})D_n(x_k)-\beta_n(x_k)D_n(x_{k-1}]\varphi_n(y_k)+D_n(x_k)\frac{\Omega(x_{k-1})}{(y_k-y_{k-1})^2}\varphi_n(y_{k-1})=0 \tag{63}$$

and

$$D_{n-1}(x_{k-1})\psi_n(y_{k+1})+[\alpha_{n-1}(x_{k-1})D_{n-1}(x_k)-\beta_{n-1}(x_k)D_{n-1}(x_{k-1}]\psi_n(y_k)+D_{n-1}(x_k)\frac{\Omega(x_{k-1})}{(y_k-y_{k-1})^2}\psi_n(y_{k-1})=0$$

showing clearly[8] that ψ_n is a solution of the equation for φ_{n-1}.

Shall we conclude $\psi_n^{(1)}=\varphi_{n-1}^{(1)}$ and $\psi_n^{(2)}=\varphi_{n-1}^{(2)}$? We can do that for a single value of n, say $n=0$. Then, from the various identities for α_{n+1}, β_{n+1}, and D_{n+1} above, we see that

$$\begin{aligned}(\eta_2-b_n)\varphi_n(\eta_2)-a_n\psi_n(\eta_2) &= (\eta_2-b_n)\beta_n\varphi_n(\eta_1)+(\eta_2-b_n)a_nD_n\psi_n(\eta_1)+a_n^2D_{n-1}\varphi_n(\eta_1)+a_n\alpha_n\psi_n(\eta_1)\\ &= \beta_{n+1}[(\eta_1-bn)\varphi_n(\eta_1)-a_n\psi_n(\eta_1)]+[(\eta_2-b_n)\beta_n+(\eta_1-b_n)\alpha_n+(\eta_1-b_n)(\eta_2-b_n)D_n+a_n^2D_{n-1}]\varphi_n(\eta_1)\\ &= \beta_{n+1}[(\eta_1-bn)\varphi_n(\eta_1)-a_n\psi_n(\eta_1)]+a_{n+1}^2D_{n+1}\varphi_n(\eta_1)\end{aligned}$$

and

$$\begin{aligned}\varphi_n(\eta_2) &= \beta_n\varphi_n(\eta_1)+a_nD_n\psi_n(\eta_1)\\ &= -D_n[(\eta_1-b_n)\varphi_n(\eta_1)-a_n\psi_n(\eta_1)]+[(\eta_1-b_n)D_n+\beta_n]\varphi_n(\eta_1)\\ &= -D_n[(\eta_1-b_n)\varphi_n(\eta_1)-a_n\psi_n(\eta_1)]-\alpha_{n+1}\varphi_n(\eta_1)\end{aligned}$$

so, we recover the recurrence

$$a_{n+1}\varphi_{n+1}(x)=(x-b_n)\varphi_n(x)-a_n\varphi_{n-1}(x) \qquad (64)$$

and summarize the whole thing in the

Theorem. *There exist sequences of functions* $\{\varphi_n^{(1)}\}$ *and* $\{\varphi_n^{(2)}\}$, $n\in\mathbb{Z}$, *satisfying simultaneously the difference equation* (63) *and the recurrence relation* (64) *such that*

$$p_n(x)=\frac{1}{\sqrt{\mu_0}}\frac{\varphi_n^{(1)}(x)\varphi_{-1}^{(2)}(x)-\varphi_n^{(2)}(x)\varphi_{-1}^{(1)}(x)}{\varphi_0^{(1)}(x)\varphi_{-1}^{(2)}(x)-\varphi_0^{(2)}(x)\varphi_{-1}^{(1)}(x)},$$

$$\tilde{p}_{n-1}(x)=\frac{\sqrt{\mu_0}}{a_0}\frac{\varphi_n^{(1)}(x)\varphi_{-1}^{(2)}(x)-\varphi_n^{(2)}(x)\varphi_{-1}^{(1)}(x)}{\varphi_0^{(1)}(x)\varphi_{-1}^{(2)}(x)-\varphi_0^{(2)}(x)\varphi_{-1}^{(1)}(x)},$$

associated polynomials:

$$p_{n-r}^{(r)}(x)=K_r\frac{\varphi_n^{(1)}(x)\varphi_{r-1}^{(2)}(x)-\varphi_n^{(2)}(x)\varphi_{r-1}^{(1)}(x)}{\varphi_0^{(1)}(x)\varphi_{-1}^{(2)}(x)-\varphi_0^{(2)}(x)\varphi_{-1}^{(1)}(x)}.$$

The factors ϕ and ρ in (61) have been replaced by their expressions in (62), in such a way that $p_{-1}=\tilde{p}_{-1}=p_{-1}^{(r)}=0$.

Hahn's theory.

We just had a little taste of a formidable field of investigation worked by W. Hahn:

Hahn, Wolfgang:

Ueber lineare Differentialgleichungen, deren Loesungen einer Rekursionsformel genuegen. *Math. Nachr.* **4**, 1-11 (1951).

Ueber lineare Differentialgleichungen, deren Loesungen einer Rekursionsformel genuegen. II. *Math. Nachr.* **7**, 85-104 (1952).

summarized (?) in

Hahn, Wolfgang

[8] After performing $\alpha_n(x_k)D_{n-1}(x_{k-1})-\beta_n(x_{k-1})D_{n-1}(x_k)=[\alpha_{n-1}(x_{k-1})+(y_k-b_{n-1})D_{n-1}(x_{k-1})]D_{n-1}(x_k)-[\beta_{n-1}(x_k)+(y_k-b_{n-1})D_{n-1}(x_k)]D_{n-1}(x_{k-1})=\alpha_{n-1}(x_{k-1})D_{n-1}(x_k)-\beta_{n-1}(x_k)D_{n-1}(x_{k-1})$.

Kennzeichnung der linearen Differentialgleichungen, die durch Orthogonalpolynome befriedigt werden. Bericht No.54. Graz: Mathematisch-statistische Sektion im Forschungszentrum Graz. 16 S. (1976).

where we find:

Theorem. *Let the sequence $\{y_n(x)\}$, $n \in \mathbb{Z}$, of functions of the complex variable x satisfy simultaneously*

1. *Each y_n is solution of a linear homogeneous differential equation of order k*

$$L_n y := \sum_{j=0}^{k} c_{n,j}(x) \frac{d^{k-j} y}{dx^{k-j}}(x) = 0 \tag{65}$$

in x, where the coefficients $c_{n,j}$ are polynomials with degrees bounded with respect to n; y_n can not be written as solution of such an equation of order smaller than k.

2. *The recurrence relation*

$$y_n(x) = A_n(x) y_{n-1}(x) + B_n(x) y_{n-2}(x) \tag{66}$$

holds, with coefficients which are polynomials with uniformly bounded degrees.

3. *Neither the differential equation (65) nor any of the two linear differential equations whose $A_n y_{n-1}$ and $B_n y_{n-2}$ are solutions have any common solution.*

Then,

1. *k must be even: $k = 2r$,*
2. *The differential equation (65) has a fundamental system of solutions of the form*

$$\phi^{(i)} \varphi_n^{(1)}, \phi^{(i)} \varphi_n^{(2)}, \qquad i = 1, \ldots, r,$$

where $\varphi_n^{(1)}$ and $\varphi_n^{(2)}$ are solutions of a differential equations of order 2 with polynomial coefficients of bounded degree, and where $\{\phi^{(1)}, \ldots, \phi^{(r)}\}$ is a fundamental system of solutions of a linear differential equation of order r.

3. *If we want y_n to be a polynomial of degree n, we can only have $k = 2$ or $k = 4$, and the degrees of A_n and B_n be 1 and 0.*

When $k = 4$, one has $y_n = \varphi_n^{(1)} \varphi_{-1}^{(2)} - \varphi_n^{(2)} \varphi_{-1}^{(1)}$.

Challenge: extend this to the $\mathscr{D}$ operator!!

Stieltjes function: if we succeed to have $\varphi_n^{(2)}(x)/\varphi_n^{(1)}(x) \to 0$ when $n \to \infty$,

$$\mathcal{S}(x) = \lim_{n \to \infty} \frac{\tilde{p}_{n-1}(x)}{p_n(x)} = \frac{\varphi_0^{(2)}(x)}{\varphi_{-1}^{(2)}(x)},$$

i.e., a solution of a Riccati equation as ratio of solutions of linear second order difference equations, as it should.

See also Gruenbaum, F.Alberto; Haine, Luc: Some functions that generalize the Askey-Wilson polynomials. *Commun. Math. Phys.* **184**, No.1, 173-202 (1997).

The author's abstract: We determine all biinfinite tridiagonal matrices for which some family of eigenfunctions are also eigenfunctions of a second order q-difference operator. The solution is described in terms of an arbitrary solution of a q-analogue of Gauss hypergeometric equation depending on five free parameters and extends the four dimensional family of solutions given by the Askey-Wilson polynomials. There is some evidence that this bispectral problem, for an arbitrary order q-difference operator, is intimately related with some q-deformation of the Toda lattice hierarchy and its Virasoro symmetries. When tridiagonal matrices are replaced by the Schroedinger operator, and $q = 1$, this statement holds with Toda replaced by KdV. In this context, this paper determines the analogs of the Bessel and Airy potentials.

Gruenbaum, F.Alberto; Haine, Luc: On a q-analogue of Gauss equation and some q-Riccati equations. Ismail, Mourad E. H. (ed.) et al., *Special functions, q-series and related topics.* Providence, RI: American Mathematical Society, *Fields Inst. Commun.* **14**, 77-81 (1997).

Gruenbaum, F.Alberto; Haine, Luc A theorem of Bochner, revisited. Fokas, A. S. (ed.) et al., *Algebraic aspects of integrable systems: in memory of Irene Dorfman.* Boston, MA: Birkhaeuser, *Prog. Nonlinear Differ. Equ. Appl.* **26**, 143-172 (1997).

We pose a bispectral problem associated with the Toda lattice hierarchy and describe explicitly all its "basic solutions" from which all other solutions are expected to be obtained by repeated application of the Darboux process. We identify these basic solutions as fixed points of (in general) some (sub)algebras of the Virasoro algebra formed by the Toda master symmetries. Finally, we illustrate by four simple examples our contention that repeated applications of the Darboux transformation from some of these fixed points will provide integral manifolds of the Toda hierarchy or its master symmetries.