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Special topics in approximation theory.

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Elliptic lattices.

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Sept. 2006 - etc.
This version: May 11, 2012 (incomplete and unfinished)
The present file is http://perso.uclouvain.be/alphonse.magnus/num3/m32006.pdf

Ars Longa[1]

Contents

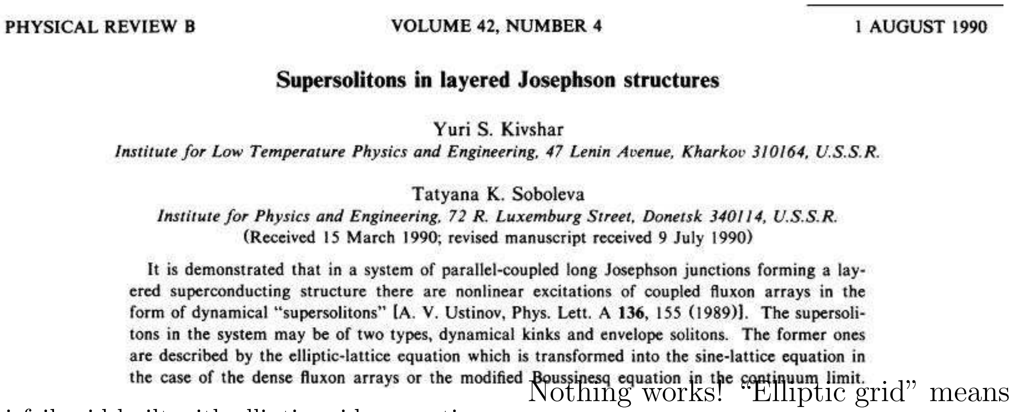
[1] The author of the following thesis: *Sur les champs stationnaires des intégrales multiples du calcul des variations. Transformation de Haar-Carathéodory. Espaces généralisés de Cartan.*, Louvain : UCL, 1942, choose *Ars Longa* as author's name. These things happened in 1942. I really hope that he/she had (or still has?) a *Vita Longa*.

1. The name of the game.

1.1. Elliptic grid, lattice, sequence.

Have a look at this, on what is called "elliptic sequences":
`http://www.research.att.com/~njas/sequences/A006721`
Greetings from The On-Line Encyclopedia of Integer Sequences!
A006721 Somos-5 sequence: $a(n) = (a(n-1)a(n-4) + a(n-2)a(n-3))/a(n-5)$. (Formerly M0735) 1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, 22833, 165713, 1249441, 9434290, 68570323, 1013908933, 11548470571, 142844426789, 2279343327171, 57760865728994, 979023970244321
REFERENCES ...
David Gale, "The strange and surprising saga of the Somos sequence", Math. Intelligencer 13(1) (1991), pp. 40-42.
...
A. J. van der Poorten, Elliptic curves and continued fractions http://arXiv.org/abs/math.NT/0403225
A. J. van der Poorten, Recurrence relations for elliptic sequences... http://arXiv.org/abs/math.NT/0412293
J. Propp, The Somos Sequence Site http://www.math.wisc.edu/~propp/somos.html
...
D. Zagier, Problems posed at the St Andrews Colloquium, 1996 http://www-groups.dcs.st-and.ac.uk/~john/Zagier/Prob

Consider (see somos.html above) recurrence relations linking products of cosines: $\cos((n+1)\theta)\cos((n-1)\theta) = \cos^2(n\theta)\cos^2\theta - \sin^2(n\theta)\sin^2\theta$, $\cos((n+2)\theta)\cos((n-2)\theta) = \cos^2(n\theta)\cos^2(2\theta) - \sin^2(n\theta)\sin^2(2\theta)$, and eliminate $\sin^2(n\theta)$: $c_{n+1}c_{n-1}/\sin^2\theta - c_{n+2}c_{n-2}/\sin^2(2\theta) = c_n^2(\cot^2\theta - \cot^2(2\theta))$, yielding a family of recurrence relations which showed sometimes unsuspected integer particular solutions. Experts made the connection with integral invariants of elliptic curves and these remarkable sequences have been called "elliptic".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Supersolitons in layered Josephson structures

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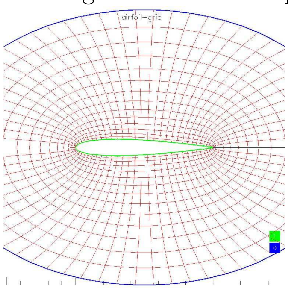
It is demonstrated that in a system of parallel-coupled long Josephson junctions forming a layered superconducting structure there are nonlinear excitations of coupled fluxon arrays in the form of dynamical "supersolitons" [A. V. Ustinov, Phys. Lett. A **136**, 155 (1989)]. The supersolitons in the system may be of two types, dynamical kinks and envelope solitons. The former ones are described by the elliptic-lattice equation which is transformed into the sine-lattice equation in the case of the dense fluxon arrays or the modified Boussinesq equation in the continuum limit.

Airfoil grid built with elliptic grid generation.



William D. Henshaw 2006-06-17
http://www.llnl.gov/CASC/Overture/henshaw/overtureFigures/node5.html

Nothing works! "Elliptic grid" means a convenient mesh for discretizing over ellipses, "elliptic difference operator" is a partial difference operator extending partial differential operator of elliptic type, an "elliptic lattice" is a set of complex numbers $\{m\omega_1 + n\omega_2\}$, m and $n \in \mathbb{Z}$ of periods of elliptic functions, or simply the support of a partial difference operator, or also a special digital filter…

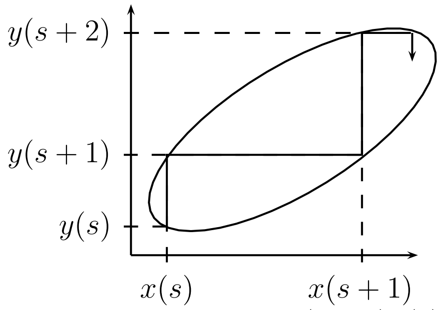
What is meant here is a new rule for building lattices $\{x_n\}, n \in \mathbb{Z}$, which may be useful in difference calculus.

1.2. **Known lattices.**

(1) Arithmetic progression $x_n = an + b$,
(2) Geometric progression $x_n = aq^n + b$,
(3) Double geometric progression ANSUW (Askey- Nikiforov- Suslov- Uvarov- Wilson [3, 5, 36, 37, 42–44])

$$x_n = aq^n + bq^{-n} + c. \tag{1}$$

Why this latter formula? My favorite way to introduce it is that the first order divided difference of a polynomial on such a lattice yields a polynomial of lower degree. So, $\dfrac{p(x_{n+1}) - p(x_n)}{x_{n+1} - x_n}$ is a polynomial of degree $k-1$ if p has degree k. With $p(x) = x^2$, $x_{n+1} + x_n = \alpha x_n + \beta$. No, this is too easy, we recover the simple geometric $q-$lattice, with $q = \alpha - 1$. It must have been a central divided difference $\dfrac{p(x_{n+1/2}) - p(x_{n-1/2})}{x_{n+1/2} - x_{n-1/2}}$. Then, still with $p(x) = x^2$, $x(s + 1/2) + x(s - 1/2) = \alpha x(s) + \beta$. Entering (1), assumed to be valid for integer as well as half-integer s, one finds indeed that it fits, with $q^{1/2} + q^{-1/2} = \alpha$, certainly the fastest derivation.



But why choose a difference operator built on $x-$values? A (seemingly) more general setting uses an (apparently) independent lattice of values: $\dfrac{p(y(s+1))-p(y(s))}{y(s+1)-y(s)}$ which must be a polynomial of degree lower than the degree of p:
$p(y)=y^2:\ y(s+1)+y(s)=\alpha x(s)+\beta$,
$p(y)=y^3:\ y(s+1)^2+y(s+1)y(s)+y(s)^2=$ some polynomial of degree 2, from which we subtract the square of the first result, to get $\quad y(s+1)y(s)=\gamma x(s)^2+\delta x(s)+\epsilon$.
These two equations already tell what $y(s)$ and $y(s+1)$ are, as functions of

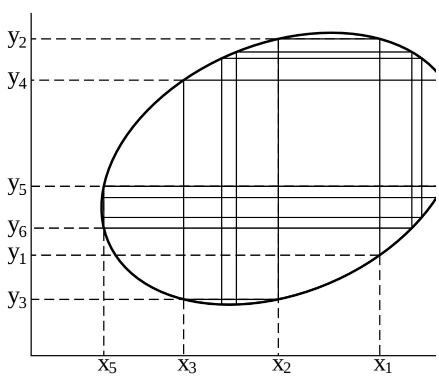
$$y(s), y(s+1)=\frac{\alpha x(s)+\beta}{2}\pm\sqrt{\left(\frac{\alpha x(s)+\beta}{2}\right)^2-\gamma x(s)^2-\delta x(s)-\epsilon},$$

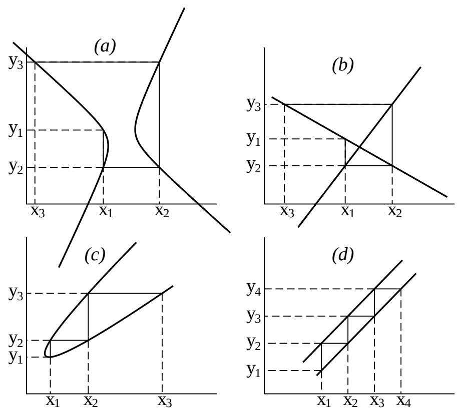
so, the (x,y) are on a conic $y^2-\alpha xy+\gamma x^2-\beta y+\delta x+\epsilon=0$. But $y(s+1)$ must be the ordinate of a second point with absciss $x(s+1)$! A rigid law rules therefore the $x-$ and $y-$sequences, $y(s+2)+y(s+1)=\alpha x(s+1)+\beta=\alpha\left[\dfrac{\alpha y(s+1)-\delta}{\gamma}-x(s)\right]+\beta=$ $\alpha\dfrac{\alpha y(s+1)-\delta}{\gamma}-y(s)-y(s+1)+2\beta$.

One recovers [36, 37] x_n and y_n as combinations of q^n and q^{-n}, where

$$q+\frac{1}{q}=\frac{\alpha^2-2\gamma}{\gamma}.$$

The slopes of the asymptotes of the conic above are s_1 and s_2 such that $s_1+s_2=\alpha$ and $s_1s_2=\gamma$, or $\dfrac{s_1}{s_2}+\dfrac{s_2}{s_1}=\dfrac{\alpha^2-2\gamma}{\gamma}$: ***q is the ratio of the slopes of the asymptotes of the conic.***





Figures from [37]
Remark also that $q^{1/2}+q^{-1/2}=\alpha/\sqrt{\gamma}$.

2. **The new lattice, at last**

It will be introduced through four equivalent points of view:

2.1. **Points** $(x_n, y_n), (x_n, y_{n+1})$ **on a biquadratic curve.**

From [39]:

Simplest difference equations relate two values of the unknown fuction f: say, $f(\varphi(x))$ and $f(\psi(x))$.

Most instances [41] are $(\varphi(x), \psi(x)) = (x, x+h)$, or the more symmetric $(x - h/2, x + h/2)$, or also (x, qx) in $q-$difference equations [21]. Recently, more complicated forms $(r(x) - \sqrt{s(x)}, r(x) + \sqrt{s(x)})$ have appeared [5,12,28,29,43,44], where r and s are rational functions.

This latter trend will be examined here: we need, for each x, two values $f(\varphi(x))$ and $f(\psi(x))$ for f.
A first-order difference equation is $\mathcal{F}\left(x, f(\varphi(x)), f(\psi(x))\right) = 0$, or $f(\varphi(x)) - f(\psi(x)) = \mathcal{G}\left(x, f(\varphi(x)), f(\psi(x))\right)$ if we want to emphasize the difference of f. There is of course some freedom in this latter writing. Only symmetric forms in φ and ψ will be considered here:

$$(\mathcal{D}f)(x) = \mathscr{F}\left(x, f(\varphi(x)), f(\psi(x))\right), \tag{2}$$

where $\mathcal{D}$ is the divided difference operator

$$(\mathcal{D}f)(x) = \frac{f(\psi(x) - f(\varphi(x)}{\psi(x) - \varphi(x)}, \tag{3}$$

and where $\mathscr{F}$ is a symmetric function of its two last arguments.

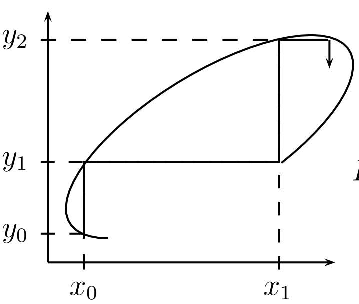
For instance, a linear difference equation of first order may be written as

$$a(x)f(\varphi(x)) + b(x)f(\psi(x)) + c(x) = 0,$$

as well as

$$\alpha(x)(\mathcal{D}f)(x) = \beta(x)[f(\varphi(x)) + f(\psi(x))] + \gamma(x),$$

with $\alpha(x) = [b(x) - a(x)][\psi(x) - \varphi(x)]/2$, $\beta(x) = -[a(x) + b(x)]/2$, and $\gamma(x) = -c(x)$.



The simplest choice for φ and ψ is to take the two determinations of an algebraic function of degree 2, i.e., the two $y-$roots of

$$F(x, y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \tag{4a}$$

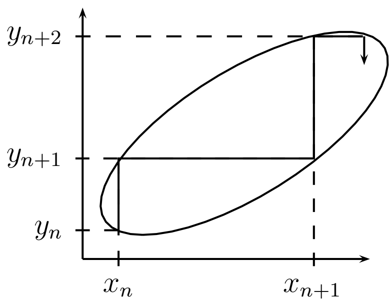
where X_0, X_1, and X_2 are rational functions.

But difference equations must allow the recovery of f on a whole set of points! An initial-value problem for a first order difference equation starts with a value for $f(y_0)$ at $x = x_0$, where y_0 is one root of (4a) at $x = x_0$. The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where y_1 is the second root of (4a) at x_0. We need x_1 such that y_1 is one of the two roots of (4a) at x_1, so for one of the roots of $F(x,y_1)=0$ which is not x_0. Here again, the simplest case is when F is of degree 2 in x:

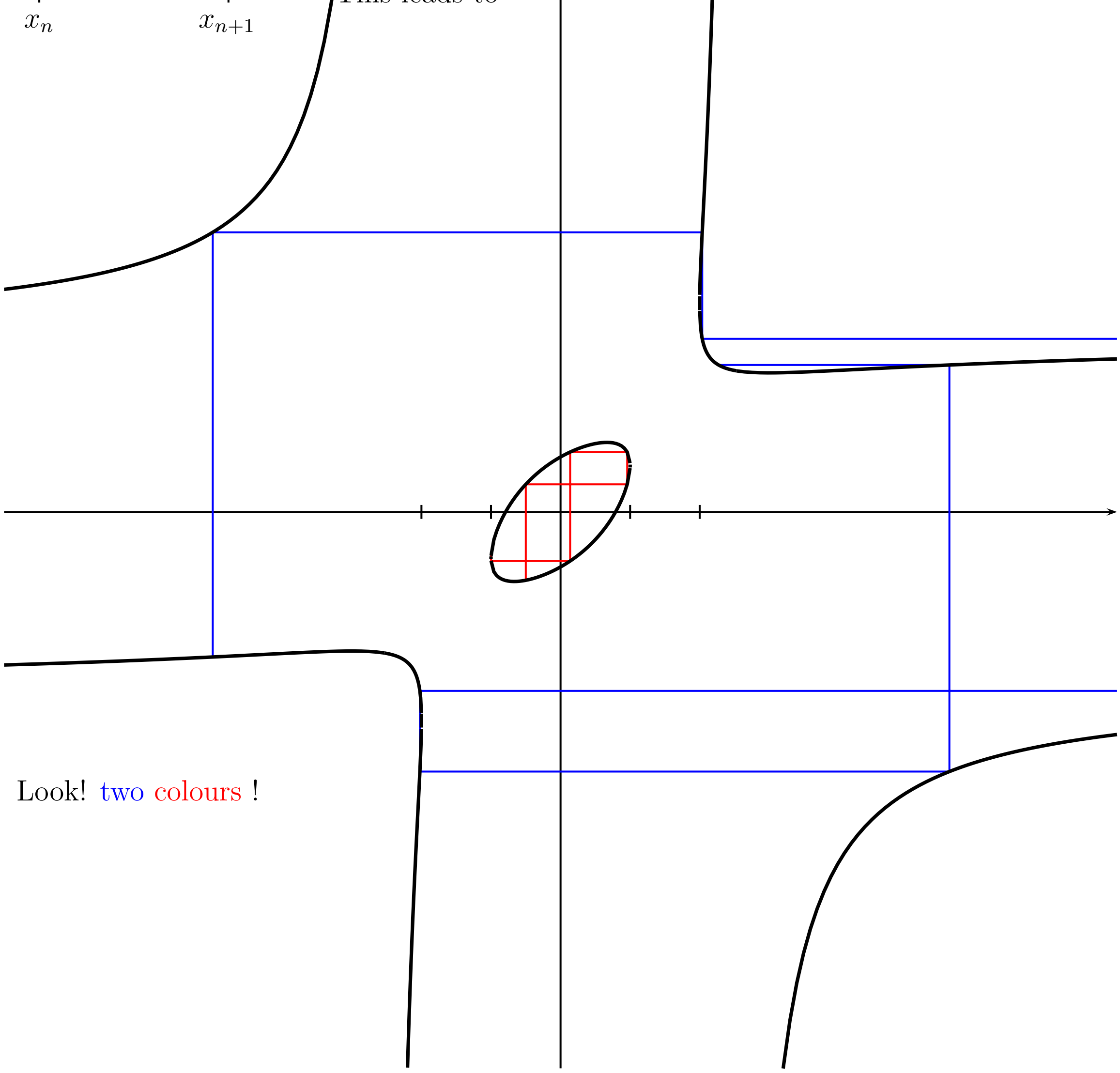
$$F(x,y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \tag{4b}$$

Both forms (4a) and (4b) hold simultaneously when F is ***biquadratic***:

$$F(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}x^i y^j. \tag{5}$$



We again look for a sequence $\{(x_n,y_n\}_{n\in\mathbb{Z}}$ on an algebraic curve $F(x,y)=0$ such that 1) at $x=x_n$ correspond two ordinates y_n and y_{n+1} given by an equation of degree two with coefficients which are rational functions of x_n; 2) at $y=y_{n+1}$ correspond two abscissae x_n and x_{n+1} given by an equation of degree two with coefficients which are rational functions of y_{n+1}. This leads to



Definition 1. *A sequence $\{\dots,x_{-1},x_0,x_1,\dots\}$ is an elliptic lattice if there exists a sequence $\{\dots,y_{-1},y_0,y_1,\dots\}$ and a biquadratic polynomial F (see* (4a), (4b), (5)*) $F(x,y)=$*

$\sum_{i=0}^2 \sum_{j=0}^2 c_{i,j} x^i y^j = X_0(x) + X_1(x)y + X_2(x)y^2 = Y_0(y) + Y_1(y)x + Y_2(y)x^2$, *such that* $F(x_n, y_n) = 0$ *and* $F(x_n, y_{n+1}) = 0$, *for* $n \in \mathbb{Z}$.

As y_n and y_{n+1} are the two roots in t of $F(x_n, t) = X_0(x_n) + X_1(x_n)t + X_2(x_n)t^2 = 0$, useful identities are

$$y_n + y_{n+1} = -\frac{X_1(x_n)}{X_2(x_n)} = -\frac{c_{2,1}x_n^2 + c_{1,1}x_n + c_{0,1}}{c_{2,2}x_n^2 + c_{1,2}x_n + c_{0,2}}, y_n y_{n+1} = \frac{X_0(x_n)}{X_2(x_n)} = \frac{c_{2,0}x_n^2 + c_{1,0}x_n + c_{0,0}}{c_{2,2}x_n^2 + c_{1,2}x_n + c_{0,2}}. \tag{6}$$

and the direct formula

$$y_n \text{ and } y_{n+1} = \frac{-X_1(x_n) \pm \sqrt{P(x_n)}}{2X_2(x_n)}, \tag{7}$$

where

$$P = X_1^2 - 4X_0X_2 \tag{8}$$

is a polynomial of degree 4.

Also, as x_{n-1} and x_n are the two roots in t of $F(t, y_n) = 0$,

$$x_n + x_{n-1} = -\frac{Y_1(y_n)}{Y_2(y_n)} = -\frac{c_{1,2}y_n^2 + c_{1,1}y_n + c_{1,0}}{c_{2,2}y_n^2 + c_{2,1}y_n + c_{2,0}}, x_n x_{n-1} = \frac{Y_0(y_n)}{Y_2(y_n)} = \frac{c_{0,2}y_n^2 + c_{0,1}y_n + c_{0,0}}{c_{2,2}y_n^2 + c_{2,1}y_n + c_{2,0}}. \tag{9}$$

Factorizations:

$$F(x, y) = X_2(x)(y - \varphi(x))(y - \psi(x)) \quad = Y_2(y)(x - \varphi^{-1}(y))(x - \psi^{-1}(y)). \tag{10}$$

$F(x_n, y) = X_2(x_n)(y - y_n)(y - y_{n+1}), \qquad F(x, y_n) = Y_2(y_n)(x - x_n)(x - x_{n-1})$.

The construction above is called "T-algorithm" in [59, Theorem 6].

Exercise. Is $\{x_n\} = \{1, 2, 3, \dots\}$ an elliptic sequence? Yes, of course, with $y_n = n$, so $F(x, y) = (y - x)(y - x - 1)$. What are the other admissible $y-$sequences? And $\{1, 4, \dots, n^2, \dots\}$? Yes. And $\{1, \sqrt{2}, \sqrt{3}, \dots\}$? No. Answers are very easy through other equivalent definitions, see next section, § 2.2.

There are 9 coefficients in (5), actually 8 degrees of freedom. Transformations $x \to \dfrac{\alpha' + \beta' x}{\gamma' + \delta' x}, y \to \dfrac{\alpha'' + \beta'' y}{\gamma'' + \delta'' y}$, with $\alpha'\delta' - \beta'\gamma'$ and $\alpha''\delta'' - \beta''\gamma'' \neq 0$ lead to basically equivalent (x_n, y_n) sequences. Two essential degrees of freedom remain. See Spiridonov & Zhedanov [59]

The first one is the ***modulus*** k. It appears from the cross ratios $R_{a,b,c,d} = \dfrac{\dfrac{z_a - z_c}{z_a - z_d}}{\dfrac{z_b - z_c}{z_b - z_d}}$, where $z_1, \dots, z_4$ are the zeros of P, and where (a, b, c, d) is a permutation of $(1, 2, 3, 4)$. The important thing is that the cross ratio is left unchanged under a $x \to (\alpha' + \beta' x)/(\gamma' + \delta' x)$ transformation. Among other choices, we may send the four zeros of P to a symmetric set $\{-1/k, -1, 1, 1/k\}$. The 6 values of R under the 24 possible permutations of the indexes happen to be $\left(\dfrac{1+k}{1-k}\right)^2, \dfrac{(1+k)^2}{4k}, -\dfrac{(1-k)^2}{4k}$ and their inverses. If R_1 is one of these values, the other ones are generated by applying one or several times $R \to 1/R$ and $R \to 1 - R$

(modular group (**?**)). Then the various possible values of the modulus are u such that $R=\left(\dfrac{1+u}{1-u}\right)^2$ and are, if k is one root, $\pm k$, $\pm 1/k$, $\pm\left(\dfrac{1\pm\sqrt{k}}{1\mp\sqrt{k}}\right)^2$,

The same cross ratio would be obtained with the four zeros of $Q=Y_1^2-4Y_0Y_2$, relevant from $F(x,y)=0\Rightarrow x=(-Y_1(y)\pm\sqrt{Q(y)})/(2Y_2(y))$, see (32) p. 35 for a case where the degree of Q is reduced to 3, see also Appel & Goursat p.293, where it is stated that the four tangents to the cubic meeting a given point of the curve define the same cross ratio.

A vanishing modulus corresponds to a polynomial P with a double zero. We then recover the ANSUW case, up to rational transformations of coordinates.

The reduced form with two parameters which will be used here is

$$\begin{aligned}F(x,y)&=k^2(1-k^2z'^2)x^2y^2-k^2(1-z'^2)(x^2+y^2)-2k(1-k^2)z'xy+1-k^2z'^2\\&=k^2(1-k^2z'^2)x^2y^2-k^2(1-z'^2)(x+y)^2-2k(z'-k)(1+kz')xy+1-k^2z'^2\end{aligned}\tag{11}$$

The second parameter z' is investigated now. With (11), (7) is $y=\dfrac{k(1-k^2)z'x\pm\sqrt{P(x)}}{k^2[(1-k^2z'^2)x^2+z'^2-1]}$, with $P(x)=k^2(1-z'^2)(1-k^2z'^2)(1-x^2)(1-k^2x^2)$.

We already see that, if z'^2 is close to 1 or k^{-2}, P is almost the zero polynomial, the curve in the $x-y$ plane has almost no width, any x_n leads to y_n and y_{n+1} close together, and to $x_{n\pm1}$ close together too, and close to x_n.

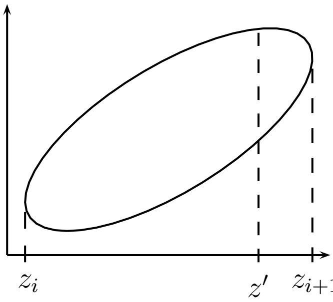
Well, let us take the example $z'=k^{-1}+$ a small ε. Then, y_{n+1} and $y_n=x_n\pm\sqrt{\dfrac{2k\varepsilon(1-x_n^2)(1-k^2x_n^2)}{1-k^2}}+O(\varepsilon)$, and $x_{n\pm1}=x_n\pm2\sqrt{\dfrac{2k\varepsilon(1-x_n^2)(1-k^2x_n^2)}{1-k^2}}+O(\varepsilon)$, we see that z' is linked to a ***step*** of the $x-$lattice, and that this step is small if z' is close to ±1 or $\pm k^{-1}$.

Baxter [10, § 15.10] uses $x^2y^2+c(x^2+y^2)+2dxy+1$, so that $P(x)=4[d^2x^2-(c+x^2)(cx^2+1)]=-4c[x^4-(k+k^{-1})x^2+1]=-4c(1-k^2(x^2/k))(1-(x^2/k))$ if $k+k^{-1}=(d^2-c^2-1)/c$, or $(k+c)(1+kc)=kd^2$.

Baxter's form follows from (11) in the variables $k^{1/2}x$ and $k^{1/2}y$: $(k^{1/2}x)^2(k^{1/2}y)^2-\dfrac{k(1-z'^2)}{1-k^2z'^2}[(k^{1/2}x)^2+(k^{1/2}y)^2]-\dfrac{2(1-k^2)z'}{1-k^2z'^2}k^{1/2}xk^{1/2}y+1=0$. Check that $\dfrac{d^2-c^2-1}{c}=-\dfrac{(1-k^2)^2z'^2-k^2(1-z'^2)^2}{k(1-k^2z'^2)(1-z'^2)}+\dfrac{1-k^2z'^2}{k(1-z'^2)}=k+k^{-1}$ indeed.

Then, Baxter finds that $\sqrt{-1/(kc)}$ is the elliptic sine of the relevant step [10, eq. (15.10.8)]. Here, $\sqrt{-\dfrac{1}{kc}} = \sqrt{\dfrac{1-k^2z'^2}{k^2(1-z'^2)}} = k^{-1}\mathrm{dc}(\zeta)$ $=\mathrm{sn}(\zeta+K+iK')$ if z' is the elliptic sine z' $=\mathrm{sn}(\zeta)$. From [1, p.572]:

		u	$u+K+iK'$
16.8.1	sn	sn u	$m^{-1/2}$dc u



The second parameter z' is a fixed point of the relation between x_n and x_{n+1}, i.e., $E(z',z')=0$ (see next section for E). In the $F(x,y)=0$ setting, the fixed points $x_n = x_{n+1}$ occur at the $y-$extrema, so when $\partial F/\partial x = 0$.
With (11), at $y=\pm 1$ or $\pm 1/k$, $x = \pm 1/(kz')$ or $\pm z'$, $\partial F/\partial x =$ $2k^2(1-k^2z'^2)xy^2 - 2k^2(1-z'^2)x - 2k(1-k^2)z'y$
$= 2k^2x[z'^2(1-k^2y^2)-(1-y^2)] - 2k(1-k^2)z'y$
$= 2k(1-k^2)z'(kz'x\mp 1)$ or $2(1-k^2)(x\mp z') = 0$.
The four abscissae of interest are therefore $\pm 1/(kz')$ and $\pm z'$.

A cross ratio is $\dfrac{\dfrac{1/(kz')+z'}{1/(kz')+1/(kz')}}{\dfrac{z'+z'}{z'+1/(kz')}} = \dfrac{(1+kz'^2)^2}{4kz'^2}$. So, calculations similar to the operations defining the modulus lead here to kz'^2.

With Baxter's form, the abscissae corresponding to $y=\pm k^{1/2}$ and $y=\pm k^{-1/2}$ are $x: x^2y^2+c(x^2+y^2)+2dxy+1 = (k+c)x^2 \pm 2dk^{1/2}x + ck + 1 = (k+c)^{-1}[(k+c)x\pm dk^{1/2}]^2 = 0 \Rightarrow x = \mp dk^{1/2}/(k+c) = \mp(1+kc)/(dk^{1/2})$ and $(k^{-1}+c)x^2 \pm 2dk^{-1/2}x + k^{-1}c + 1 = k^{-1}(1+ck)^{-1}[(1+kc)x\pm dk^{1/2}]^2 \Rightarrow x = \mp dk^{1/2}/(1+kc) = \mp(k+c)/(dk^{1/2})$. The cross ratio is now $\dfrac{\dfrac{k+c+1+kc}{k+c+k+c}}{\dfrac{1+kc+1+kc}{1+kc+k+c}} = \dfrac{(1+c)^2(1+k)^2}{4(k+c)(1+kc)}$

2.2. **Symmetric biquadratic relation between x_n and $x_{n\pm 1}$.**

The companion sequence $\{y_n\}$ is not needed in the definition of an elliptic lattice, but the definition above is best suited to the description of difference equations.

A relation involving only x_n and x_{n+1} is obtained by the elimination of y_{n+1} through the resultant of the two polynomials in y_{n+1} from (9) $P_1(y_{n+1}) = (x_n+x_{n+1})Y_2(y_{n+1}) + Y_1(y_{n+1})$ and $P_2(y_{n+1}) = x_nx_{n+1}Y_2(y_{n+1}) - Y_0(y_{n+1})$.

The form of this resultant is most easily found through interpolation at the two zeros u and v of Y_2: let $Y_2(t) = \alpha(t-u)(t-v)$, $Y_1(t) = \beta(t-u)(t-v)+\beta' t+\beta''$, $Y_0(t) = \gamma(t-u)(t-v)+\gamma' t+\gamma''$, then, using the basis $\{1, t, (t-u)(t-v), t(t-u)(t-v)\}$, the resultant is built with the coefficients of $tP_1(t), P_1(t), tP_2(t)$, and $P_2(t)$:

$$E(x_n,x_{n+1}) = \begin{vmatrix} (x_n+x_{n+1})\alpha+\beta & \beta' & (u+v)\beta'+\beta'' & -uv\beta' \\ 0 & (x_n+x_{n+1})\alpha+\beta & \beta' & \beta'' \\ x_nx_{n+1}\alpha-\gamma & -\gamma' & -(u+v)\gamma'-\gamma'' & uv\gamma' \\ 0 & x_nx_{n+1}\alpha-\gamma & -\gamma' & -\gamma'' \end{vmatrix} = 0,$$

which is clearly a symmetric polynomial of degree 2 in x_n+x_{n+1} and x_nx_{n+1}

$$(\beta'u+\beta'')(\beta'v+\beta'')(\alpha x_nx_{n+1}-\gamma)^2 \\ +[(\beta'u+\beta'')(\gamma'v+\gamma'')+(\beta'v+\beta'')(\gamma'u+\gamma'')](\alpha(x_n+x_{n+1})+\beta)(\alpha x_nx_{n+1}-\gamma) \\ +(\gamma'u+\gamma'')(\gamma'v+\gamma'')(\alpha(x_n+x_{n+1})+\beta)^2 \\ +(\beta''\gamma'-\beta'\gamma'')[\beta'(\alpha x_nx_{n+1}-\gamma)+\gamma'(\alpha(x_n+x_{n+1})+\beta)]=0,$$

or, with $S=x_n+x_{n+1}$ and $\Pi=x_nx_{n+1}$,

$$[(\gamma'v+\gamma'')(\alpha S+\beta)+(\beta'v+\beta'')(\alpha\Pi-\gamma)][(\gamma'u+\gamma'')(\alpha S+\beta)+(\beta'u+\beta'')(\alpha\Pi-\gamma)] \\ +(\beta''\gamma'-\beta'\gamma'')[\gamma'(\alpha S+\beta)+\beta'(\alpha\Pi-\gamma)]=0. \quad (12)$$

Example. With the canonical form (11), $\alpha=k^2(1-k^2z'^2)$, $\beta=0$, $\beta'=-2k(1-k^2)z'$, $\beta''=0$, $\gamma=-k^2(1-z'^2)$, $\gamma'=0$, $\gamma''=1-k^2z'^2-k^2\dfrac{(1-z'^2)^2}{1-k^2z'^2}=\dfrac{(1-k^2)(1-k^2z'^4)}{1-k^2z'^2}$, so $u+v=0$, $uv=-\dfrac{1-z'^2}{1-k^2z'^2}$,

$$E(x_n,x_{n+1})=\beta'^2uv(\alpha x_nx_{n+1}-\gamma)^2+\gamma''^2\alpha^2(x_n+x_{n+1})^2-\beta'^2\gamma''(\alpha x_nx_{n+1}-\gamma)$$
$$=\alpha^2\beta'^2uvx_n^2x_{n+1}^2+\gamma''^2\alpha^2(x_n+x_{n+1})^2-\alpha\beta'^2\underbrace{(2\gamma uv+\gamma'')}_{[1+k^2-4k^2z'^2+k^2(1+k^2)z'^4]/(1-k^2z'^2)}x_nx_{n+1}+\beta'^2\gamma\underbrace{(uv\gamma+\gamma'')}_{1-k^2z'^2}$$
$$=-4k^6(1-k^2)^2z'^2(1-z'^2)(1-k^2z'^2)x_n^2x_{n+1}^2+k^4(1-k^2)^2(1-k^2z'^4)^2(x_n+x_{n+1})^2 \\ -4k^4(1-k^2)^2z'^2[1+k^2-4k^2z'^2+k^2(1+k^2)z'^4]x_nx_{n+1}-4k^4(1-k^2)^2z'^2(1-z'^2)(1-k^2z'^2)$$
$$=-4k^4(1-k^2)^2z'^2(1-z'^2)(1-k^2z'^2)\left\{k^2x_n^2x_{n+1}^2-\frac{(1-k^2z_0^4)^2}{4z'^2(1-k^2z'^2)(1-z'^2)}(x_n+x_{n+1})^2 \right. \\ \left. +\frac{1+k^2-4k^2z'^2+k^2(1+k^2)z'^4}{(1-z'^2)(1-k^2z'^2)}x_nx_{n+1}+1\right\}$$

which is again of the form (11), with the same k, but with z' replaced by z'':

$$E(x,y)=k^2(1-k^2z''^2)x^2y^2-k^2(1-z''^2)(x+y)^2+2k(z''-k)(1+kz'')xy+1-k^2z''^2=0 \quad (13)$$

such that $\dfrac{1-z''^2}{1-k^2z''^2}=\dfrac{(1-k^2z'^4)^2}{4k^2z'^2(1-k^2z'^2)(1-z'^2)}$ and $2\dfrac{z''-k}{k(1-kz'')}=-\dfrac{1+k^2-4k^2z'^2+k^2(1+k^2)z'^4}{k^2(1-z'^2)(1-k^2z'^2)}$,

$$\text{from which}\quad z''=-\frac{1-2k^2z'^2+k^2z'^4}{k(1-2z'^2+k^2z'^4)}. \quad (14)$$

Interesting identities are $z''\pm1=-\dfrac{(1\mp k)(1\pm kz'^2)^2}{k(1-2z'^2+k^2z'^4)}$, $1-kz''=\dfrac{2(1-z'^2)(1-k^2z'^2)}{1-2z'^2+k^2z'^4}$, $1+kz''=-\dfrac{2(1-k^2)z'^2}{1-2z'^2+k^2z'^4}$.

Definition 2. *An elliptic lattice, or grid, is a sequence satisfying a symmetric biquadratic relation* [59, Theorem 5]

$$E(x_n,x_{n+1}) = d_{0,0} + d_{0,1}(x_n+x_{n+1}) + d_{0,2}(x_n+x_{n+1})^2 + d_{1,1}x_nx_{n+1} \\ + d_{1,2}x_nx_{n+1}(x_n+x_{n+1}) + d_{2,2}x_n^2x_{n+1}^2 = 0. \quad (15)$$

Remark that the coefficient of the product x_nx_{n+1} is $d_{1,1}+2d_{0,2}$.

From symmetry of E, $E(x_n,x_{n-1}) = E(x_{n-1},x_n) = (15)$ at $n-1$ vanishes too. So, x_{n-1} and x_{n+1} are the two $y-$roots of $E(x_n,y)=0$, and

$$x_{n-1}+x_{n+1} = -\frac{d_{0,1}+(d_{1,1}+2d_{0,2})x_n+d_{1,2}x_n^2}{d_{0,2}+d_{1,2}x_n+d_{2,2}x_n^2}, x_{n-1}x_{n+1} = \frac{d_{0,0}+d_{0,1}x_n+d_{0,2}x_n^2}{d_{0,2}+d_{1,2}x_n+d_{2,2}x_n^2}. \quad (16)$$

$$x_{n-1} \text{ and } x_{n+1} = \frac{-(d_{0,1}+(d_{1,1}+2d_{0,2})x_n+d_{1,2}x_n^2) \pm \sqrt{P(x_n)}}{2(d_{0,2}+d_{1,2}x_n+d_{2,2}x_n^2)}, \quad (17)$$

with the same P as before (times a constant), as from (9) at $n+1$, $x_{n+1} = -\dfrac{Y_1(y_{n+1})}{Y_2(y_{n+1})} - x_n = -\dfrac{Y_1}{Y_2} - x_n$ at $\dfrac{-X_1(x_n)\pm\sqrt{P(x_n)}}{2X_2(x_n)}$, from (7).

It becomes much easier to check if a given sequence is an elliptic one, as $x_{n-1}+x_{n+1}$ and $x_{n-1}x_{n+1}$ must be rational functions of degree $\leqslant 2$ of x_n.

Example. Let $x_n = 1/\sin(n\theta)$, then $x_{n-1}+x_{n+1} = \dfrac{1}{\cos\theta\sin(n\theta)-\sin\theta\cos(n\theta)}$ $+\dfrac{1}{\cos\theta\sin(n\theta)+\sin\theta\cos(n\theta)} = \dfrac{2\cos\theta/x_n}{\cos^2\theta/x_n^2-\sin^2\theta(1-1/x_n^2)} = \dfrac{2\cos\theta x_n}{1-\sin^2\theta x_n^2}$; $x_{n-1}x_{n+1} = \dfrac{1}{[\cos\theta\sin(n\theta)-\sin\theta\cos(n\theta)][\cos\theta\sin(n\theta)+\sin\theta\cos(n\theta)]} = \dfrac{1}{\cos^2\theta/x_n^2-\sin^2\theta(1-1/x_n^2)} = \dfrac{x_n^2}{1-\sin^2\theta x_n^2}$, so

$E(x_n,x_{n\pm1}) = (1-x_n^2\sin^2\theta)x_{n\pm1}^2 - 2x_nx_{n+1}\cos\theta + x_n^2 = 0$,

$E(x,y) = -\sin^2\theta\, x^2y^2 + x^2 + y^2 - 2\cos\theta\, xy$.

$E(x_n,x_{n+1})=0$ gives x_{n+1} as a function of x_n, a procedure quite common in numerical analysis. The ***fixed points*** $z_0 : E(z_0,z_0)=0$ are especially interesting. May a sequence x_n converge towards some fixed point? Normally, no: from the symmetry of E, $dx_{n+1}/dx_n \to -1$ near a fixed point. But what is “normally”?

2.3. Coefficients of continued fraction expansion of an algebraic function.

We consider the algorithm for the coefficients of a continued fraction expansion of an algebraic function involving $\sqrt{P}$, where P is a polynomial of degree $\leqslant 4$ (but a part of the algorithm is valid for any degree [hyperelliptic case, see § 4.1, p. 42]), and let $z_1,\dots,z_4$ be the zeros of P ($z_4=\infty$ if degree $P<4$).

We choose a continuous $\sqrt{P(z)}$ outside a system of two cuts, say $[z_1,z_2]$ and $[z_3,z_4]$. So, instead of dealing with the two-sheeted Riemann surface, we will sometimes emphasize a sign ε in $\varepsilon\sqrt{P(z)}$, with $\varepsilon=+1$ or -1.

Let also z_0 different from $z_1,\dots,z_4$, and $\gamma+\delta(z-z_0)$ be the two first terms of the Taylor expansion of $\sqrt{P(z)}$ about $z=z_0$.

We look at continued fraction expansions about z_0 of functions involving $\sqrt{P}$.

Start with $\sqrt{P}$ itself, to see how things are going:
$\sqrt{P(z)} = \gamma+\delta(z-z_0) - \dfrac{(z-z_0)^2}{\text{Tayl}_1 - \text{new cf}}$, where "Tayl"$_1$ means the two first terms of the Taylor expansion of what is needed there, i.e., of $\dfrac{(z-z_0)^2}{\gamma+\delta(z-z_0)-\sqrt{P(z)}}$, and "new cf" is the remainder $\text{Tayl}_1(z) - \dfrac{(z-z_0)^2}{\gamma+\delta(z-z_0)-\sqrt{P(z)}} = \text{Tayl}_1(z) - \dfrac{\gamma+\delta(z-z_0)+\sqrt{P(z)}}{(z-z_0)^{-2}[(\gamma+\delta(z-z_0))^2-P(z)]}$, where the denominator is a polynomial of degree $\leqslant 2$. So, intermediate steps will involve $\dfrac{S_n-\sqrt{P}}{Z_n}$, where Z_n and S_n are polynomials, with $S_n(z)-\sqrt{P(z)} = O((z-z_0)^2)$ near z_0, so that $S_n^2(z)-P(z) =$ a polynomial multiple of $(z-z_0)^2 Z_n(z)$, say $(z-z_0)^2 W_n(z)Z_n(z)$. We keep at each step the Taylor expansion of degree 1 and consider the remainder (associate continued fraction, see Perron [51, § 20], etc.)

(1) $$\xi_0 = \frac{\sqrt{D}+P_0}{Q_0},$$

wo P_0, $Q_0 (\neq 0)$, D ganze Zahlen sind, und zwar D positiv, aber kein Quadrat; umgekehrt ist jede solche Zahl eine quadratische Irrationalzahl. Dabei können wir ohne Beschränkung der Allgemeinheit $\sqrt{D}$ positiv annehmen, weil ja

$$\frac{-\sqrt{D}+P_0}{Q_0} = \frac{\sqrt{D}-P_0}{-Q_0}.$$

Ferner dürfen wir voraussetzen, daß

(2) $$\frac{D-P_0^2}{Q_0} = Q_{-1}$$

eine ganze Zahl ist; sollte dies nämlich nicht von vorn herein der Fall
...

$$P_1 = b_0 Q_0 - P_0,$$

$$Q_1 = \frac{D-(b_0Q_0-P_0)^2}{Q_0} = \frac{D-P_0^2+2b_0Q_0P_0-b_0^2Q_0^2}{Q_0} = Q_{-1}+2b_0P_0-b_0^2Q_0.$$

$$\frac{S_n(z)-\sqrt{P(z)}}{Z_n(z)} = \frac{(z-z_0)^2}{\dfrac{(z-z_0)^2}{[S_n(z)-\sqrt{P(z)}]/Z_n(z)}} = \frac{(z-z_0)^2}{\dfrac{[S_n(z)+\sqrt{P(z)}]/Z_n(z)}{(z-z_0)^{-2}[S_n^2(z)-P(z)]/Z_n^2 = W_n(z)/Z_n}},$$

and $\dfrac{S_n(z)+\sqrt{P(z)}}{W_n(z)} = \text{Tayl}_1(z) - \dfrac{S_{n+1}(z)-\sqrt{P(z)}}{Z_{n+1}(z)}$, so, $Z_{n+1}(z) = W_n(z)$, $S_{n+1}(z) = -S_n(z)+W_n(z)\text{Taylor}_1(z)$, and check that $S_{n+1}^2(z)-P(z)$ is a polynomial multiple of $(z-z_0)^2 Z_{n+1}(z)$: $S_{n+1}^2(z)-P(z) = \underbrace{S_n^2(z)-P(z)}_{(z-z_0)^2 Z_n(z)W_n(z)} -2S_n(z)W_n(z)\text{Tayl}_1(z)+W_n^2(z)\text{Tayl}_1^2(z)$, so that $W_{n+1}(z)(=Z_{n+2}(z)) = Z_n(z) - \mathrm{Tayl}_1(z)\dfrac{2S_n(z) - W_n(z)\mathrm{Tayl}_1(z)}{(z-z_0)^2}$ which is still a polynomial, as $S_n(z) - \sqrt{P(z)}$, and therefore, $2S_n(z)/W_n(z) - \mathrm{Tayl}_1(z)$ are $O((z-z_0)^2)$. See also that $Z_{n+2}(z) = Z_n(z) - \mathrm{Tayl}_1(z)\dfrac{S_n(z) - S_{n+1}(z)}{(z-z_0)^2}$

Let $\mathrm{Tayl}_1(z)$= two first Taylor terms of $(S_n + \sqrt{P})/Z_{n+1}$ be written $[1-\beta_n(x-z_0)]/\alpha_n$, so that $\dfrac{S_n(z) - \sqrt{P(z)}}{Z_n(z)} = \dfrac{(z-z_0)^2}{\dfrac{S_n(z)+\sqrt{P(z)}}{Z_{n+1}(z)}} = \dfrac{\alpha_n(x-z_0)^2}{1-\beta_n(x-z_0)+O((x-z_0)^2)}$

$$\begin{bmatrix} S_{n+1}(z) \\ Z_{n+1}(z) \\ Z_{n+2}(z) \end{bmatrix} = \begin{bmatrix} -1 & 0 & [1-\beta_n(z-z_0)]/\alpha_n \\ 0 & 0 & 1 \\ -\dfrac{2(1-\beta_n(z-z_0)}{\alpha_n(z-z_0)^2} & 1 & \dfrac{(1-\beta_n(z-z_0))^2}{\alpha_n^2(z-z_0)^2} \end{bmatrix} \begin{bmatrix} S_n(z) \\ Z_n(z) \\ Z_{n+1}(z) \end{bmatrix} \tag{18}$$

Example. $P(z) = 1+z^2+z^4, z_0 = 0, S_0(z) =$ Taylor of $\sqrt{P} \equiv 1, Z_0(z) \equiv 1 \Rightarrow Z_1(z) = (S_0^2(z) - P(z))/z^2 = -1-z^2$, new Taylor of $(S_0+\sqrt{P})/Z_1 = -2, S_1(z) = -S_0(z) +$ Tayl $Z_1(z) = 1+2z^2, Z_2(z) = (S_1^2(z) - P(z))/(z^2Z_1(z)) = -3$, new Taylor of $(S_1+\sqrt{P})/Z_2 = -2/3, S_2(z) = 1-2z^2$ and $Z_3(z) = 5/3 - z^2$ etc. Degree of $S_n = 2$, degrees of Z_n alternatively 0 and 2.

We will only consider now functions $(S_n - \sqrt{P})/Z_n$ with S_n of degree 2 and Z_n of degree 1.

Why the negative sign before $\sqrt{P}$? Perron uses a positive sign. But the same Perron showed how to retrieve the measure from the Stieltjes transform $\int_S (z-x)^{-1}d\mu(x)$: when z has a small ***positive*** imaginary part $z = x^* + i\varepsilon$, $\varepsilon > 0$, the imaginary part of the Stieltjes transform is close to $-\pi i\mu'(x_0)$, see also Haydock, Haydox & Nex [22]. Then, formulas with $\mu' > 0$ and $Z_n(z) > 0$ for large z contain a negative sign before the square root.

Examples (with $z_0 = \infty$, so that we consider expansions with negative powers about ∞)

1: $\displaystyle\int_a^b \frac{\sqrt{(x-a)(b-x)}\,dx}{\pi(z-x)} = \frac{(b-a)^2}{8}\left(\frac{1}{z} + \frac{(a+b)/2}{z^2} + \cdots\right)$
$= z - (a+b)/2 - \sqrt{(z-a)(z-b)}$.

2: $\displaystyle\int_a^b + \int_c^d \frac{\sqrt{(x-a)(x-b)(x-c)(d-x)}\,dx}{|x-x_0|(z-x)}$
$= \text{const.}\ \dfrac{S(z) = (z-x_0)^2 - (a+b+c+d-4x_0)(z-x_0)/2 + S(x_0) - \sqrt{(z-a)(z-b)(z-c)(z-d)}}{z-x_0}$

if $a<b<c<d$ are real, the measure is positive if z_0 is in the gap $[b,c]$. The numerator must vanish at $z = x_0$. If that does not happen, one must add to the measure a Dirac mass at $x = x_0$.

Finally, let the continued fraction step be

$$f_n(z) := \frac{S_n(z) - \sqrt{P(z)}}{(z-z_0)Z_n(z)} = \frac{(z-z_0)}{\alpha_n + \beta_n(z-z_0) - (z-z_0)f_{n+1}(z)}. \tag{19}$$

The coefficients α_n and β_n are immediately found from $\dfrac{S_n(z)+\sqrt{P(z)}}{Z_{n+1}(z)}$
$= \dfrac{2\gamma + 2\delta(z-z_0) + O((z-z_0)^2)}{Z_{n+1}(z_0) + Z'_{n+1}(z_0)(z-z_0) + \cdots} = \alpha_n + \beta_n(z-z_0) + O((z-z_0)^2)$

with P of degree 3 or 4, asks for a sequence of polynomials $\{S_n(x) = \gamma + \delta(x - z_0) + \xi_n(x - z_0)^2, Z_n(x) = \zeta_n(x - x_n)\}$, $n = 0, 1, \dots$, such that $S_n^2(x) - P(x) = \alpha_n(x - z_0)^2 Z_n(x) Z_{n+1}(x)$, see § 3.4.1 p. 37.

Definition 3. $\{x_n\}$ in the continued fraction arrangement $f_n(x) = \dfrac{S_n(x) - \sqrt{P(x)}}{(x - z_0)\zeta_n(x - x_n)}$ $= \dfrac{\alpha_n(x - z_0)}{1 - \beta_n(x - z_0) - (x - z_0) f_{n+1}(x)} = \cfrac{\alpha_n(x - z_0)}{1 - \beta_n(x - z_0) - \cfrac{\alpha_{n+1}(x - z_0)^2}{1 - \beta_{n+1}(x - z_0) - \ddots}}$ is an instance of ***elliptic lattice***.

2.4. **Values of elliptic functions with arguments in arithmetic progression.**

Jacobi and Abel related the continued fraction above to a closed formula for x_n through the Jacobi inversion problem, see (38) p. 40, which brings us to

Definition 4. *An elliptic lattice is a sequence $x_n = \mathcal{E}(nh + u_0)$, where $\mathcal{E}$ is any elliptic function of order 2 (i.e., with 2 zeros and 2 poles in a fundamental parallelogram of periods).*

A first way [59] to establish this definition is to recognize (15) as an addition formula for elliptic functions.

One may also establish, see p. 17, that the biquadratic curve $F(x, y) = 0$ in (5) has genus 1 and a parametric representation

$$x = \mathcal{E}_1(s), \quad y = \mathcal{E}_2(s),$$

with $\mathcal{E}_1$ and $\mathcal{E}_2$ elliptic functions of order 2, inverting an elliptic integral of the first kind

$$s = \int^x \frac{du}{\sqrt{P(u)}} = \int^y \frac{dv}{\sqrt{Q(v)}}.$$

with a continuous determination of the square root along the path of integration.

Now, let s_n and s'_n correspond to the two points (x_n, y_n) and (x_n, y_{n+1}). As $\mathcal{E}_1(s_n) = \mathcal{E}_1(s'_n)$ with y_{n+1} normally different from y_n, $s_n + s'_n =$ a constant, say γ_1 (as s_n and s'_n are integrals involving the square root of a polynomial on two paths with the same endpoints [the second endpoint being x_n], the square roots are opposite on a part of the paths). Similarly, $s'_n + s_{n+1} =$ another constant, say γ_2. Therefore, $s_{n+1} = s_n + h$, with $h = \gamma_2 - \gamma_1$.

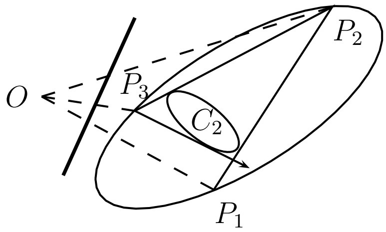
The essential parameters in the description of an elliptic sequence is the modulus k and the step h (actually, the ratio h/ω one one period). The modulus is also related to the ratio ω_1/ω_2 of periods. Finally, in a multiplicative setting, the main parameters are the nome p and the multiplier q, which are basically (i.e., up to multiplication by constants) the exponentials of the periods ratio and the step.

The modulus and the step depend only on F in (5) (or E in (15)). For each starting point (x_0, y_0), or $s_0 = h_0$, there is a different elliptic lattice with the same k and h.

It is always possible to relate $\mathcal{E}_2$ to $\mathcal{E}_1$ through a rational transformation of first degree $\mathcal{E}_2(s) = \dfrac{\alpha\mathcal{E}_1(s + h/2) + \beta}{\gamma\mathcal{E}_1(s + h/2) + \delta}$ [59, p. 298].

2.5. **Relations with other problems.**

and there are relations with
Poncelet problem (see [**?**, 15, 16])



One considers two conics C_1 and C_2. From a point P_1 on C_1, one constructs a tangent to C_2 meeting C_1 at P_2, and so on. The coordinates of the P_i's, or the (central or parallel) projections of these points on any straight line, make an elliptic lattice.
From P_2 onwards, there is no double choice left for the next point, as one of the two possible tangents leads to the preceding point. At the first point, the other possible tangent simply leads to P_0, and continues with P_{-1}, etc.

From Burskii and Zhedanov: let $\{X = R_1(x), Y = R_2(x)\}$ and $\{X = S_1(y), Y = S_2(y)\}$ be parametric representations of the two conics C_1 and C_2. R_1, etc. are rational functions of degree $\leqslant 2 : R_i = r_i/u, S_i = s_i/v$. The line P_nQ_n, with parameters x_n and y_n, is tangent at Q_n, so

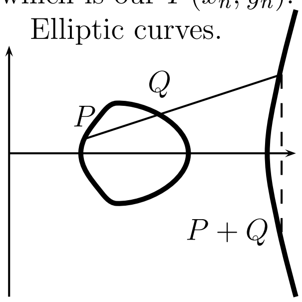
$$\frac{Y_n^{(1)} - Y_n^{(2)}}{X_n^{(1)} - X_n^{(2)}} = \frac{R_2(x_n) - S_2(y_n)}{R_1(x_n) - S_1(y_n)} = \frac{S_2'(y_n)}{S_1'(y_n)},$$

or

$$\begin{vmatrix} R_1(x_n) - S_1(y_n) & R_2(x_n) - S_2(y_n) \\ S_1'(y_n) & S_2'(y_n) \end{vmatrix} = \begin{vmatrix} 1 & R_1(x_n) & R_2(x_n) \\ 1 & S_1(y_n) & S_2(y_n) \\ 0 & S_1'(y_n) & S_2'(y_n) \end{vmatrix} = \begin{vmatrix} u(x_n) & r_1(x_n) & r_2(x_n) \\ v(y_n) & s_1(y_n) & s_2(y_n) \\ v'(y_n) & s_1'(y_n) & s_2'(y_n) \end{vmatrix} = 0,$$

which is our $F(x_n, y_n)$.

Elliptic curves.



An elliptic curve is neither an ellipse nor a curve. It is a set of points which happens to be a group with respect to a peculiar addition rule. The geometric picture may seem strange: let P be a polynomial of third degree, follow the line joining two points of the (cubic!) curve $y = \sqrt{P(x)}$ until it intersects a third one, and take the symmetric point with respect to the $x-$axis.

Excerpts of Appell & Goursat [4]
`http://www.archive.org/details/theoriefonctions00apperich`
Theory of algebraic functions and their integrals: study of analytic functions on a Riemann surface.

Contents:

1. Two-sheeted Riemann surfaces $u^2 = z, u^2 = A(z-e_1)(z-e_2)(z-e_3)(z-e_4), u^2 = A(z-e_1)\cdots(z-e_n)$. Uniform function on a Riemann surface: zeros, singular points, poles, orders. Rational functions of z and u: characteristic properties. Genus. 1–54.

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From chap. 6, one may also establish that the biquadratic curve $F(x, y) = 0$ in (5) has genus 1 and a parametric representation

$$x = \mathcal{E}_1(s), \quad y = \mathcal{E}_2(s),$$

with $\mathcal{E}_1$ and $\mathcal{E}_2$ elliptic functions of order 2.

Indeed, from chapter 6 of Appell & Goursat's book [4], a birational transformation $(x, y) \leftrightarrow (\xi, \eta)$ sending the biquadratic curve (5) $F(x, y) = 0$ to the canonical elliptic curve $\eta^2 = P_3(\xi)$, where P_3 is a polynomial of third degree, see [4, p.292]: from (7) $y = (-X_1(x) + \sqrt{P(x)})/(2X_2(x))$, choose $w =$ a square root of $P(x)$, so that $y = (-X_1(x) + w)/(2X_2(x)) \leftrightarrow w = X_1(x) + 2yX_2(x)$, and $x = z_1 + 1/\xi$, where z_1 is one of the four roots of $P(x) = 0$. Then, with $P(z_1 + 1/\xi) = P_3(\xi)/\xi^4$ and $\eta = w\xi^2$, $\eta^2 = \xi^4 P(z_1 + 1/\xi) = P_3(\xi)$.

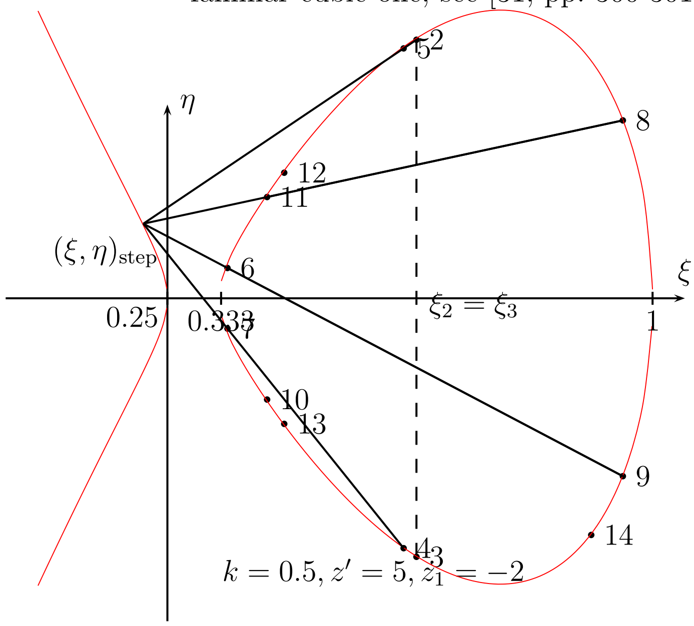
$$\begin{aligned}
x &= z_1 + \frac{1}{\xi}, & \xi &= \frac{1}{x - z_1}, \\
y &= \frac{-X_1(z_1 + 1/\xi) + \eta/\xi^2}{2X_2(z_1 + 1/\xi)}, & \eta &= [X_1(x) + 2yX_2(x)]/(x - z_1)^2 \\
&= \frac{-\xi^2 X_1(z_1 + 1/\xi) + \eta}{2\xi^2 X_2(z_1 + 1/\xi)} \\
F(x, y) &= X_0(x) + X_1(x)y + X_2(x)y^2 = 0 & \eta^2 &- P_3(\xi) = 0.
\end{aligned} \tag{20}$$

Then, the Weierstrass representation holds

$$\begin{gathered}
\xi = \wp(hu + u_0), \ \ \eta = \wp'(hu + u_0). \\
\text{So, } x = z_1 + 1/\wp, \ \ y = (-X_1 + \wp'/\wp^2)/(2X_2).
\end{gathered} \tag{21}$$

The birational transformation, and the Weirstrass function representation are also described by Burskii & Zhedanov [16].

However, the authors of [31] recommend the biquadratic setting instead of the more familiar cubic one, see [31, pp. 300-301].



What are the (ξ, η) corresponding to the points (x_n, y_n) and (x_n, y_{n+1})? Let us index them as $(\xi, \eta)_{2n}$ and $(\xi, \eta)_{2n+1}$. The ξs are obviously enough $\xi_{2n} = \xi_{2n+1} = 1/(x_n - z_1)$. Then, η_{2n} and η_{2n+1} are the two opposite square roots of $P_3(\xi_{2n})$. The next point is taken on the line $y =$ constant $= y_{n+1}$, so on the curve $\dfrac{-\xi^2 X_1(z_1 + 1/\xi) + \eta}{2\xi^2 X_2(z_1 + 1/\xi)} =$ constant $= \dfrac{-\xi_{2n}^2 X_1(z_1 + 1/\xi_{2n}) + \eta_{2n+1}}{2\xi_{2n}^2 X_2(z_1 + 1/\xi_{2n})} = y_{n+1}$, which is a parabola $\eta =$ a second degree polynomial in ξ, up to the intersection with $\eta^2 = P_3(\xi)$: the equation for ξ_{2n+2} is $\xi^4[X_1(z_1 + 1/\xi) + 2y_{n+1}X_2(z_1 + 1/\xi)]^2 = P_3(\xi) = \xi^4 P(z_1 + 1/\xi) = \xi^4[X_1(z_1 + 1/\xi)^2 - 4X_0(z_1 + 1/\xi)X_2(z_1 + 1/\xi)$, so a second degree equation $y_{n+1}X_1(z_1 + 1/\xi) + y_{n+1}^2 X_2(z_1 + 1/\xi)) = -X_0(z_1 + 1/\xi)$, which is our $F(x, y_{n+1}) = 0$. The two $\xi-$roots of $\xi^2 F(z_1 + 1/\xi, y_{n+1}) = (1 + \xi z_1)^2 Y_2(y_{n+1}) + \xi(1 + \xi z_1)Y_1(y_{n+1}) + \xi^2 Y_0(y_{n+1}) = 0$ are therefore ξ_{2n} and ξ_{2n+2}, whence $\xi_{2n} + \xi_{2n+2} = -(2z_1 Y_2(y_{n+1}) + Y_1(y_{n+1}))/F(z_1, y_{n+1})$ and $\xi_{2n}\xi_{2n+2} = Y_2(y_{n+1})/F(z_1, y_{n+1})$.
As $F(z_1, y) = X_2(z_1)y^2 + X_1(z_1)y + X_0(z_1) = 0$ at the double root $y = y(z_1) = -X_1(z_1)/(2X_2(z_1))$ when $X_1^2(z_1) - 4X_0(z_1)X_2(z_1) = 0$, $F(z_1, y) \equiv (2X_2(z_1)y + X_1(z_1))^2/(4X_2(z_1))$.

Remark that $(\xi_{2n+2}-\xi_{2n})^2 = [(2z_1Y_2(y_{n+1})+Y_1(y_{n+1}))^2-4Y_2(y_{n+1})F(z_1,y_{n+1})]/F^2(z_1,y_{n+1}) = [Y_1^2(y_{n+1}) - 4Y_0(y_{n+1})Y_2(y_{n+1})]/F^2(z_1,y_{n+1})$.
Also, $\eta_{2n} + \eta_{2n+2} = -\eta_{2n+1} + \eta_{2n+2} = -[2\xi_{2n}^2 X_2(z_1 + 1/\xi_{2n})]y_{n+1} - \xi_{2n}^2 X_1(z_1 + 1/\xi_{2n}) + [2\xi_{2n+2}^2 X_2(z_1 + 1/\xi_{2n+2})]y_{n+1} + \xi_{2n+2}^2 X_1(z_1 + 1/\xi_{2n+2}) = 2[(\xi_{2n+2}^2 - \xi_{2n}^2)X_2(z_1) + (\xi_{2n+2} - \xi_{2n})X_2'(z_1)]y_{n+1} + (\xi_{2n+2}^2 - \xi_{2n}^2)X_1(z_1) + (\xi_{2n+2} - \xi_{2n})X_1'(z_1)$
$\xi_{2n+2}\eta_{2n}+\xi_{2n}\eta_{2n+2} = -\xi_{2n+2}\eta_{2n+1}+\xi_{2n}\eta_{2n+2} = -\xi_{2n+2}[2\xi_{2n}^2 X_2(z_1+1/\xi_{2n})]y_{n+1}-\xi_{2n+2}\xi_{2n}^2 X_1(z_1+1/\xi_{2n})+\xi_{2n}[2\xi_{2n+2}^2 X_2(z_1+1/\xi_{2n+2})]y_{n+1}+\xi_{2n}\xi_{2n+2}^2 X_1(z_1+1/\xi_{2n+2}) = (\xi_{2n+2}-\xi_{2n})[2\xi_{2n}\xi_{2n+2}X_2(z_1)y_{n+1} - X_2''y_{n+1} + \xi_{2n}\xi_{2n+2}X_1(z_1) - X_1''/2] = (\xi_{2n+2} - \xi_{2n})\left[4X_2\dfrac{Y_2(y_{n+1})}{2X_2y_{n+1} + X_1} - X_2''y_{n+1} - X_1''/2\right]$.

According to the addition rule of an elliptic curve, each new point $(\xi,\eta)_{2n+2}$ is the result of adding a fixed step to $(\xi,\eta)_{2n}$, which means that the straight line joining $(\xi,\eta)_{\text{step}}$ to (ξ_{2n},η_{2n}) must meet the cubic $\eta^2 = P_3(\xi)$ at the opposite of (ξ_{2n+2},η_{2n+2}), which happens to be $(\xi_{2n+2},-\eta_{2n+2}) = (\xi,\eta)_{2n+3}$.The straight line $\eta = -\dfrac{\eta_{2n}+\eta_{2n+2}}{\xi_{2n+2}-\xi_{2n}}\xi + \dfrac{\xi_{2n+2}\eta_{2n}+\xi_{2n}\eta_{2n+2}}{\xi_{2n+2}-\xi_{2n}}$ meets the cubic $\eta^2 - \xi^4P(z_1+1/\xi) = \left[\dfrac{\eta_{2n}+\eta_{2n+2}}{\xi_{2n+2}-\xi_{2n}}\right]^2\xi^2 + \cdots - P'(z_1)\xi^3 - P''(z_1)\xi^2/2 - \cdots = \{2[(\xi_{2n} + \xi_{2n+2})X_2(z_1) + X_2'(z_1)]y_{n+1} + (\xi_{2n} + \xi_{2n+2})X_1(z_1) + X_1'(z_1)\}^2\xi^2 + \cdots - P'(z_1)\xi^3 - P''(z_1)\xi^2/2 - \cdots$ at a third point $\xi^{(\text{step},1)}$ together with $\xi = \xi_{2n}$ and ξ_{2n+2}, so that $P'(z_1)\xi^{(\text{step},1)} = \{2[(\xi_{2n} + \xi_{2n+2})X_2(z_1) + X_2'(z_1)]y_{n+1} + (\xi_{2n} + \xi_{2n+2})X_1(z_1) + X_1'(z_1)\}^2 - P''(z_1)/2 - P'(z_1)(\xi_{2n} + \xi_{2n+2}) = [(\xi_{2n} + \xi_{2n+2})(2X_2(z_1)y_{n+1} + X_1(z_1)) + 2X_2'(z_1)y_{n+1} + X_1'(z_1)]^2 - P''(z_1)/2 - P'(z_1)(\xi_{2n}+\xi_{2n+2}) = \left[-4X_2(z_1)\dfrac{2z_1Y_2(y_{n+1}) + Y_1(y_{n+1})}{2y_{n+1}X_2(z_1) + X_1(z_1)} + 2X_2'(z_1)y_{n+1} + X_1'(z_1)\right]^2 - P''(z_1)/2 + P'(z_1)4X_2(z_1)\dfrac{2z_1Y_2(y_{n+1}) + Y_1(y_{n+1})}{(2y_{n+1}X_2(z_1) + X_1(z_1))^2}$ must be independent of y_{n+1}!!! Also, for all x, y, $\partial F(x,y)/\partial x = X_2'y^2 + X_1'y + X_0' = 2Y_2(y)x + Y_1(y)$. At $x = z_1$, $P'(z_1) = 2X_1X_1' - 4X'0X_2 - 4X_0X_2' = -4X_2[X_2'(y(z_1))^2 + X_1'y(z_1) + X_0']$, where $y(z_1) = -X_1(z_1)/(2X_2(z_1))$, as $X_0(z_1) = X_1^2(z_1)/(4X_2(z_1))$. So, $P'(z_1) = -4X_2(z_1)[2z_1Y_2(y(z_1)) + Y_1(y(z_1))]$, and

$$P'(z_1)\xi^{(\text{step},1)} = \left\{-2\frac{X_2'y^2 + X_1'y + X_0'}{y - y(z_1)} + 2X_2'y + X_1'\right\}^2 - \frac{P''(z_1)}{2} - 4[X_2'y^2(z_1)+X_1'y(z_1)+X_0']\frac{X_2'y^2 + X_1'y + X_0'}{(y - y(z_1))^2}$$

$$= \left\{-2\frac{[X_2'y(z_1) + X_1']y + X_0'}{y - y(z_1)} + X_1'\right\}^2 - \frac{P''(z_1)}{2} - 4[X_2'y^2(z_1)+X_1'y(z_1)+X_0']\frac{X_2'y^2 + X_1'y + X_0'}{(y - y(z_1))^2}$$

$$= \left\{-2X_2'y(z_1) - X_1' + \frac{R}{y - y(z_1)}\right\}^2 - \frac{P''(z_1)}{2} + 2R\frac{X_2'y^2 + X_1'y + X_0'}{(y - y(z_1))^2}$$ where R is the residue $-2[X_2'y^2(z_1) + X_1'y(z_1) + X_0'] = P'(z_1)/(2X_2(z_1))$.Then, $P'(z_1)\xi^{(\text{step},1)} = (2X_2'y(z_1) + X_1')^2$

$$-2(2X_2'y(z_1)+X_1')\frac{R}{y - y(z_1)} + \frac{R^2}{(y - y(z_1))^2} - \frac{P''(z_1)}{2} + 2R\frac{-R/2 + (2X_2'y(z_1) + X_1')(y - y(z_1)) + X_2'(y - y(z_1))^2}{(y - y(z_1))^2}$$

$= (2X_2'y(z_1) + X_1')^2 - P''(z_1)/2 + 2RX_2' = X_1'^2 - P''(z_1)/2 - 4X_2'X_0' = -X_1X_1'' + 2X_0''X_2 + 2X_0X_2'' = 2X_2[X_0'' + X_1''y(z_1) + X_2''y^2(z_1)]$.

$$\xi^{(\text{step},1)} = \frac{2X_2(z_1)[X_0'' + X_1''y(z_1) + X_2''y^2(z_1)]}{P'(z_1)} = -\frac{1}{2}\,\frac{X_0'' + X_1''y(z_1) + X_2''y^2(z_1)}{X_0' + X_1'y(z_1) + X_2'y^2(z_1)}$$

$$= -\frac{1}{2}\,\frac{\partial^2F(x,y)/\partial x^2 \text{ at } (z_1,y(z_1))}{\partial F(x,y)/\partial x \text{ at } (z_1,y(z_1))} = -\frac{Y_2(y(z_1))}{2z_1Y_2(y(z_1)) + Y_1(z_1)}.$$

$$\eta^{(\text{step},1)} = -\frac{\eta_{2n}+\eta_{2n+2}}{\xi_{2n+2}-\xi_{2n}}\xi^{(\text{step},1)} + \frac{\xi_{2n+2}\eta_{2n}+\xi_{2n}\eta_{2n+2}}{\xi_{2n+2}-\xi_{2n}} = \left[-2X_2'y(z_1)-X_1'+\frac{R=P'/(2X_2)}{y_{n+1}-y(z_1)}\right]\xi^{(\text{step},1)}$$

$$+4X_2\frac{Y_2(y_{n+1})}{2X_2y_{n+1}+X_1}-X_2''y_{n+1}-X_1''/2 = \left[-2X_2'y(z_1)-X_1'+\frac{P'/(2X_2)}{y_{n+1}-y(z_1)}\right]\xi^{(\text{step},1)}+\frac{\overbrace{2Y_2(y_{n+1})}^{\partial^2F(x,y_{n+1})/\partial x^2}-X_2''y_{n+1}^2-}{y_{n+1}-y(z_1)}$$

$$X_1''/2 = \left[-2X_2'y(z_1)-X_1'+\frac{P'/(2X_2)}{y_{n+1}-y(z_1)}\right]\xi^{(\text{step},1)}+\frac{(X_2''y(z_1)+X_1'')y_{n+1}+X_0''}{y_{n+1}-y(z_1)}-X_1''/2 =$$

$$(2X_2'y(z_1)+X_1')\xi^{\text{step},1}+\frac{P'(z_1)\xi^{(\text{step},1)}/(2X_2(z_1))+X_2''y^2(z_1)+X_1''y(z_1)+X_0''=0}{y_{n+1}-y(z_1)}+X_2''y(z_1)+$$

$$X_1''/2$$

$$= (2X_2'y(z_1)+X_1')\xi^{\text{step},1}+X_2''y(z_1)+X_1''/2$$

$$= \frac{1}{2}\,\frac{(X_1'X_2''-X_1''X_2')y^2(z_1)+2(X_0'X_2''-X_0''X_2')y(z_1)+X_0'X_1''-X_0''X_1'}{X_2'y^2(z_1)+X_1'y(z_1)+X_0'}$$

Returning to (x,y), $x^{(\text{step},1)} = z_1-2\dfrac{X_0'+X_1'y(z_1)+X_2'y^2(z_1)}{X_0''+X_1''y(z_1)+X_2''y^2(z_1)} = z_1-\dfrac{2z_1Y_2(y(z_1))+Y_1(z_1)}{Y_2(y(z_1))} = -z_1-\dfrac{Y_1(z_1)}{Y_2(y(z_1))} = z_1'$, the second $x-$root of $F(x,y)=Y_2(y)x^2+Y_1(y)x+Y_0(y)=0$ at $y=y(z_1)$;

$$y^{(\text{step},1)} = \frac{-X_1(z_1+1/\xi^{(\text{step},1)})+\eta^{(\text{step},1)}/(\xi^{(\text{step},1)})^2}{2X_2(z1+1/\xi^{(\text{step},1)})} = \frac{-X_1\xi^2-X_1'\xi-X_1''/2+\eta}{2[X_2\xi^2+X_2'\xi+X_2''/2]}$$

$$= \frac{-X_1\xi^2+2X_2'y(z_1)\xi+X_2''y(z_1)}{2[X_2\xi^2+X_2'\xi+X_2''/2]} = y(z_1) \text{ as } -X_1(z_1)=2X_2(z_1)y(z_1).$$

To multiples of $(\xi,\eta)_{(\text{step},1}$ according to the addition rule of the elliptic curve correspond iterates of $(x,y)^{(\text{step},1)}$ on the biquadratic curve $F(x,y)=0$. Let $x_n^{(\text{step},1)}$ and $y_n^{(\text{step},1)}$ be the abscissae and ordinates so constructed. To the neutral element $(\xi,\eta)=(\infty,\infty)$ of the elliptic curve correspond the starting point $(x,y)_0^{(\text{step},1)}=(z_1,y(z_1))$. Remark that $x_{-n}^{(\text{step},1)}=x_n^{(\text{step},1)}, y_{-n}^{(\text{step},1)}=y_{n+1}^{(\text{step},1)}$.

Of course, the same may be done with any z_i, $i=1,\dots,4$, not just z_1.

More: goto (44), p. 48

New notation: let (x_n,y_n) correspond to (ξ_n,η_n), and (x_n,y_{n+1}) correspond to $(\xi_n,-\eta_n)$: $x_n=z_1+\dfrac{1}{\xi_n}$, $y_n=\dfrac{-X_1(z_1+1/\xi_n)+\eta_n/\xi_n^2}{2X_2(z_1+1/\xi_n)}$, $y_{n+1}=\dfrac{-X_1(z_1+1/\xi_n)-\eta_n/\xi_n^2}{2X_2(z_1+1/\xi_n)}$, then, (ξ_{n+1},η_{n+1}) is the second intersection of the line $y=$ constant $=y_{n+1}$, i.e. the parabola $\eta=\xi^2X_1(z_1+1/\xi)+2y_{n+1}\xi^2X_2(z_1+1/\xi)$ and the cubic $\eta^2-\xi^4(X_1^2-4X_0X_2)=0$, so the equation for $\xi=\xi_{n+1}$ is $y_{n+1}X_1(z_1+1/\xi)+y_{n+1}^2X_2(z_1+1/\xi)+X_0(z_1+1/\xi)=0$, [2]

$$\frac{-X_1(z_1+1/\xi_n)-\eta_n/\xi_n^2}{2X_2(z_1+1/\xi_n)}X_1(z_1+1/\xi)+\frac{[X_1(z_1+1/\xi_n)+\eta_n/\xi_n^2]^2}{4X_2^2(z_1+1/\xi_n)}X_2(z_1+1/\xi)+X_0(z_1+1/\xi)=0,$$

$$-X_1(z_1+1/\xi_n)X_1(z_1+1/\xi)+\frac{X_1^2(z_1+1/\xi_n)}{2X_2(z_1+1/\xi_n)}X_2(z_1+1/\xi)+(\eta_n/\xi_n^2)\left[\frac{X_1(z_1+1/\xi_n)}{X_2(z_1+1/\xi_n)}X_2(z_1+1/\xi)-X_1(z_1+1/\xi)\right]+$$

$$\frac{X_1^2(z_1+1/\xi_n)-4X_0(z_1+1/\xi_n)X_2(z_1+1/\xi_n)}{2X_2(z_1+1/\xi_n)}X_2(z_1+1/\xi)+2X_0(z_1+1/\xi)X_2(z_1+1/\xi_n)=0$$

$$\left(\frac{\eta_n}{\xi_n^2}+X_1(z_1+1/\xi_n)\right)\left[\frac{X_1(z_1+1/\xi_n)}{X_2(z_1+1/\xi_n)}X_2(z_1+1/\xi)-X_1(z_1+1/\xi)\right]+2[X_0(z_1+1/\xi)X_2(z_1+1/\xi_n)-X_0(z_1+$$

[2] Of course, this is $F(z_1+1/\xi,y_{n+1})=0$.

$1/\xi_n)X_2(z_1+1/\xi)]=0,$ two roots: $\xi=\xi_n$ and
$\left(\dfrac{\eta_n}{\xi_n^2}+X_1(z_1+1/\xi_n)\right)\left[\dfrac{X_1(z_1+1/\xi_n)}{X_2(z_1+1/\xi_n)}X_2'(z_1+1/\xi_n)-X_1'(z_1+1/\xi_n)+\left(\dfrac{1}{\xi}-\dfrac{1}{\xi_n}\right)\left[\dfrac{X_1(z_1+1/\xi_n)}{X_2(z_1+1/\xi_n)}X_2''-X_1''\right]/2\right]+$
$2[X_0'(z_1+1/\xi_n)X_2(z_1+1/\xi_n)-X_0(z_1+1/\xi_n)X_2'(z_1+1/\xi_n)]+\left(\dfrac{1}{\xi}-\dfrac{1}{\xi_n}\right)[X_0''X_2(z_1+1/\xi_n)-X_0(z_1+$
$1/\xi_n)X_2'']=0$

$$\frac{1}{\xi_{n+1}}=\frac{1}{\xi_n}-\frac{\left(\dfrac{\eta_n}{\xi_n^2}+X_1\right)\left[\dfrac{X_1}{X_2}X_2'-X_1'\right]+2[X_0'X_2-X_0X_2']}{\left(\dfrac{\eta_n}{\xi_n^2}+X_1\right)\left[\dfrac{X_1}{X_2}X_2''-X_1''\right]/2+X_0''X_2-X_0X_2''}$$ at $z_1+1/\xi_n:X_j(z_1)=$
$X_j-X_j'/\xi_n+X_j''/(2\xi_n^2)$:

$$\frac{1}{\xi_{n+1}}=\frac{\left(\dfrac{\eta_n}{\xi_n}+X_1\xi_n\right)\left[\dfrac{X_1}{X_2}[X_2(z_1)-X_2]-[X_1(z_1)-X_1]\right]+2\xi_n[[X_0(z_1)-X_0]X_2-X_0[X_2(z_1)-X_2]]}{\left(\dfrac{\eta_n}{\xi_n^2}+X_1\right)\left[\dfrac{X_1}{X_2}X_2''-X_1''\right]/2+X_0''X_2-X_0X_2''}$$

$$=\frac{\left(\dfrac{\eta_n}{\xi_n}+X_1\xi_n\right)\left[\dfrac{X_1}{X_2}X_2(z_1)-X_1(z_1)\right]+2\xi_n[X_0(z_1)X_2-X_0X_2(z_1)]}{\left(\dfrac{\eta_n}{\xi_n^2}+X_1\right)\left[\dfrac{X_1}{X_2}X_2''-X_1''\right]/2+X_0''X_2-X_0X_2''}$$

also, $\xi_{n+1}-\xi_n=\xi_n\dfrac{\left(\dfrac{\eta_n}{\xi_n^2}+X_1\right)\left[\dfrac{X_1}{X_2}X_2'-X_1'\right]+2[X_0'X_2-X_0X_2']}{\left(\dfrac{\eta_n}{\xi_n}+X_1\xi_n\right)\left[\dfrac{X_1}{X_2}X_2(z_1)-X_1(z_1)\right]+2\xi_n[X_0(z_1)X_2-X_0X_2(z_1)]}$

$\eta_{n+1}=\xi_{n+1}^2X_1(z_1+1/\xi_{n+1})-2\dfrac{\eta_n/\xi_n^2+X_1}{2X_2}\xi_{n+1}^2X_2(z_1+1/\xi_{n+1})$

$$=-\frac{\xi_{n+1}^2}{\xi_n^2}\eta_n+\xi_{n+1}^2X_1'\left(\frac{1}{\xi_{n+1}}-\frac{1}{\xi_n}\right)+\xi_{n+1}^2(X_1''/2)\left(\frac{1}{\xi_{n+1}}-\frac{1}{\xi_n}\right)^2-\xi_{n+1}^2\frac{\dfrac{\eta_n}{\xi_n^2}+X_1}{X_2}\left[X_2'\left(\frac{1}{\xi_{n+1}}-\frac{1}{\xi_n}\right)+(X_2''/2)\left(\frac{1}{\xi_{n+}}\right.\right.$$

where X_1, X_1', etc. are the values at z_1+1/ξ_n.

Also, from $y_{n+1}=\dfrac{-X_1(z_1+1/\xi_n)-\eta_n/\xi_n^2}{2X_2(z_1+1/\xi_n)}$, $\eta_{n+1}=\xi_{n+1}^2X_1(z_1+1/\xi_{n+1})+2y_{n+1}\xi_{n+1}^2X_2(z_1+$
$1/\xi_{n+1})$.

And now, the constant difference of $(\xi,\eta)_{n+1}$ and $(\xi,\eta)_n$ according to the addition rule of elliptic curves: look at where the line $\eta=\eta_n-\dfrac{\eta_{n+1}+\eta_n}{\xi_{n+1}-\xi_n}(\xi-\xi_n)$ meets the cubic $\eta^2-\xi^4P(z_1+1/\xi)=\eta^2-P'(z_1)\xi^3-(P''(z_1)/2)\xi^2-\cdots=0:$

$$\xi_{\text{step}}=\frac{1}{P'(z_1)}\left[\left(\frac{\eta_{n+1}+\eta_n}{\xi_{n+1}-\xi_n}\right)^2-P''(z_1)/2\right]-\xi_n-\xi_{n+1}$$

$$=\frac{1}{P'(z_1)}\left[\left(\frac{\left(1-\dfrac{\xi_{n+1}^2}{\xi_n^2}\right)\eta_n+\xi_{n+1}^2X_1'\left(\dfrac{1}{\xi_{n+1}}-\dfrac{1}{\xi_n}\right)+\xi_{n+1}^2(X_1''/2)\left(\dfrac{1}{\xi_{n+1}}-\dfrac{1}{\xi_n}\right)^2-\xi_{n+1}^2\dfrac{\dfrac{\eta_n}{\xi_n^2}+X_1}{X_2}\left[X_2'\left(\dfrac{1}{\xi_{n+1}}-\dfrac{1}{\xi_n}\right)+(X_2''/2)\left(\dfrac{1}{\xi_{n+1}}-\right.\right.}{\xi_{n+1}-\xi_n}\right.\right.$$

$\xi_n-\xi_{n+1}$

$$= \frac{1}{P'(z_1)}\left[\left(-\frac{\xi_n+\xi_{n+1}}{\xi_n^2}\eta_n - \frac{\xi_{n+1}}{\xi_n}X_1' + \frac{\xi_{n+1}-\xi_n}{\xi_n^2}X_1''/2 - \xi_{n+1}^2\frac{\frac{\eta_n}{\xi_n^2}+X_1}{X_2}\left[-\frac{X_2'}{\xi_n\xi_{n+1}} + \frac{(X_2''/2)}{\xi_n^2\xi_{n+1}^2}(\xi_{n+1}-\xi_n)\right]\right)^2 - P''(z_1)/2\right] - \xi_n - \xi_{n+1}$$

we isolate η_n through $\eta_n = (\eta_n+\xi_n^2X_1) - \xi_n^2X_1 = -2X_2\xi_n^2 y_{n+1} - \xi_n^2X_1$

$$= \frac{1}{P'(z_1)}\left[\left(-2X_2y_{n+1}\left[\frac{\xi_{n+1}X_2'}{\xi_nX_2} - \frac{(X_2''/2)}{\xi_n^2X_2}(\xi_{n+1}-\xi_n) - \xi_n - \xi_{n+1}\right] + (\xi_n+\xi_{n+1})X_1 - \frac{\xi_{n+1}}{\xi_n}X_1' + \frac{\xi_{n+1}-\xi_n}{\xi_n^2}X_1''/2\right)^2 - P''(z_1)/2\right] - \xi_n - \xi_{n+1}$$

$$= \frac{1}{P'(z_1)}\left[\left(-2y_{n+1}\left[\frac{(X_2''/2)}{\xi_n} - \xi_n - \xi_{n+1}X_2(z_1)\right] + \xi_{n+1}X_1(z_1) + \xi_nX_1 - \frac{X_1''/2}{\xi_n}\right)^2 - P''(z_1)/2\right] - \xi_n - \xi_{n+1}$$

$$= \frac{1}{P'(z_1)}\left[\{2y_{n+1}\left[X_2'(z_1) + (\xi_n+\xi_{n+1})X_2(z_1)\right] + (\xi_n+\xi_{n+1})X_1(z_1) + X_1'(z_1)\}^2 - P''(z_1)/2\right] - \xi_n - \xi_{n+1}$$

which depends only on y_{n+1} as $\xi_n+\xi_{n+1} = \dfrac{1}{x_n-z_1} + \dfrac{1}{x_{n+1}-z_1} = \dfrac{-Y_1(y_{n+1})/Y_2(y_{n+1}) - 2z_1}{[Y_0(y_{n+1}) + z_1Y_1(y_{n+1}) + z_1^2Y_2(y_{n+1})]/Y_2(y_{n+1})} = -\dfrac{\partial F/\partial x}{F}$ at (z_1, y_{n+1}), so,

$$\xi_{\text{step}} = \frac{1}{P'(z_1)}\left[\left\{\underbrace{2y_{n+1}\left[X_2'(z_1) - \frac{\partial F/\partial x}{F}X_2(z_1)\right] - \frac{\partial F/\partial x}{F}X_1(z_1) + X_1'(z_1)}_{F\partial^2\log F/\partial x\partial y}\right\}^2 - P''(z_1)/2\right] + \frac{\partial F/\partial x}{F}$$

with $F(x,y) = X_2(x)(y-\varphi(x))(y-\psi(x))$, where φ and $\psi = (-X_1 \pm \sqrt{P})/(2X_2)$, $\partial^2\log F/\partial x\partial y = \dfrac{\partial}{\partial x}\left[\dfrac{1}{y-\varphi} + \dfrac{1}{y-\psi}\right] = \dfrac{\varphi'}{(y-\varphi)^2} + \dfrac{\psi'}{(y-\psi)^2} = \dfrac{X_2^2}{F^2}[(\varphi'+\psi')[(y-(\varphi+\psi)/2)^2 + (\psi-\varphi)^2/4] + 2(y-(\varphi+\psi)/2)(\psi-\varphi)(\psi'-\varphi')/2] = \dfrac{X_2^2}{F^2}\left[-\left(\dfrac{X_1}{X_2}\right)'\left[\left(y+\dfrac{X_1}{2X_2}\right)^2 + \dfrac{P}{4X_2^2}\right] + \left(y+\dfrac{X_1}{2X_2}\right)\left(\dfrac{P}{2X_2^2}\right)'\right]$.

At $x=z_1$, $P=0$, $\varphi=\psi=y(z_1)=-X_1/(2X_2)$, $F(z_1,y) = X_2(y-y(z_1))^2$, $P'(z_1) = 2X_1X_1' - 4X_0X_2' - 4X_0'X_2 = -4X_2(X_1'y(z_1) + X_2'y^2(z_1) + X_0') = -4X_2\partial F/\partial x$ at $(z_1, y(z_1))$, $P'/X_2^2 = (P/X_2^2)' = (X_1^2/X_2^2)' - 4(X_0/X_2)'$, $P''(z_1)/2 = X_1X_1'' + (X_1')^2 - 2X_0X_2'' - 4X_0'X_2' - 2X_0''X_2 = -2X_2\partial^2F/\partial x^2 + (X_1')^2 - 4X_0'X_2'$, and

$$\xi_{\text{step}} = \frac{1}{P'(z_1)}\left[\left\{\frac{X_2}{y-y(z_1)}\left[-\left(\frac{X_1}{X_2}\right)'(y-y(z_1)) + \frac{P'}{2X_2^2}\right]\right\}^2 - P''(z_1)/2\right] + \frac{X_2'}{X_2} + \frac{(X_1/X_2)'y + (X_0/X_2)'}{(y-y(z_1))^2}$$

$$= \frac{1}{P'(z_1)}\left[\left\{-X_2\left(\frac{X_1}{X_2}\right)' + X_2\frac{P'/(2X_2^2)}{y-y(z_1)}\right\}^2 - P''(z_1)/2\right] + \frac{X_2'}{X_2} + \frac{(X_1/X_2)'}{y-y(z_1)} - \frac{P'/(4X_2^2)}{(y-y(z_1))^2}$$

$$= \frac{1}{P'(z_1)}\left[\left\{-X_2\left(\frac{X_1}{X_2}\right)'\right\}^2 - P''(z_1)/2\right] + \frac{X_2'}{X_2}$$

$$= -\frac{2X_2\partial^2F/\partial x^2 - (X_1')^2 + 4X_0'X_2' + (X_1')^2 - 2\dfrac{X_1X_1'X_2'}{X_2} + \dfrac{X_1^2(X_2')^2}{X_2^2} + \dfrac{X_2'}{X_2}\left[2X_1X_1' - 4X_0X_2' - 4X_0'X_2\right]}{4X_2\partial F/\partial x}$$

$$= -\frac{\partial^2F/\partial x^2}{2\partial F/\partial x} \text{ at } (x,y) = (z_1, y(z_1)).$$

$$\eta_{\rm step} = \eta_n - \left(\frac{\eta_{n+1}+\eta_n}{\xi_{n+1}-\xi_n}\right)(\xi_{\rm step}-\xi_n) = -\eta_{n+1} - \left(\frac{\eta_{n+1}+\eta_n}{\xi_{n+1}-\xi_n}\right)(\xi_{\rm step}-\xi_{n+1})$$
$$= \frac{\eta_n-\eta_{n+1}}{2} - \left(\frac{\eta_{n+1}+\eta_n}{\xi_{n+1}-\xi_n}\right)\left(\xi_{\rm step} - \frac{\xi_n+\xi_{n+1}}{2}\right)$$
$$= \frac{1}{2}\left[\eta_n - \xi_{n+1}^2 X_1(z_1+1/\xi_{n+1}) - 2y_{n+1}\xi_{n+1}^2 X_2(z_1+1/\xi_{n+1})\right] - \left(\frac{\eta_{n+1}+\eta_n}{\xi_{n+1}-\xi_n}\right)\left(\xi_{\rm step} - \frac{\xi_n+\xi_{n+1}}{2}\right)$$
$$= -\xi_n^2 y_{n+1} X_2(z_1+1/\xi_n) - \xi_n^2 X_1(z_1+1/\xi_n)/2 - \xi_{n+1}^2 X_1(z_1+1/\xi_{n+1})/2 - y_{n+1}\xi_{n+1}^2 X_2(z_1+1/\xi_{n+1})$$
$$- \{2y_{n+1}\left[X_2'(z_1)+(\xi_n+\xi_{n+1})X_2(z_1)\right] + (\xi_n+\xi_{n+1})X_1(z_1) + X_1'(z_1)\}\left(\xi_{\rm step} - \frac{\xi_n+\xi_{n+1}}{2}\right)$$
$$= -y_{n+1}[(\xi_n^2+\xi_{n+1}^2)X_2(z_1) + (\xi_n+\xi_{n+1})X_2'(z_1) + X_2''] - (\xi_n^2+\xi_{n+1}^2)X_1(z_1)/2$$
$$- (\xi_n+\xi_{n+1})X_1'(z_1)/2 - X_1''/2 - \{2y_{n+1}\left[X_2'(z_1)+(\xi_n+\xi_{n+1})X_2(z_1)\right] + (\xi_n+\xi_{n+1})X_1(z_1) + X_1'(z_1)\}\left(\xi_{\rm step} - \frac{\xi_n+\xi_{n+1}}{2}\right)$$
$$= (\xi_n+\xi_{n+1})^2[\underbrace{-y_{n+1}X_2 - X_1/2 - (2y_{n+1}X_2+X_1)(-1/2)}_{=0}] + \xi_n\xi_{n+1}[2y_{n+1}X_2+X_1] + (\xi_n+\xi_{n+1})\{-y_{n+1}X_2' - X_1'/2 - (2y_{n+1}X_2+X_1)\xi_{\rm step} + y_{n+1}X_2' + X_1'/2\} - y_{n+1}X_2'' - X_1''/2 - (2y_{n+1}X_2'+X_1')\xi_{\rm step}$$
$$= [2y_{n+1}X_2+X_1][\xi_n\xi_{n+1} - (\xi_n+\xi_{n+1})\xi_{\rm step}] - y_{n+1}X_2'' - X_1''/2 - (2y_{n+1}X_2'+X_1')\xi_{\rm step}$$
$$= [2y_{n+1}X_2+X_1]\frac{Y_2(y_{n+1}) + (Y_1(y_{n+1})+2z_1Y_2(y_{n+1}))\xi_{\rm step}}{F(z_1,y_{n+1}) = X_2(y_{n+1}-y(z_1))^2} - y_{n+1}X_2'' - X_1''/2 - (2y_{n+1}X_2'+X_1')\xi_{\rm step}$$
$$= [\overbrace{2y_{n+1}X_2+X_1}^{2X_2(z_1)\,[y_{n+1}-y(z_1)]}]\frac{\dfrac{Y_2(y_{n+1})Y_1(y(z_1)) - Y_1(y_{n+1})Y_2(y(z_1))}{2z_1Y_2((z_1)) + Y_1(y(z_1)) = -Y_2(y(z_1))/\xi_{\rm step}}}{F(z_1,y_{n+1}) = X_2(y_{n+1}-y(z_1))^2} - y_{n+1}X_2'' - X_1''/2 - (2y_{n+1}X_2'+X_1')\xi_{\rm step}$$
$$= \frac{2}{y_{n+1}-y(z_1)}\left[Y_1(y_{n+1}) - \frac{Y_1(y(z_1))}{Y_2(y(z_1))}Y_2(y_{n+1})\right]\xi_{\rm step} - y_{n+1}X_2'' - X_1''/2 - (2y_{n+1}X_2'+X_1')\xi_{\rm step}$$
.

Now, for any x and y, $\partial F/\partial x = 2Y_2(y)x + Y_1(y) = X_2'(x)y^2 + X_1'(x)y + X_0'$. Of special interest is $F(x, y(z_1))$ which is a quadratic polynomial in x, vanishing at $x = z_1$ and $x = z_1'$ (see p.), so that $\partial F/\partial x$ vanishes at $(x,y) = ((z_1+z_1')/2, y(z_1))$, so

$$\xi_{\rm step} = -\frac{\partial^2 F/\partial x^2}{2\partial F/\partial x}(z_1, y(z_1)) = -\frac{Y_2(y(z_1))}{2z_1Y_2(y(z_1)) + Y_1(y(z_1))} = \frac{1}{z_1'-z_1} \qquad (22a)$$

Remark that $x_{\rm step} = z_1 + \dfrac{1}{\xi_{\rm step}} = z_1'$. Return to $\eta_{\rm step}$:

$$\eta_{\rm step} = \frac{2\left[Y_1(y_{n+1}) + (z_1+z_1')Y_2(y_{n+1})\right]\xi_{\rm step}}{y_{n+1}-y(z_1)} - y_{n+1}X_2'' - X_1''/2 - (2y_{n+1}X_2'+X_1')\xi_{\rm step}$$
$$= \frac{2[X_2'((z_1+z_1')/2)y_{n+1}^2 + X_1'((z_1+z_1')/2)y_{n+1} + X_0'((z_1+z_1')/2)]\xi_{\rm step}}{y_{n+1}-y(z_1)} - 2\xi_{\rm step}[y_{n+1}X_2'((z_1+z_1')/2) + X_1'((z_1+z_1')/2)/2]$$
$$= 2\xi_{\rm step}[X_1'((z_1+z_1')/2)/2 + X_2'((z_1+z_1')/2)y(z_1)]$$
$$= \xi_{\rm step}\frac{X_1(z_1') - X_1(z_1) - [X_2(z_1') - X_2(z_1)]\dfrac{X_1(z_1)}{X_2(z_1)}}{z_1'-z_1}$$

$$\eta_{\rm step} = \xi_{\rm step}^2[X_1(z_1') + 2y(z_1)X_2(z_1')] \qquad (22b)$$

leading to $y_{\text{step}} = \dfrac{-X_1(z_1') + \eta_{\text{step}}/\xi_{\text{step}}^2}{2X_2(z_1')} = y(z_1)$.

From cubic to biquadratic, following Appell & Goursat [4, p.293]: take (ξ_0, η_0) on the cubic $\eta^2 = P_3(\xi)$ and intersect with the line $\eta - \eta_0 = t(\xi - \xi_0)$. For a given t, the equation for ξ is $[\eta_0 + t(\xi - \xi_0)]^2 - P_3(\xi) = 0$ solved by $\xi = \xi_0$ and the two roots of $2\eta_0 t + t^2(\xi - \xi_0) - \dfrac{P_3(\xi) - P_3(\xi_0)}{\xi - \xi_0} = 0$:

$$-(P_3'''/6)(\xi - \xi_0)^2 + (t^2 - P_3''(\xi_0)/2)(\xi - \xi_0) + 2\eta_0 t - P_3'(\xi_0) = 0,$$

$$\xi = \xi_0 + \frac{t^2 - P_3''(\xi_0)/2 \pm \sqrt{\mathcal{Q}(t) := (t^2 - P_3''(\xi_0)/2)^2 + 2P_3'''(2\eta_0 t - P_3'(\xi_0))/3}}{P_3'''/6}$$

The four zeros of the quartic $\mathcal{Q}$ are the slopes of lines leading to a double intersection, i.e., of the four tangents to the cubic issued from the point (ξ_0, η_0). We show that the cross ratio of these four tangents is independent of the point (ξ_0, η_0) of the cubic curve: let (ξ_1, η_1) be another point on the cubic curve, then $\mathcal{Q}_1(t) = (t^2 - P_3''(\xi_1)/2)^2 + 2P_3'''(2\eta_1 t - P_3'(\xi_1))/3$ An interesting point is $\xi \to \infty$, then, $P_3''(\xi) \sim P_3'''\xi$, $P_3'(\xi) \sim P_3'''\xi^2/2$, $\eta \sim (P_3'''\xi^3/6)^{1/2}$, and $\mathcal{Q}_\infty(t\sqrt{6P_3'''\xi}) \sim (P_3''')^2\xi^2[(6t^2 - 1/2)^2 + 2(2t - 1/2)/3]$.

3. **Proofs of equivalence.**

3.1. **1. and 2.**

3.1.1. *From 1. to 2.* : as seen in (12) in § 2.2, p. 9, the elimination of y_n from (9) yields a relation of degree 2 in $x_n + x_{n-1}$ and $x_n x_{n-1}$, which is (15).

But one can also find (16) directly from (6) and (9). Indeed, the sum of $x_n + x_{n-1}$ and $x_{n+1} + x_n$ from (9) yields $x_{n-1} + 2x_n + x_{n+1}$ as $-Y_1(y_n)/Y_2(y_n) - Y_1(y_{n+1})/Y_2(y_{n+1})$, which is a symmetric rational function of y_n and y_{n+1}, therefore, from (6), a rational function of x_n, allowing to recover (16), although the final rational function seems liable to be of fourth degree, but wait.

A expansion in simple fractions leads to $x_n + x_{n-1} = -\dfrac{Y_1(y_n)}{Y_2(y_n)} = -\dfrac{c_{1,2}}{c_{2,2}} - \dfrac{Y_1(u)/(u-v)}{c_{2,2}(y_n - u)} - \dfrac{Y_1(v)/(v-u)}{c_{2,2}(y_n - v)}$, where u and v are the two roots of $Y_2(y) = 0$ (ordinates of the horizontal asymptotes of the curve $F(x, y) = 0$). Then,

$$\begin{aligned}
&x_{n-1} + 2x_n + x_{n+1} \\
&= -2\frac{c_{1,2}}{c_{2,2}} - \frac{Y_1(u)}{c_{2,2}(u-v)} \frac{y_n + y_{n+1} - 2u}{y_n y_{n+1} - (y_n + y_{n+1})u + u^2} - \frac{Y_1(v)}{c_{2,2}(v-u)} \frac{y_n + y_{n+1} - 2v}{y_n y_{n+1} - (y_n + y_{n+1})v + v^2} \\
&\qquad = -2\frac{c_{1,2}}{c_{2,2}} + \frac{Y_1(u)}{c_{2,2}(u-v)} \frac{X_1(x_n) + 2uX_2(x_n)}{X_0(x_n) + X_1(x_n)u + X_2(x_n)u^2 = F(x_n, u)} \\
&\qquad\qquad + \frac{Y_1(v)}{c_{2,2}(v-u)} \frac{X_1(x_n) + 2vX_2(x_n)}{X_0(x_n) + X_1(x_n)v + X_2(x_n)v^2 = F(x_n, v)}
\end{aligned}$$

of small degree, as $F(x_n,u) = Y_0(u) + Y_1(u)x_n$ and $F(x_n,v) = Y_0(v) + Y_1(v)x_n$. The numerators are $\partial F(x_n,y)/\partial y = Y_0'(y) + Y_1'(y)x_n + Y_2'(y)x_n^2$ at $y = u$ and $y = v$, so

$$x_{n-1} + x_{n+1} = -2\frac{c_{1,2}}{c_{2,2}} + \frac{1}{c_{2,2}(u-v)}\left[\frac{Y_0'(u) + Y_1'(u)x_n + Y_2'(u)x_n^2}{x_n + Y_0(u)/Y_1(u)} - \frac{Y_0'(v) + Y_1'(v)x_n + Y_2'(v)x_n^2}{x_n + Y_0(v)/Y_1(v)}\right] - 2x_n$$

which must reduce to

$$x_{n-1} + x_{n+1} = -\frac{d_{0,1} + (2d_{0,2} + d_{1,1})x_n + d_{1,2}x_n^2}{d_{2,2}[x_n + Y_0(u)/Y_1(u)][x_n + Y_0(v)/Y_1(v)]} \tag{23a}$$

Remark that, as $Y_2'(y) = 2c_{2,2}y + c_{2,1}$, the $-2x_n$ term cancels with the contribution of $(Y_2'(u) - Y_2'(v))/(c_{2,2}(u-v))$.

The equation (23a) is a ***recurrence relation*** joining x_{n-1}, x_n, and x_{n+1}. Remark that ρ_1 and ρ_2, the two roots of $d_{2,2}\rho^2 + d_{1,2}\rho + d_{0,2} = 0$ must be $-Y_0(u)/Y_1(u)$ and $-Y_0(v)/Y_1(v)$.

Also, $x_n x_{n-1} = \dfrac{Y_0(y_n)}{Y_2(y_n)} = \dfrac{c_{0,2}}{c_{2,2}} + \dfrac{Y_0(u)/(u-v)}{c_{2,2}(y_n-u)} + \dfrac{Y_0(v)/(v-u)}{c_{2,2}(y_n-v)}$,

$$\begin{aligned}
&(u-v)^2[c_{2,2}x_nx_{n-1} - c_{0,2}][c_{2,2}x_nx_{n+1} - c_{0,2}]\\
&= \frac{Y_0^2(u)}{(y_n-u)(y_{n+1}-u)} - \frac{Y_0(u)Y_0(v)}{(y_n-u)(y_{n+1}-v)} - \frac{Y_0(u)Y_0(v)}{(y_{n+1}-u)(y_n-v)} + \frac{Y_0^2(v)}{(y_n-v)(y_{n+1}-v)}\\
&= \frac{Y_0^2(u)(y_n-v)(y_{n+1}-v) - Y_0(u)Y_0(v)[(y_{n+1}-u)(y_n-v) + (y_n-u)(y_{n+1}-v)] + Y_0^2(v)(y_n-u)(y_{n+1}-u)}{(y_n-u)(y_{n+1}-u)(y_n-v)(y_{n+1}-v)}\\
&= X_2(x_n)\frac{Y_0^2(u)F(x_n,v) - 2Y_0(u)Y_0(v)[X_0(x_n) + (u+v)X_1(x_n) + uvX_2(x_n)] + Y_0^2(v)F(x_n,u)}{F(x_n,u)F(x_n,v) = [Y_0(u) + Y_1(u)x_n][Y_0(v) + Y_1(v)x_n]}
\end{aligned}$$

$$\begin{aligned}
&x_{n-1}x_{n+1}\\
&= \frac{X_2(x_n)}{(u-v)^2c_{2,2}^2x_n^2}\,\frac{Y_0^2(u)F(x_n,v) - 2Y_0(u)Y_0(v)[X_0(x_n) + (u+v)X_1(x_n) + uvX_2(x_n)] + Y_0^2(v)F(x_n,u)}{[Y_0(u) + Y_1(u)x_n][Y_0(v) + Y_1(v)x_n]}\\
&\qquad + \frac{c_{0,2}}{c_{2,2}x_n}\,\frac{\sigma_0 + \sigma_1x_n + \sigma_2x_n^2}{[x_n + Y_0(u)/Y_1(u)][x_n + Y_0(v)/Y_1(v)]} - \frac{c_{0,2}^2}{c_{2,2}^2x_n^2}
\end{aligned}$$

which must also reduce to

$$x_{n-1}x_{n+1} = \frac{d_{0,0} + d_{0,1}x_n + d_{0,2}x_n^2}{d_{2,2}[x_n + Y_0(u)/Y_1(u)][x_n + Y_0(v)/Y_1(v)]}, \tag{23b}$$

but it does not seem useful to follow this track any longer.

3.1.2. *From 2. to 1.* : From a sequence satisfying (15), let us construct a valid sequence $\{y_n\}$. We must construct $F(x,y) = \alpha(y-u)(y-v)x^2 + [\beta(y-u)(y-v) + \beta'y + \beta'']x + \gamma(y-u)(y-v) + \gamma'y + \gamma''$ such that the resultant above is exactly (15).

With $S = x_n + x_{n+1}$ and $\Pi = x_nx_{n+1}$, (15) is

$$d_{0,0} + d_{0,1}S + d_{0,2}S^2 + d_{1,1}\Pi + d_{1,2}\Pi S + d_{2,2}\Pi^2 = 0. \tag{24}$$

We see that (12) holds if $(S,\Pi)=(-\beta/\alpha,\gamma/\alpha)$, so, one chooses $(-\beta/\alpha,\gamma/\alpha)=(S,\Pi)$ as a point[3] on the conic (24). We now have

$$\alpha^2E = d_{0,2}(\alpha S+\beta-\beta)^2+d_{1,2}(\alpha\Pi-\gamma+\gamma)(\alpha S+\beta-\beta)+d_{2,2}(\alpha\Pi-\gamma+\gamma)^2+d_{0,1}\alpha(\alpha S+\beta-\beta)$$
$$+d_{1,1}\alpha(\alpha\Pi-\gamma+\gamma)+d_{0,0}\alpha^2 = d_{0,2}(\alpha S+\beta)^2-2d_{0,2}\beta(\alpha S+\beta)+d_{0,2}\beta^2$$
$$+d_{1,2}(\alpha\Pi-\gamma)(\alpha S+\beta)-d_{1,2}\beta(\alpha\Pi-\gamma)+d_{1,2}\gamma(\alpha S+\beta)-d_{1,2}\beta\gamma+d_{2,2}(\alpha\Pi-\gamma)^2$$
$$+2d_{2,2}\gamma(\alpha\Pi-\gamma)+d_{2,2}\gamma^2+d_{0,1}\alpha(\alpha S+\beta)-d_{0,1}\alpha\beta+d_{1,1}\alpha(\alpha\Pi-\gamma)+d_{1,1}\alpha\gamma+\alpha^2d_{0,0}$$
$$=d_{0,2}(\alpha S+\beta)^2+d_{1,2}(\alpha\Pi-\gamma)(\alpha S+\beta)+d_{2,2}(\alpha\Pi-\gamma)^2+d'_{0,1}(\alpha S+\beta)+d'_{1,1}(\alpha\Pi-\gamma)=0,$$
$$\text{if } \; d_{0,2}\beta^2-d_{1,2}\beta\gamma+d_{2,2}\gamma^2-d_{0,1}\alpha\beta+d_{1,1}\alpha\gamma+\alpha^2d_{0,0}=0, \tag{25}$$

with $d'_{0,1}=d_{0,1}\alpha-2d_{0,2}\beta+d_{1,2}\gamma$ and $d'_{1,1}=d_{1,1}\alpha-d_{1,2}\beta+2d_{2,2}\gamma$. Let ρ_1 and ρ_2 be the two roots of $d_{2,2}\rho^2+d_{1,2}\rho+d_{0,2}=0$, then, $d_{2,2}[\alpha\Pi-\gamma-\rho_1(\alpha S+\beta)][\alpha\Pi-\gamma-\rho_2(\alpha S+\beta)]+d'_{0,1}(\alpha S+\beta)+d'_{1,1}(\alpha\Pi-\gamma)=0$, and we multiply by some constant C and compare with (12):

$$\beta'u+\beta''=C_1, \qquad \gamma'u+\gamma''=-\rho_1C_1,$$
$$\beta'v+\beta''=C_2, \qquad \gamma'v+\gamma''=-\rho_2C_2,$$
$$(\beta''\gamma'-\beta'\gamma'')\beta'=Cd'_{1,1}, \qquad (\beta''\gamma'-\beta'\gamma'')\gamma'=Cd'_{0,1},$$

where $C_1C_2=Cd_{2,2}$,

so, $\dfrac{\gamma'u+\gamma''}{\beta'u+\beta''}$ and $\dfrac{\gamma'v+\gamma''}{\beta'v+\beta''}$ are $-\rho_1$ and $-\rho_2$. We subtract $\gamma'/\beta'=d'_{0,1}/d'_{1,1}$:

$$\beta'u+\beta''=\frac{\beta''\gamma'-\beta'\gamma''}{\beta'(\rho_1+d'_{0,1}/d'_{1,1})}, \quad \beta'v+\beta''=\frac{\beta''\gamma'-\beta'\gamma''}{\beta'(\rho_2+d'_{0,1}/d'_{1,1})}, \quad \frac{\beta'u+\beta''}{\beta'v+\beta''}=\frac{d'_{1,1}\rho_2+d'_{0,1}}{d'_{1,1}\rho_1+d'_{0,1}}$$

$$\beta''=\beta'\frac{d'_{1,1}(u\rho_1-v\rho_2)+d'_{0,1}(u-v)}{d'_{1,1}(\rho_2-\rho_1)}$$

$$\gamma''=\gamma'\frac{d'_{1,1}(v-u)\rho_1\rho_2+d'_{0,1}(v\rho_1-u\rho_2)}{d'_{0,1}(\rho_2-\rho_1)}$$

$$\beta''\gamma'-\beta'\gamma''=\frac{\beta'\gamma'(u-v)(d'_{1,1}\rho_1+d'_{0,1})(d'_{1,1}\rho_2+d'_{0,1})}{d'_{1,1}d'_{0,1}(\rho_2-\rho_1)}$$

$$\beta'u+\beta''=\frac{\gamma'(u-v)(d'_{1,1}\rho_2+d'_{0,1})}{d'_{0,1}(\rho_2-\rho_1)}, \quad \beta'v+\beta''=\frac{\gamma'(u-v)(d'_{1,1}\rho_1+d'_{0,1})}{d'_{0,1}(\rho_2-\rho_1)}$$

$$C=(\beta'u+\beta'')(\beta'v+\beta'')/d_{2,2}=\frac{1}{d_{2,2}}\frac{\gamma'(u-v)(d'_{1,1}\rho_2+d'_{0,1})}{d'_{0,1}(\rho_2-\rho_1)}\frac{\gamma'(u-v)(d'_{1,1}\rho_1+d'_{0,1})}{d'_{0,1}(\rho_2-\rho_1)}$$

or $\dfrac{\beta'}{d'_{1,1}}=\dfrac{\gamma'}{d'_{0,1}}=\dfrac{C}{\beta''\gamma'-\beta'\gamma''}=\dfrac{u-v}{d_{2,2}(\rho_2-\rho_1)}$

The degrees of freedom are therefore u, v, and $(\beta,\gamma)/\alpha$ on the conic $R=0$ of (24).

[3]The importance of this conic has been stressed by A.Ronveaux [52].

$$
\begin{aligned}
F(x,y) &= \alpha(y-u)(y-v)x^2 + [\beta(y-u)(y-v)+\beta' y+\beta'']x + \gamma(y-u)(y-v)+\gamma' y+\gamma'' \\
&= (y-u)(y-v)(\alpha x^2+\beta x+\gamma) + \frac{u-v}{(\rho_2-\rho_1)d_{2,2}}\left[d'_{1,1}x\left(y+\frac{\beta''}{\beta'}\right)+d'_{0,1}\left(y+\frac{\gamma''}{\gamma'}\right)\right] \\
= (y-u)(y-v)(\alpha x^2+\beta x+\gamma) &+ \frac{u-v}{(\rho_2-\rho_1)d_{2,2}}\left[d'_{1,1}xy + \frac{d'_{1,1}(u\rho_1-v\rho_2)+d'_{0,1}(u-v)}{\rho_2-\rho_1}x\right. \\
&\left. + d'_{0,1}y + \frac{d'_{1,1}(v-u)\rho_1\rho_2+d'_{0,1}(v\rho_1-u\rho_2)}{\rho_2-\rho_1}\right] \\
= (y-u)(y-v)(\alpha x^2+\beta x+\gamma) &+ \frac{u-v}{(\rho_2-\rho_1)^2 d_{2,2}}[d'_{1,1}[\rho_2(y-v)-\rho_1(y-u)]x + d'_{0,1}(u-v)x \\
&+ d'_{0,1}[\rho_2(y-u)-\rho_1(y-v)] + d'_{1,1}\rho_1\rho_2(v-u)]
\end{aligned} \tag{26}
$$

Where is is recalled that $d'_{0,1} = d_{0,1}\alpha - 2d_{0,2}\beta + d_{1,2}\gamma$ and $d'_{1,1} = d_{1,1}\alpha - d_{1,2}\beta + 2d_{2,2}\gamma$.

The two numbers u and v are the values of y making x infinite, we write (26) (divided by $y-v$) as a function of $Y = \dfrac{y-u}{y-v}$, using $y = \dfrac{vY-u}{Y-1}$ and $v-u = y-u-(y-v)$:

$$
\frac{Y}{Y-1}(\alpha x^2+\beta x+\gamma) - \frac{1}{(\rho_2-\rho_1)^2 d_{2,2}}[d'_{1,1}[\rho_2-\rho_1 Y]x + d'_{0,1}(1-Y)x + d'_{0,1}[\rho_2 Y-\rho_1] + d'_{1,1}\rho_1\rho_2(Y-1)] = 0
$$

Now, u and v have disappeared, they are completely hidden in Y, very satisfactory.
$(d'_{1,1}\rho_1+d'_{0,1})(x-\rho_2)Y^2 + [(\rho_2-\rho_1)^2 d_{2,2}(\alpha x^2+\beta x+\gamma) - (d'_{1,1}\rho_2+d'_{0,1})(x-\rho_1) - (d'_{1,1}\rho_1+d'_{0,1})(x-\rho_2)]Y + (d'_{1,1}\rho_2+d'_{0,1})(x-\rho_1) = 0$

$$Y = \frac{-(\rho_2-\rho_1)^2 d_{2,2}(\alpha x^2+\beta x+\gamma) + (d'_{1,1}\rho_2+d'_{0,1})(x-\rho_1) + (d'_{1,1}\rho_1+d'_{0,1})(x-\rho_2) \pm \sqrt{P(x)}}{2(d'_{1,1}\rho_1+d'_{0,1})(x-\rho_2)}$$

with $P(x) = [(\rho_2-\rho_1)^2 d_{2,2}(\alpha x^2+\beta x+\gamma) - (d'_{1,1}\rho_2+d'_{0,1})(x-\rho_1) - (d'_{1,1}\rho_1+d'_{0,1})(x-\rho_2)]^2 - (d'_{1,1}\rho_1+d'_{0,1})(d'_{1,1}\rho_2+d'_{0,1})(x-\rho_1)(x-\rho_2)$

With the example of p.11, $\alpha^2 E(x,y) = \sin^2\theta\, x^2y^2 - x^2 - y^2 + 2\cos\theta\, xy = \sin^2\theta\Pi^2 - S^2 + 2(1+\cos\theta)\Pi = [\sin\theta\, xy + x + y][\sin\theta\, xy - (x+y)] + 2(1+\cos\theta)xy : \alpha = \sin\theta, d_{2,2} = 1, d_{0,2} = -1/\sin^2\theta, d_{1,1} = 2(1+\cos\theta)/\sin^2\theta, \rho_1,\rho_2 = \mp 1/\sin\theta, d'_{1,1} = 2(1+\cos\theta+\gamma\sin\theta)/\sin\theta, d'_{0,1} = 2\beta/\sin^2\theta$.

Then, $F(x,y) = (y-u)(y-v)(\sin\theta\, x^2+\beta x+\gamma) + ((u-v)\sin\theta)/2)\{[d'_{1,1}x + d'_{0,1}](y-(u+v)/2) + (u-v)[d'_{0,1}x\sin\theta + d'_{1,1}/\sin\theta]/2\}$, with $\alpha^2\gamma^2 - \beta^2 + 2(1+\cos\theta)\alpha\gamma = (1+\cos\theta+\gamma\sin\theta)^2 - (1+\cos\theta)^2 - \beta^2 = 0$.

Also,

$$\begin{aligned}
&\frac{2(Y-1)F}{(u-v)(y-v)}\\
&= -2Y(\sin\theta x^2+\beta x+\gamma)+(Y-1)\sin\theta\{[d'_{1,1}x+d'_{0,1}](Y+1)+[d'_{0,1}x\sin\theta+d'_{1,1}/\sin\theta](1-Y)\}/2\\
&= -2Y(\sin\theta x^2+\beta x+\gamma)+\{[d'_{1,1}\sin\theta\, x+d'_{0,1}\sin\theta](Y^2-1)-[d'_{0,1}x\sin^2\theta+d'_{1,1}](Y-1)^2\}/2\\
&= [d'_{1,1}\sin\theta-d'_{0,1}\sin^2\theta](x-1/\sin\theta)Y^2/2-[2\sin\theta\, x^2+(2\beta-d'_{0,1}\sin^2\theta)x+2\gamma-d'_{1,1}]Y\\
&\qquad -[d'_{1,1}\sin\theta+d'_{0,1}\sin^2\theta](x+1/\sin\theta)/2\\
&= [1+\cos\theta-\beta+\gamma\sin\theta](x-1/\sin\theta)Y^2-2[\sin\theta\, x^2-(1+\cos\theta)/\sin\theta]Y\\
&\qquad -[1+\cos\theta+\beta+\gamma\sin\theta](x+1/\sin\theta)
\end{aligned}$$

$$Y=\frac{\sin\theta\, x^2-(1+\cos\theta)/\sin\theta\pm x\sin\theta\sqrt{x^2-1}}{[1+\cos\theta-\beta+\gamma\sin\theta](x-1/\sin\theta)}=\frac{(1+\cos\theta)[-x\cos\theta\pm\sqrt{x^2-1}][x\pm\sqrt{x^2-1}]}{\sin\theta[1+\cos\theta-\beta+\gamma\sin\theta](x-1/\sin\theta)}$$

$=\dfrac{(1+\cos\theta)[(1-\cos\theta)R^2-1-\cos\theta]}{\sin\theta[1+\cos\theta-\beta+\gamma\sin\theta](R+1/R-2/\sin\theta)}$ where $R=x\pm\sqrt{x^2-1}$, so that $x=(R+1/R)/2$. Remark that $\dfrac{2}{\sin\theta}=\dfrac{1+\cos\theta}{\sin\theta}+\dfrac{1-\cos\theta}{\sin\theta}=\dfrac{1+\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}$ and $\dfrac{1+\cos\theta}{1-\cos\theta}=\left(\dfrac{1+\cos\theta}{\sin\theta}\right)^2=\cot^2(\theta/2)$, $Y=\dfrac{\sin\theta\,(R+\cot(\theta/2))}{[1+\cos\theta-\beta+\gamma\sin\theta](1-R^{-1}\tan(\theta/2))}$.

If $x=1/\sin(n\theta)$, $R=(1\pm\cos(n\theta))/\sin(n\theta)$, $R^{-1}=(1\mp\cos(n\theta))/\sin(n\theta)$,

$$Y=\frac{\sin\theta\,(1\pm\cos(n\theta)+\cot(\theta/2)\sin(n\theta))}{[1+\cos\theta-\beta+\gamma\sin\theta](\sin(n\theta)-(1\mp\cos(n\theta))\tan(\theta/2))}=\frac{2\cos^2(\theta/2)\,(\sin(\theta/2)+\sin((n\pm1/2)\theta))}{[1+\cos\theta-\beta+\gamma\sin\theta](\sin((n\pm1/2)\theta)-\cos(\theta/2)}$$

Note that

$$\begin{aligned}
E(x,y)&=d_{0,0}+d_{0,1}(x+y)+d_{0,2}(x+y)^2+d_{1,1}xy+d_{1,2}xy(x+y)+d_{2,2}x^2y^2\\
&=(d_{0,2}+d_{1,2}x+d_{2,2}x^2)y^2+(d_{0,1}+(2d_{0,2}+d_{1,1})x+d_{1,2}x^2)y+d_{0,0}+d_{0,1}x+d_{0,2}x^2\\
&=d_{2,2}(x-\rho_1)(x-\rho_2)y^2+(d_{0,1}+(2d_{0,2}+d_{1,1})x+d_{1,2}x^2)y+d_{0,0}+d_{0,1}x+d_{0,2}x^2
\end{aligned}$$

$\alpha^2E(x,y)$ is

$d_{2,2}[\alpha xy-\gamma-\rho_1(\alpha(x+y)+\beta)][\alpha xy-\gamma-\rho_2(\alpha(x+y)+\beta)]+[\underbrace{-2d_{0,2}\beta+d_{1,2}\gamma+d_{0,1}\alpha}_{d'_{0,1}}](\alpha(x+y)+\beta)+[\underbrace{-d_{1,2}\beta+2d_{2,2}\gamma+d_{1,1}\alpha}_{d'_{1,1}}](\alpha xy-\gamma)$

3.2. **Half-integer indexes (new in 2011) .**

Spiridonov and Zhedanov have a very ingenious argument [59, p. 298] amounting to build a possible $y-$ sequence as half-integer entries of the $x-$sequence: $y_n=x_{n-1/2}$. They rely on elliptic functions identities, but here is an elementary derivation: can we find a quadratic algebraic function $r\pm\sqrt{s}$, with r and s rational functions, such that two iterations of this function to x_n gives x_{n+1}? The result must be a quadratic algebraic functions whose two determinations are x_{n-1} and x_{n+1}, as explained in (17). Similarly, the two choices of the sign of the square root in $r(x_n)\pm\sqrt{s(x_n)}$ must yield $x_{n-1/2}$ and $x_{n+1/2}$. A new application of $r\pm\sqrt{s}$ represents a new half-step either forward or backward, so that the four determinations $r(r(x_n)+\sigma_1\sqrt{s(x_n)})+\sigma_2\sqrt{s(r(x_n)+\sigma_1\sqrt{s(x_n)})}$ with independent signs σ_1 and σ_2, must be x_{n+1}, x_{n-1}, AND x_n twice.

$$\prod_{\sigma_1=\pm 1}\prod_{\sigma_2=\pm 1}(x-r(r(x_n)+\sigma_1\sqrt{s(x_n)})-\sigma_2\sqrt{s(r(x_n)+\sigma_1\sqrt{s(x_n)})} \equiv (x-x_{n-1})(x-x_{n+1})(x-x_n)^2$$

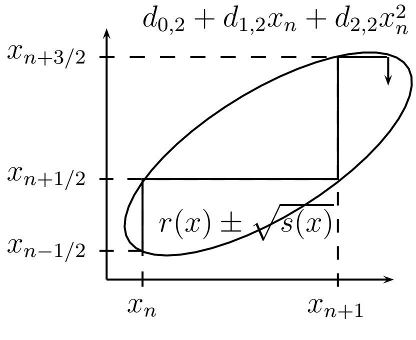
$$[[x-r(r(x_n)+\sqrt{s(x_n)})]^2-s(r(x_n)+\sqrt{s(x_n)})][[x-r(r(x_n)-\sqrt{s(x_n)})]^2-s(r(x_n)-\sqrt{s(x_n)})]$$
$$= x^4-2[r(r(x_n)+\sqrt{s(x_n)})+r(r(x_n)-\sqrt{s(x_n)})]x^3+[r^2(r(x_n)+\sqrt{s(x_n)})$$
$$+4r(r(x_n)+\sqrt{s(x_n)})r(r(x_n)-\sqrt{s(x_n)})+r^2(r(x_n)-\sqrt{s(x_n)})+s(r(x_n)+\sqrt{s(x_n)})+s(r(x_n)-\sqrt{s(x_n)})]x^2+\cdots$$
$$\equiv \left(x^2+\frac{d_{0,1}+(2d_{0,2}+d_{1,1})x_n+d_{1,2}x_n^2}{d_{0,2}+d_{1,2}x_n+d_{2,2}x_n^2}x+\frac{d_{0,0}+d_{0,1}x_n+d_{0,2}x_n^2}{d_{0,2}+d_{1,2}x_n+d_{2,2}x_n^2}\right)(x-x_n)^2 \text{ from (16)}$$
$$\equiv \frac{E(x,x_n)}{d_{0,2}+d_{1,2}x_n+d_{2,2}x_n^2}(x-x_n)^2. \quad ?$$

Example: $x_n=\cos(n\theta) \Rightarrow x_{n\pm 1/2}=\cos(\theta/2)\cos(n\theta)\mp\sin(\theta/2)\sin(n\theta)$: $r(x)=x\cos(\theta/2)$ and $s(x)=(1-x^2)\sin^2(\theta/2)$, $s(r(x_n)\pm\sqrt{s(x_n)})=[1-x_n^2\cos^2(\theta/2)\mp 2x_n\cos(\theta/2)\sin(\theta/2)\sqrt{1-x_n^2}-(1-x_n^2)\sin^2(\theta/2)]\sin^2(\theta/2)=[x_n\sin(\theta/2)-\cos(\theta/2)\sqrt{1-x_n^2}]^2\sin^2(\theta/2)$ etc. So, the rational function s must contain P and $s(r\pm\sqrt{s})$ must be the square of an algebraic expression containing $\sqrt{P}$.

From $x_{n+1}=x_n+x_{n+1}-x_n=$ sum of roots of $F(x,y_{n+1})=0$ minus $x_n=-\dfrac{Y_1(y_{n+1})}{Y_2(y_{n+1})}-x_n$

$$=-\frac{Y_1\left(\dfrac{-X_1(x_n)+\sqrt{P(x_n)}}{2X_2(x_n)}\right)}{Y_2\left(\dfrac{-X_1(x_n)+\sqrt{P(x_n)}}{2X_2(x_n)}\right)}-x_n \text{ which must be the same as}$$

$$\frac{-(d_{0,1}+(2d_{0,2}+d_{1,1})x_n+d_{1,2}x_n^2)+\text{ const. }\sqrt{P(x_n)}}{d_{0,2}+d_{1,2}x_n+d_{2,2}x_n^2=d_{2,2}(x_n-\rho_1)(x_n-\rho_2)} \text{ Seems to lead nowhere.}$$

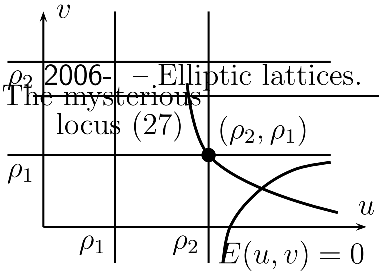


Try something else. A half step forward from x_n must be the same than a half step backward from x_{n+1}: $F(x_n,x_{n+1/2})=0$, as well as $F(x_{n+1},x_{n+1/2})=0$, holding for half-integer indexes as well, $F(x_{n+1/2},x_n)=F(x_{n+1/2},x_{n+1})=0$, amounting to F to be ***symmetric***, so, in (26), u and v are the two roots of $\alpha x^2+\beta x+\gamma=0$, whence $E(u,v)=E(v,u)=0$, as $-\beta/\alpha$ and γ/α are the sum S and the product Π of the coordinates of a point of the symmetric quartic (15) in the form (24).

The equality of the coefficients of x and y in (26) yields $\dfrac{d'_{1,1}(u\rho_1-v\rho_2)+d'_{0,1}(u-v)}{\rho_2-\rho_1}=d'_{0,1}$,

or $d'_{1,1}\dfrac{u+v}{2}+d'_{0,1}=(u-v)\dfrac{d'_{1,1}(\rho_2+\rho_1)/2+d'_{0,1}}{\rho_2-\rho_1}=(u-v)\dfrac{-d'_{1,1}d_{1,2}/(2d_{2,2})+d'_{0,1}}{\rho_2-\rho_1}$,

$$\frac{d'_{1,1}\dfrac{u+v}{2}+d'_{0,1}}{u-v}=\frac{d'_{1,1}\dfrac{\rho_2+\rho_1}{2}+d'_{0,1}}{\rho_2-\rho_1} \qquad (27)$$



Funny symmetry between the two pairs (u,v) and (ρ_1,ρ_2) ...but do we have $E(\rho_1,\rho_2)=0$??? Of course no, ρ_1 and ρ_2 are the abscissae and ordinates of the asymptotes of the curve $E=0$.

Now, (27) is

$$\frac{[-d_{1,2}\beta+2d_{2,2}\gamma+d_{1,1}\alpha]\dfrac{-\beta}{2\alpha}-2d_{0,2}\beta+d_{1,2}\gamma+d_{0,1}\alpha}{u-v}=\frac{[-d_{1,2}\beta+2d_{2,2}\gamma+d_{1,1}\alpha]\dfrac{-d_{1,2}}{2d_{2,2}}-2d_{0,2}\beta+d_{1,2}\gamma+d_{0,1}\alpha}{\rho_2-\rho_1}$$

$$\frac{[-d_{1,2}\beta+2d_{2,2}\gamma+d_{1,1}\alpha]\dfrac{-\beta}{2\alpha}-2d_{0,2}\beta+d_{1,2}\gamma+d_{0,1}\alpha}{u-v}=\frac{[-d_{1,2}\beta+d_{1,1}\alpha]\dfrac{-d_{1,2}}{2d_{2,2}}-2d_{0,2}\beta+d_{0,1}\alpha}{\rho_2-\rho_1}$$

as $d'_{0,1}=d_{0,1}\alpha-2d_{0,2}\beta+d_{1,2}\gamma$ and $d'_{1,1}=d_{1,1}\alpha-d_{1,2}\beta+2d_{2,2}\gamma$.

And as ρ_1 and ρ_2 be the two roots of $d_{2,2}\rho^2+d_{1,2}\rho+d_{0,2}=0$, remark that $(\rho_2-\rho_1)^2=(d_{1,2}^2-4d_{2,2}d_{0,2})/d_{2,2}^2$.

$$\frac{[-d_{1,2}\beta+2d_{2,2}\gamma+d_{1,1}\alpha]\dfrac{-\beta}{2\alpha}-2d_{0,2}\beta+d_{1,2}\gamma+d_{0,1}\alpha}{u-v}=\frac{\left[d_{0,1}-\dfrac{d_{1,1}d_{1,2}}{2d_{2,2}}\right]\alpha+(\rho_2-\rho_1)^2d_{2,2}\beta/2}{\rho_2-\rho_1}$$

With $u+v=-\beta/\alpha$ and $uv=\gamma/\alpha$,

$$\frac{[d_{1,2}(u+v)+2d_{2,2}uv+d_{1,1}](u+v)-4d_{0,2}(u+v)+2d_{1,2}uv+2d_{0,1}}{2(u-v)}=\frac{d_{0,1}-\dfrac{d_{1,1}d_{1,2}}{2d_{2,2}}-(\rho_2-\rho_1)^2d_{2,2}(u+v)/2}{\rho_2-\rho_1},$$

a second equation for (u,v), together with $E(u,v)=0$.

With the example above, $E(u,v)=-\sin^2\theta\ u^2v^2+u^2+v^2-2\cos\theta\ uv=0$, $d'_{0,1}=d_{0,1}+2d_{0,2}(u+v)+d_{1,2}uv=2(u+v)$, $d'_{1,1}=d_{1,1}+d_{1,2}(u+v)+2d_{2,2}uv=-2(1+\cos\theta)-2\sin^2\theta\ uv=-4\cos^2(\theta/2)[1+2\sin^2(\theta/2)\ uv]$,

$$F(x,y)=(y-u)(y-v)(x^2-(u+v)x+uv)+\frac{u-v}{(\rho_2-\rho_1)d_{2,2}=-2\sin\theta}$$
$$\left[d'_{1,1}xy+\frac{d'_{1,1}(u\rho_1-v\rho_2)+d'_{0,1}(u-v)}{\rho_2-\rho_1}x+d'_{0,1}y+\frac{d'_{1,1}(v-u)\rho_1\rho_2+d'_{0,1}(v\rho_1-u\rho_2)}{\rho_2-\rho_1}\right]$$
$$=(y-u)(y-v)(x-u)(x-v)+\frac{u-v}{-2\sin\theta}$$
$$\left[-2[1+\cos\theta+\sin^2\theta\ uv]xy+\frac{2[1+\cos\theta+\sin^2\theta\ uv](u+v)/\sin\theta+2(u+v)(u-v)}{2/\sin\theta}x+2(u+v)y\right.$$
$$\left.+\frac{2[1+\cos\theta+\sin^2\theta\ uv](v-u)/\sin^2\theta-2(u+v)^2/\sin\theta}{2/\sin\theta}\right]$$
$$=(y-u)(y-v)(x-u)(x-v)+\frac{u-v}{-2\sin\theta}$$
$$\left\{[1+\cos\theta+\sin^2\theta\ uv][-2xy+(u+v)x+(v-u)/\sin\theta]+(u+v)(u-v)x\sin\theta+2(u+v)y-(u+v)^2\right\}$$

and $[1+\cos\theta+\sin^2\theta\ uv](u+v)+(u+v)(u-v)\sin\theta=2(u+v)$ so,

1. either $u+v=0$, $u-v\neq 0$, and $E(u,-u)=-\sin^2\theta\ u^4+2(1+\cos\theta)u^2=4\cos^2(\theta/2)[-\sin^2(\theta/2)u^2+1]u^2=0: u=-v=1/\sin(\theta/2)$, $F(x,y)=(y^2-u^2)(x^2-u^2)$

$$+\frac{2u}{-2\sin\theta}\left\{-2[1+\cos\theta-\sin^2\theta\ u^2]xy+2[1+\cos\theta-\sin^2\theta\ u^2]/(\sin(\theta/2)\sin\theta)\right\}$$

$$= (y^2-1/\sin^2(\theta/2))(x^2-1/\sin^2(\theta/2)) - \frac{\overbrace{[1+\cos\theta-\sin^2\theta\ /\sin^2(\theta/2)]}^{-2\cos^2(\theta/2)}[2/(\sin(\theta/2)\sin\theta)-2xy]}{-\sin\theta\sin(\theta/2)}$$

$$= -\frac{1}{\sin^2(\theta/2)}\left[-\sin^2(\theta/2)x^2y^2+x^2+y^2-\underbrace{\frac{4\sin^2(\theta/2)\cos^2(\theta/2)}{\sin\theta\sin(\theta/2)}}_{2\cos(\theta/2)}xy\underbrace{-\frac{1}{\sin^2(\theta/2)}+\frac{4\sin^2(\theta/2)\cos^2(\theta/2)}{\sin^2\theta\sin^2(\theta/2)}}_{=0}\right]$$

which is $E(xy)$ with parameter $\theta/2$ instead of θ!

2. or $\sin^2\theta\ \ uv+\cos\theta-1=(v-u)\sin\theta$, hyperbola with asymptotes $u=1/\sin\theta$ and $v=-1/\sin\theta$, part of the asymptotes of $E(u,v)=0$, is that meaningful? One puts $v=\dfrac{\cos\theta-1+u\sin\theta}{\sin\theta(1-u\sin\theta)}$ in $E(u,v)=-\sin^2\theta\ \ u^2v^2+u^2+v^2-2\cos\theta\ \ uv=0$: $\dfrac{(1+u\sin\theta)(\cos\theta-1+u\sin\theta)^2}{\sin^2\theta(1-u\sin\theta)}+u^2-2\dfrac{u\cos\theta(\cos\theta-1+u\sin\theta)}{\sin\theta(1-u\sin\theta)}=\dfrac{(\cos\theta-1)^2}{\sin^2\theta}=0$, so what??

Could the TWO sides of (27) vanish simultaneously? Alas, no: with $E(x_n,x_{n+1})=x_{n+1}^2x_n^2+0.1x_{n+1}x_n(x_n+x_{n+1})-(x_{n+1}^2+x_n^2)+2x_{n+1}x_n+0.25(x_n+x_{n+1})+1$, one finds

β/α	γ/α	left side	right side
-0.50000	-0.22900	0.11586	-0.47566
-0.40000	-0.24788	0.12493	-0.37553
-0.30000	-0.26137	0.14072	-0.27541
-0.20000	-0.26928	0.16199	-0.17528
-0.10000	-0.27150	0.18741	-0.07516
-0.024938	-0.26938	0.20839	0
0	-0.26795	0.21560	0.02497
0.10000	-0.25862	0.24511	0.12509
0.20000	-0.24355	0.27442	0.22522
0.26814	-0.23004	0.29344	0.29344
0.30000	-0.22284	0.30190	0.32534
0.40000	-0.19663	0.32582	0.42547
0.50000	-0.16513	0.34432	0.52559

then, at equality of the two sides, $u=0.36394$, $v=-0.63208$ ($\rho_1=-1.0512, \rho_2=0.95125$), and

$F(x,y)=x^2y^2+0.26814xy(x+y)-0.23004(x^2+y^2)+1.8193xy+0.31796(x+y)+0.98238$

It seems that we have to do it the hard way. From the $R-$conic (24), γ is a root of an equation of degree 2

$d_{2,2}\gamma^2-(d_{1,2}\beta-d_{1,1}\alpha)\gamma+d_{0,2}\beta^2-d_{0,1}\alpha\beta+d_{0,0}\alpha^2=0$

and also the square of (27), using $(u-v)^2=(\beta^2-4\alpha\gamma)/\alpha^2$:

$$\left[[-d_{1,2}\beta+2d_{2,2}\gamma+d_{1,1}\alpha]\frac{-\beta}{2\alpha}-2d_{0,2}\beta+d_{1,2}\gamma+d_{0,1}\alpha\right]^2$$

$$= \left[\frac{\left[d_{0,1} - \dfrac{d_{1,1}d_{1,2}}{2d_{2,2}}\right]\alpha + (\rho_2-\rho_1)^2 d_{2,2}\beta/2}{\rho_2-\rho_1} \right]^2 (\beta^2 - 4\alpha\gamma)/\alpha^2$$

$(d_{1,2} - d_{2,2}\beta/\alpha)^2\gamma^2 + 2(d_{1,2} - d_{2,2}\beta/\alpha)[d_{0,1}\alpha - (2d_{0,2} + d_{1,1}/2)\beta + d_{1,2}\beta^2/\alpha]\gamma + [d_{0,1}\alpha - (2d_{0,2} + d_{1,1}/2)\beta + d_{1,2}\beta^2/\alpha]^2$

$$= \left[\frac{\left[d_{0,1} - \dfrac{d_{1,1}d_{1,2}}{2d_{2,2}}\right]^2\alpha^2}{(d_{1,2}^2 - 4d_{2,2}d_{0,2})/d_{2,2}^2} + \left[d_{0,1} - \frac{d_{1,1}d_{1,2}}{2d_{2,2}}\right]\alpha\beta d_{2,2} + (d_{1,2}^2 - 4d_{2,2}d_{0,2})\beta^2/4 \right] (\beta^2 - 4\alpha\gamma)/\alpha^2$$

after elimination of γ^2:

$(d_{1,2} - d_{2,2}\beta/\alpha)^2[(d_{1,2}\beta - d_{1,1}\alpha)\gamma - d_{0,2}\beta^2 + d_{0,1}\alpha\beta - d_{0,0}\alpha^2]/d_{2,2}$
$+2(d_{1,2} - d_{2,2}\beta/\alpha)[d_{0,1}\alpha - (2d_{0,2} + d_{1,1}/2)\beta + d_{1,2}\beta^2/(2\alpha)]\gamma + [d_{0,1}\alpha - (2d_{0,2} + d_{1,1}/2)\beta + d_{1,2}\beta^2/(2\alpha)]^2$

$$= \left[\frac{\left[d_{0,1} - \dfrac{d_{1,1}d_{1,2}}{2d_{2,2}}\right]^2\alpha^2}{(d_{1,2}^2 - 4d_{2,2}d_{0,2})/d_{2,2}^2} + \left[d_{0,1} - \frac{d_{1,1}d_{1,2}}{2d_{2,2}}\right]\alpha\beta d_{2,2} + (d_{1,2}^2 - 4d_{2,2}d_{0,2})\beta^2/4 \right] (\beta^2 - 4\alpha\gamma)/\alpha^2$$

$$\Big\{(d_{1,2} - d_{2,2}\beta/\alpha)[(d_{1,2} - d_{2,2}\beta/\alpha)(d_{1,2}\beta - d_{1,1}\alpha)/d_{2,2} + 2d_{0,1}\alpha - 2(2d_{0,2} + d_{1,1}/2)\beta + d_{1,2}\beta^2/\alpha]$$
$$\left. +4\frac{[d_{0,1}d_{2,2} - d_{1,1}d_{1,2}/2]^2\alpha}{d_{1,2}^2 - 4d_{2,2}d_{0,2}} + 4[d_{0,1}d_{2,2} - d_{1,1}d_{1,2}/2]\beta + (d_{1,2}^2 - 4d_{2,2}d_{0,2})\beta^2/\alpha \right\}\gamma$$
$$= (d_{1,2} - d_{2,2}\beta/\alpha)^2(d_{0,2}\beta^2 - d_{0,1}\alpha\beta + d_{0,0}\alpha^2)/d_{2,2} - [d_{0,1}\alpha - (2d_{0,2} + d_{1,1}/2)\beta + d_{1,2}\beta^2/(2\alpha)]^2 +$$
$$\frac{[d_{0,1}d_{2,2} - d_{1,1}d_{1,2}/2]^2\beta^2}{d_{1,2}^2 - 4d_{2,2}d_{0,2}} + [d_{0,1}d_{2,2} - d_{1,1}d_{1,2}/2]\beta^3/\alpha + (d_{1,2}^2 - 4d_{2,2}d_{0,2})\beta^4/(4\alpha^2)$$

$$\left\{ [d_{1,2}^3/d_{2,2} - 4d_{1,2}d_{0,2} - d_{1,1}d_{1,2} + 2d_{0,1}d_{2,2}]\beta + \left[4\frac{[d_{0,1}d_{2,2} - d_{1,1}d_{1,2}/2]^2}{d_{1,2}^2 - 4d_{2,2}d_{0,2}} - d_{1,2}^2 d_{1,1}/d_{2,2} + 2d_{0,1}d_{1,2} \right]\alpha \right\}\gamma$$
$$= \left[d_{1,2}^2 d_{0,2}/d_{2,2} + 2d_{1,2}d_{0,1} + d_{2,2}d_{0,0} - d_{0,1}d_{2,2} - (2d_{0,2} + d_{1,1}/2)^2 + \frac{[d_{0,1}d_{2,2} - d_{1,1}d_{1,2}/2]^2}{d_{1,2}^2 - 4d_{2,2}d_{0,2}} \right]\beta^2 +$$
$$\left[-2d_{1,2}d_{0,0} - d_{1,2}^2 d_{0,1}/d_{0,0} + 2d_{0,1}(2d_{0,2} + d_{1,1}/2)\right]\alpha\beta + [d_{1,2}^2 d_{0,0}/d_{2,2} - d_{0,1}^2]\alpha^2$$

Amazing (13 Feb. 2011), it reduces to $\gamma = \dfrac{\text{degree 2}}{\text{degree 1}}$!

Putting this formula for γ in $E(u,v) = d_{2,2}\gamma^2 - (d_{1,2}\beta - d_{1,1}\alpha)\gamma + d_{0,2}\beta^2 - d_{0,1}\alpha\beta + d_{0,0}\alpha^2 = 0$ amounts to a polynomial equation of degree 4 for β/α. With the example $E(u,v) = u^2v^2 + 0.1uv(u+v) - (u^2+v^2) + 2uv + 0.25(u+v) + 1 = (\gamma/\alpha)^2 - 0.1\beta\gamma/\alpha^2 - \beta^2/\gamma^2 + 4\gamma/\alpha - 0.25\beta/\gamma + 1$, one finds the four roots

β/α	γ/α	u	v	left side	right side
-2.2172	-4.9068	3.5857	-1.3685	-2.1950	-2.1950
-0.18928	-0.26979	-0.43333	0.62261	-0.16454	-0.16454
0.0035158	-3.7319	-1.9336	1.9301	0.028489	0.028489
0.26814	-0.23004	0.36394	-0.63208	0.29344	0.29344

each one leading to a valid F:

$[x^2-2.2172x-4.9068]y^2-[2.2172x^2+8.9183x+0.68660]y-4.9068x^2-0.68660x+1.2428$

$[x^2-0.18928x-0.26979]y^2-[0.18928x^2+1.7989x-0.13308]y-0.26979x^2+0.13308x+1.0303$

$[x^2+0.0035158x-3.7319]y^2+[0.0035158x^2+6.6839x+0.21100]y-3.7319x^2+0.21100x+1.0101$

$[x^2+0.26814x-0.23004]y^2+[0.26814x^2+1.8193x+0.31796]y-0.23004x^2+0.31796x+0.98238$

For a given sequence $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$ satisfying $E(x_n, x_{n\pm1}) = 0$, we find four valid sequences $\{\cdots, x_{-3/2}, x_{-1}, x_{-1/2}, x_0, x_{1/2}, x_1, \cdots\}$ we solve $F(x_0, y) = 0$ in y and evaluate $-\dfrac{Y_1(y)}{Y_2(y)} - x_0$: from (9), the result must be x_1 (and y is then $x_{1/2}$) or x_{-1} (and y is then $x_{-1/2}$). One then proceeds with repeated use of (6) and (9).

With the example above, starting with $x_0 = 0$:

x_{-1}	$x_{-1/2}$	x_0	$x_{1/2}$	x_1	$x_{3/2}$	x_2
1.1328	-0.57808	0	0.43815	-0.88278	2.0770	-4.6529
1.1328	2.2163	0	-1.7231	-0.88278	-0.60265	-4.6529
1.1328	0.54929	0	-0.49275	-0.88278	-1.4312	-4.6529
1.1328	-1.4879	0	2.8701	-0.88278	0.58236	-4.6529

Is it a surprise that there are 4 interpolants $\{\cdots, x_{n-1/2}, x_n, x_{n+1/2}, x_{n+1}, \cdots\}$ to a given elliptic sequence $\{x_n\}$? If we accept that $x_n = \mathcal{E}(an+b)$ with a periodic function of period p and integer n, then $\mathcal{E}(an+b+a/2)$ AND $\mathcal{E}(an+b+a/2+p/2)$ are both valid instances of $x_{n+1/2}$, as a new translation of $a/2$ (resp. $a/2+p/2$), will indeed yield x_{n+1}. And if the function $\mathcal{E}$ has TWO periods, say p and p', then there are four possible half-step translations which are: $a/2, a/2+p/2, a/2+p'/2$, AND $a/2+(p+p')/2$.

So, elementary recovery of half-integer indexes gives a hint that two-periodic functions will be somehow involved. Honestly, I knew it.

More on halving-doubling step in § 4.2, p. 45

Same experiment with the form (13) $E(x,y) = k^2(1-k^2z''^2)x^2y^2 - k^2(1-z''^2)(x+y)^2 + 2k(z''-k)(1+kz'')xy + 1 - k^2z''^2 = 0$, $d'_{0,1} = d_{0,1}\alpha - 2d_{0,2}\beta + d_{1,2}\gamma = 2k^2(1-z''^2)\beta$ and $d'_{1,1} = d_{1,1}\alpha - d_{1,2}\beta + 2d_{2,2}\gamma = 2k(z''-k)(1+kz'')\alpha + 2k^2(1-k^2z''^2)\gamma$, $\rho_1, \rho_2 = \mp\sqrt{(1-z''^2)/(1-k^2z''^2)}$, (27) is

$$\frac{[2k(z''-k)(1+kz'')\alpha + 2k^2(1-k^2z''^2)\gamma](-\beta/2) + 2k^2(1-z''^2)\alpha\beta}{u-v} = \frac{2k^2(1-z''^2)\alpha\beta}{\rho_2-\rho_1},\text{ or}$$

$$\frac{[-(z''-k)(1+kz'') + 2k(1-z''^2)]\alpha\beta - k(1-k^2z''^2)\beta\gamma}{\alpha^{-1}\sqrt{\beta^2-4\alpha\gamma}} = \pm k\sqrt{(1-z''^2)(1-k^2z''^2)}\alpha\beta.$$

Again, $\beta = 0$ is an obvious solution! But we do it by the book: squaring and eliminating γ^2 through $k^2(1-k^2z''^2)\gamma^2 - k^2(1-z''^2)\beta^2 + 2k(z''-k)(1+kz'')\alpha\gamma + (1-k^2z''^2)\alpha^2 = 0$, $[-(z''-k)(1+kz'') + 2k(1-z''^2)]^2\alpha^2\beta^2 - 2k(1-k^2z''^2)[-(z''-k)(1+kz'') + 2k(1-z''^2)]\alpha\beta^2\gamma + (1-k^2z''^2)\beta^2[k^2(1-z''^2)\beta^2 - 2k(z''-k)(1+kz'')\alpha\gamma - (1-k^2z''^2)\alpha^2] = (\beta^2-4\alpha\gamma)k^2(1-z''^2)(1-k^2z''^2)\beta^2$. Amazing (again): a lot of terms cancel, and we have $\beta = 0$ as only solution! Then, $E = k^2(1-k^2z''^2)\gamma^2 + 2k(z''-k)(1+kz'')\alpha\gamma + (1-k^2z''^2)\alpha^2 = (1-k^2z''^2)\left(k^2\gamma^2 + 2k\dfrac{z''-k}{1-kz''}\alpha\gamma + \alpha^2\right) = 0 \Rightarrow$ an equation for $uv = -u^2 = \gamma/\alpha = k^{-1}\dfrac{-z''+k \pm \sqrt{(1-k^2)(z''^2-1)}}{1-kz''}$, and F is from (26), $(y^2-\gamma/\alpha)(x^2-\gamma/\alpha) +$

$\dfrac{u-v}{(\rho_2-\rho_1)^2 d_{2,2}}[d'_{1,1}[\rho_2(y-v)-\rho_1(y-u)]x + d'_{0,1}(u-v)x + d'_{0,1}[\rho_2(y-u)-\rho_1(y-v)] +$
$d'_{1,1}\rho_1\rho_2(v-u)] = (y^2-\gamma/\alpha)(x^2-\gamma/\alpha) + \dfrac{2ud'_{1,1}[2\rho_2 xy + 2u\rho_2^2]}{4\rho_2^2 d_{2,2}}$
$$= (y^2-\gamma/\alpha)(x^2-\gamma/\alpha) + \frac{d'_{1,1}\left[\pm\sqrt{-\dfrac{\gamma(1-z''^2)}{\alpha(1-k^2z''^2)}}xy - \dfrac{\gamma(1-z''^2)}{\alpha(1-k^2z''^2)}\right]}{((1-z''^2)/(1-k^2z''^2))k^2(1-k^2z''^2)}$$
now, from (14) $z'' = -\dfrac{1-2k^2z'^2+k^2z'^4}{k(1-2z'^2+k^2z'^4)}$, $z''-k = \dfrac{-1-k^2+4k^2z'^2-k^2(1+k^2)z'^4}{k(1-2z'^2+k^2z'^4)}$,
$z''-k^{-1} = \dfrac{-2+2(1+k^2)z'^2-2k^2z'^4}{k(1-2z'^2+k^2z'^4)}$, $z''^2-1 = (1-k^2)\dfrac{(1-k^2z'^4)^2}{k^2(1-2z'^2+k^2z'^4)^2}$,
$1-k^2z''^2 = 4\dfrac{(1-k^2)z'^2(1-k^2z'^2)(1-z'^2)}{(1-2z'^2+k^2z'^4)^2}$,
$\dfrac{\gamma}{\alpha} = uv = \dfrac{(1+k^2)(1+k^2z'^4)\pm(1-k^2)(1-k^2z'^4)-4k^2z'^2}{2k^2(1-(1+k^2)z'^2+k^2z'^4)} = \dfrac{(1-k^2z'^2)^2 \text{ and } k^2(1-z'^2)^2}{k^2(1-k^2z'^2)(1-z'^2)} =$
$\dfrac{1}{k}\left(\dfrac{1-k^2z'^2}{k(1-z'^2)}\right)^{\pm1}$, $\dfrac{\gamma(1-z''^2)}{\alpha(1-k^2z''^2)} = -\dfrac{(1-k^2z'^4)^2}{k^2z'^2(1-z'^2)^2}$ and $-\dfrac{(1-k^2z'^4)^2}{z'^2(1-k^2z'^2)^2}$,
$d'_{1,1} = 2k\alpha[(z''-k)(1+kz'')+k(1-k^2z''^2)\gamma/\alpha]$
$= -2k\alpha\dfrac{2(1-k^2)z'^2}{1-2z'^2+k^2z'^4}\left[\dfrac{-1-k^2+4k^2z'^2-k^2(1+k^2)z'^4}{k(1-2z'^2+k^2z'^4)} + \dfrac{-2+2(1+k^2)z'^2-2k^2z'^4}{1-2z'^2+k^2z'^4}\dfrac{\gamma}{\alpha}\right]$,"

3.3. **(1. or 2.) and 3.**

3.3.1. *From 3. to 2. and 1.* Let $P(z) = c(z-z_1)(z-z_2)(z-z_3)(z-z_4)$, whose square root has the Taylor expansion about z_0 $\sqrt{P(z)} = \gamma+\delta(x-z_0)+\cdots$ which is matched by the two first Taylor coefficients of $S_n(z)$, so that only one unknown remains in S_n:

$$S_n(x) = \gamma+\delta(x-z_0)+\xi_n(x-z_0)^2.$$

We will need further coefficients: $P(x) = \gamma^2+2\gamma\delta(x-z_0)+P''(z_0)(x-z_0)^2/2+P^{(3)}(z_0)(x-z_0)^3/6+P^{(4)}(z_0)(x-z_0)^4/24 \Rightarrow$

$$\sqrt{P(x)} = \gamma+\delta(x-z_0)+\underbrace{\frac{P''(z_0)/2-\delta^2}{2\gamma}}_{\epsilon}(x-z_0)^2+\underbrace{\frac{P'''(z_0)/6-\dfrac{\delta}{\gamma}\left(\overbrace{P''(z_0)/2-\delta^2}^{2\gamma\epsilon}\right)}{2\gamma}}_{\eta}(x-z_0)^3+\cdots$$

$\dfrac{S_n(x)+\sqrt{P(x)}}{Z_{n+1}(x)} = \dfrac{1-\beta_n(x-z_0)}{\alpha_n}+O((x-z_0)^2) \Rightarrow \alpha_n = \dfrac{\zeta_{n+1}(z_0-x_{n+1})}{2\gamma}$, $\beta_n = -\dfrac{\delta}{\gamma}+$
$\dfrac{1}{z_0-x_{n+1}}$, and $\dfrac{S_n(z)-\sqrt{P(z)}}{Z_n(z)} = \dfrac{\alpha_n(x-z_0)^2}{1-\beta_n(x-z_0)+O((x-z_0)^2)} \Rightarrow \alpha_n = \dfrac{\xi_n-\epsilon}{\zeta_n(z_0-x_n)}$, $\beta_n =$
$-\dfrac{1}{z_0-x_n}-\dfrac{\eta}{\xi_n-\epsilon}$, so

$$\frac{1}{(z_0-x_n)(z_0-x_{n+1})}=\frac{\zeta_n\zeta_{n+1}}{2\gamma(\xi_n-\epsilon)},\qquad \frac{1}{z_0-x_n}+\frac{1}{z_0-x_{n+1}}=\frac{\delta}{\gamma}-\frac{\eta}{\xi_n-\epsilon}$$

We will only need $Z_n(x)=$ const. $(x-x_n)$ and $W_n(x)=Z_{n+1}(x)=$ const. $(x-x_{n+1})$.

We now expand $S_n^2(x)-P(x)=(x-z_0)^2W_n(x)Z_n(x)=$ const. $(x-z_0)^2(x-x_n)(x-x_{n+1})=$ const. $[(x-z_0)^4+(2z_0-x_n-x_{n+1})(x-z_0)^3+(z_0-x_n)(z_0-x_{n+1})(x-z_0)^2]$:

$\xi_n^2-P^{(4)}(z_0)/24$ is the constant ($=\zeta_n\zeta_{n+1}$), the coeff. of $(x-z_0)^4$,

$$\begin{aligned}2\xi_n\delta-P^{(3)}(z_0)/6&=2(\xi_n\delta-\delta\epsilon-\gamma\eta)=[\xi_n^2-P^{(4)}(0)/24](2z_0-x_n-x_{n+1}),\\ \delta^2+2\gamma\xi_n-P''(z_0)/2&=2\gamma(\xi_n-\epsilon)=[\xi_n^2-P^{(4)}(z_0)/24](z_0-x_n)(z_0-x_{n+1}),\end{aligned}\tag{28}$$

from which x_n+x_{n+1} and x_nx_{n+1} are rational functions of ξ_n.

x_n and x_{n+1} are the roots of $(\xi_n^2-P^{(4)}(0)/24)(z_0-x)^2-2(\xi_n\delta-\delta\epsilon-\gamma\eta)(z_0-x)+2\gamma(\xi_n-\epsilon)=0$. Do we have $\xi_n=y_{n+1}$? Then, $\xi_{n-1}+\xi_n$ must be the rational function $[2\delta(z_0-x_n)-2\gamma]/(z_0-x_n)^2$. From (18), $\xi_n+\xi_{n-1}=$ coefficient of z^2 of $S_n(z)+S_{n-1}(z)=[1-\beta_{n-1}(z-z_0)]Z_n(z)/\alpha_{n-1}$, so $\xi_n+\xi_{n-1}=-\beta_{n-1}\zeta_n/[\zeta_n(z_0-x_n)/(2\gamma)]=-2\beta_{n-1}\gamma/(z_0-x_n)=2[\delta-\gamma/(z_0-x_n)]/(z_0-x_n)$ OK.

So $F(x,y)=(y^2-P^{(4)}(0)/24)(z_0-x)^2-2(y\delta-\delta\epsilon-\gamma\eta)(z_0-x)+2\gamma(y-\epsilon)$.

The elimination of ξ_n yields a symmetric algebraic relation between x_n and x_{n+1}: let $S=2z_0-x_n-x_{n+1}$ and $\Pi=(z_0-x_n)(z_0-x_{n+1})$,

$$\xi_n=\frac{\dfrac{P'''(z_0)/6}{S}+\dfrac{\delta^2-P''(z_0)/2=-2\gamma\epsilon}{\Pi}}{\dfrac{2\delta}{S}-\dfrac{2\gamma}{\Pi}},\ 2\delta\xi_n-P'''(z_0)/6=\frac{2\gamma^2\eta S}{\delta\Pi-\gamma S},$$

$$\frac{2\gamma^2\eta S}{\delta\Pi-\gamma S}=S\left[\frac{P'''(z_0)\Pi/6-2\gamma\epsilon S}{2\delta\Pi-2\gamma S}\right]^2-SP^{(4)}(z_0)/24,$$

$$8\gamma^2\eta(\delta\Pi-\gamma S)=[P'''(z_0)\Pi/6-2\gamma\epsilon S]^2-P^{(4)}(z_0)[\delta\Pi-\gamma S]^2/6,$$

$$[(P'''(z_0)/6)^2-P^{(4)}(z_0)/6]\Pi^2-\gamma[2\epsilon P'''(z_0)-\delta P^{(4)}(z_0)]\Pi S/3+\gamma^2[4\epsilon^2-P^{(4)}(z_0)/6]S^2-8\gamma^2\eta(\delta\Pi-\gamma S)=0.\tag{29}$$

3.3.2. *From 3. to 1.* There is also a direct way to construct a valid $y-$sequence, simply from $y_{n+1}=\xi_n$ in (28):

$$F(x,y)=(y^2-c)(x-z_0)^2+(2\delta y-P^{(3)}(z_0)/6)(x-z_0)+\delta^2+2\gamma y-P''(z_0)/2.\tag{30}$$

From (9) and (28), the two $x-$roots of $F(x,y)=0$ at $y=y_{n+1}=\xi_n$ must indeed be x_n and x_{n+1}.

As a function of y, $X_2(x)y^2+X_1(x)y+X_0(x)$:

$$F(x,y)=(x-z_0)^2y^2+2(\delta(x-z_0)+\gamma)y-c(x-z_0)^2-(P^{(3)}(z_0)/6)(x-z_0)+\delta^2-P''(z_0)/2.\tag{31}$$

Check that $X_1^2-4X_0X_2=$ const. P: indeed, $X_1(x)^2-4X_0(x)X_2(x)=4c(x-z_0)^4+4(P^{(3)}(z_0)/6)(x-z_0)^3+4P''(z_0)/2(z-z_0)^2+8\gamma\delta(x-z_0)+4\gamma^2=4P(x)$.

Of special interest is $Q(y)=Y_1(y)^2-4Y_0(y)Y_2(y)$ from (30):

$$Q(y)=[2\delta y-P^{(3)}(z_0)/6)]^2-4(y^2-c)[\delta^2+2\gamma y-P''(z_0)/2]$$

which is only of third degree

$$-8\gamma y^3+2P''(z_0)y^2+4[2c+P^{(3)}(z_0)/6]y+[P^{(3)}(z_0)/6]^2+4c[\delta^2-P''(z_0)/2],$$

and happens to be basically a cubic resolvent [11,63] of $P(x) = c(x-z_1)\cdots(x-z_4)$! Indeed, with $\delta = P'(z_0)/(2\gamma)$ and $\gamma^2 = P(z_0)$, we find
$-\dfrac{Q(y)}{\gamma^4} =$

$$\left(\frac{2y}{\gamma}\right)^3 - \frac{P''(z_0)}{2P(z_0)}\left(\frac{2y}{\gamma}\right)^2 - \left(\frac{4c}{P(z_0)} - \frac{P'(z_0)P^{(3)}(z_0)}{6P^2(z_0)}\right)\frac{2y}{\gamma} - \left(\frac{P^{(3)}(z_0)}{6P(z_0)}\right)^2 - \frac{c}{P(z_0)}\left(\left(\frac{P'(z_0)}{P(z_0)}\right)^2 - 2\frac{P''(z_0)}{P(z_0)}\right)$$

whose coefficients depend only on the four values $\rho_j := 1/(z_0 - z_j)$, $j = 1, \dots, 4$, as $P'(z_0)/P(z_0) = \sum_1^4 \rho_j$; $P''(z_0)/(2P(z_0))$ is the sum of the 6 products $\rho_i\rho_j$ for $1 \leqslant i < j \leqslant 4$; $P^{(3)}(z_0)/(6P(z_0))$ is the sum of the 4 possible products $\rho_i\rho_j\rho_k$, $1 \leqslant i < j < k \leqslant 4$; and $c/P(z_0) = \rho_1\rho_2\rho_3\rho_4$.

Then, the coefficient of $(2y/\gamma)^2$ is (-1) times the sum of the 6 products $\rho_i\rho_j$ above; the coefficient of $2y/\gamma$ is the sum of the 12 products $\rho_i\rho_j\rho_k\rho_\ell$ with one repeated factor; and the last coefficient is (-1) times the sum of the four terms of the form $(\rho_i\rho_j\rho_k)^2$ and the sum of the four terms $\rho_i^3\rho_j\rho_k\rho_\ell$.

This amounts to the zeros of Q as

$$\frac{2y}{\gamma} = \rho_1\rho_2 + \rho_3\rho_4,\ \ \rho_1\rho_3 + \rho_2\rho_4,\ \text{ and } \rho_1\rho_4 + \rho_2\rho_3,$$

and ∞, if Q is to be considered a fourth-degree polynomial.

Check:

$$\frac{\rho_1\rho_2 + \rho_3\rho_4 - (\rho_1\rho_3 + \rho_2\rho_4)}{\rho_1\rho_2 + \rho_3\rho_4 - (\rho_1\rho_4 + \rho_2\rho_3)} = \frac{\dfrac{z_3 - z_2}{z_3 - z_1}}{\dfrac{z_4 - z_2}{z_4 - z_1}}. \tag{32}$$

The general F is $(1+Cy)^2 F\left(x, \dfrac{A+By}{1+Cy}\right)$, with the F of (30), and the new Q is $(1+Cy)^4 Q\left(\dfrac{A+By}{1+Cy}\right)$, whose four zeros are given by $\dfrac{A+By}{1+Cy} = \dfrac{\gamma}{2}(\rho_j\rho_4 + \rho_k\rho_\ell)$, and $y = -1/C$.

The choice of A, B, and C ensuring that $Q =$ constant P is such that $\dfrac{A+Bz_j}{1+Cz_j} = \dfrac{\gamma}{2}(\rho_j\rho_4 + \rho_k\rho_\ell)$, $j = 1,2,3$, and $z_4 = -1/C$, in order to keep the cross-ratio in (32), so $A + Bz_j = \dfrac{\gamma}{2z_4}(\rho_j\rho_4 + \rho_k\rho_\ell)\left(\dfrac{1}{\rho_j} - \dfrac{1}{\rho_4}\right)$, $j = 1,2,3$, $A + Bz_1 = A + Bz_0 - \dfrac{B}{\rho_1} = \dfrac{\gamma}{2z_4}(\rho_1\rho_4 + \rho_2\rho_3)\left(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_4}\right)$, $A + Bz_0 - \dfrac{B}{\rho_2} = \dfrac{\gamma}{2z_4}(\rho_2\rho_4 + \rho_1\rho_3)\left(\dfrac{1}{\rho_2} - \dfrac{1}{\rho_4}\right)$, $A + Bz_0 - \dfrac{B}{\rho_3} = \dfrac{\gamma}{2z_4}(\rho_3\rho_4 + \rho_1\rho_2)\left(\dfrac{1}{\rho_3} - \dfrac{1}{\rho_4}\right)$, then, $A + Bz_0 = \gamma(-\rho_1 - \rho_2 - \rho_3 + \rho_4)/(2z_4)$ and $B = \gamma(-\rho_1\rho_2 - \rho_1\rho_3 - \rho_2\rho_3 + \rho_1\rho_2\rho_3/\rho_4)/(2z_4)$.

For the F of (11), $x_n + x_{n+1} = \dfrac{2(1-k^2)z_0 y_{n+1}}{k[(1-k^2z_0^2)y_{n+1}^2 - 1 + z_0^2]}$, $x_n x_{n+1} = \dfrac{1 - k^2z_0^2 - k^2(1-z_0^2)y_{n+1}^2}{k^2[(1-k^2z_0^2)y_{n+1}^2 - 1 + z_0^2]}$,

$$E(x_n, x_{n+1}) = \frac{(1-k^2z_0^4)^2}{4z_0^2}(x_n+x_{n+1})^2 - k^2(1-z_0^2)(1-k^2z_0^2)(x_nx_{n+1})^2$$
$$- [(1+k^2)(1+k^2z_0^4) - 4k^2z_0^2]x_nx_{n+1} - 1 - k^2z_0^4 + (1+k^2)z_0^2 \quad (33)$$

Fixed points $x_n = x_{n+1}$ satisfy $(1-z_0^2)(1-k^2z_0^2)(x^2-z_0^2)(k^2x^2-z_0^{-2}) = 0$.

3.3.3. *From 2. to 3.* Can we find P, z_0,etc. from (15)? If $z_0=\infty$, $d_{2,2}=0$; when $z_0=0$, $d_{0,0}=0$. As $S=\Pi=0$ satisfies (29), so does $x=y=z_0$, i.e., z_0 is one of the four roots of

$$d_{2,2}z_0^4 + 2d_{1,2}z_0^3 + (4d_{0,2}+d_{1,1})z_0^2 + 2d_{0,1}z_0 + d_{0,0} = 0.$$

Then, with $x = z_0-(z_0-x_n)$ and $y = z_0-(z_0-x_{n+1})$, $x+y = 2z_0-S$, $xy = z_0^2-z_0S+\Pi$, and (15) becomes

$$d_{2,2}\Pi^2 - (2d_{2,2}z_0+d_{1,2})S\Pi + (d_{2,2}z_0^2+(d_{1,2}+d_{1,1})z_0+d_{0,2})S^2 + (2d_{2,2}z_0^2+2(d_{1,2}+d_{1,1})z_0)\Pi$$
$$- (2d_{2,2}z_0^3 + (3d_{1,2}+2d_{1,1})z_0^2 + 4d_{0,2}Cz_0 + d_{0,1})S = 0,$$

must match (29)

Another way: (15) yields $y = x_{n\pm1}$ as a quadratic function of $x = x_n$ as

$$y = \frac{-d_{1,2}x^2 - (2d_{0,2}+d_{1,1})x - d_{0,1} \pm \sqrt{(d_{1,2}x^2+(2d_{0,2}+d_{1,1})x+d_{0,1})^2 - 4(d_{2,2}x^2+d_{1,2}x+d_{0,2})(d_{0,2}x^2+d_{0,1}x+d_{0,0})}}{2(d_{2,2}x^2+d_{1,2}x+d_{0,2})}$$

Let us look now at the (S_n, P, Z_n) construction as a way to find x_{n+1} from x_n: if x_n is known, $S_n^2 - P$ must vanish at $x = x_n \Rightarrow S_n(x_n) = \gamma + \delta(x_n - z_0) + \xi_n(x_n-z_0)^2 = \pm\sqrt{P(x_n)}$, giving two possible values for ξ_n. Then, we factor $S_n^2 - P$:

$$\left[\gamma + \delta(x-z_0) + \frac{\pm\sqrt{P(x_n)} - \gamma - \delta(x_n-z_0)}{(x_n-z_0)^2}(x-z_0)^2\right]^2 - \gamma^2 - 2\gamma\delta(x-z_0) - \frac{P''(z_0)}{2}(x-z_0)^2 - \frac{P'''(z_0)}{6}(x-z_0)^3 - \frac{P^{(4)}(z_0)}{24}$$

what remains is constant times $(x-z_0)^2(x-x_n)$ times a last factor which must be a constant times $x - x_{n+1}$, yielding for x_{n+1} an expression containing $\sqrt{P(x_n)}$, so that

$$(d_{1,2}x^2+(2d_{0,2}+d_{1,1})x+d_{0,1})^2 - 4(d_{2,2}x^2+d_{1,2}x+d_{0,2})(d_{0,2}x^2+d_{0,1}x+d_{0,0}) = \text{ const. } P(x).$$

Generalization.

Generalization of Padé approximation and continued fraction (recurrence relations) constructions: see [7, 8, 47, 61]

3.4. **3. and 4.**

3.4.1. *3. to 4.: Algorithm for square root of a polynomial.*

Basically from Perron [51, § 20]

We may as well consider the general hyperelliptic case, with P of degree $2m$ [45].

Let $P(z) = c(z-z_1)\cdots(z-z_{2m})$, S_0 a polynomial of degree m such that $S_0(z_0) = \gamma$ at a fixed point z_0, where γ is the value at z_0 of a well defined definition of the square root of P (we will need this square root only in a neigbourhood of z_0), and Z_0 and W_0 polynomials of degree $m-1$ which are a factors of $P - S_0^2$.

As a matter of fact, this strange latter condition comes from the consideration of a function f which is the root of

$$(x-z_0)Z_0(x)f^2(x) - 2S_0(x)f(x) + (x-z_0)W_0(x) = 0,$$

vanishing at $x = z_0$. If $z_0 = \infty$, the property of f is $f(x) = O(1/x)$ for large x. We have

$$f(x) = \frac{S_0(x) - \sqrt{P(x)}}{(x-z_0)Z_0(x)},$$

where $P(x) = S_0(x)^2 - (x-z_0)^2 Z_0(x)W_0(x)$.

We now look at a continued fraction expansion to f of the form

$$f(x) = \cfrac{\alpha_0(x-z_0)}{1-\beta_0(x-z_0) - \cfrac{\alpha_1(x-z_0)^2}{1-\beta_1(x-z_0)-\cdots}},$$

or

$$f_n(x) = \frac{\alpha_n(x-z_0)}{1-\beta_n(x-z_0)-(x-z_0)f_{n+1}(x)}\ , \quad n = 0, 1, \ldots$$

Remark that the two first Taylor terms of the expansion of f_n are

$$f_n(x) = \alpha_n(x-z_0) + \alpha_n\beta_n(x-z_0)^2 + \cdots$$

The form of f is kept in all the f_n's, as

$$f_n(x) = \frac{S_n(x) - \sqrt{P(x)}}{(x-z_0)Z_n(x)} \tag{34}$$

$$\Rightarrow f_{n+1}(x) = \frac{1}{x-z_0} - \beta_n - \frac{\alpha_n}{f_n(x)} = \frac{1-\beta_n(x-z_0)}{x-z_0} - \frac{\alpha_n(x-z_0)Z_n(x)[S_n(x)+\sqrt{P(x)}]}{S_n^2(x) - P(x)}$$

which, if f_n is a root of $(x-z_0)Z_n(x)f_n^2(x) - 2S_n(x)f_n(x) + (x-z_0)W_n(x) = 0$, with $S_n(x)^2 - (x-z_0)^2 Z_n(x)W_n(x) = P(x)$, turns as $f_{n+1}(x) = \dfrac{[1-\beta_n(x-z_0)]W_n(x)/\alpha_n - S_n(x) - \sqrt{P(x)}}{(x-z_0)W_n(x)/\alpha_n}$,

so that

$Z_{n+1}(x) = W_n(x)/\alpha_n$,

$S_{n+1}(x) = [1-\beta_n(x-z_0)]Z_{n+1}(x) - S_n(x)$,

and, as we want

$$S_{n+1}^2(x) - P(x) = (x-z_0)^2 Z_{n+1}(x)W_{n+1}(x) = (x-z_0)^2\alpha_{n+1}Z_{n+1}(x)Z_{n+2}(x), \tag{35}$$

remarking that $S_{n+1}^2(x) = [1-\beta_n(x-z_0)]Z_{n+1}(x)\{[1-\beta_n(x-z_0)]Z_{n+1}(x) - 2S_n(x)\} + S_n^2(x) = [1-\beta_n(x-z_0)]Z_{n+1}(x)(S_{n+1}(x) - S_n(x)) + S_n^2(x)$,

$$W_{n+1}(x) = \alpha_n W_n(x) + [1-\beta_n(x-z_0)]\frac{S_{n+1}(x)-S_n(x)}{(x-z_0)^2}.$$

which is a polynomial, as S_n and S_{n+1} match $\sqrt{P}$ up to second order, so that the degree of the polynomial W_{n+1} remains $\leqslant m-1$.

(Hyper)elliptic functions and all that.

The Green function.

For any n_0 and p, from (34):

$$f_{n_0}f_{n_0+1}\dots f_{n_0+p-1} = (x-z_0)^{-p}\frac{S_{n_0}-\sqrt{P}}{Z_{n_0}}\frac{S_{n_0+1}-\sqrt{P}}{Z_{n_0+1}}\cdots\frac{S_{n_0+p-1}-\sqrt{P}}{Z_{n_0+p-1}},$$

$$\begin{aligned}[f_{n_0}f_{n_0+1}\dots f_{n_0+p-1}]^2 &= (x-z_0)^{-2p}\frac{[S_{n_0}-\sqrt{P}]^2[S_{n_0+1}-\sqrt{P}]^2\cdots[S_{n_0+p-1}-\sqrt{P}]^2}{Z_{n_0}\,(Z_{n_0}Z_{n_0+1})\,(Z_{n_0+1}Z_{n_0+2})\,\cdots(Z_{n_0+p-2}Z_{n_0+p-1})\,Z_{n_0+p-1}}\\
&= \frac{(-1)^p\alpha_{n_0}\dots\alpha_{n_0+p-1}[S_{n_0}-\sqrt{P}]^2[S_{n_0+1}-\sqrt{P}]^2\cdots[S_{n_0+p-1}-\sqrt{P}]^2 Z_{n_0+p}}{Z_{n_0}[S_{n_0}^2-P][S_{n_0+1}^2-P]\cdots[S_{n_0+p-1}^2-P]}\\
&\text{(from (35)}\\
&= \frac{Z_{n_0+p}}{Z_{n_0}}(-1)^p\alpha_{n_0}\dots\alpha_{n_0+p-1}\frac{[S_{n_0}-\sqrt{P}][S_{n_0+1}-\sqrt{P}]\cdots[S_{n_0+p-1}-\sqrt{P}]}{[S_{n_0}+\sqrt{P}][S_{n_0+1}+\sqrt{P}]\cdots[S_{n_0+p-1}+\sqrt{P}]}\end{aligned}$$

When p is (almost) a period, $Z_{n_0+p}\approx Z_{n_0}$, we have the limit

$$\exp(\mathcal{G}(z)) = \exp(g(z)+ih(z)) = \lim_{n\to\infty}\left(\frac{a_1\cdots a_n}{f_0(z)\cdots f_{n-1}(z)}\right)^{1/n} \approx \left(\frac{(S_{n_0}+\sqrt{P})\cdots(S_{n_0+p-1}+\sqrt{P})}{(S_{n_0}-\sqrt{P})\cdots(S_{n_0+p-1}-\sqrt{P})}\right)^{1/(2p)}$$

When P has real roots, $\exp(\mathcal{G}(z))$ is the ratio of complex conjugate numbers if $z\in$ the support S, the locus where $P(z)\leqslant 0$. $g(z)$ is the harmonic function vanishing on S, positive outside S (minimum principle), behaving like $-\log|z-z_0|+$ constant near z_0 ($\log|z|$+const. for large z if $z_0=\infty$): the ***Green function*** singular at z_0 of S. When $z_0=\infty$, the constant is the ***Robin constant***[4] = -log of the ***logarithmic capacity*** of S.

What if P has general complex zeros $z_1,\dots,z_{2m}$? $\exp\mathcal{G}(z)$ is already on the unit circle, i.e., $g(z)=0$ at the zeros of P. We still can think of a path S where $g(z)=0$, i.e., $\mathcal{G}(z)$ is pure imaginary, and this path is characterized by the differential condition

$$\mathcal{G}'(z)\,dz \text{ pure imaginary}$$

[4]About Robin, see K. Gustafson, T. Abe: "The third boundary condition- was it Robin's?", *Math. Intelligencer* vol. **20** (1998), nr. 1, pp. 63-71; "(Victor) Gustave Robin: 1855-1897", *ibid.*, vol. **20**, nr. 2, pp. 47-53.

(or the *quadratic differential* $(\mathcal{G}')^2(z)\,dz^2 \leqslant 0$). We must know more on $\mathcal{G}'$:

$$\mathcal{G}' \approx \frac{1}{2p}\sum\left[\frac{S'+\dfrac{P'}{2\sqrt{P}}}{S+\sqrt{P}} - \frac{S'-\dfrac{P'}{2\sqrt{P}}}{S-\sqrt{P}}\right]$$

$$\approx \frac{1}{2p}\sum \frac{-2S'\sqrt{P}+2\dfrac{P'S}{2\sqrt{P}}}{S^2-P=(x-z_0)^2\alpha_m Z_m Z_{m+1}}$$

is therefore a rational function divided by $\sqrt{P}$, with possible poles at the zeros of the Z polynomials. But let us look again at $\exp(\mathcal{G}(z)) \approx \left(\dfrac{(S_{n_0}+\sqrt{P})\cdots(S_{n_0+p-1}+\sqrt{P})}{(S_{n_0}-\sqrt{P})\cdots(S_{n_0+p-1}-\sqrt{P})}\right)^{1/(2p)}$ if some $S_m \pm \sqrt{P}$ vanishes, so does $S_m^2 - P$, so, either Z_m or Z_{m+1}. If it is Z_m, look at $S_m = -S_{m-1} + [1-\beta_{m-1}(x-z_0)]Z_m$, which shows that $S_{m-1} \mp \sqrt{P}$ vanishes as well; and if it is Z_{m+1}, $S_{m+1} = -S_m + [1-\beta_m(x-z_0)]Z_{m+1} = -S_m$ at a zero of Z_{m+1}. So, $\exp\mathcal{G}$ has no pole nor zero in the finite complex plane, and we must have

$$\mathcal{G}'(z) = \frac{Q(z)}{\sqrt{P(z)}}, \tag{36}$$

where $Q(z) = z^{m-1} + \cdots$ is a monic polynomial of degree $m-1$.

When $m > 1$, Q is completely determined by the conditions

$$\text{periods } = 2\int_{z_1}^{z_k} \frac{Q(z)}{\sqrt{P(z)}}\,dz \quad \text{pure imaginary.}$$

Importance of the Green function $\mathcal{G}$ on the behaviour of the denominator B_n.

Remember: $\dfrac{A_1(x)}{B_1(x)} = \dfrac{\alpha_0(x-z_0)}{1-\beta_0(x-z_0)}$,

$\dfrac{A_2(x)}{B_2(x)} = \dfrac{\alpha_0(x-z_0)}{1-\beta_0(x-z_0)-\dfrac{\alpha_1(x-z_0)^2}{1-\beta_1(x-z_0)}} = \dfrac{\alpha_0(x-z_0)[1-\beta_1(x-z_0)]}{[1-\beta_0(x-z_0)][1-\beta_1(x-z_0)]-\alpha_1(x-z_0)^2}$,

$A_{n+1}(x) = [1-\beta_n(x-z_0)]A_n(x) - \alpha_n(x-z_0)^2 A_{n-1}(x)$,

$B_{n+1}(x) = [1-\beta_n(x-z_0)]B_n(x) - \alpha_n(x-z_0)^2 B_{n-1}(x)$, $A_0(x) = 0$, $A_1(x) = \alpha_0(x-z_0)$, $B_0(x) = 1$, $B_1(x) = 1-\beta_0(x-z_0)$.

$$\begin{bmatrix} A_n(x) & B_n(x) \\ A_{n+1}(x) & B_{n+1}(x)\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha_n(x-z_0)^2 & 1-\beta_n(x-z_0)\end{bmatrix}\begin{bmatrix} A_{n-1}(x) & B_{n-1}(x) \\ A_n(x) & B_n(x)\end{bmatrix} \tag{37}$$

$\begin{bmatrix} A_{-1}(x) & B_{-1}(x) \\ A_0(x) & B_0(x)\end{bmatrix} = \begin{bmatrix} -(x-z_0)^{-1} & 0 \\ 0 & 1\end{bmatrix}$

The products $(x-z_0)^n f_0(x)\dots f_n(x)$ satisfy the same recurrence relation, from $f_{n+1}(x) = \dfrac{1}{x-z_0} - \beta_n - \dfrac{\alpha_n}{f_n(x)}$, $f_0(x) = \dfrac{S_0(x)-\sqrt{P(x)}}{(x-z_0)Z_0(x)}$,

$$(x-z_0)f_0(x)f_1(x) = (x-z_0)\frac{S_0(x)-\sqrt{P(x)}}{(x-z_0)Z_0(x)}\frac{[1-\beta_0(x-z_0)]Z_1(x)-S_0(x)-\sqrt{P(x)}}{(x-z_0)Z_1(x)}$$
$$= \frac{B_1(x)S_0(x)Z_1(x)-S_0^2(x)+P(x)-B_1(x)Z_1(x)\sqrt{P(x)}}{(x-z_0)Z_0(x)Z_1(x)} = B_1(x)f(x)-A_1(x).$$

Let n be a quasi-period, then

$$B_{n-1}(x)f(x)-A_{n-1}(x) = \frac{B_{n-1}(x)S_0(x)-A_{n-1}(x)Z_0(x)-B_{n-1}(x)\sqrt{P(x)}}{Z_0(x)}$$

$= (z-z_0)^{n-1}f_0(x)\cdots f_{n-1}(x) \approx (-1)^n a_1\cdots a_n e^{-n\mathcal{G}}$. Now, take the other determination of $\sqrt{P}$. $\mathcal{G}$ becomes $-\mathcal{G}$ (as seen form the definition of $\exp(\mathcal{G})$ through a product of $\dfrac{X_k+\sqrt{Y}}{X_k-\sqrt{Y}}$, or frm the quadratic differential), and

$$B_{n-1}f_{\text{conjugate}}-A_{n-1} = \frac{B_{n-1}S_0-A_{n-1}Z_0+B_{n-1}\sqrt{P}}{Z_0} \approx (-1)^n a_1\cdots a_n e^{n\mathcal{G}},$$

whence

$$B_{n-1} \approx (-1)^n a_1\cdots a_n Z\frac{e^{n\mathcal{G}}-e^{-n\mathcal{G}}}{2\sqrt{Y}}.$$

showing that most of the zeros of B_{n-1} are on the locus $S = \{z : \operatorname{Re}\mathcal{G}(z) = g(z) = 0\}$.

Remark: "Pell's equation". Make the product

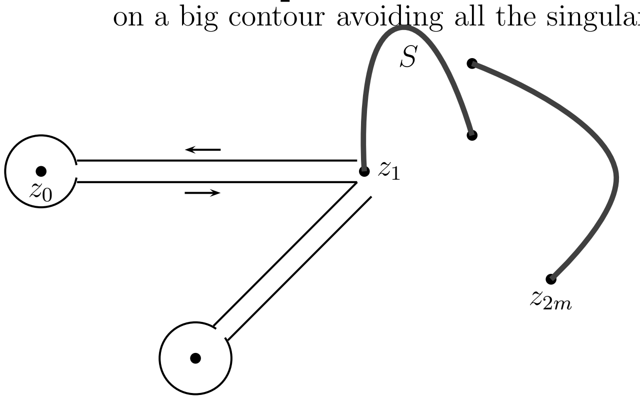
$$\begin{aligned}[B_{n-1}S_0-A_{n-1}Z_0]^2-B_{n-1}^2P &= Z_0^2(x-z_0)^{2n}f_0\dots f_{n-1}f_{0,\text{conj.}}\cdots f_{n-1,\text{conj.}}\\ &= Z_0^2\frac{S_0-\sqrt{P}}{Z_0}\cdots\frac{S_{n-1}-\sqrt{P}}{Z_{n-1}}\frac{S_0+\sqrt{P}}{Z_0}\cdots\frac{S_{n-1}+\sqrt{P}}{Z_{n-1}}\\ &= \alpha_0^2\dots\alpha_{n-1}^2(x-z_0)^{2n}Z_0Z_n.\end{aligned}$$

What is (are) the period(s)?

If $m>1$, let us consider the integrals of

$$\frac{t^k}{2\pi i\sqrt{P(t)}}\log\left[\frac{B_{n-1}(t)S_0(t)-A_{n-1}(t)Z_0(t)+B_{n-1}(t)\sqrt{P(t)}}{B_{n-1}(t)S_0(t)-A_{n-1}(t)Z_0(t)-B_{n-1}(t)\sqrt{P(t)}}\right]dt,\quad k=0,\dots,m-2,$$

on a big contour avoiding all the singularities, so that the value $=0$,



On the other hand, making the contour shrink about S and the zeros of U and $V = B_{n-1}(t)S_0(t)-A_{n-1}(t)Z_0(t)\pm B_{n-1}(t)\sqrt{P(t)}$ (which are z_0 and the zeros of Z_0 and Z_n), one finds

$$0 = -2n\int_{z_0}^{z_1}\frac{t^k}{\sqrt{P(t)}}dt + \sum_{\text{zeros of }Z_0,Z_n}\pm\int_{z_1}^{\text{zero}}\frac{t^k}{\sqrt{P(t)}}dt + \sum_j N_j\int_{z_j}^{z_{j+1}}\frac{t^k}{\sqrt{P(t)}}dt,\qquad k=0,\dots,m-2, \tag{38}$$

where the first term accounts for the zero of multiplicity $2n$ at z_0 of $V = B_{n-1}(t)S_0(t) - A_{n-1}(t)Z_0(t) - B_{n-1}(t)\sqrt{P(t)}$, so that the logarithm of U/V on the lower path from z_0 to z_1 is the value on the upper path minus $4n\pi i$; the $\pm$ signs tell if the zero is a zero of U or V; and where N_j are (unknown) integers. There are only $m-1$ integrals $\displaystyle\int_{z_j}^{z_{j+1}} \frac{t^k}{\sqrt{P(t)}}\,dt$ to consider, in the "gaps" of S. Integrals on the two sides of S vanish, as $\sqrt{P}_+ = -\sqrt{P}_-$, so that the logarithm takes opposite values on the sides of S.

It happens that, knowing n, P, and Z_0, (38) allows to find the remaining unknowns, including the $\pm$ signs (***Jacobi problem***, [2,6,32,45,47,48]).

There is absolutely no need for n to be an integer in the description (38) of the Jacobi problem. Let $x_n^{(1)},\dots,x_n^{(m-1)}$ be the unknown zeros of Z_n. To see how these x_n^{ℓ}'s are functions of n, we . . . take the derivative of (38) with respect to n (!!):

$$\nu_k = \sum_{\ell=1}^{m-1} s_\ell \frac{(x_n^{(\ell)})^k}{\sqrt{P(x_n^{(\ell)})}}\frac{dx_n^{(\ell)}}{dn} \qquad\qquad k=0,\dots,m-2,$$

where $\nu_k = 2\displaystyle\int_{z_0}^{z_1}\frac{t^k}{\sqrt{P(t)}}\,dt$. We have a system of differential equations for the $x_n^{(\ell)}$'s. A single derivative is isolated through Lagrange interpolation polynomials:

$$\frac{dx_n^{(\ell)}}{dn} = 2s_\ell\sqrt{P(x_n^{(\ell)})}\int_{z_0}^{z_1}\prod_{\substack{q=1\\ q\neq\ell}}\frac{t-x_n^{(q)}}{x_n^{(\ell)}-x_n^{(q)}}\,\frac{dt}{\sqrt{P(t)}}.$$

An initial condition consists of a set $x_n^{(1)},\dots,x_n^{(m-1)}$ ***and*** the signs $s_1,\dots,s_{m-1}$ (= a set of places on the ***Riemann surface*** of $\sqrt{P}$).

The vector $[x_n^{(1)},\dots,x_n^{(m-1)}]$ is a well defined function (***Jacobi-Abel function***) of the left-hand side $[n\nu_0,\dots,n\nu_{m-2}]$.

Periodicity: $x_n^{(1)},\dots,x_n^{(m-1)}$ are kept unchanged if the left-hand side of (38) is augmented by integers times the integrals $2\displaystyle\int_{z_j}^{z_{j+1}}\frac{t^k}{\sqrt{Y(t)}}\,dt$ (***periods***)

It figures: in the canonical Jacobi setting, the x_k's are ±1 and $\pm k^{-1}$. Then, (38) becomes

$$2n\mathsf{K} + \int_{x_1=-k^{-1}}^{z_0}\frac{dt}{k\sqrt{P(t)}} = \pm\int_{x_1=-k^{-1}}^{z_n}\frac{dt}{k\sqrt{P(t)}} + 2N_n i\mathsf{K}', \tag{39}$$

as the phases of $\sqrt{P(t)} = \sqrt{(t^2-1)(t^2-k^{-2})} = k^{-1}\sqrt{(1-t^2)(1-k^2t^2)}$ are $+1$, $-i$, -1, $+i$, and $+1$ on $\mathbb{R}+i\varepsilon$, and the relevant definite integrals of $dt/\sqrt{(1-t^2)(1-k^2t^2)}$ are

on	$(-\infty,-k^{-1})$	$(-k^{-1},-1)$	$(-1,1)$	$(1,k^{-1})$	$(k^{-1},+\infty)$
	K	$-i\mathsf{K}'$	$-2\mathsf{K}$	$i\mathsf{K}'$	K,

where $\mathsf{K} = \displaystyle\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_{k^{-1}}^{\infty}\frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$ (take $t=1/ku$), $\mathsf{K}' = \displaystyle\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-(1-k^2)t^2)}} = \int_1^{k^{-1}}\frac{du}{\sqrt{(u^2-1)(1-k^2u^2)}}$ (take $t = \dfrac{1}{u}\sqrt{\dfrac{u^2-1}{1-k^2}} \Rightarrow u = [1-(1-k^2)t^2]^{-1/2}$)

(see Milne-Thomson, in Abramowitz & Stegun:)

	$F(\varphi\backslash\alpha)$	Equivalent Inverse Jacobian Elliptic Function	φ	t Substitution
	17.4.41 $a\int_0^x \frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$	$\mathrm{sc}^{-1}\left(\frac{x}{b}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\tan\varphi=\frac{x}{b}$	$t=b\ \mathrm{sc}\ v$
$\cos\alpha=b/a$, $a>b$, $m=(a^2-b^2)/a^2$	**17.4.42** $a\int_x^\infty \frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$	$\mathrm{cs}^{-1}\left(\frac{x}{a}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\tan\varphi=\frac{a}{x}$	$t=a\ \mathrm{cs}\ v$
	17.4.43 $a\int_b^x \frac{dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$	$\mathrm{nd}^{-1}\left(\frac{x}{b}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\sin^2\varphi=\frac{a^2(x^2-b^2)}{x^2(a^2-b^2)}$	$t=b\ \mathrm{nd}\ v$

We clean a little bit (39): the integral from $-k^{-1}$ to -1 is $-i\mathsf{K}'$, so that we throw in the right-hand side an *even* integer multiple of $i\mathsf{K}'$ (the periods of the Jacobi elliptic integral of first kind are $4\mathsf{K}$ and $2i\mathsf{K}'$); the integral from -1 to 0 is $-\mathsf{K}$. We have

$$(2n-1)\mathsf{K} + \int_0^{z_0} \frac{dt}{k\sqrt{Y(t)}} = \mp\mathsf{K} \pm \int_0^{z_n} \frac{dt}{k\sqrt{Y(t)}} + \text{ periods.}$$

We now see in an elementary way how the Jacobi problem is solved for z_n, including the $\pm$ sign: when $n=0$, the upper sign is the obvious choice; when $n=1$, take the lower sign, the integral from 0 to z_1 is the opposite of the integral from 0 to z_0, so $z_1=-z_0$; when $n=2$, take the upper sign again, but perform the integral to z_2 by adding a complete loop about $[-1,1]$ weighting $4\mathsf{K}$ (or add $4\mathsf{K}$ to the "periods" term), etc.

With elliptic sine: $z_n = \mathrm{sn}(2n\mathsf{K} + \arg\mathrm{sn} z_0) = (-1)^n z_0$.

Now we deal with a symmetric set, BUT $x=\infty$ is sent to some value $t=-1/\gamma$, (39) is changed into

$$2n\int_{-\infty}^{-1/\gamma} \frac{dt}{k\sqrt{Y(t)}} + \int_{x_1=-k^{-1}}^{z_0} \frac{dt}{k\sqrt{Y(t)}} = \pm\int_{x_1=-k^{-1}}^{z_n} \frac{dt}{k\sqrt{Y(t)}} + 2N_n i\mathsf{K}',$$

and $z_n = \mathrm{sn}(an+b)$, with $a = \displaystyle\int_{-\infty}^{-1/\gamma} \frac{dt}{k\sqrt{Y(t)}}$, leading to a true periodic sequence whether a/K is a rational number or not.

4. **Further identities and formulas.**

4.1. **Hyperelliptic excursions.**

So, P is a polynomial of degree $2m=2g+2$, of zeros $z_1,\dots,z_{2m}$. We choose a continuous $\sqrt{P(z)}$ outside a system of cuts. So, instead of dealing with the two-sheeted Riemann surface, we will sometimes emphasize a sign ε in $\varepsilon\sqrt{P(z)}$, with $\varepsilon=+1$ or -1.

Let also z_0 different from $z_1,\dots,z_{2m}$, and $\gamma+\delta(z-z_0)$ be the two first terms of the Taylor expansion of $\sqrt{P(z)}$ about $z=z_0$.

We look at continued fraction expansions about z_0 of functions involving $\sqrt{P}$. We follow § 2.3, p. 11.

So, consider $f_n(z) := \dfrac{S_n(z) - \sqrt{P(z)}}{(z-z_0)R_n(z)}$, with polynomials S_n and R_n such that $S_n(z) = \gamma+\delta(z-z_0)+O((z-z_0)^2$, so that $f_n(z_0)=0$, $\dfrac{S_n^2(z)-P(z)}{(z-z_0)^2R_n(z)}$ = the new polynomial $R_{n+1}(z)$, and the continued fraction step $f_n(z) := \dfrac{S_n(z) - \sqrt{P(z)}}{(z-z_0)R_n(z)} = \dfrac{(z-z_0)}{\alpha_n+\beta_n(z-z_0)-(z-z_0)f_{n+1}(z)}$, as in (**??**), in p. **??**. We still have $\dfrac{S_n(z)+\sqrt{P(z)}}{R_{n+1}(z)} = \mathrm{Tayl}_1(z) - \dfrac{S_{n+1}(z)-\sqrt{P(z)}}{R_{n+1}(z)}$, so, $S_{n+1}(z) = -S_n(z) + R_{n+1}(z)\mathrm{Taylor}_1(z)$, $R_{n+2}(z) = R_n(z) - \mathrm{Tayl}_1(z)\dfrac{S_n(z)-S_{n+1}(z)}{(z-z_0)^2}$

What about the degrees? The degree of $S_n^2 - P$ can normally not be smaller than $2m = 2g+2$, so the smallest choice for the degrees of the R_ns is degree$(R_n) = g$.

The Rs and the Ss will be the xs and ys of the hyperelliptic billiard.

Let us choose X_0 = such a polynomial of degree g, Y_0 of degree $g+1$, with the two first Taylor coefficients as $\sqrt{P}$: $Y_0(z) = \gamma+\delta(z-z_0)+O((z-z_0)^2)$, and such that $P(z)-Y_0^2(z)$ is a polynomial multiple of $(z-z_0)^2X_0(z)$, $P(z)-Y_0^2(z) = (z-z_0)^2X_0(z)X_1(z)$, according to what has been found above about Q_n.

Given X_0, Y_0 is constructed through interpolation of $\sqrt{P}$ at the zeros $x_0^{(1)},\dots,x_0^{(g)}$ of X_0, here is how any choice of g signs $\varepsilon_1,\dots,\varepsilon_g$ appears:

$$Y_0(z) = \gamma+\delta(z-z_0)+(z-z_0)^2\left[\text{polynomial interpolant of degree } g-1 \text{ of value } \frac{\varepsilon_j\sqrt{P(x_0^{(j)}}-\gamma-\delta(x_0^{(j)}-z_0)}{(x_0^{(j)}-z_0)^2} \text{ at } x_0^{(j)}\right]$$

where each $\varepsilon_j = +1$ or -1. So, there are 2^g possible Y_0s for a given X_0.

The continued fraction construction may proceed with

$$\frac{\sqrt{P(z)}+Y_n(z)}{X_{n+1}(z)} = \mathrm{Taylor}_1 + \cfrac{(z-z_0)^2}{\cfrac{\sqrt{P(z)}+Y_{n+1}(z)}{X_{n+2}(z)}},$$

but the use of $f_n(z) = -\dfrac{(z-z_0)X_{n+1}(z)}{\sqrt{P(z)}+Y_n(z)} = \dfrac{Y_n(z)-\sqrt{P(z)}}{(z-z_0)X_n(z)}$ will be preferred here.

The algebraic function f_n is the root of

$$(z-z_0)X_n(z)f_n^2(z) - 2Y_n(z)f_n(z) - (z-z_0)X_{n+1}(z) = 0 \tag{40}$$

vanishing at $z=z_0$. The Xs are our object of study, they are polynomials of degree g, of which g elements are considered, such as the zeros $x_n^{(1)},\dots,x_n^{(g)}$ of X_n. Y_n is a polynomial of degree $m = g+1$ with the constraint that

$$Y_n^2(z) + (z-z_0)^2X_n(z)X_{n+1}(z) = P(z), \tag{41}$$

constraint which is satisfied if $Y_n(z) = \gamma+\delta(z-z_0)+O((z-z_0)^2)$. Then, Y_n depends on g elements too.

Finally, let the continued fraction step be

$$f_n(z) = \frac{\alpha_n(z-z_0)}{1+\beta_n(z-z_0)+\alpha_n(z-z_0)f_{n+1}(z)}. \tag{42}$$

The coefficients α_n and β_n are immediately related to the two first Taylor coefficients of f_n, which follow from (40), knowing that f_n vanishes at z_0:

$$f_n(z) = \underbrace{\alpha_n(z-z_0) - \alpha_n\beta_n(z-z0)^2 + \cdots}_{\text{from (42)}} = \underbrace{-\frac{X_{n+1}(z_0)}{2Y_n(z_0)=2\gamma}(z-z_0) - \left(\frac{X_{n+1}}{2Y_n}\right)'(z_0)(z-z0)^2 + \cdots}_{\text{from (40)}}$$

Billiard: first, recurrence relations from (42):

$$\begin{aligned} f_{n+1}(z) &= \frac{1}{f_n(z)} + \frac{1}{\alpha_n(z-z_0)} + \frac{\beta_n}{\alpha_n} \\ &= \frac{(z-z_0)X_n(z)}{Y_n(z)-\sqrt{P(z)}} + \frac{1}{\alpha_n(z-z_0)} + \frac{\beta_n}{\alpha_n} \\ &= -\frac{Y_n(z)+\sqrt{P(z)}}{(z-z_0)X_{n+1}(z)} \ (\text{from}(41)) + \frac{1}{\alpha_n(z-z_0)} + \frac{\beta_n}{\alpha_n} \end{aligned}$$

which must be $\dfrac{Y_{n+1}(z)-\sqrt{P(z)}}{(z-z_0)X_{n+1}(z)}$, so

$$Y_{n+1}(z) = \frac{1+\beta_n(z-z_0)}{\alpha_n}X_{n+1}(z) - Y_n(z), \tag{43a}$$

and we must check (41) at the next step: $Y_{n+1}^2(z) - P(z) = [(1+\beta_n(z-z_0))/\alpha_n]^2 X_{n+1}(z) - 2[(1+\beta_n(z-z_0))/\alpha_n]Y_n(z) - (z-z_0)^2 X_n(z)$, so that

$$\begin{aligned} X_{n+2}(z) &= [(1+\beta_n(z-z_0))/\alpha_n]\frac{2Y_n(z) - [(1+\beta_n(z-z_0))/\alpha_n]X_{n+1}(z)}{(z-z_0)^2} + X_n(z) \\ &= [(1+\beta_n(z-z_0))/\alpha_n]\frac{Y_n(z)-Y_{n+1}(z)}{(z-z_0)^2} + X_n(z) \end{aligned} \tag{43b}$$

which is clearly a polynomial, as Y_n and Y_{n+1} share the two first Taylor coefficients about z_0.

So, a x and a y of the hyperelliptic game are vectors of $\mathbb{C}^g$, as the zeros $(x_n^{(1)},\ldots,x_n^{(g)})$ of X_n, and the values $(\varepsilon_{n,1}\sqrt{P(x_n^{(1)})},\ldots,\varepsilon_{n,g}\sqrt{P(x_n^{(g)})}$ of Y_n at the zeros of X_n.
The g choices of the εs have to be made only at $n=0$, all the subsequent operations (43a)-(43b) being well defined. What about other choices of the signs εs? One case is clear: when ***all*** the g signs are changed, (43a) at $n-1$ shows that Y_n becomes the old Y_{n-1}, i.e., we are exploring the past! This is confirmed by (38) in the form $0 = -2n\displaystyle\int_{z_0}^{z_1}\frac{t^k}{\sqrt{P(t)}}\,dt + \sum_j \pm \int_{x_0^{(j)}}^{x_n^{(j)}}\frac{t^k}{\sqrt{P(t)}}\,dt +$ periods, showing that changing all the $\pm$s (are they the same thing as the $\varepsilon_{n,j}$ of above?) is the same thing as replacing n by $-n$.

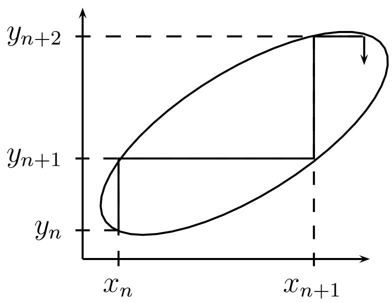
And here is a possible difference operator

$$(\mathcal{D}f)(x_n^{(1)},\ldots,x_n^{(g)}) =$$
$$\left(\frac{f(\varepsilon_{n,1}\sqrt{P(x_n^{(1)})},0,\ldots,0)-f(-\varepsilon_{n,1}\sqrt{P(x_n^{(1)})},0,\ldots,0)}{2\varepsilon_{n,1}\sqrt{P(x_n^{(1)})}},\ldots,\frac{f(0,\ldots,0,\varepsilon_{n,g}\sqrt{P(x_n^{(g)})})-f(0,\ldots,0,-\varepsilon_{n,g}\sqrt{P(x_n^{(g)})})}{2\varepsilon_{n,g}\sqrt{P(x_n^{(g)})}}\right)$$

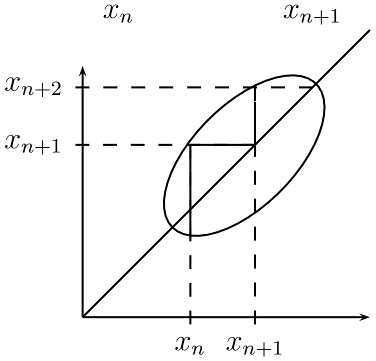
which looks like a kind of ***gradient***. Other combinations out of exterior algebra furnish all other difference operators, such as ***curls***, etc. (oh no, not me, please!).

And what may $F(X_n, Y_n)$ be here? Considering how Y_n are obtained from X_n, with the $\varepsilon_{n,1},\ldots,\varepsilon_{n,g}$ degrees of freedom, F must be an enormous contraption of degree 2^g, but such that equations may be solved by a sequence of quadratic operations, perhaps something like a chain of squares $(a+(b+(c+\cdots)^2)^2)^2$?

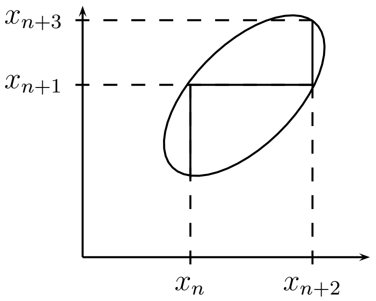
4.2. **Addition and duplication formulas.**



We start from the usual construction of $\{x_n\}$ and $\{y_n\}$ sequences, with $F(x_n, y_n) = 0$, and $F(x_n, y_{n+1}) = 0 \Rightarrow x_n$ and x_{n+1} are the two roots in x of $F(x, y_{n+1}) = 0$.



We bypassed (section 2.2, p. 9) the ys by constructing E such that $E(x_n, x_{n+1}) = 0$. By symmetry of E, the two roots of $E(x_n, y) = 0$ are $y = x_{n+1}$ and $y = x_{n-1}$. The horizontal $y = x_{n+1}$ would cut the curve $E = 0$ at a second point of abscissa x_{n+2}... If we want to see the familiar spiderweb showing $x_n, x_{n+1},\ldots$ we stop the horizontal when it meets the bisector $x = y$.



But yes! Let us keep the same construction as before!

Let us go further: bypassing the $x_{n\pm1}$-step in (15),

$$x_{n-1}+x_{n+1} = -\frac{d_{1,2}x_n^2+(2d_{0,2}+d_{1,1})x_n+d_{0,1}}{d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2}},\ x_{n-1}x_{n+1} = \frac{d_{0,2}x_n^2+d_{0,1}x_n+d_{0,0}}{d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2}},$$

$$x_n+x_{n+2} = -\frac{d_{1,2}x_{n+1}^2+(2d_{0,2}+d_{1,1})x_{n+1}+d_{0,1}}{d_{2,2}x_{n+1}^2+d_{1,2}x_{n+1}+d_{0,2}},\ x_nx_{n+2} = \frac{d_{0,2}x_{n+1}^2+d_{0,1}x_{n+1}+d_{0,0}}{d_{2,2}x_{n+1}^2+d_{1,2}x_{n+1}+d_{0,2}},$$

$$x_{n-2}+x_n = -\frac{d_{1,2}x_{n-1}^2+(2d_{0,2}+d_{1,1})x_{n-1}+d_{0,1}}{d_{2,2}x_{n-1}^2+d_{1,2}x_{n-1}+d_{0,2}},\ x_{n-2}x_n = \frac{d_{0,2}x_{n-1}^2+d_{0,1}x_{n-1}+d_{0,0}}{d_{2,2}x_{n-1}^2+d_{1,2}x_{n-1}+d_{0,2}},$$

$$x_{n-2}+x_{n+2} = -\frac{d_{1,2}x_{n-1}^2+(2d_{0,2}+d_{1,1})x_{n-1}+d_{0,1}}{d_{2,2}x_{n-1}^2+d_{1,2}x_{n-1}+d_{0,2}} - \frac{d_{1,2}x_{n+1}^2+(2d_{0,2}+d_{1,1})x_{n+1}+d_{0,1}}{d_{2,2}x_{n+1}^2+d_{1,2}x_{n+1}+d_{0,2}} - 2x_n$$
$$= -\{2d_{1,2}d_{2,2}x_{n-1}^2x_{n+1}^2 + [d_{2,2}(2d_{0,2}+d_{1,1})+d_{1,2}^2]x_{n-1}x_{n+1}(x_{n-1}+x_{n+1})$$
$$+ (d_{0,2}d_{1,2}+d_{0,1}d_{2,2})(x_{n-1}^2+x_{n+1}^2) + (2d_{0,2}+d_{1,1})d_{1,2}x_{n-1}x_{n+1}$$
$$+ [(2d_{0,2}+d_{1,1})d_{0,2}+d_{0,1}d_{1,2}](x_{n+1}+x_{n-1}) + 2d_{0,1}d_{0,2}\}$$
$$/[(d_{2,2}x_{n-1}^2+d_{1,2}x_{n-1}+d_{0,2})(d_{2,2}x_{n+1}^2+d_{1,2}x_{n+1}+d_{0,2})] - 2x_n$$

where we replace $x_{n-1}+x_{n+1}$ and $x_{n-1}x_{n+1}$ by the rational functions of second degree in x_n recalled above.

Fourth degree?

No: the denominator is the product of the four factors $(x_{n\pm1}-\rho_i)$ rearranged as the product of $(x_{n-1}-\rho_i)(x_{n+1}-\rho_i) = x_{n-1}x_{n+1} - \rho_i(x_{n-1}+x_{n+1}) + \rho_i^2 = \dfrac{d_{0,2}x_n^2+d_{0,1}x_n+d_{0,0}}{d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2}} + \dfrac{d_{1,2}x_n^2+(2d_{0,2}+d_{1,1})x_n+d_{0,1}}{d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2}}\rho_i + \rho_i^2 = \dfrac{E(x_n,\rho_i)}{d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2}}$

$$= -\frac{\overbrace{(d_{2,2}\rho_i^2+d_{1,2}\rho_i+d_{0,2})}^{=0}x_n^2 + (d_{1,2}\rho_i^2+(2d_{0,2}+d_{1,1})\rho_i+d_{0,1})x_n + d_{0,2}\rho_i^2+d_{0,1}\rho_i+d_{0,0}}{d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2}},$$ so the degree of the final expression will remain 2.

Remark that the new ρ_i is $\rho_i^{(2)} = -\dfrac{d_{0,2}\rho_i^2+d_{0,1}\rho_i+d_{0,0}}{d_{1,2}\rho_i^2+(2d_{0,2}+d_{1,1})\rho_i+d_{0,1}}$

After some struggle with a symbolic calculator, one has
denominator $= [d_{01}^2d_{22}^2 + d_{02}d_{22}d_{11}^2 + 4d_{0,2}^2d_{22}d_{11}$ etc. awful

Try again the resultant trick, from § 2.2: eliminate $x_{n\pm1}$ from (15)
$E(x_n,x_{n\pm1}) = d_{0,0}+d_{0,1}(x_n+x_{n\pm1})+d_{0,2}(x_n+x_{n\pm1})^2+d_{1,1}x_nx_{n\pm1}+d_{1,2}x_nx_{n\pm1}(x_n+x_{n\pm1})+d_{2,2}x_n^2x_{n\pm1}^2 = [d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2}]x_{n\pm1}^2 + [d_{1,2}x_n^2+(d_{1,1}+2d_{0,2})x_n+d_{0,1}]x_{n\pm1} + d_{0,2}x_n^2 + d_{0,1}x_n + d_{0,0} = 0$, and $E(x_{n\pm2},x_{n\pm1}) = 0$. I don't even try to be bright, the resultant is

$$\begin{vmatrix} d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2} & d_{1,2}x_n^2+(d_{1,1}+2d_{0,2})x_n+d_{0,1} & d_{0,2}x_n^2+d_{0,1}x_n+d_{0,0} & 0 \\ 0 & d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2} & d_{1,2}x_n^2+(d_{1,1}+2d_{0,2})x_n+d_{0,1} & d_{0,2}x_n^2+d_{0,1}x_n+d_{0,0} \\ d_{2,2}x_{n\pm2}^2+d_{1,2}x_{n\pm2}+d_{0,2} & d_{1,2}x_{n\pm2}^2+(d_{1,1}+2d_{0,2})x_{n\pm2}+d_{0,1} & d_{0,2}x_{n\pm2}^2+d_{0,1}x_{n\pm2}+d_{0,0} & 0 \\ 0 & d_{2,2}x_{n\pm2}^2+d_{1,2}x_{n\pm2}+d_{0,2} & d_{1,2}x_{n\pm2}^2+(d_{1,1}+2d_{0,2})x_{n\pm2}+d_{0,1} & d_{0,2}x_{n\pm2}^2+d_{0,1}x_{n\pm2}+d_{0,0} \end{vmatrix}$$

$= (x_{n\pm2}-x_n)^2$ times

$$\begin{vmatrix} d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2} & d_{1,2}x_n^2+(d_{1,1}+2d_{0,2})x_n+d_{0,1} & d_{0,2}x_n^2+d_{0,1}x_n+d_{0,0} & 0 \\ 0 & d_{2,2}x_n^2+d_{1,2}x_n+d_{0,2} & d_{1,2}x_n^2+(d_{1,1}+2d_{0,2})x_n+d_{0,1} & d_{0,2}x_n^2+d_{0,1}x_n+d_{0,0} \\ d_{2,2}(x_{n\pm2}+x_n)+d_{1,2} & d_{1,2}(x_{n\pm2}+x_n)+d_{1,1}+2d_{0,2} & d_{0,2}(x_{n\pm2}+x_n)+d_{0,1} & 0 \\ 0 & d_{2,2}(x_{n\pm2}+x_n)+d_{1,2} & d_{1,2}(x_{n\pm2}+x_n)+d_{1,1}+2d_{0,2} & d_{0,2}(x_{n\pm2}+x_n)+d_{0,1} \end{vmatrix}$$

$$= \left|\begin{array}{cccc} d_{2,2}(s-d)^2/4+d_{1,2}(s-d)/2+d_{0,2} & d_{1,2}(s-d)^2/4+(d_{1,1}+2d_{0,2})(s-d)/2+d_{0,1} & d_{0,2}(s-d)^2/4+d_{0,1}(s-d)/2+d_{0,0} & \\ 0 & d_{2,2}(s-d)^2/4+d_{1,2}(s-d)/2+d_{0,2} & d_{1,2}(s-d)^2/4+(d_{1,1}+2d_{0,2})(s-d)/2+d_{0,1} & d_{0,2} \\ d_{2,2}s+d_{1,2} & d_{1,2}s+d_{1,1}+2d_{0,2} & d_{0,2}s+d_{0,1} & \\ 0 & d_{2,2}s+d_{1,2} & d_{1,2}s+d_{1,1}+2d_{0,2} & \end{array}\right.$$

where $s = x_{n\pm2}+x_n$ and $d = x_{n\pm2}-x_n$. Remark that $s^2-d^2 = 4p$, where $p = x_{n\pm2}x_n$.

$$= \left|\begin{array}{cccc} d_{2,2}(s^2+d^2)/4+d_{1,2}s/2+d_{0,2} & d_{1,2}(s^2+d^2)/4+(d_{1,1}+2d_{0,2})s/2+d_{0,1} & d_{0,2}(s^2+d^2)/4+d_{0,1}s/2+d_{0,0} & 0 \\ 0 & d_{2,2}(s^2+d^2)/4+d_{1,2}s/2+d_{0,2} & d_{1,2}(s^2+d^2)/4+(d_{1,1}+2d_{0,2})s/2+d_{0,1} & d_{0,2}(s^2+d^2)/4 \\ d_{2,2}s+d_{1,2} & d_{1,2}s+d_{1,1}+2d_{0,2} & d_{0,2}s+d_{0,1} & 0 \\ 0 & d_{2,2}s+d_{1,2} & d_{1,2}s+d_{1,1}+2d_{0,2} & d_{0,2}s \end{array}\right.$$

$$= \begin{vmatrix} -d_{2,2}p + d_{0,2} & -d_{1,2}p + d_{0,1} & -d_{0,2}p + d_{0,0} & 0 \\ 0 & -d_{2,2}p + d_{0,2} & -d_{1,2}p + d_{0,1} & -d_{0,2}p + d_{0,0} \\ d_{2,2}s + d_{1,2} & d_{1,2}s + d_{1,1} + 2d_{0,2} & d_{0,2}s + d_{0,1} & 0 \\ 0 & d_{2,2}s + d_{1,2} & d_{1,2}s + d_{1,1} + 2d_{0,2} & d_{0,2}s + d_{0,1} \end{vmatrix}$$

$$= \begin{vmatrix} -d_{2,2}p + d_{0,2} & d_{0,1} - d_{1,2}d_{0,2}/d_{2,2} & d_{0,0} - d_{0,2}^2/d_{2,2} & 0 \\ 0 & d_{0,2} - d_{2,2}d_{0,0}/d_{0,2} & d_{0,1} - d_{1,2}d_{0,0}/d_{0,2} & -d_{0,2}p + d_{0,0} \\ d_{2,2}s + d_{1,2} & d_{1,1} + 2d_{0,2} - d_{1,2}^2/d_{2,2} & d_{0,1} - d_{0,2}d_{1,2}/d_{2,2} & 0 \\ 0 & d_{1,2} - d_{2,2}d_{0,1}/d_{0,2} & d_{1,1} + 2d_{0,2} - d_{1,2}d_{0,1}/d_{0,2} & d_{0,2}s + d_{0,1} \end{vmatrix}$$

$$= (d_{2,2}p - d_{0,2})(d_{0,2}p - d_{0,0}) \begin{vmatrix} d_{1,1} + 2d_{0,2} - d_{1,2}^2/d_{2,2} & d_{0,1} - d_{0,2}d_{1,2}/d_{2,2} \\ d_{1,2} - d_{2,2}d_{0,1}/d_{0,2} & d_{1,1} + 2d_{0,2} - d_{1,2}d_{0,1}/d_{0,2} \end{vmatrix}$$

$$-(d_{2,2}p - d_{0,2})(d_{0,2}s + d_{0,1}) \begin{vmatrix} d_{0,2} - d_{2,2}d_{0,0}/d_{0,2} & d_{0,1} - d_{1,2}d_{0,0}/d_{0,2} \\ d_{1,1} + 2d_{0,2} - d_{1,2}^2/d_{2,2} & d_{0,1} - d_{0,2}d_{1,2}/d_{2,2} \end{vmatrix}$$

$$-(d_{2,2}s + d_{1,2})(d_{0,2}p - d_{0,0}) \begin{vmatrix} d_{0,1} - d_{1,2}d_{0,2}/d_{2,2} & d_{0,0} - d_{0,2}^2/d_{2,2} \\ d_{1,2} - d_{2,2}d_{0,1}/d_{0,2} & d_{1,1} + 2d_{0,2} - d_{1,2}d_{0,1}/d_{0,2} \end{vmatrix}$$

$$+(d_{2,2}s + d_{1,2})(d_{0,2}s + d_{0,1}) \begin{vmatrix} d_{0,1} - d_{1,2}d_{0,2}/d_{2,2} & d_{0,0} - d_{0,2}^2/d_{2,2} \\ d_{0,2} - d_{2,2}d_{0,0}/d_{0,2} & d_{0,1} - d_{1,2}d_{0,0}/d_{0,2} \end{vmatrix}$$

Hmm

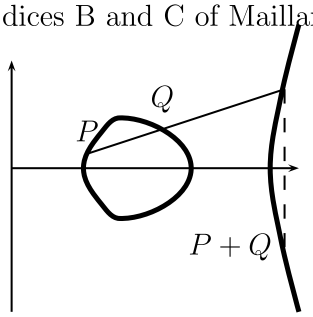
With

$$f_n(z) := \frac{S_n(z) - \sqrt{P(z)}}{(z - z_0)Z_n(z)} = \frac{(z - z_0)}{\alpha_n + \beta_n(z - z_0) - (z - z_0)f_{n+1}(z)}$$

$$= \cfrac{(z - z_0)}{\alpha_n + \beta_n(z - z_0) - \cfrac{(z - z_0)^2}{\alpha_{n+1} + \beta_{n+1}(z - z_0) - \cfrac{S_{n+2}(z) - \sqrt{P(z)}}{Z_{n+2}(z)}}}$$

etc.

Multiplication formulas on a canonical form are investigated in § 5.2 (p.307) and appendices B and C of Maillard & Boukraa's paper [31].



Finally, I return to the birational transformation to the Weierstrass cubic (20) of p. 17. $(x, y)_{n+m}$ is the result of adding to $(x, y)_n$ m times the step $(x, y)^{(\text{step},i)}$, according to the addition rule of elliptic curves. We go to the (ξ, η) plane with, say $(\tilde{\xi}, \tilde{\eta})$ for $(x, y)_n$, and $(\xi, \eta)^*$ for $(x, y)^{(\text{step},i)}$:
$\tilde{\xi} = \dfrac{1}{x_n - z_i}, \tilde{\eta} = \dfrac{X_1(x_n) + 2y_n X_2(x_n)}{(x_n - z_i)^2} = \tilde{\xi}^2[X_1(z_i + 1/\tilde{\xi}) + 2y_n X_2(z_i + 1/\tilde{\xi})]$,

$\xi^* = \dfrac{1}{x_m^{(\text{step},i)} - z_i}, \eta^* = [X_1(x_m^{(\text{step},i)}) + 2y_m^{(\text{step},i)} X_2(x_m^{(\text{step},i)})]/(x_m^{(\text{step},i)} - z_i)^2 = (\xi^*)^2[X_1(z_i + 1/\xi^*) + 2y_m^{(\text{step},i)} X_2(z_i + 1/\xi^*)]$. The straight line $\eta = \dfrac{\eta^* - \tilde{\eta}}{\xi^* - \tilde{\xi}}\xi + \dfrac{\xi^*\tilde{\eta} - \tilde{\xi}\eta^*}{\xi^* - \tilde{\xi}}$ meets the cubic

$\eta^2-\xi^4P(z_i+1/\xi)=\left[\dfrac{\eta^*-\tilde\eta}{\xi^*-\tilde\xi}\right]^2\xi^2+\cdots-P'(z_i)\xi^3-P''(z_i)\xi^2/2-\cdots$
$=\left[\dfrac{(\xi^*)^2[X_1(x_m^{(\text{step},i)})+2y_m^{(\text{step},i)}X_2(x_m^{(\text{step},i)})]-\tilde\xi^2[X_1(x_n)+2y_nX_2(x_n)]}{\xi^*-\tilde\xi}\right]^2\xi^2+\cdots-P'(z_i)\xi^3-$
$P''(z_i)\xi^2/2-\cdots$ at a third point (ξ,η): $P'(z_i)\xi=\{\ ...\}^2-P''(z_i)/2-P'(z_i)(\xi^*+\tilde\xi)$,
$\eta=\dfrac{\eta^*-\tilde\eta}{\xi^*-\tilde\xi}\xi+\dfrac{\xi^*\tilde\eta-\tilde\xi\eta^*}{\xi^*-\tilde\xi}=\dfrac{\eta^*-\tilde\eta}{\xi^*-\tilde\xi}[\{\ ...\}^2-P''(z_i)/2-P'(z_i)(\xi^*+\tilde\xi)]/P'(z_i)+\dfrac{\xi^*\tilde\eta-\tilde\xi\eta^*}{\xi^*-\tilde\xi}$
$=\dfrac{\eta^*-\tilde\eta}{\xi^*-\tilde\xi}[\{\ ...\}^2-P''(z_i)/2]/P'(z_i)+\dfrac{\tilde\xi\tilde\eta-\xi^*\eta^*}{\xi^*-\tilde\xi}$

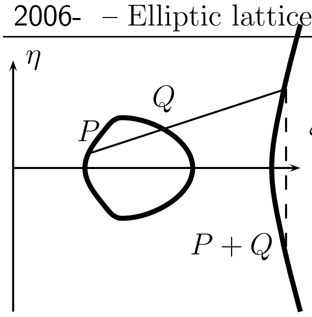
Whence an addition formula looking quite like the formula for the Weierstrass $\wp$ function:

$$\frac{P'(z_i)}{x_{m+n}-z_i}=$$
$$\left[\frac{(x_n-z_i)[X_1(x_m^{(\text{step},i)})+2y_m^{(\text{step},i)}X_2(x_m^{(\text{step},i)})]}{(x_m^{(\text{step},i)}-z_i)(x_n-x_m^{(\text{step},i)})}-\frac{(x_m^{(\text{step},i)}-z_i)[X_1(x_n)+2y_nX_2(x_n)]}{(x_n-z_i)(x_n-x_m^{(\text{step},i)})}\right]^2$$
$$-P''(z_i)/2-P'(z_i)\left[\frac{1}{x_m^{(\text{step},i)}-z_i}+\frac{1}{x_n-z_i}\right]$$
$$y_{m+n}=\frac{-(x_{m+n}-z_i)^2\eta-X_1(x_{m+n})}{2X_2(x_{m+n})}=\frac{-(x_{m+n}-z_i)^2\left[\dfrac{\eta^*-\tilde\eta}{\xi^*-\tilde\xi}\xi+\dfrac{\xi^*\tilde\eta-\tilde\xi\eta^*}{\xi^*-\tilde\xi}\right]-X_1(x_{m+n})}{2X_2(x_{m+n})}\tag{44}$$

With $2X_2y=-X_1+\sqrt{P}$, $\dfrac{P'(z_i)}{x_{m+n}-z_i}=\left[\dfrac{(x_n-z_i)\sqrt{P(x_m^{(\text{step},i)})}}{(x_m^{(\text{step},i)}-z_i)(x_n-x_m^{(\text{step},i)})}-\dfrac{(x_m^{(\text{step},i)}-z_i)\sqrt{P(x_n)}}{(x_n-z_i)(x_n-x_m^{(\text{step},i)})}\right]^2-$
$P''(z_i)/2-P'(z_i)\left[\dfrac{1}{x_m^{(\text{step},i)}-z_i}+\dfrac{1}{x_n-z_i}\right]$
$=\dfrac{(x_n-z_i)^2[P(x_m^{(\text{step},i)})=P'(z_i)(x_m^{(\text{step},i)}-z_i)+(P''(z_i)/2)(x_m^{(\text{step},i)}-z_i)^2+\cdots]}{(x_m^{(\text{step},i)}-z_i)^2(x_n-x_m^{(\text{step},i)})^2}-2\dfrac{\sqrt{P(x_m^{(\text{step},i)})}\sqrt{P(x_n)}}{(x_n-x_m^{(\text{step},i)})^2}+$
$\dfrac{(x_m^{(\text{step},i)}-z_i)^2[P(x_n)=P'(z_i)(x_n-z_i)+(P''(z_i)/2)(x_n-z_i)^2+\cdots]}{(x_n-z_i)^2(x_n-x_m^{(\text{step},i)})^2}-P''(z_i)/2$
$-P'(z_i)\left[\dfrac{1}{x_m^{(\text{step},i)}-z_i}+\dfrac{1}{x_n-z_i}\right]=$

Check: from (9), $x_{n-1}=-\dfrac{Y_1(y_n)}{Y_2(y_n)}-x_n=$

Hmm, other construction: I want x_{m+n} to be the sum of two elements $\tilde x_n$ and $\tilde x_m$ of the ***same*** sequence. It will be more convenient to add a translation and consider x_{m+n+j} with some fixed integer j (which will be set to -1 later on). From the birational transformation sending the biquadratic on a cubic



$\tilde{\xi}_n = \dfrac{1}{\tilde{x}_n - z_i}, \tilde{\eta}_n = \dfrac{X_1(\tilde{x}_n) + 2\tilde{y}_n X_2(\tilde{x}_n)}{(\tilde{x}_n - z_i)^2}$,
$\tilde{\xi}_m = \dfrac{1}{\tilde{x}_m - z_i}, \tilde{\eta}_m = \dfrac{X_1(\tilde{x}_m) + 2\tilde{y}_m X_2(\tilde{x}_m)}{(\tilde{x}_m - z_i)^2}$. Remark that $(\tilde{x}_n, \tilde{y}_{n+1})$ corresponds to $(\tilde{\xi}_n, -\tilde{\eta}_n)$.
$(\xi_{m+n+j}\eta_{m+n+j})$ is (constants times) the Weierstrass elliptic functions $\wp$ and $\wp'$ at $a(m+n+j)+b_i$. The constant term b_i, which we don't have to evaluate, fortunately, depends on i as the birational transformation contains z_i. Well, if we want $a(m+n+j)+b_i$ to be the sum of two arguments with the same constant term, the arguments must be $am + (b_i + aj)/2$ and $an + (b_i + aj)/2$, so $(\tilde{\xi}_n, \tilde{\eta}_n) =$ cts $\times$ $\wp$ and $\wp'$ at $an + (b_i + aj)/2$.
Now, $(\xi_{m+n+j}, \eta_{m+n+j})$ is the sum of $(\tilde{\xi}_m, \tilde{\eta}_m)$ and $(\tilde{\xi}_n, \tilde{\eta}_n)$ according to the addition rule of elliptic curves. The straight line $\eta = \dfrac{\tilde{\eta}_n - \tilde{\eta}_m}{\tilde{\xi}_n - \tilde{\xi}_m}\xi + \dfrac{\tilde{\xi}_n\tilde{\eta}_m - \tilde{\xi}_m\tilde{\eta}_n}{\tilde{\xi}_n - \tilde{\xi}_m}$ meets the cubic $0 = \eta^2 - \xi^4 P(z_i + 1/\xi) = \left[\dfrac{\tilde{\eta}_n - \tilde{\eta}_m}{\tilde{\xi}_n - \tilde{\xi}_m}\right]^2 \xi^2 + \cdots - P'(z_i)\xi^3 - P''(z_i)\xi^2/2 - \cdots$
$= \left[\dfrac{(\tilde{x}_m - z_i)^2[X_1(\tilde{x}_n) + 2\tilde{y}_n X_2(\tilde{x}_n)] - (\tilde{x}_n - z_i)^2[X_1(\tilde{x}_m) + 2\tilde{y}_m X_2(\tilde{x}_m)]}{(\tilde{x}_m - z_i)(\tilde{x}_n - z_i)(\tilde{x}_m - \tilde{x}_n)}\right]^2 \xi^2 + \cdots - P'(z_i)\xi^3 - P''(z_i)\xi^2/2 - \cdots$ at a third point (ξ, η): $P'(z_i)\xi = \{\ \ldots\}^2 - P''(z_i)/2 - P'(z_i)(\tilde{\xi}_n + \tilde{\xi}_m)$,
$\eta = \dfrac{\tilde{\eta}_n - \tilde{\eta}_m}{\tilde{\xi}_n - \tilde{\xi}_m}\xi + \dfrac{\tilde{\xi}_n\tilde{\eta}_m - \tilde{\xi}_m\tilde{\eta}_n}{\tilde{\xi}_n - \tilde{\xi}_m} = \dfrac{\tilde{\eta}_n - \tilde{\eta}_m}{\tilde{\xi}_n - \tilde{\xi}_m}[\{\ \ldots\}^2 - P''(z_i)/2 - P'(z_i))(\tilde{\xi}_n + \tilde{\xi}_m)]/P'(z_i) + \dfrac{\tilde{\xi}_n\tilde{\eta}_m - \tilde{\xi}_m\tilde{\eta}_n}{\tilde{\xi}_n - \tilde{\xi}_m}$

Whence

$$\frac{P'(z_i)}{x_{m+n+j} - z_i} = \left[\frac{(\tilde{x}_m - z_i)[X_1(\tilde{x}_n) + 2\tilde{y}_n X_2(\tilde{x}_n)]}{(\tilde{x}_n - z_i)(\tilde{x}_m - \tilde{x}_n)} - \frac{(\tilde{x}_n - z_i)[X_1(\tilde{x}_m) + 2\tilde{y}_m X_2(\tilde{x}_m)]}{(\tilde{x}_m - z_i)(\tilde{x}_m - \tilde{x}_n)}\right]^2 - P''(z_i)/2 - P'(z_i)\left[\frac{1}{\tilde{x}_m - z_i} + \frac{1}{\tilde{x}_n - z_i}\right] \quad (45)$$

With $2X_2 y = -X_1 + \sqrt{P}$, $\dfrac{P'(z_i)}{x_{m+n+j} - z_i} = \left[\dfrac{(\tilde{x}_m - z_i)\sqrt{P(\tilde{x}_n)}}{(\tilde{x}_n - z_i)(\tilde{x}_m - \tilde{x}_n)} - \dfrac{(\tilde{x}_n - z_i)\sqrt{P(\tilde{x}_m)}}{(\tilde{x}_m - z_i)(\tilde{x}_m - \tilde{x}_n)}\right]^2 - P''(z_i)/2 - P'(z_i)\left[\dfrac{1}{\tilde{x}_n - z_i} + \dfrac{1}{\tilde{x}_m - z_i}\right]$

Check: $x_n = \sin(\theta_0 + nh)$, y_n and $y_{n+1} = x_{n\mp 1/2} \Rightarrow y = x\cos(h/2) \pm \sin(h/2)\sqrt{1 - x^2}$: $y^2 - 2\cos(h/2)xy + x^2 - \sin^2(h/2) = 0, P(x) = X_1^2(x) - 4X_0(x)X_2(x) = 4\sin^2(h/2)(1 - x^2)$.
With $z_i = 1$,

$$\xi_n = \frac{1}{\sin(\theta_0 + nh) - 1} = -\frac{1}{2\sin^2(\pi/4 - \theta_0/2 - nh/2)}, \eta_n = \frac{-2\cos(h/2)\sin(\theta_0 + nh) + 2\sin(\theta_0 + (n - 1/2)h)}{(\sin(\theta_0 + nh) - 1)^2} =$$

$\dfrac{-2\sin(h/2)\cos(\theta_0+nh)}{(\sin(\theta_0+nh)-1)^2} = \dfrac{-\sin(h/2)\cos(\pi/4-\theta_0/2-nh/2)}{\sin^3(\pi/4-\theta_0/2-nh/2)}$; $\eta^2 = 4\sin^2(h/2)\xi^4(1-x^2) = 4\sin^2(h/2)\xi^4[1-(1+1/\xi)^2] = 4\sin^2(h/2)\xi^2(-2\xi-1)$. Also, let $\tilde{x}_n = \sin(\tilde{\theta}_0+nh)$. Let $\theta = \tilde{\theta}_0+nh$ and $\varphi = \tilde{\theta}_0+mh$, then we expect $x_{m+n} = \sin(\theta+\varphi)$. Is $\dfrac{P'(1)}{x_{m+n}-1} = \dfrac{8\sin^2(h/2)}{1-\sin(\theta_0+(m+n)h)}$ the right-hand side of (45)? The big square above is

$$\left[\frac{(\sin\varphi-1)[-2\cos(h/2)\sin\theta+2\sin(\theta-h/2)]}{(\sin\theta-1)(\sin\varphi-\sin\theta)} - \frac{(\sin\theta-1)[-2\cos(h/2)\sin\varphi+2\sin(\varphi-h/2)]}{(\sin\varphi-1)(\sin\varphi-\sin\theta)}\right]^2 =$$

$$4\sin^2(h/2)\left[\frac{-\cos\theta(\sin\varphi-1)}{(\sin\theta-1)(\sin\varphi-\sin\theta)} + \frac{\cos\varphi(\sin\theta-1)}{(\sin\varphi-1)(\sin\varphi-\sin\theta)}\right]^2$$

$$= \frac{4\sin^2(h/2)}{(\sin\varphi-\sin\theta)^2}\left[\frac{-(\sin\theta+1)(\sin\varphi-1)^2}{\sin\theta-1} - \frac{(\sin\varphi+1)(\sin\theta-1)^2}{\sin\varphi-1} - 2\cos\theta\cos\varphi\right]$$

$$= 4\sin^2(h/2)\left[-\frac{\sin\theta+1}{\sin\theta-1} + 2 + \frac{\cos^2\theta-2\cos\theta\cos\varphi+\cos^2\varphi}{(\sin\varphi-\sin\theta)^2} - \frac{\sin\varphi+1}{\sin\varphi-1}\right]$$

to which we add $-P''(z_i)/2 - P'(z_i)\left[\dfrac{1}{\tilde{x}_n-z_i} + \dfrac{1}{\tilde{x}_m-z_i}\right] = 4\sin^2(h/2) + 8\sin^2(h/2)\left[\dfrac{1}{\sin\theta-1} + \dfrac{1}{\sin\varphi-1}\right]$ which makes $4\sin^2(h/2)\left[1+\dfrac{\cos^2\theta-2\cos\theta\cos\varphi+\cos^2\varphi}{(\sin\varphi-\sin\theta)^2}\right] = 4\sin^2(h/2)\dfrac{2-2\cos(\theta-\varphi)}{(\sin\varphi-\sin\theta)^2} = \dfrac{4\sin^2(h/2)}{\cos^2((\theta+\varphi)/2)} = \dfrac{8\sin^2(h/2)}{1+\cos(\theta+\varphi)} = \dfrac{8\sin^2(h/2)}{1-\sin(\theta+\varphi-\pi/2)}$, so that $\theta_0+(m+n)h = 2\tilde{\theta}_0+(m+n)h-\pi/2$. Where is this $\pi/2$ coming from??

Ooh, that's because the addition formula associated to the straight line in the (ξ,η) plane is related to the Weierstrass function which has a double pole at the origin. So, we must choose the parameter in $\xi = (x-z_i)^{-1}$ such that the origin of the parameter occurs at $x = z_i$. This means here $x = \sin(\pi/2+(\theta-\pi/2))$, so that $\theta-\pi/2$ is the parameter of interest. YES! Then, $\tilde{x}_n = \sin(\tilde{\theta}_0+nh) = \sin(\theta_0/2+\pi/4+nh) = \sin(\pi/2+(\theta_0/2-\pi/4)+nh)$ and, indeed, $\theta_0+(m+n)h-\pi/2$ is the sum of $\theta_0/2+mh-\pi/4$ and $\theta_0+nh-\pi/4$.

Return to (45) with a generic polynomial P.

Let $n^{(i)}$ be such that $x_{n^{(i)}} = z_i$. The natural parameter with respect to the addition rule is then $n-n^{(i)}$, so $m+n-n^{(i)} = [m-n^{(i)}/2]+[n-n^{(i)}/2]$, and we have $\tilde{x}_n = x_{n^{(i)}+[n-n^{(i)}/2]} = x_{n^{(i)}/2+n}$. Actually, one must write $\tilde{x}_n^{(i)}$, as this construction depends on i. But how is it done?

We don't need $n^{(i)}$, just that $\tilde{x}_0^{(i)}$ produces x_0 by duplication: $\tilde{x}_0 = x_{n^{(i)}+[-n^{(i)}/2]}$ duplicates into $x_{n^{(i)}+2[-n^{(i)}/2]} = x_0$ of course.

A wonderful simplification holds when $m=n$. Then, $\sqrt{P(\tilde{x}_m)} = \sqrt{P(\tilde{x}_n)} + (\tilde{x}_m-\tilde{x}_n)P'(\tilde{x}_n)/(2\sqrt{P(\tilde{x}_n)}) + \cdots$, and

$$\frac{P'(z_i)}{x_{2n+j}-z_i} = \left[\frac{2\sqrt{P(\tilde{x}_n)}}{\tilde{x}_n-z_i} - \frac{P'(\tilde{x}_n)}{2\sqrt{P(\tilde{x}_n)}}\right]^2 - \frac{P''(z_i)}{2} - \frac{2P'(z_i)}{\tilde{x}_n-z_i} = \frac{4P(\tilde{x}_n)}{(\tilde{x}_n-z_i)^2} - 2\frac{P'(\tilde{x}_n)+P'(z_i)}{\tilde{x}_n-z_i} + \frac{(P'(\tilde{x}_n))^2}{4P(\tilde{x}_n)} - \frac{P''(z_i)}{2}.$$

In the (ξ,η) plane: the tangent to the cubic $\eta^2 = P_3(\xi)$ at $(\tilde{\xi}_n, \tilde{\eta}_n)$ meets the curve at $(\xi_{2n+j}, -\eta_{2n+j})$: $\eta = \tilde{\eta}_n + \dfrac{P_3'(\tilde{\xi}_n)}{2\sqrt{P_3(\tilde{\xi}_n)} = 2\tilde{\eta}_n}(\xi - \tilde{\xi}_n)$ and $\eta^2 = P_3(\xi) \Rightarrow \dfrac{(P_3'(\tilde{\xi}_n))^2}{4P_3(\tilde{\xi}_n)}(\xi-\tilde{\xi}_n)^2 +$ $P_3'(\tilde{\xi}_n)(\xi-\tilde{\xi}_n) + \tilde{\eta}_n^2 - (P_3'''/6)(\xi-\tilde{\xi}_n)^3 - (P_3''(\tilde{\xi}_n)/2)(\xi-\tilde{\xi}_n)^2 - P_3'(\tilde{\xi}_n)(\xi-\tilde{\xi}_n) - P_3(\tilde{\xi}_n) = 0 \Rightarrow$ $\xi_{2n+j} = \tilde{\xi}_n + \left[\dfrac{(P_3'(\tilde{\xi}_n))^2}{4P_3(\tilde{\xi}_n)} - (P_3''(\tilde{\xi}_n)/2)\right]/(P_3'''/6) = \dfrac{(3\tilde{\xi}_n^2 + 2\alpha\tilde{\xi}_n + \beta)^2}{4\tilde{\xi}_n^3 + 4\alpha\tilde{\xi}_n^2 + 4\beta\tilde{\xi}_n + 4\gamma} - \alpha - 2\tilde{\xi}_n =$ $\dfrac{\tilde{\xi}_n^4 - 2\beta\tilde{\xi}_n^2 - 8\gamma\tilde{\xi}_n + \beta^2 - 4\alpha\gamma}{4\tilde{\xi}_n^3 + 4\alpha\tilde{\xi}_n^2 + 4\beta\tilde{\xi}_n + 4\gamma = 4\tilde{\eta}_n^2}$ if $P_3(x) = x^3 + \alpha x^2 + \beta x + \gamma$,

does not look like a square, ***BUT***(2 Jan. 2012)

$$\xi_{2n+j} - \lambda = \frac{\tilde{\xi}_n^4 - 2\beta\tilde{\xi}_n^2 - 8\gamma\tilde{\xi}_n + \beta^2 - 4\alpha\gamma}{4\tilde{\xi}_n^3 + 4\alpha\tilde{\xi}_n^2 + 4\beta\tilde{\xi}_n + 4\gamma = 4\tilde{\eta}_n^2} - \lambda = \frac{\tilde{\xi}_n^4 - 4\lambda\tilde{\xi}_n^3 - (4\alpha\lambda + 2\beta)\tilde{\xi}_n^2 - (4\beta\lambda + 8\gamma)\tilde{\xi}_n + \beta^2 - 4\alpha\gamma - 4\gamma\lambda}{4\tilde{\eta}_n^2}$$

$= \dfrac{(\tilde{\xi}_n^2 - 2\lambda\tilde{\xi}_n - 2\lambda^2 - 2\alpha\lambda - \beta)^2}{4\tilde{\eta}_n^2}$ if $4\lambda(2\lambda^2 + 2\alpha\lambda + \beta) = -4\beta\lambda - 8\gamma$ and $(2\lambda^2 + 2\alpha\lambda + \beta)^2 =$ $\beta^2 - 4\alpha\gamma - 4\gamma\lambda \Leftrightarrow P_3(\lambda) = 0 \Leftrightarrow \lambda = \lambda_{i,k} = -1/(z_i-$ some other $z_k) \Leftrightarrow x_{2n+j} = z_i$

$$+\frac{1}{\xi_{2n+j} = -\dfrac{1}{z_i - z_k} + \dfrac{(\tilde{\xi}_n^2 - 2\lambda\tilde{\xi}_n - 2\lambda^2 - 2\alpha\lambda - \beta)^2}{4\tilde{\eta}_n^2}} \Leftrightarrow x_{2n+j} - z_k = \frac{(z_i - z_k)(\tilde{\xi}_n^2 - 2\lambda\tilde{\xi}_n - 2\lambda^2 - 2\alpha\lambda - \beta)^2}{4\xi_{2n+j}\tilde{\eta}_n^2}.$$

It figures: from (21) p. 17, ξ is a Weierstrass $\wp$ function, and the three differences $\wp - e_k = \xi - \lambda_{i,k}$ are the squares of the fundamental elliptic functions of Halphen (see Appendix C of Robert Conte & Micheline Musette's *The Painlevé Handbook*, Springer 2008).

Strange: squares are recovered with $x_{2n+j} - z_k$, $k \neq i$, although $\xi_{2n+j} = (x_{2n+j} - z_i)^{-1}$, $\tilde{\xi}_n$, and $(\alpha,\beta,\gamma) = (P''(z_i)/2, P'''(z_i)/6, P''''/24)/P'(z_i)$ depend on i. Incidentally, $2\lambda^2 + 2\alpha\lambda + \beta = -\beta - 2\gamma/\lambda_{i,k} = [-P'''(z_i)/6 - 2P''''/(24\lambda_{i,k})]/P'(z_i) = [-4z_i + \sum_1^4 z_r + 2(z_i - z_k)]/((z_i - z_k)(z_i - z_m)(z_i - z_p)) = [-z_i - z_k + z_m + z_p]/((z_i - z_k)(z_i - z_m)(z_i - z_p))$, where k, m, p are the three indexes $\neq i$.

$$\frac{x_{2n+j} - z_k}{x_{2n+j} - z_i} = \frac{(z_i - z_k)(\tilde{\xi}_n^2 - 2\lambda\tilde{\xi}_n - 2\lambda^2 - 2\alpha\lambda - \beta)^2}{4\tilde{\eta}_n^2}, \text{ with } \lambda = \lambda_{i,k} = \frac{1}{z_k - z_i}$$

$$\frac{x_{2n+j} - z_k}{x_{2n+j} - z_m} = \frac{(z_i - z_k)(\tilde{\xi}_n^2 - 2\lambda_{i,k}\tilde{\xi}_n - 2\lambda_{i,k}^2 - 2\alpha\lambda_{i,k} - \beta)^2}{(z_i - z_m)(\tilde{\xi}_n^2 - 2\lambda_{i,m}\tilde{\xi}_n - 2\lambda_{i,m}^2 - 2\alpha\lambda_{i,m} - \beta)^2}$$

$$\eta_{2n+j} = -\tilde{\eta}_n - \frac{P_3'(\tilde{\xi}_n)}{2\tilde{\eta}_n}\left[\frac{(P_3'(\tilde{\xi}_n))^2}{4P_3(\tilde{\xi}_n)} - (P_3''(\tilde{\xi}_n)/2)\right]/(P_3'''/6),$$

From (9), $x_{2n+j+1} = -\dfrac{Y_1(y_{2n+j+1})}{Y_2(y_{2n+j+1})} - x_{2n+j}$

For any $k \in \{1,2,3,4\}$, $x_{2n+j+1} - z_k = -\dfrac{Y_1(y_{2n+j+1}) + 2Y_2(y_{2n+j+1})z_k}{Y_2(y_{2n+j+1})} - (x_{2n+j} - z_k) =$ $\dfrac{Y_0(y_{2n+j+1}) + Y_1(y_{2n+j+1})z_k + Y_2(y_{2n+j+1})z_k^2}{Y_2(y_{2n+j+1})(x_{2n+j} - z_k)}$,

Hmm, x_{2n+j+1} follows from x_{2n+j} by adding the step in (22a)-(22b), p. 22, so $(\xi_{2n+j+1}, \eta_{2n+j+1}) =$ $(\xi_{2n+j}, \eta_{2n+j}) + (\xi_{\text{step}}, \eta_{\text{step}}) = (\tilde{\xi}_n, \tilde{\eta}_n) + (\tilde{\xi}_n, e\tilde{t}a_n) + (\xi_{\text{step}}, \eta_{\text{step}}) = (\tilde{\xi}_n, \tilde{\eta}_n) + (\tilde{\xi}_{n+1}, \tilde{\eta}_{n+1})$

$$\frac{\tilde\eta_{n+1}-\tilde\eta_n}{\tilde\xi_{n+1}-\tilde\xi_n}=\frac{-\eta_{2n+j+1}-\tilde\eta_n}{\xi_{2n+j+1}-\tilde\xi_n},\ (\eta_{2n+j+1})^2=P_3(\xi_{2n+j+1})=\left[\tilde\eta_n+(\xi_{2n+j+1}-\tilde\xi_n)\frac{\tilde\eta_{n+1}-\tilde\eta_n}{\tilde\xi_{n+1}-\tilde\xi_n}\right]^2,$$

$$\xi_{2n+j+1}=\frac{1}{P_3'''/6}\left[-P_3''(0)/2+\left(\frac{\tilde\eta_{n+1}-\tilde\eta_n}{\tilde\xi_{n+1}-\tilde\xi_n}\right)^2\right]-\tilde\xi_{n+1}-\tilde\xi_n$$

Also, $(\tilde\xi_{n+1},\tilde\eta_{n+1})=(\tilde\xi_n,\tilde\eta_n)+(\xi_{\text{step}},\eta_{\text{step}})$:
$\dfrac{\tilde\eta_n-\eta_{\text{step}}}{\tilde\xi_n-\xi_{\text{step}}}=\dfrac{-\tilde\eta_{n+1}-\eta_{\text{step}}}{\tilde\xi_{n+1}-\xi_{\text{step}}}$

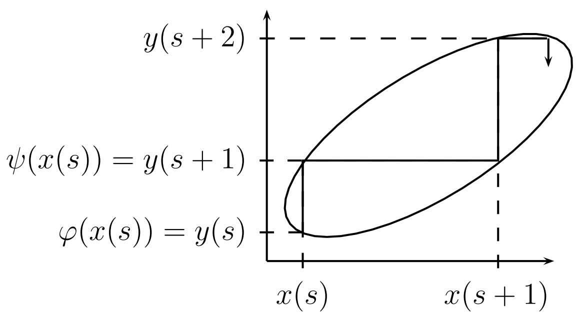
No, there is nothing strange: $\wp-e_i$ is the square of an elliptic function because it has a double zero AND a double pole. Now, $x_{2n+j}-z_i$ has a double zero, but... Ah, when $x\to\infty$, and $F(x,y)=Y_2(y)x^2+Y_1(y)x+Y_0(y)=0\Rightarrow y=$ the roots u and v of $Y_2(y)=0$.

5. General difference operator

$$\mathcal{D}f(x)=\frac{f(\psi(x))-f\varphi(x))}{\psi(x)-\varphi(x))}$$

must be a rational function of degree 2 for $f(x)=1/(x+c)$:

$$-\frac{1}{(\psi(x)+c)(\varphi(x)+c)}=-\frac{1}{\psi(x)\varphi(x)+c(\psi(x)+\varphi(x))+c^2}$$



New notation: $F(x,y)=X_2(x)y^2+X_1(x)y+X_0(x)=0$, where the X's are polynomials of degree $\leqslant 2$. So, $\varphi+\psi=-X_1/X_2$ and $\varphi\psi=X_0/X_2\Rightarrow$ a (nonsymmetric) biquadratic relation between x and $\varphi(x)$ or $\psi(x)$. We now are sure to have two elliptic lattices for the x's and the y's.

Relation between y_n and y_{n+1}: use $y_n+y_{n+1}=-X_1(x_n)/X_2(x_n)$ and $y_ny_{n+1}=X_0(x_n)/X_2(x_n)$, and consider the resultant of the two polynomials in x_n
$(y_n+y_{n+1})X_2(x_n)+X_1(x_n)$ and $y_ny_{n+1}X_2(x_n)-X_0(x_n)$,
which is the full symmetric biquadratic relation between y_n and y_{n+1}.

Of course, the same holds for x_n and x_{n+1}, just by considering $F(x,y)=Y_2(y)x^2+Y_1(y)x+Y_0(y)$. The polynomials are now
$(x_n+x_{n+1})Y_2(y_{n+1})+Y_1(y_{n+1})$ and $x_nx_{n+1}Y_2(y_{n+1})-Y_0(y_{n+1})$.

Exemple: with $F(x,y)=(\underbrace{x^2/2+x+1}_{X_2(x)})y^2+(\underbrace{2x^2-2x-3}_{X_1(x)})y+x^2+x+1$, one has
$(49/4*x_{n+1}^2+59/2*x_{n+1}+71/4)*x_n^2+(59/2*x_{n+1}^2+85/2*x_{n+1}+17/2)*x_n+71/4*x_{n+1}^2+17/2*x_{n+1}+1$
and $(37/4*y_{n+1}^2+3*y_{n+1}+1/4)*y_n^2+(3*y_{n+1}^2-55/2*y_{n+1}-9/2)*y_n+1/4*y_{n+1}^2-9/2*y_{n+1}+21$.

Recurrence relations of the x's and the y's:
$x_{n-1}+2x_n+x_{n+1}=-\dfrac{Y_1(y_n)}{Y_2(y_n)}-\dfrac{Y_1(y_{n+1})}{Y_2(y_{n+1})}$ is a symmetric rational function of y_n and y_{n+1}, so, a rational function of x_n.

For the y's: $y_{n-1}+y_n+y_{n+1} = -\dfrac{X_1(x_{n-1})}{X_2(x_{n-1})} - \dfrac{X_1(x_n)}{X_2(x_n)}$ is a symmetric rational function of x_{n-1} and x_n, so, a rational function of y_n.

6. Recurrences of biorthogonal rational functions.

From excerpts of Spiridonov & Zhedanov [57], also
A. Zhedanov, Biorthogonal rational functions and generalized eigenvalue problem, *J. Approx. Theory* **101** (1999), no. 2, 303–329, and [70].
Also Brezinski, Iserles, Ismail, Masson, Norsett.

6.1. **Padé and interpolatory continued fractions.**

6.1.1. *Padé.* $\dfrac{\alpha_0}{x-\beta_0+\dfrac{\alpha_1}{x-\beta_1+\dfrac{\ddots}{+\dfrac{\alpha_{n-1}}{x-\beta_{n-1}}}}}$ matches a given Laurent expansion $c_0/x+$ $c_1/x^2+\cdots$ at ∞ up to the c_{2n-1}/x^{2n} term. Numerators and denominators satisfy the recurrence relation $P_{n+1}(x)=(x-\beta_n)P_n(x)+\alpha_n P_{n-1}(x)$, suggesting some kind of (formal?) orthogonality. This is even more obvious in the matrix-vector setting

$$\begin{bmatrix} \beta_0 & \sqrt{-\alpha_1} & & & \\ \sqrt{-\alpha_1} & \beta_1 & \sqrt{-\alpha_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ \vdots \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ \vdots \end{bmatrix}.$$

If one wants to approximate a Taylor expansion about the origin, just take $z=1/x$ and rewrite the continued fraction as $\dfrac{\alpha_0 z}{1-z\beta_0+\dfrac{\alpha_1 z^2}{1-\beta_1 z+\dfrac{\ddots}{+\dfrac{\alpha_{n-1}z^2}{1-\beta_{n-1}z}}}}$ which matches a given Taylor-Maclaurin expansion up to the z^{2n} term.

A slower progression is achieved with the *corresponding continued fraction* to a given expansion

$$\frac{\alpha_0' z}{1+\dfrac{\alpha_1' z}{1+\dfrac{\ddots}{+\alpha_n' z}}}$$

6.1.2. *Interpolation.* Rational interpolations to a given set of values at $x=y_0,y_1,\ldots$ (yes, the relevant set will be a $y-$lattice) are achieved by

$$\frac{q_n(x)}{p_n(x)} = \alpha'_0 + \cfrac{x-y_0}{\alpha'_1 + \cfrac{x-y_1}{\ddots \, \cfrac{}{\alpha'_{n-1} + \cfrac{x-y_{n-1}}{\alpha'_n}}}}$$

(Thiele) which agree with a given set up to $x = y_n$.

Remark that the degrees are n/n for q_{2n}/p_{2n}, $(n+1)/n$ for q_{2n+1}/p_{2n+1}. Contract the recurrences

$p_{2n}(x) = \alpha'_{2n} p_{2n-1}(x) + (x-y_{2n-1})p_{2n-2}(x), p_{2n+1}(x) = \alpha'_{2n+1}p_{2n}(x)+(x-y_{2n})p_{2n-1}(x), p_{2n+2}(x) = \alpha'_{2n+2}p_{2n+1}(x) + (x-y_{2n+1})p_{2n}(x)$,

into $p_{2n+2}(x) = \alpha'_{2n+2}\left[\alpha'_{2n+1}p_{2n}(x) + (x-y_{2n})\dfrac{p_{2n}(x)-(x-y_{2n-1})p_{2n-2}(x)}{\alpha'_{2n}}\right] + (x-y_{2n+1})p_{2n}(x)$,

so, a recurrence relation for $P_n = \text{const. } p_{2n}$:

$$P_{n+1}(x) = (x-\beta_n)P_n(x) + \alpha_n(x-y_{2n-1})(x-y_{2n})P_{n-1}(x).$$

Consider now rational functions $R_n(x) = \dfrac{P_n(x)}{(x-y_2)(x-y_4)\cdots(x-y_{2n})}$:

$$(x-y_{2n+2})R_{n+1}(x) = (x-\beta_n)R_n(x) + \alpha_n(x-y_{2n-1})R_{n-1}(x),$$

so that the matrix-vector setting is now

$$\begin{bmatrix} \beta_0 & -y_2 & & & \\ \alpha_1 y_1 & \beta_1 & -y_4 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} R_0(x) \\ R_1(x) \\ \vdots \\ \vdots \end{bmatrix} = x \begin{bmatrix} 1 & -1 & & & \\ \alpha_1 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} R_0(x) \\ R_1(x) \\ \vdots \\ \vdots \end{bmatrix}.$$

So, $\{R_0, R_1, \dots\}$ is a right eigenvector which is in some way biorthogonal to the set of left eigenvectors $\{T_0, T_1, \dots\}$ satisfying the recurrence

$$\alpha_{n+1}(x-y_{2n+1})T_{n+1}(x) = -(x-\beta_n)T_n(x) + (x-y_{2n})T_{n-1}(x),$$

which is of the same structure that the recurrence of the R_n's, but with the odd x's interchanged with the even x's. Actually, $T_n(x)$ is a constant times the same $P_n(x)$ as before, divided by $(x-y_1)(x-y_3)\dots(x-y_{2n-1})$. $P_n(x)$ is a constant times the determinant

$$\begin{vmatrix} x-\beta_0 & -(x-y_2) & & & \\ \alpha_1(x-y_1) & x-\beta_1 & -(x-y_4) & & \\ & \ddots & \ddots & \ddots & \\ & & & \alpha_{n-1}(x-y_{2n-3}) & x-\beta_{n-1} \end{vmatrix}$$

6.2. **Biorthogonality and orthogonality.**

But what is the bilinear form exhibiting the biorthogonality condition? Could it be a linear form $\mathscr{L}$ applied to a product? Let us try, for $k < n$,

$$0 = \mathscr{L}(R_n T_k) = \mathscr{L}\left(\frac{P_n(x)}{(x-y_2)\dots(x-y_{2n})}\frac{P_k(x)}{(x-y_1)\dots(x-y_{2k-1})}\right)$$
$$= \mathscr{L}\left(\frac{P_n(x)P_k(x)(x-y_{2k+1})\dots(x-y_{2n-1})}{(x-y_1)(x-y_2)\dots(x-y_{2n})}\right)$$

showing that P_n is plainly orthogonal with respect to the scalar product $\mathscr{L}\left(\dfrac{g_1(x)g_2(x)}{(x-y_1)(x-y_2)\dots(x-y_{2n})}\right)$.

Confirmation: as Q_n/P_n interpolates f at $y_0,\dots,y_{2n}$,

$$P_n(x)f(x) - Q_n(x) = 0 \quad \text{at } y_0,\dots,y_{2n}$$

$$P_n(x)g(x)f(x) - g(x)Q_n(x) = 0 \quad \text{at } y_0,\dots,y_{2n}$$

for any polynomial g of degree $< n$. Perform the divided difference of order $2n$ on $y_0,\dots,y_{2n}$:

$$[y_0,\dots,y_{2n}]_{P_n g f} = 0, \quad \forall \text{ polynomial } g \text{ of degree } < n.$$

The divided difference is $\displaystyle\sum_{j=0}^{2n}\frac{P_n(y_j)g(y_j)\,f(y_j)}{\omega'_{2n}(y_j)}$, where $\omega_{2n}(x) = (x-y_0)\dots(x-y_{2n})$.

If f is holomorphic in a domain containing $y_0,\dots,y_{2n}$, the divided difference is

$$\frac{1}{2\pi i}\int_C \frac{P_n(t)g(t)\,f(t)\,dt}{(t-y_0)\dots(t-y_{2n})}$$

Even simpler: if $f(x) = \displaystyle\sum_0^{N-1}\frac{\rho_k}{x-x'_k}$,

$$\langle g_1, g_2\rangle = \sum_0^{N-1}\rho_k g_1(x'_k)g_2(x'_k).$$

6.3. **Example: the exponential function (Iserles).**

Rational interpolations to e^{ax} at $x = 0, h, 2h, \dots$

$e^{ax} = \alpha'_0/(1+\cdots)$ at $x = 0 : \alpha'_0 = 1$,

$e^{-ax} = 1 + \alpha'_1 x/(1+\cdots)$ at $x = h : \alpha'_1 = (e^{-ah}-1)/h$, $\dfrac{\alpha'_1 x}{e^{-ax}-1} = 1 + \alpha'_2(x-h)$ at $x = 2h : \alpha'_2 = \dfrac{1-e^{-ah}}{h(1+e^{-ah})}$,

$\alpha'_{2j-1} = \dfrac{1-e^{ah}}{h(j-1+je^{ah})(1+e^{ah})}$, $\alpha'_{2j} = \dfrac{e^{ah}(e^{ah}-1)}{h(j-1+je^{ah})(1+e^{ah})}$

7. Elliptic Riccati equations.

An elliptic Riccati equation is

$$a(x)\frac{f(\psi(x))-f(\varphi(x))}{\psi(x)-\varphi(x)} = b(x)f(\varphi(x))f(\psi(x)) + c(x)(f(\varphi(x))+f(\psi(x))) + d(x) \qquad (46)$$

with φ and ψ as in § 5, p. 52, and where a, b, c, and d are rational functions.

We consider now rational interpolation according to the setting of § 6.1.2 above. Why is this relevant?

A first-order difference equation of the kind (46) relates $f(y_0)$ to $f(y_1)$ when $x = x_0$; $f(y_1)$ to $f(y_2)$ when $x = x_1$, etc. So, we have information to introduce in the interpolation: from $f_n(x) = \alpha'_n + \dfrac{x-y_n}{f_{n+1}(x)}$, $f_n(y_n) = \alpha'_n$ and $f_n(y_{n+1}) = \alpha'_n + (y_{n+1}-y_n)/\alpha'_{n+1}$, so $\alpha'_{n+1} = (y_{n+1}-y_n)/(f_n(y_{n+1})-f_n(y_n))$.

Now, from (46) at $x = x_n$, so $\varphi(x) = y_n$ and $\psi(x) = y_{n+1}$, $\dfrac{a_n(x_n)}{\alpha'_{n+1}} = b_n(x_n)\alpha'_n\left[\alpha'_n + \dfrac{y_{n+1}-y_n}{\alpha'_{n+1}}\right] + c_n(x_n)\left[2\alpha'_n + \dfrac{y_{n+1}-y_n}{\alpha'_{n+1}}\right] + d_n(x_n)$, or

$$\alpha'_{n+1} = \frac{a_n(x_n) - (y_{n+1}-y_n)(\alpha'_n b_n(x_n) + c_n(x_n)}{b_n(x_n)\alpha'^2_n + 2c_n(x_n)\alpha'_n + d_n(x_n)},\ n = 0, 1, \dots \qquad (47)$$

In [39] (Luminy 2007), I considered rational approximants to

$$f_0(x) = \cfrac{x-y_0}{\alpha_0 x + \beta_0 - \cfrac{(x-y_1)(x-y_2)}{\ddots\ \cfrac{}{\alpha_{n-2}x + \beta_{n-2} + \cfrac{(x-y_{2n-3})(x-y_{2n-2})}{\alpha_{n-1}x + \beta_{n-1} + \cdots}}}}$$

making clear that the n^{th} approximant (stopped at, and including the $\alpha_{n-1}x + \beta_{n-1}$ term) is the rational function of degree n interpolating f_0 at $x = y_0, y_1, \dots, y_{2n}$. So, $f_n(x) = \dfrac{x-y_{2n}}{\alpha_n x + \beta_n - (x-y_{2n+1})f_{n+1}(x)}$, $\alpha_n x + \beta_n$ is the polynomial interpolant of degree 1 to $(x-y_{2n})/f_n(x)$ at y_{2n+1} and y_{2n+2}.

Furthermore, the ***Riccati*** form is well suited to continued fraction progression:

from the Luminy paper [39]:

Let $f_0(x) = f(x) - f(y_0)$ be expanded in an interpolatory continued fraction ($R_{II}-$fraction [24, 25, 59, 69], or contracted Thiele's continued fraction [41, Chap. 5])

$$f_0(x) = \cfrac{x-y_0}{\alpha_0 x + \beta_0 - \cfrac{(x-y_1)(x-y_2)}{\ddots\ \cfrac{}{\alpha_{n-2}x + \beta_{n-2} + \cfrac{(x-y_{2n-3})(x-y_{2n-2})}{\alpha_{n-1}x + \beta_{n-1} + \cdots}}}}$$

making clear that the n^{th} approximant (stopped at, and including the $\alpha_{n-1}x+\beta_{n-1}$ term) is the rational function of degree n interpolating f_0 at $x = y_0, y_1, \dots, y_{2n}$.

Let p_n be the denominator of this n^{th} approximant. If f is a Stieltjes transform, it is known [25, 59] [69, § 5] [70] that $\{p_n(x)/((x-y_0)(x-y_2)\dots(x-y_{2n})\}$ and $\{p_m(x)/((x-y_1)(x-y_3)\dots(x-y_{2m+1})\}$ are biorthogonal sequences of rational functions.

From $f_n(x)=\dfrac{x-y_{2n}}{\alpha_n x+\beta_n-(x-y_{2n+1})f_{n+1}(x)}$, $\alpha_n x+\beta_n$ is the polynomial interpolant of degree 1 to $(x-y_{2n})/f_n(x)$ at y_{2n+1} and y_{2n+2}, so we need $f_n(y_{2n+1})$ and $f_n(y_{2n+2})$ in order to find α_n and β_n.

If f_n satisfies a difference equation of first order $\mathcal{F}_n\left(x, f_n(\varphi(x)), f_n(\psi(x))\right)=0$, we find $f_n(y_{2n+1})$ from the equation at x_{2n}, as $\varphi(x_{2n})=y_{2n}$, $\psi(x_{2n})=y_{2n+1}$, and $f_n(y_{2n})=0$. Next, the equation at $x=x_{2n+1}$ yields $f_n(y_{2n+2})$.

As seen in the section **??** on the Pearson equation, a linear difference equation of first order easily involves a weight function useful in orthogonality considerations. However, Riccati equations are better suited to continued fraction constructions [21, 34]. Of course, linear difference equations of first order are special cases of Riccati equations, that is why the coefficients in (**??**) are written $a(x), c(x)$, and $d(x)$, whereas $b(x)$ is the coefficient of the nonlinear part of a Riccati equation.

So, if f_n satisfies the Riccati equation

$$a_n(x)\frac{f_n(\psi(x))-f_n(\varphi(x))}{\psi(x)-\varphi(x)}=b_n(x)f_n(\varphi(x))f_n(\psi(x)) \\ +c_n(x)(f_n(\varphi(x))+f_n(\psi(x)))+d_n(x), \quad (48)$$

one finds at $x=x_{2n}$, $\varphi(x)=y_{2n}$, $\psi(x)=y_{2n+1}$, and knowing that $f_n(y_{2n})=0$, $a_n(x_{2n})\dfrac{f_n(y_{2n+1})}{y_{2n+1}-y_{2n}}=c_n(x_{2n})f_n(y_{2n+1})+d_n(x_{2n})$ yields $f_n(y_{2n+1})=\dfrac{d_n(x_{2n})}{\dfrac{a_n(x_{2n})}{y_{2n+1}-y_{2n}}-c_n(x_{2n})}$ and

$$\alpha_n y_{2n+1}+\beta_n=\frac{y_{2n+1}-y_{2n}}{f_n(y_{2n+1})}=\frac{a_n(x_{2n})-(y_{2n+1}-y_{2n})c_n(x_{2n})}{d_n(x_{2n})}, \quad (49)$$

and at $x=x_{2n+1}$, $a_n(x_{2n+1})\dfrac{\dfrac{y_{2n+2}-y_{2n}}{\alpha_n y_{2n+2}+\beta_n}-\dfrac{y_{2n+1}-y_{2n}}{\alpha_n y_{2n+1}+\beta_n}}{y_{2n+2}-y_{2n+1}}=$
$b_n(x_{2n+1})\dfrac{y_{2n+2}-y_{2n}}{\alpha_n y_{2n+2}+\beta_n}\dfrac{y_{2n+1}-y_{2n}}{\alpha_n y_{2n+1}+\beta_n}+c_n(x_{2n+1})\left[\dfrac{y_{2n+2}-y_{2n}}{\alpha_n y_{2n+2}+\beta_n}+\dfrac{y_{2n+1}-y_{2n}}{\alpha_n y_{2n+1}+\beta_n}\right]+d_n(x_{2n+1})$, or

$$a_n(x_{2n+1})(\alpha_n y_{2n}+\beta_n)=b_n(x_{2n+1})(y_{2n+2}-y_{2n})(y_{2n+1}-y_{2n}) \\ +c_n(x_{2n+1})[(y_{2n+2}-y_{2n})(\alpha_n y_{2n+1}+\beta_n)+(y_{2n+1}-y_{2n})(\alpha_n y_{2n+2}+\beta_n)] \\ +d_n(x_{2n+1})(\alpha_n y_{2n+1}+\beta_n)(\alpha_n y_{2n+2}+\beta_n). \quad (50)$$

which shows how to extract α_n and β_n from $a_n,\dots$ at x_{2n} and x_{2n+1}.
Remark also that at $x=x_{2n-1}$, knowing that $f_n(\psi(x_{2n-1}))=f_n(y_{2n})=0$, (48) yields

$$\left[\frac{a_n(x_{2n-1})}{y_{2n}-y_{2n-1}}+c_n(x_{2n-1})\right]f_n(y_{2n-1})+d_n(x_{2n-1})=0. \quad (51)$$

And here is how the ***Riccati*** form is well suited to continued fraction progression:

7.1. **Theorem.**

If f_n satisfies the Riccati equation (48) *with rational coefficients a_n, b_n, c_n, and d_n, and if* $f_n(x) = \dfrac{x - y_{2n}}{\alpha_n x + \beta_n - (x - y_{2n+1})f_{n+1}(x)}$, *then f_{n+1} satisfies an equation of same complexity (degree of the rational functions) of its coefficients.*

(Actually, the degrees of a_n, etc. will at most exceed the degrees at $n = 0$ by 3 units).

Proof. Let us start with (48) at $n = 0$ with polynomial coefficients a_0, b_0, c_0, and d_0. Suppose that, at the n^{th} step, a_n, etc. are polynomials with b_n and d_n containing the factors X_2 and $x - x_{2n-1}$, and c_n containing the factor X_2 (from (4a) and (5), X_2 is a polynomial of degree $\leqslant 2$).

Of course, if the initial coefficients a_0, etc. do not contain such factors, we may have to multiply the four coefficients of (48) at $n = 0$ by one or several factors of $(x - x_{-1})X_2(x)$, that's why the degrees are liable to have to be augmented by up to 3 units, but this operation has to be done only at $n = 0$.

We suppose therefore that $b_n(x) = (x - x_{2n-1})X_2(x)\tilde{b}_n(x)$, $c_n(x) = X_2(x)\tilde{c}_n(x)$, and $d_n(x) = (x - x_{2n-1})X_2(x)\tilde{d}_n(x)$ in (48), where $\tilde{b}_n$, $\tilde{c}_n$, and $\tilde{d}_n$ are polynomials, so that (48) is now

$$a_n(x)\frac{f_n(\psi(x)) - f_n(\varphi(x))}{\psi(x) - \varphi(x)} = (x - x_{2n-1})X_2(x)\tilde{b}_n(x)f_n(\varphi(x))f_n(\psi(x))$$
$$+ X_2(x)\tilde{c}_n(x)(f_n(\varphi(x)) + f_n(\psi(x))) + (x - x_{2n-1})X_2(x)\tilde{d}_n(x), \quad (52)$$

in which we enter $f_n(x) = \dfrac{x - y_{2n}}{\alpha_n x + \beta_n - (x - y_{2n+1})f_{n+1}(x)}$.

After multiplication by $(\alpha_n\varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi))(\alpha_n\psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi))$:

$$a_n\frac{(\psi - y_{2n})[\alpha_n\varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)] - (\varphi - y_{2n})[\alpha_n\psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]}{\psi - \varphi}$$
$$= a_n\left[\alpha_n y_{2n} + \beta_n + \left(\varphi\psi - \frac{(y_{2n} + y_{2n+1})(\varphi + \psi)}{2} + y_{2n}y_{2n+1}\right)\frac{f_{n+1}(\psi) - f_{n+1}(\varphi)}{\psi - \varphi}\right.$$
$$\left. + \frac{y_{2n+1} - y_{2n}}{2}(f_{n+1}(\varphi) + f_{n+1}(\psi)\right]$$
$$= (x - x_{2n-1})X_2\tilde{b}_n(\varphi - y_{2n})(\psi - y_{2n})$$
$$+ X_2\tilde{c}_n[(\psi - y_{2n})[\alpha_n\varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)] + (\varphi - y_{2n})[\alpha_n\psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]]$$
$$+ (x - x_{2n-1})X_2\tilde{d}_n[\alpha_n\varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)][\alpha_n\psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]$$

which is the Riccati equation for f_{n+1}, $\hat{a}_{n+1}\dfrac{f_{n+1}(\psi)-f_{n+1}(\varphi)}{\psi-\varphi} = \hat{b}_{n+1}f_{n+1}(\varphi)f_{n+1}(\psi) + \hat{c}_{n+1}(f_{n+1}(\varphi)+f_{n+1}(\psi)) + \hat{d}_{n+1}$, where $\hat{a}_{n+1}$ etc. are symmetric functions of φ and ψ, so are rational functions thanks to (6) (where $y_n=\varphi(x_n)$ and $y_{n+1}=\psi(x_n)$):

$$\hat{a}_{n+1} = \left(\varphi\psi - \frac{(y_{2n}+y_{2n+1})(\varphi+\psi)}{2} + y_{2n}y_{2n+1}\right)a_n + \frac{y_{2n+1}-y_{2n}}{2}(\psi-\varphi)^2X_2\tilde{c}_n$$
$$+\frac{\alpha_n y_{2n+1}+\beta_n}{2}(\psi-\varphi)^2(x-x_{2n-1})X_2\tilde{d}_n = \frac{\left(X_0+\dfrac{(y_{2n}+y_{2n+1})X_1}{2}+y_{2n}y_{2n+1}X_2\right)}{X_2}a_n$$
$$+\frac{[y_{2n+1}-y_{2n}]\tilde{c}_n + (\alpha_n y_{2n+1}+\beta_n)(x-x_{2n-1})\tilde{d}_n}{2}\,\frac{X_1^2-4X_0X_2}{X_2} \quad (53a)$$

$$\hat{b}_{n+1} = (x-x_{2n-1})X_2\tilde{d}_n(\varphi-y_{2n+1})(\psi-y_{2n+1}) = (x-x_{2n-1})\tilde{d}_n(X_0+y_{2n+1}X_1+y_{2n+1}^2X_2) \quad (53b)$$

$$\hat{c}_{n+1} = -(y_{2n+1}-y_{2n})a_n - \left(\varphi\psi - \frac{(y_{2n}+y_{2n+1})(\varphi+\psi)}{2}+y_{2n}y_{2n+1}\right)X_2\tilde{c}_n$$
$$-\left(\alpha_n\left(\varphi\psi - y_{2n+1}\frac{\varphi+\psi}{2}\right)+\beta_n\left(\frac{\varphi+\psi}{2}-y_{2n+1}\right)\right)(x-x_{2n-1})X_2\tilde{d}_n$$
$$= -(y_{2n+1}-y_{2n})a_n - \left(X_0+\frac{(y_{2n}+y_{2n+1})X_1}{2}+y_{2n}y_{2n+1}X_2\right)\tilde{c}_n$$
$$-\left(\alpha_n\left(X_0+y_{2n+1}\frac{X_1}{2}\right)-\beta_n\left(\frac{X_1}{2}+y_{2n+1}X_2\right)\right)(x-x_{2n-1})\tilde{d}_n \quad (53c)$$

$$\hat{d}_{n+1} = -(\alpha_n y_{2n}+\beta_n)a_n + (x-x_{2n-1})X_2\tilde{b}_n(\varphi-y_{2n})(\psi-y_{2n})$$
$$+X_2\tilde{c}_n[(\psi-y_{2n})(\alpha_n\varphi+\beta_n)+(\varphi-y_{2n})(\alpha_n\psi+\beta_n)]$$
$$+(x-x_{2n-1})X_2\tilde{d}_n(\alpha_n\varphi+\beta_n)(\alpha_n\psi+\beta_n)$$
$$= -(\alpha_n y_{2n}+\beta_n)a_n + (x-x_{2n-1})\tilde{b}_n(X_0+y_{2n}X_1+y_{2n}^2X_2)$$
$$+\tilde{c}_n[\alpha_n(2X_0+y_{2n}X_1)-\beta_n(X_1+2y_{2n}X_2)] + (x-x_{2n-1})\tilde{d}_n(\alpha_n^2X_0-\alpha_n\beta_nX_1+\beta_n^2X_2) \quad (53d)$$

The first coefficient $\hat{a}_{n+1}$ is not a polynomial, but a rational function of denominator X_2. We recover polynomials by multiplying the four coefficients $\hat{a}_{n+1}$, $\hat{b}_{n+1}$, $\hat{c}_{n+1}$, and $\hat{d}_{n+1}$ by X_2. Moreover, this already restores the factor X_2 in $X_2\hat{b}_{n+1}$, $X_2\hat{c}_{n+1}$, and $X_2\hat{d}_{n+1}$!

However, the degrees of the new coefficients are higher than before. The problem is settled by seeing that the four polynomials $X_2\hat{a}_{n+1}$, $\hat{b}_{n+1}$, $\hat{c}_{n+1}$, and $\hat{d}_{n+1}$ vanish at $x=x_{2n-1}$ and at $x=x_{2n}$. Then, we will simply divide the four of them by $(x-x_{2n-1})(x-x_{2n})$.

(1) The most obvious case is (53b): $\hat{b}_{n+1}(x) = (x-x_{2n-1})\tilde{d}_n(x)(X_0(x)+y_{2n+1}X_1(x)+y_{2n+1}^2X_2(x)) = (x-x_{2n-1})\tilde{d}_n(x)F(x,y_{2n+1}) = (x-x_{2n-1})\tilde{d}_n(x)Y_2(y_{2n+1})(x-x_{2n})(x-x_{2n+1})$, shows indeed the factors $x-x_{2n-1}$ and $x-x_{2n}$, as well as $x-x_{2n+1}$, so that $\tilde{b}_{n+1}(x) = \dfrac{\hat{b}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})(x-x_{2n+1})} = Y_2(y_{2n+1})\tilde{d}_n(x)$.

(2) Next, from (53d), $\hat{d}_{n+1}(x_{2n-1}) = (\alpha_n y_{2n}+\beta_n)[-a_n(x_{2n-1})-(y_{2n}-y_{2n-1})c_n(x_{2n-1})] = 0$ from (51), knowing that $d_n(x_{2n-1})=0$,

$\hat{d}_{n+1}(x_{2n}) = (\alpha_n y_{2n}+\beta_n)[-a_n(x_{2n})+(y_{2n+1}-y_{2n})c_n(x_{2n})+(\alpha_n y_{2n+1}+\beta_n)d_n(x_{2n})] = (\alpha_n y_{2n}+\beta_n)(y_{2n+1}-y_{2n})\left[-\dfrac{a_n(x_{2n})}{y_{2n+1}-y_{2n}}+c_n(x_{2n})+\dfrac{d_n(x_{2n})}{f_n(y_{2n+1})}\right] = 0$, as the last factor is the Riccati equation (48) at $x=x_{2n}$ divided by $f_n(y_{2n+1})$ (see also (49)).
$\hat{d}_{n+1}(x_{2n+1}) = -(\alpha_n y_{2n}+\beta_n)a_n(x_{2n+1}) + b_n(x_{2n+1})(y_{2n+1}-y_{2n})(y_{2n+2}-y_{2n}) + c_n(x_{2n+1})[(y_{2n+2}-y_{2n})(\alpha_n y_{2n+1}+\beta_n)+(y_{2n+1}-y_{2n})(\alpha_n y_{2n+2}+\beta_n)]]+d_n(x_{2n+1})(\alpha_n y_{2n+1}+\beta_n)(\alpha_n y_{2n+2}+\beta_n) = 0$, from (50).

(3) In order to show that $\hat{a}_{n+1}$ and $\hat{c}_{n+1}$ both vanish at $x=x_{2n-1}$ and x_{2n}, we consider $\dfrac{a_n}{\psi-\varphi}\pm c_n$: $\dfrac{\hat{a}_{n+1}(x)}{\psi(x)-\varphi(x)}+\hat{c}_{n+1}(x) =$
$(\varphi(x)-y_{2n+1})\left[(\psi(x)-y_{2n})\left(\dfrac{a_n(x)}{\psi(x)-\varphi(x)}-c_n(x)\right)-(\alpha_n\psi(x)+\beta_n)d_n(x)\right]$ vanishes at $x=x_{2n}$, as the big factor is $a_n(x_{2n})-(y_{2n+1}-y_{2n})c_n(x_{2n})-(\alpha_n y_{2n+1}+\beta_n)d_n(x_{2n}) = 0$ from (49). At $x=x_{2n-1}$, we already encountered the condition $a_n(x_{2n-1})+(y_{2n}-y_{2n-1})c_n(x_{2n-1})=0$ from (51) and $d_n(x_{2n-1})=0$.

The obvious vanishing of the first factor at $x=x_{2n+1}$ will allow the same condition at x_{2n+1}: $a_{n+1}(x_{2n+1})+(y_{2n+2}-y_{2n+1})c_{n+1}(x_{2n+1})=0$, and this will imply $f_{n+1}(y_{2n+2})=0$ at the next step.

(4) $\dfrac{\hat{a}_{n+1}(x)}{\psi(x)-\varphi(x)}-\hat{c}_{n+1}(x) =$
$(\psi(x)-y_{2n+1})\left[(\varphi(x)-y_{2n})\left(\dfrac{a_n(x)}{\psi(x)-\varphi(x)}+c_n(x)\right)+(\alpha_n\varphi(x)+\beta_n)d_n(x)\right]$ obviously vanishes now at $x=x_{2n}$; at $x=x_{2n-1}$, the big factor is $-a_n(x_{2n-1})-(y_{2n}-y_{2n-1})c_n(x_{2n-1})=0$ as already encountered (in (51), together with $d_n(x_{2n-1})=0$).

We proceed therefore with $a_{n+1}(x)=\dfrac{X_2(x)\hat{a}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})}$, $b_{n+1}(x)=\dfrac{X_2(x)\hat{b}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})}$, $c_{n+1}(x)=\dfrac{X_2(x)\hat{c}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})}$, $d_{n+1}(x)=\dfrac{X_2(x)\hat{d}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})}$. □

A very interesting identity is $\hat{a}_{n+1}^2-(\psi-\varphi)^2\hat{c}_{n+1}^2 = (\varphi-y_{2n})(\psi-y_{2n})(\varphi-y_{2n+1})(\psi-y_{2n+1})(a_n^2-(\psi-\varphi)^2c_n^2)-(\psi-\varphi)^2\hat{b}_{n+1}[\hat{d}_{n+1}-(\varphi-y_{2n})(\psi-y_{2n})b_n]$, or $\hat{a}_{n+1}^2-(\psi-\varphi)^2(\hat{c}_{n+1}^2-\hat{b}_{n+1}\hat{d}_{n+1}) = (\varphi-y_{2n})(\psi-y_{2n})(\varphi-y_{2n+1})(\psi-y_{2n+1})(a_n^2-(\psi-\varphi)^2(c_n^2-b_nd_n))$,
$a_{n+1}^2-(\psi-\varphi)^2(c_{n+1}^2-b_{n+1}d_{n+1}) = \left[\dfrac{X_2(x)}{(x-x_{2n-1})(x-x_{2n})}\right]^2\dfrac{F(x,y_{2n})}{X_2(x)}\dfrac{F(x,y_{2n+1})}{X_2(x)}(a_n^2-(\psi-\varphi)^2(c_n^2-b_nd_n)) = Y_2(y_{2n})Y_2(y_{2n+1})\dfrac{x-x_{2n+1}}{x-x_{2n-1}}(a_n^2-(\psi-\varphi)^2(c_n^2-b_nd_n))$,

$$\frac{c_n^2(x)-b_n(x)d_n(x)}{X_2^2(x)}P(x)-a_n^2(x) = C_n\frac{x-x_{2n-1}}{x-x_{-1}}\left[\frac{c_0^2(x)-b_0(x)d_0(x)}{X_2^2(x)}P(x)-a_0^2(x)\right], \quad (54)$$

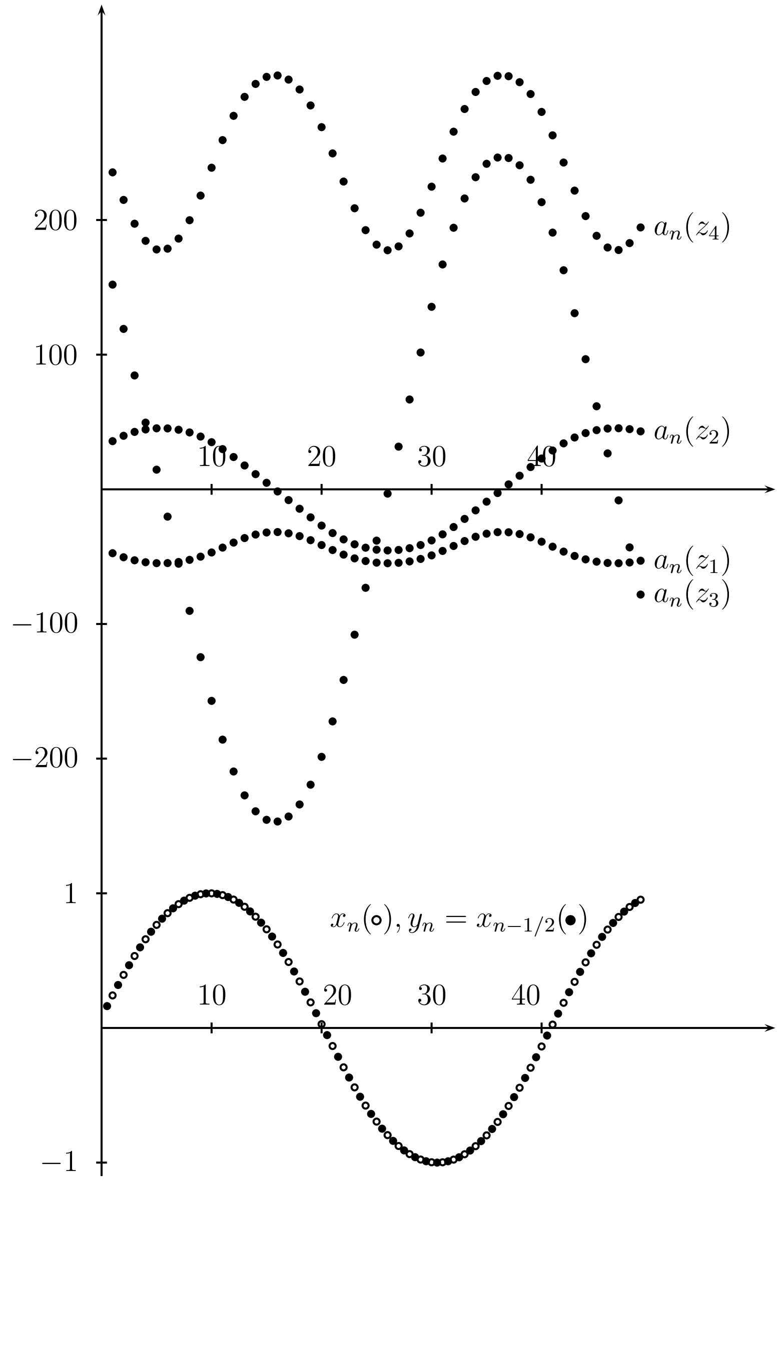
where $C_n = Y_2(y_0)Y_2(y_1)\cdots Y_2(y_{2n-2})Y_2(y_{2n-1})$ ((probably a) mistake in [39, p. 800, after (23)], and P from (8).

8. Classical?

a_n is a cubic polynomial, so is determined through the four values $a_n(z_i)$, $i = 1, \ldots, 4$, at the zeros $z_1, \ldots, z_4$ of P.

```
ellhyper Tue10May2011
k= 0.5 ; P= x^4 - 5.0*x^2 + 4.0 ; z0= 2.005000 ;
z= [-2.000000, -1.000000, 1.000000, 2.000000]~
F= (-0.001251562*y^2 + 0.7550062)*x^2 - 1.503750*y*x + (0.7550062*y^2 - 0.005006250)
X0= 0.7550062*x^2 - 0.005006250 ; X1= -1.503750*x ; X2= -0.001251562*x^2 + 0.7550062 ;
Y0= 0.7550062*y^2 - 0.005006250 ; Y1= -1.503750*y ; Y2= -0.001251562*y^2 + 0.7550062
zprime=[-2.020159, -0.9900209, 0.9900209, 2.020159] X2Y2zi=[0.5624812, 0.5681508, 0.5681508, 0.5624812]
zeros X2=[-24.56117 + 0.E-404*I, 24.56117 + 0.E-404*I]~
 x0= 0.08142935 , x-1= -0.08142935 y0=0 y-1=-0.1621851
```

$\{x_n, y_n, a_n(z_1), \ldots, a_n(z_4)\}$ for $n = 1, \ldots, 50$:



at $x = z_i$ where $P(x) = X_1^2(x) - 4X_0(x)X_2(x) = 0$, $F(z_i, y) = X_0(z_i) + X_1(z_i)y + X_2(z_i)y^2 = 0$ has a double root $y = \varphi(z_i) = \psi(z_i) = -\dfrac{X_1(z_i)}{2X_2(z_i)} = -\dfrac{2X_0(z_i)}{X_1(z_i)}$.

$$\hat{a}_{n+1}(z_i) = \frac{\left(X_0(z_i) + \dfrac{(y_{2n}+y_{2n+1})X_1(z_i)}{2} + y_{2n}y_{2n+1}X_2(z_i)\right)}{X_2(z_i)} a_n(z_i)$$

$$= \frac{\left(X_0(z_i) - \dfrac{X_1(x_{2n})X_1(z_i)}{2X_2(x_{2n})} + \dfrac{X_0(x_{2n})}{X_2(x_{2n})}X_2(z_i)\right)}{X_2(z_i)} a_n(z_i) = \frac{F(x_{2n}, \varphi(z_i))}{X_2(x_{2n})} a_n(z_i)$$

$$= \frac{Y_2(\varphi(z_i))(x_{2n} - z_i)(x_{2n} - z_i')}{X_2(x_{2n})} a_n(z_i) \qquad \text{from (10) with “}y_n\text{”} = \varphi(z_i),\ \text{so that “}x_n\text{” and “}x_{n-1}\text{” are } z_i$$

$$a_{n+1}(z_i) = \frac{X_2(z_i)\hat{a}_{n+1}(z_i)}{(z_i - x_{2n-1})(z_i - x_{2n})} = \frac{X_2(z_i)Y_2(\varphi(z_i))(z_i' - x_{2n})}{X_2(x_{2n})(z_i - x_{2n-1})} a_n(z_i),$$

The $a_n(z_i)$ of the picture above are $a_n(z_i)/\sqrt{Y_2(y_0)Y_2(y_1)\cdots Y_2(y_{2n-1})}$

(1) **Gauss hypergeometric ratio and classical orthogonal polynomials.**
$\varphi(x) = \psi(x) = x$,
$a_n(x) = x(1 - ax)$, $b_n = b_0 - n$, $d_n(x) = -\alpha_n b_n x$,
$c_n(x) = [-a/4 + (c_0' + a/4)(-1)^n]x + b_n/2$,
$\alpha_{n+1} = [a + \alpha_n b_n + 2c_n']/b_{n+2}$.
Orthogonal polynomials (Jacobi, etc.) are $x^n P_n(1/x)$.

(2) **Discrete orthogonal and bi-orthogonal polynomials.**
$\varphi(x) = x$, $\psi(x) = x + h$,
(a) Orthogonal: limit case $y_0 = y_1 = \cdots = \infty$,
(b) Bi-orthogonal $y_j = jh$

(3) **a*S*key-*N*ikiforov-s*U*slov-*U*varov-wi*L*son.**

(4) **And now?**

We try to keep the degrees as low as possible.

8.1. **“Elliptic logarithm”.**

We extend $f(x) = \log\dfrac{x-a}{x-b}$ which satisfies $f'(x) = \dfrac{a-b}{(x-a)(x-b)}$ by looking for a function whose divided difference is a rational function of low degree.

Answer: work out the difference of

$$\Lambda_N(x) = \sum_0^N \frac{\rho_k}{x - y_k'}, \tag{55}$$

where $y_k' = y(s_1 + k)$ is an elliptic lattice in the same family, but with $s_1 - s_0$ not an integer.

Simplest way to look at the difference is to suppose that x is some x_n. Then,

$$\mathcal{D}\Lambda_N(x_n) = \frac{\Lambda_N(y_{n+1}) - \Lambda_N(y_n)}{y_{n+1} - y_n} = -\sum_0^N \frac{1}{(y_n - y_k')(y_{n+1} - y_k')},$$

As y_n and y_{n+1} are the two ordinates associated to x_n, $y_n + y_{n+1} = -X_1(x_n)/X_2(x_n)$, $y_n y_{n+1} = X_0(x_n)/X_2(x_n)$, $(y_n - y_k')(y_{n+1} - y_k') = (X_0(x_n) + X_1(x_n)y_k' + X_2(x_n)y_k'^2)/X_2(x_n) = F(x_n, y_k')/X_2(x_n) = Y_2(y_k')(x_n - x_{k-1}')(x_n - x_k')/X_2(x_n)$,

$$\mathcal{D}\Lambda_N(x_n) = -X_2(x_n)\sum_0^N \frac{\rho_k/Y_2(y'_k)}{(x_n - x'_{k-1})(x_n - x'_k)}$$

$$= X_2(x_n)\left[\frac{\dfrac{\rho_0}{(x'_0 - x'_{-1})Y_2(y'_0)}}{x_n - x'_{-1}} + \sum_0^{N-1}\frac{\dfrac{\rho_{k+1}}{(x'_{k+1} - x'_k)Y_2(y'_{k+1})} - \dfrac{\rho_k}{(x'_k - x'_{k-1})Y_2(y'_k)}}{x_n - x'_k} - \frac{\dfrac{\rho_N}{(x'_N - x'_{N-1})Y_2(y'_N)}}{x_n - x'_N}\right],$$

so that, when $\rho_k = (x'_k - x'_{k-1})Y_2(y'_k)$,

$$\mathcal{D}\Lambda_N(x) = \frac{(x'_{-1} - x'_N)X_2(x)}{(x - x'_{-1})(x - x'_N)} = (x'_{-1} - x'_N)c_{2,2} + \frac{X_2(x'_{-1})}{x - x'_{-1}} - \frac{X_2(x'_N)}{x - x'_N}.$$

We can solve $\mathcal{D}f(x) = \dfrac{1}{x - A}$ with $f(y_0) = 0$, if an infinity of x'_ns, starting with $x'_{-1} = A$, come as close as we want to a zero, say ζ, of X_2, then

$$f(x) = -\frac{c_{2,2}(A - \zeta)}{X_2(A)}(x - y_0) + \frac{1}{X_2(A)}\lim_{x'_N \to \zeta}[\Lambda_N(x) - \Lambda_N(y_0)]$$

Here is an example where $f(1)$ is estimated whenever x' comes close to ζ which happens to be -2.529822 here:
$F(x, y) = -0.3125x^2y^2 + 2x^2 - 2.25xy + 2y^2 - 1.25, A = -7,$

n	x'	y'	res	f(1)
-1	-7	-2.168439		
0	2.006819	3.351538	1.021789	-0.02471402
1	-19.96266	2.738246	-0.5662515	0.09425273
2	-2.068051	-2.371686	-0.3255921	0.1349691
3	4.274935	-4.641409	2.254690	0.04885997
4	2.433328	2.049464	0.09509338	0.004647754
5	-2.642949	34.53302	-141.3408	0.1267041
...				
13	-2.560190	-2.024363	-0.06545824	0.1276923
...				
56	-2.525475	-2.029866	-0.07278552	0.1281233
...				
238	-2.536399	550.1306	-35945.00	0.1279866

which is easy to check, as 1 is itself close to an infinity of ys, so that we estimate a value of f by running the difference equation in the form $f(y_{n+1}) = f(y_n) + \dfrac{y_{n+1} - y_n}{x_n - A}$:

n	x	y	f
0	0.7905694	0	0
1	0.5167233	0.9856450	0.1265177
2	-0.9529158	-0.3790235	-0.05503327
3	-0.1575301	-0.8702581	-0.1362682
4	0.9994804	0.6923469	0.09210034

```
  5  -0.2301987   0.6400372   0.08556120
  6  -0.9309638  -0.9011730  -0.1420984
  7   0.5766530  -0.3102076  -0.04472454
  8   0.7466518   0.9944962   0.1274760
  9  -0.8303816  -0.07436214 -0.01050081
 10  -0.4536000  -0.9726185  -0.1560943
 11   0.9706641   0.4453679   0.06051121
...
 69   0.6604197   0.9999706   0.1280656
...
431   0.6621363   0.9999845   0.1280671
...
793   0.6638488   0.9999940   0.1280681
```

We therefore start the process with $a_0(x) = (x - x'_0)(x - x'_N)$, $b_0 = c_0 = 0$, $d_0(x) = X_2(x)$.

Managing to keep ***polynomials***, it appears that b_n, c_n, and d_n are X_2 times a polynomial of degree 1 or 2, whereas a_n has degree 3 or 4.

Here is an example with

$$F(x,y) = (\underbrace{x^2/2 + x + 1}_{X_2(x)})y^2 + (\underbrace{2x^2 - 2x - 3}_{X_1(x)})y + x^2 + x + 1,$$

and starting at $x_{-1} = -0.45$:

k	-1	0	1	2	3	4	5	6	7	8	9
x_k	-0.45000	0.36869	-0.48027	0.42834	-0.50390	0.47655	-0.52104	0.51127	-0.53184	0.53096	-0.53639
y_k	2.0348	0.56784	1.8444	0.64066	1.6551	0.72731	1.4731	0.82873	1.3027	0.94572	1.1465

The function f satisfies $\mathcal{D}f(x) = \dfrac{X_2(x)}{(x - x'_0)(x - x'_9)}$, where x'_0 now starts at 0.25:

k	0	1	2	3	4	5	6	7	8	9
x'_k	0.25000	-0.38410	0.17205	-0.33547	0.091968	-0.28000	0.012074	-0.21798	-0.065644	-0.14992
y'_k	2.1599	0.47429	2.3339	0.43380	2.4852	0.40394	2.6034	0.38414	2.6800	0.37399

The first polynomial a_0 is $(x - x'_0)(x - x'_9)$, then

Remark that b_1, b_3,... are scalar multiples of X_2.

Remark also that $\alpha'_{2N+1} = \infty$, as it should: the rational function f of degree N is recovered exactly as q_{2N}/p_{2N}.

9. Linear difference relations and equations for the numerators and the denominators of the interpolants.

I try to reproduce (**??**), the recurrence relations for p_n and q_n being now (§ 6.1.2, p. 54) $p_{n+1}(x) = \alpha'_{n+1} p_n(x) + (x - y_n) p_{n-1}(x)$,

*We now consider the **linear** recurrence satisfied by combinations of such products*, i.e., by combinations of

$$p_n(\varphi)p_n(\psi), p_n(\varphi)q_{n-1}(\psi), q_{n-1}(\varphi)p_n(\psi), \text{ and } q_{n-1}(\varphi)q_{n-1}(\psi).$$

We just have to consider a product $r_n s_n$, kowing that $r_{n+1} = \alpha'_{n+1} r_n + (\varphi - y_n) r_{n-1}$, and $s_{n+1} = \alpha'_{n+1} s_n + (\psi - y_n) s_{n-1}$. Simplest is to write it as matrix-vector recurrence (*doesn't it ring a bell?*)

$$\begin{bmatrix} r_n s_{n+1} \\ r_{n+1} s_n \\ r_n s_n \\ r_{n+1} s_{n+1} \end{bmatrix} = \begin{bmatrix} & \psi - y_n & & \alpha'_{n+1} \\ \varphi - y_n & & & \alpha'_{n+1} \\ & & & 1 \\ \alpha'_{n+1}(\varphi - y_n) & \alpha'_{n+1}(\psi - y_n) & (\varphi - y_n)(\psi - y_n) & \alpha'^2_{n+1} \end{bmatrix} \begin{bmatrix} r_{n-1} s_n \\ r_n s_{n-1} \\ r_{n-1} s_{n-1} \\ r_n s_n \end{bmatrix} \tag{56}$$

slightly closer to (**??**) if the latter is put in the form

$$\begin{bmatrix} \dfrac{1}{\varphi - y_n}\left[\dfrac{\tilde{a}_{n+1}}{\psi - \varphi} - \tilde{c}_{n+1}\right] \\ \dfrac{1}{\psi - y_n}\left[\dfrac{\tilde{a}_{n+1}}{\psi - \varphi} + \tilde{c}_{n+1}\right] \\ \dfrac{\tilde{d}_{n+1}}{(\varphi - y_n)(\psi - y_n)} \\ \tilde{b}_{n+1} \end{bmatrix} = \begin{bmatrix} & \psi - y_{n-1} & & -\alpha'_n \\ \varphi - y_{n-1} & & & \alpha'_n \\ & & & -1 \\ -\alpha'_n(\varphi - y_{n-1}) & \alpha'_n(\psi - y_{n-1}) & -(\varphi - y_{n-1})(\psi - y_{n-1}) & -\alpha'^2_n \end{bmatrix} \begin{bmatrix} \dfrac{1}{\varphi - y_{n-1}}\left[\dfrac{a_n}{\psi - \varphi} - c_n\right] \\ \dfrac{1}{\psi - y_{n-1}}\left[\dfrac{a_n}{\psi - \varphi} + c_n\right] \\ \dfrac{d_n}{(\varphi - y_{n-1})(\psi - y_{n-1})} \\ b_n \end{bmatrix} \tag{57}$$

to do: curb sign errors etc. get difference relations & eq. for p_n, q_n.

10. **Hypergeometric expansions.**

From:

David R. Masson: The last of the hypergeometric continued fractions, Report-no: OP-SF 12 Sep 1994 `http://arxiv.org/abs/math.CA/9409229`

Dharma P. Gupta; David R. Masson: Contiguous relations, continued fractions and orthogonality *Trans. Amer. Math. Soc.* **350** (1998), 769-808. This article is available free of charge `http://www.ams.org/tran/1998-350-02/S0002-9947-98-01879-0/home.html`

L. M. Milne Thomson: *The Calculus Of Finite Differences*, Macmillan And Company., Limited, 1933

10.1. **Gauss hypergeometric ratio.**

also in Perron [51, chap. 8], Wall, etc. Consider the Taylor-Maclaurin expansion about the origin of

$$
\begin{aligned}
f(x) = f_0(x) = x\,\frac{{}_2F_1(a,b+1;c+1;x)}{{}_2F_1(a,b;c;x)} &= \frac{x + \dfrac{a(b+1)}{c+1}x^2 + \dfrac{a(a+1)(b+1)(b+2)}{2(c+1)(c+2)}x^3 + \cdots}{1 + \dfrac{ab}{c}x + \dfrac{a(a+1)b(b+1)}{2c(c+1)}x^2 + \cdots} \\
&= x + \frac{a(c-b)}{c(c+1)}x^2 + \frac{a(c-b)(ac+bc+c-2ab)}{c^2(c+1)(c+2)}x^3 + \cdots \\
&= \frac{x}{1 - \dfrac{a(c-b)}{c(c+1)}x - \dfrac{a(c-b)(b+1)(c-a+1)}{c(c+1)^2(c+2)}x^2 + \cdots} \\
&= \cfrac{x}{1 - \cfrac{\dfrac{a(c-b)}{c(c+1)}x}{1 - \cfrac{\dfrac{(b+1)(c-a+1)}{(c+1)(c+2)}x}{1 - \cdots}}}
\end{aligned} \tag{58}
$$

This nice continued fraction goes further with $\alpha_{2n+1} = -\dfrac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)}x$, $\alpha_{2n+2} = -\dfrac{(b+n+1)(c-a+n+1)}{(c+2n+1)(c+2n+2)}x$, $(n = 0, 1, \ldots)$ The secret is in contiguity relations relating ${}_2F_1(a,b;c;x)$ to ${}_2F_1(a'+1,b';c+1;x)$, where $(a',b') = (a,b)$ or (b,a). Then, using $\dfrac{a\cdots(a+n-1)b\cdots(b+n-1)}{c\cdots(c+n-1)} - \dfrac{(a'+1)\cdots(a'+n)b'\cdots(b'+n-1)}{(c+1)\cdots(c+n)}$
$= (a'-c)n\dfrac{(a'+1)\cdots(a'+n-1)b'\cdots(b'+n-1)}{c\cdots(c+n)}$,

$$
{}_2F_1(a,b;c;x) - {}_2F_1(a'+1,b';c+1;x) = \frac{(a'-c)b'x}{c(c+1)}\,{}_2F_1(a+1,b+1;c+2;x),
$$

or

$$
\frac{{}_2F_1(a'+1,b';c+1;x)}{{}_2F_1(a,b;c;x)} = \frac{1}{1 + \dfrac{(a'-c)b'x}{c(c+1)}\,\dfrac{{}_2F_1(a+1,b+1;c+2;x)}{{}_2F_1(a'+1,b';c+1;x)}}
$$

so that $f_n(x) = x\,\dfrac{{}_2F_1(a_{n+1},b_{n+1};c+n+1;x)}{{}_2F_1(a_n,b_n;c+n;x)}$, with $a_0 = a_1 = a, a_2 = a_3 = a+1, \ldots : a_n = a + \lfloor n/2 \rfloor, b_n = b + \lceil n/2 \rceil$.

Riccati: from Abramowitz 15.2.7, $(1-x)F' = a'_nF - a'_n(1-b'_n/c_n)F(a_{n+1},b_{n+1};c+n+1;x)$, where $(a'_n,b'_n) = (a_n,b_n)$ if $a_{n+1} > a_n$. So,

$$
a'_n(1-b'_n/c_n)\frac{f_n(x)}{x} = a'_n - (1-x)\frac{F'}{F},
$$

and use the hypergeometric differential equation $x(1-x)F'' + [c_n - (a_n+b_n+1)x]F' - a_nb_nF = 0$ (15.5.1):

to continue

10.2. Elliptic hypergeometric expansions.

Let

$$\mathcal{Y}_n(x)=\frac{(x-y_0)\cdots(x-y_{n-1})}{(x-y'_0)\cdots(x-y'_{n-1})},\mathcal{X}_n(x)=\frac{(x-x_0)\cdots(x-x_{n-1})}{(x-x'_0)\cdots(x-x'_{n-1})},\widetilde{\mathcal{X}}_n(x)=\frac{(x-x_0)\cdots(x-x_{n-1})}{(x-x'_{-1})\cdots(x-x'_n)}.$$

See that $\mathcal{D}\mathcal{Y}_n(x)=C_nX_2(x)\widetilde{\mathcal{X}}_{n-1}(x)$

Indeed, $(\varphi(x)-y_0)(\varphi(x)-y_1)\cdots(\varphi(x)-y_{n-1})$ and $(\psi(x)-y_0)(\psi(x)-y_1)\cdots(\psi(x)-y_{n-1})$ both vanish at $x=x_0,x_1,\dots,x_{n-2}$; $(\varphi(x)-y'_0)(\varphi(x)-y'_1)\cdots(\varphi(x)-y'_{n-1})$ vanishes at $x=x'_0,\dots,x'_{n-1}$, whereas $(\psi(x)-y'_0)(\psi(x)-y'_1)\cdots(\psi(x)-y'_{n-1})$ vanishes at $x=x'_{-1},\dots,x'_{n-2}$.

Simple fractions give $\mathcal{D}\dfrac{1}{x-y'_k}=-\dfrac{X_2(x)}{Y_2(y'_k)(x-x'_{k-1})(x-x'_k)}$, as seen earlier.

The constant C_n is found through particular values of x, either x_{-1} or x_{n-1}:

$$\begin{aligned}C_n&=-\frac{\mathcal{Y}_n(\varphi(x_{-1})=y_{-1})}{(y_0-y_{-1})X_2(x_{-1})\widetilde{\mathcal{X}}_{n-1}(x_{-1})}\\&=\frac{\mathcal{Y}_n(\psi(x_{n-1})=y_n)}{(y_n-y_{n-1})X_2(x_{n-1})\widetilde{\mathcal{X}}_{n-1}(x_{n-1})}\end{aligned}$$

(Of course, $C_0=0$). Check $\mathcal{Y}_1(x)=\dfrac{x-y_0}{x-y'_0}=1+\dfrac{y'_0-y_0}{x-y'_0}\Rightarrow\mathcal{D}\mathcal{Y}_1(x)=-\dfrac{(y'_0-y_0)X_2(x)}{Y_2(y'_0)(x-x'_{-1})(x-x'_0)}$

so that $C_1=\dfrac{y_0-y'_0}{Y_2(y'_0)}$ to be compared with

$-\dfrac{\mathcal{Y}_1(y_{-1})}{(y_0-y_{-1})X_2(x_{-1})\widetilde{\mathcal{X}}_0(x_{-1})}=-\dfrac{(y_{-1}-y_0)(x_{-1}-x'_{-1})(x_{-1}-x'_0)}{(y_{-1}-y'_0)(y_0-y_{-1})X_2(x_{-1})}=\dfrac{F(x_{-1},y'_0)}{(y_{-1}-y'_0)Y_2(y'_0)X_2(x_{-1})}=$
$\dfrac{(y'_0-y_{-1})(y'_0-y_0)}{(y_{-1}-y'_0)Y_2(y'_0)}$, OK.

Also,

$$\frac{(\varphi(x)-y_0)(\varphi(x)-y_1)\cdots(\varphi(x)-y_{n-1})}{(\varphi(x)-y'_1)(\varphi(x)-y'_2)\cdots(\varphi(x)-y'_n)}+\frac{(\psi(x)-y_0)(\psi(x)-y_1)\cdots(\psi(x)-y_{n-1})}{(\psi(x)-y'_1)(\psi(x)-y'_2)\cdots(\psi(x)-y'_n)}$$

$$=D_n(x)\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-2})}{(x-x'_0)(x-x'_1)\cdots(x-x'_n)},$$

where D_n is a polynomial of degree 2.

same with $\mathcal{D}^\dagger$

Riccati and 2$^{\text{nd}}$ order linear differnce eq.

From $a(x_m)\dfrac{f(y_{m+1})-f(y_m)}{y_{m+1}-y_m}=b(x_m)f(y_{m+1})f(y_m)+c(x_m)(f(y_{m+1})+f(y_m)))+d(x_m)$,

$$f(y_{m+1})=\frac{\left(\dfrac{a(x_m)}{y_{m+1}-y_m}+c(x_m)\right)f(y_m)+d(x_m)}{\dfrac{a(x_m)}{y_{m+1}-y_m}-c(x_m)-b(x_m)f(y_m)}$$

$$= -\frac{1}{b(x_n)}\left(\frac{a(x_m)}{y_{m+1}-y_m}+c(x_m)\right)+\frac{1}{b(x_m)}\frac{\dfrac{a^2(x_m)}{(y_{m+1}-y_m)^2}+b(x_m)d(x_m)-c^2(x_m)}{\dfrac{a(x_m)}{y_{m+1}-y_m}-c(x_m)-b(x_m)f(y_m)}$$

$$\frac{a(x_{m+1})}{y_{m+2}-y_{m+1}}-c(x_{m+1})-b(x_{m+1})f(y_{m+1}) = A_m+\frac{B_m}{\dfrac{a(x_m)}{y_{m+1}-y_m}-c(x_m)-b(x_m)f(y_m)},$$

10.3. **Expansion for elliptic logarithm.**

Expand (55) satisfying

$$\mathcal{D}\Lambda_N(x) = \frac{(x'_{-1}-x'_N)X_2(x)}{(x-x'_{-1})(x-x'_N)}$$

as $\sum_1^{N+1}\gamma_n\mathcal{Y}_n(x)$: $\sum_1^{N+1}\gamma_n C_n X_2(x)\widetilde{\mathcal{X}}_{n-1}(x) = \dfrac{(x'_{-1}-x'_N)X_2(x)}{(x-x'_{-1})(x-x'_N)}$,
or $\sum_1^{N+1}\gamma_n C_n\dfrac{\mathcal{X}_{n-1}(x)}{x-x'_{n-1}} = \dfrac{x'_{-1}-x'_N}{x-x'_N}$,

Residue at x'_N : $\gamma_{N+1}C_{N+1}\mathcal{X}_N(x'_N) = x'_{-1}-x'_N$;
at x'_{N-1} : $\gamma_N C_N\mathcal{X}_{N-1}(x'_{N-1})+\gamma_{N+1}C_{N+1}\dfrac{(x'_{N-1}-x_0)\cdots(x'_{N-1}-x_{N-1})}{(x'_{N-1}-x'_0)\cdots(x'_{N-1}-x'_{N-2})(x'_{N-1}-x'_N)} = \gamma_N C_N\mathcal{X}_{N-1}(x'_{N-1})+$
$\gamma_{N+1}C_{N+1}\mathcal{X}_{N-1}(x'_{N-1})\dfrac{x'_{N-1}-x_{N-1}}{(x'_{N-1}-x'_N)} = 0$; at x'_{N-2} : $\gamma_{N-1}C_{N-1}+\gamma_N C_N\dfrac{x'_{N-2}-x_{N-2}}{x'_{N-2}-x'_{N-1}}+$
$\gamma_{N+1}C_{N+1}\dfrac{(x'_{N-2}-x_{N-2})(x'_{N-2}-x_{N-1})}{(x'_{N-2}-x'_{N-1})(x'_{N-2}-x'_N)} = 0$, etc.

$$\gamma_n C_n = \frac{(x'_{n-1}-x_{n-1})(x'_N-x_n)(x'_N-x_{n+1})\cdots(x'_N-x_{N-1})}{(x'_N-x'_{n-1})(x'_N-x'_n)\cdots(x'_N-x'_{N-1})}\gamma_{N+1}C_{N+1} = \frac{(x'_{n-1}-x_{n-1})(x'_{-1}-x'_N)}{(x'_N-x_{n-1})\mathcal{X}_{n-1}(x'_N)}$$

Proof: let $S_i = \sum_{i+1}^{N+1}(x'_{n-1}-x_{n-1})\dfrac{(x-x_i)\cdots(x-x_{n-2})}{(x-x'_i)\cdots(x-x'_{n-1})}\dfrac{(x'_N-x'_i)\cdots(x'_N-x'_{n-2})}{(x'_N-x_i)\cdots(x'_N-x_{n-1})} = \dfrac{x'_i-x_i}{(x-x'_i)(x'_N-x_i)}+$
$\cdots$. Then, $S_N = \dfrac{1}{x-x'_N}$, $S_{N-1} = \dfrac{x'_{N-1}-x_{N-1}}{(x-x'_{N-1})(x'_N-x_{N-1})}+\dfrac{(x'_N-x_N)(x-x_{N-1})(x'_N-x'_{N-1})}{(x-x'_{N-1})(x-x'_N)(x'_N-x_{N-1})(x'_N-x_N)} =$
$\dfrac{x'_{N-1}-x_{N-1}+\dfrac{(x-x_{N-1})(x'_N-x'_{N-1})}{x-x'_N}}{(x-x'_{N-1})(x'_N-x_{N-1})} = \dfrac{1}{x-x'_N}$.
Show that $S_i = S_{i+1}, i = N-1, N-2, \ldots, 0$.

$$S_i = \frac{x'_i-x_i}{(x-x'_i)(x'_N-x_i)}+\frac{(x-x_i)(x'_N-x'_i)}{(x-x'_i)(x'_N-x_i)}S_{i+1} = \frac{x'_i-x_i+\dfrac{(x-x_i)(x'_N-x'_i)}{x-x'_N)}}{(x-x'_i)(x'_N-x_i)} = \frac{1}{x-x'_N} \quad \square$$

And from $C_n = -\dfrac{\mathcal{Y}_n(y_{-1})}{(y_0-y_{-1})X_2(x_{-1})\widetilde{\mathcal{X}}_{n-1}(x_{-1})}$,

$$\gamma_n = -\frac{(x'_{n-1}-x_{n-1})(x'_{-1}-x'_N)}{(x'_N-x_{n-1})\mathcal{X}_{n-1}(x'_N)}\frac{(y_0-y_{-1})X_2(x_{-1})\widetilde{\mathcal{X}}_{n-1}(x_{-1})}{\mathcal{Y}_n(y_{-1})}$$

$$= (x_{n-1}-x'_{n-1})X_2(x_{-1})\frac{(x'_N-x'_{-1})\cdots(x'_N-x'_{n-2})}{(x'_N-x_0)\cdots(x'_N-x_{n-1})}\frac{(x_{-1}-x_0)\cdots(x_{-1}-x_{n-2})(y_{-1}-y'_0)\cdots(y_{-1}-y'_{n-1})}{(x_{-1}-x'_{-1})\cdots(x_{-1}-x'_{n-1})(y_{-1}-y_1)\cdots(y_{-1}-y_{n-1})}$$

$$\frac{\gamma_{n+1}/(x_n-x'_n)}{\gamma_n/(x_{n-1}-x'_{n-1})} = \frac{(x'_N-x'_{n-1})(x_{-1}-x_{n-1})(y_{-1}-y'_n)}{(x'_N-x_n)(x_{-1}-x'_n)(y_{-1}-y_n)}$$

Return to $\mathcal{D}f(x) = \dfrac{1}{x-A}$, $x'_{-1}=A$, $f(y_0)=0$,

x'_ns, starting with $x'_{-1}=A$, come as close as we want to a zero, say ζ, of $X_2(x) = c_{2,2}(x-\zeta)(x-\zeta')$, then

$$f(x) = \underbrace{-\frac{c_{2,2}(A-\zeta)}{X_2(A)}}_{1/(\zeta'-A)}(x-y_0) + \frac{1}{X_2(A)}\lim_{x'_N\to\zeta}[\Lambda_N(x)-\Lambda_N(y_0)]$$

with the γ_ns above, with x'_N replaced by ζ.

$$\gamma_1 = -\frac{(x'_0-x_0)(x'_{-1}-\zeta)}{\zeta-x_0}\frac{(y_0-y_{-1})X_2(x_{-1})(y_{-1}-y'_0)}{(x_{-1}-A)(x_{-1}-x'_0)(y_{-1}-y_0)} = -\frac{(x'_0-x_0)(A-\zeta)}{\zeta-x_0}\frac{Y_2(y'_0)}{y'_0-y_0}$$

So, the expansion of f is

$$\sum_1^\infty \left[\frac{\gamma_n}{X_2(A)} - \frac{c_{2,2}(A-\zeta)}{X_2(A)}(y_n-y'_{n-1})\right]\mathcal{Y}_n(x)$$

(using the formal sum $x = y_0 + \sum_1^\infty (y_n-y'_{n-1})\mathcal{Y}_n(x)$).

Check $n=1$:

$$\frac{\gamma_1}{X_2(A)} - \frac{c_{2,2}(A-\zeta)}{X_2(A)}(y_1-y'_0) = \frac{(y_1-y'_0)(f(y_1)-f(y_0))}{y_1-y_0} = \frac{y_1-y'_0}{x_0-A}$$

10.4. **Convergence.**

Average behaviour: $\displaystyle\prod_1^n (x-x_k) = \prod_1^n (x-\mathcal{E}(ak+b)) \approx \Phi(x)^n$. What is $\Phi(x) = \exp\mathcal{V}_+(x)$?

Let a be a ***real*** irrational multiple of a period ω, then the same factors reappear approximately in the product after N steps if aN is close to an integer times ω. $\Phi(x)$ is the limit of the N^{th} roots of such products. The various ak, for $k=1,2,\dots,N$, modulo ω, fill uniformly the segment $[0,\omega]$:

for any j in $\{1,2,\dots,N\}$, there is a k such that ak is close to $j\omega/N$ modulo ω. Indeed, let aN be close to $M_N\omega$, with $\gcd(N,M_N)=1$.Then,

$$ak - \frac{j\omega}{N} = \omega\left(\frac{a}{\omega}-\frac{M_N}{N}\right)k + \omega\frac{kM_N-j}{N},$$

to any j, there are integers k and m such that $kM_N - mN = j$ (Bezout).

So, we rearrange the product as $\displaystyle\Phi(x) \sim \left[\prod_{j=1}^N (x-\mathcal{E}(j\omega/N+b))\right]^{1/N} \sim \exp\left[\frac{1}{\omega}\int_0^\omega \log(x-\mathcal{E}(u+b))\,du\right]$.

As $\mathcal{E}$ is the inversion of an elliptic integral of the first kind, $u+b = \int^{\mathcal{E}} \dfrac{dv}{\sqrt{P(v)}}$, we have $\Phi(x) = \exp\left[\dfrac{1}{\omega}\displaystyle\int_{\{x_n\}} \dfrac{\log(x-v)\,dv}{\sqrt{P(v)}}\right]$, where $\{x_n\}$ is the locus of the x_ns $= \{\mathcal{E}(u+b)\}, u \in [0,\omega]$. The constant $1/\omega$ is such that $\Phi(x) \sim x$ for large x: $\omega = \displaystyle\int_{\{x_n\}} \dfrac{dv}{\sqrt{P(v)}}$.

So, let the complex potential $\mathcal{V}_+(x) = \dfrac{1}{\omega}\displaystyle\int_{\{x_n\}} \dfrac{\log(x-v)\,dv}{\sqrt{P(v)}}$,

($\mathcal{V}_-$ will be used with the x'_ns)

derivative: $\mathcal{V}'_+(x) = \dfrac{1}{\omega}\displaystyle\int_{\{x_n\}} \dfrac{dv}{(x-v)\sqrt{P(v)}}$,

$P(x)\mathcal{V}'_+(x) =$ polynomial $+\dfrac{1}{\omega}\displaystyle\int_{\{x_n\}} \dfrac{\sqrt{P(v)}\,dv}{x-v}$,

$(P(x)\mathcal{V}'_+(x))' =$ (another) polynomial $+\dfrac{1}{\omega}\displaystyle\int_{\{x_n\}} \dfrac{P'(v)\,dv}{2(x-v)\sqrt{P(v)}}$,

Finally

$$P(x)\mathcal{V}''_+(x) + P'(x)\mathcal{V}'_+(x)/2 = \text{(still another) pol.}$$

$$(\sqrt{P(x)}\;\mathcal{V}'_+(x)\;)' = \frac{\text{pol.}}{\sqrt{P(x)}}$$

so, $\mathcal{V}'_+(x) =$ an incomplete elliptic integral of the second(?) kind divided by $\sqrt{P(x)}$.

With ξ such that $x = \mathcal{E}(\xi), dx/d\xi = \sqrt{P(x)}$:

$$\frac{d^2\mathcal{V}_+(x)}{d\xi^2} = \text{a pol. (in } x = \mathcal{E}(\xi))$$

What is this polynomial, by the way? $\mathcal{V}'_+(x) = x^{-1} + (\mu_{+,1}/\mu_{+,0})x^{-2} + (\mu_{+,2}/\mu_{+,0})x^{-3} + \cdots$, $\sqrt{P(x)} = \pi_0 x^2 + \pi_1 x + \pi_2 + \cdots$,

the pol. is

$$\begin{aligned}&(\pi_0 x^2 + \pi_1 x + \pi_2 + \cdots)[\pi_0 - (\pi_2 + \pi_1(\mu_{+,1}/\mu_{+,0}) + \pi_0(\mu_{+,2}/\mu_{+,0}))x^{-2} + \cdots]\\ &= \pi_0^2 x^2 + \pi_1\pi_0 x + \pi_2\pi_0 - \pi_0(\pi_2 + \pi_1(\mu_{+,1}/\mu_{+,0}) + \pi_0(\mu_{+,2}/\mu_{+,0}))\end{aligned}$$

where $\mu_{+,k} = \displaystyle\int_{\{x_n\}} \dfrac{v^k\,dv}{\sqrt{P(v)}}$ is the k^{th} moment of the contour drawn by the x_ns. The result is

$$\pi_0^2 x^2 + \pi_1\pi_0 x + \pi_2\pi_0 - \frac{\pi_0}{\mu_{+,0}} \int_{\{x_n\}} \frac{\pi_2 + \pi_1 v + \pi_0 v^2 = \sqrt{P(v)} - O(1/v)}{\sqrt{P(v)}}\,dv$$

The contour integrals on the x'_ns are the same(?? yes, see later on), so, at last

$$\frac{d^2\mathcal{V}(x)}{d\xi^2} = 0,$$

where $\mathcal{V} = \mathcal{V}_+ - \mathcal{V}_-$.

Jump of $\mathcal{V}'(x)$ when x crosses the x_n line: $\mathcal{V}'(x)_{\text{average}} \pm \pi i \dfrac{1}{\omega}\dfrac{1}{\sqrt{P(x)}}$, or in ξ:

$$\left(\frac{d\mathcal{V}}{d\xi}\right)_{\text{average}} \pm \pi i \frac{1}{\omega}$$

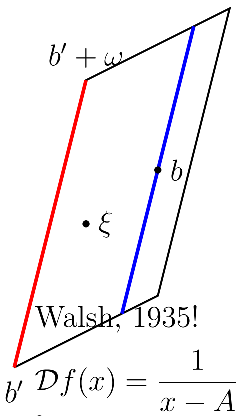
Forget it! A much faster and more complete derivation: $\mathcal{V}_+(x)$ and $\mathcal{V}_-(x)$ are contour integrals on the locii filled by $\{x_n\}$ and $\{x'_n\}$ drawn by $\mathcal{E}(an+b)$ and $\mathcal{E}(an+b')$. If x is between these two locii, the two contour integrals of $\dfrac{dv}{(x-v)\sqrt{P(v)}}$ are the same for $\mathcal{V}'_+(x)$ and $\mathcal{V}'_-(x)$, up to the residue at $v=x$:

$$\mathcal{V}'(x) = \mathcal{V}'_+(x) - \mathcal{V}'_-(x) = \frac{2\pi i}{\omega\sqrt{P(x)}} \Rightarrow \frac{d\mathcal{V}(x)}{d\xi} = \frac{2\pi i}{\omega}$$

$(\gamma_n \mathcal{Y}_n(x))^{1/n} \sim$

$$\left(\frac{(\zeta - x'_{-1})\cdots(\zeta - x'_{n-2})}{(\zeta - x_0)\cdots(\zeta - x_{n-1})} \frac{(x_{-1}-x_0)\cdots(x_{-1}-x_{n-2})(y_{-1}-y'_0)\cdots(y_{-1}-y'_{n-1})}{(x_{-1}-x'_{-1})\cdots(x_{-1}-x'_{n-1})(y_{-1}-y_1)\cdots(y_{-1}-y_{n-1})} \frac{(x-y_0)\cdots(x-y_{n-1})}{(x-y'_0)\cdots(x-y'_{n-1})} \right)^{1/n}$$

$$\sim \exp[\mathcal{V}(x_{-1}) - \mathcal{V}(\zeta) - \mathcal{W}(y_{-1}) + \mathcal{W}(x)]$$



Towards a conjecture on rate of convergence:

If the step a in $x_n = \mathcal{E}_1(an+b)$, $x'_n = \mathcal{E}_1(an+b')$, is a ***real irrational*** *multiple of a period ω, of bounded Lagrange-Markov constant(see papers and book by S.Khrushchev),*

$$|\gamma_n \mathcal{Y}_n(x)|^{1/n} \to \exp(-2\pi d(x)/|\omega|),$$

where $d(x)$ is the distance of ξ to the line $\{an+b'\}$.

$\mathcal{D}f(x) = \dfrac{1}{x-A}$ with $f(y_0)=0$, $\zeta = -2.529822$, $F(x,y) = -0.3125x^2y^2 + 2x^2 - 2.25xy + 2y^2 - 1.25$, $A=-7$,

k	x(k)	y(k)	x'(k-1)	y'(k-1)	gk	term	f(1)	f(i)
1	0.51672	0.98564;	2.0068	3.3515	;-0.05542	0.02356958	0.1285033	-0.00453082 + 0.1201190
2	-0.95291	-0.37902;	-19.9626	2.7382	;-0.21650	-0.00076035	0.1277430	-0.00000099 + 0.1495888
3	-0.15753	-0.87025;	-2.0680	-2.3716	; 0.30473	0.00043771	0.1281807	0.01064768 + 0.1357818
4	0.99948	0.69234;	4.2749	-4.6414	;-0.23940	-0.00011400	0.1280667	0.00696292 + 0.1368071
5	-0.23019	0.64003;	2.4333	2.0494	;-0.05303	0.00000740	0.1280741	0.00664251 + 0.1371257
6	-0.93096	-0.90117;	-2.6429	34.5330	;-4.06650	-0.00000609	0.1280680	0.00686082 + 0.1382964
7	0.57665	-0.31020;	-3.5481	-2.0143	; 0.77233	0.00000073	0.1280687	0.00688667 + 0.1381635
8	0.74665	0.99449;	2.1363	6.1416	;-0.07636	0.00000002	0.1280687	0.00688892 + 0.1381636
9	-0.83038	-0.07436;	9.3362	2.2354	;-0.13154	-1.411 E-10	0.1280687	0.00689104 + 0.1381629
10	-0.45360	-0.97261;	-2.0003	-3.0677	; 0.36320	1.029 E-10	0.1280687	0.00688980 + 0.1381615
15	0.90479	0.92787;	2.5007	121.2785	;-0.11543	1.975 E-15	0.1280687	0.00689019 + 0.1381625
20	-0.98422	-0.50887;	-8.1267	3.1850	;-0.28273	-2.400 E-19	0.1280687	0.00689019 + 0.1381625

Here, $b=0$, $\omega=4K$, $b'=iK'$, $k=1/2 : K=1.68575, K'=2.15652$,
rate of convergence at $x=1$ is $\exp(-\pi K'/(2K)) = 1/7.459 = 10^{-0.8727}$.

11. Interpolatory continued fraction.

$$\cfrac{\alpha_0(x-y_0)}{1+\beta_0(x-y_1)+\cfrac{\alpha_1(x-y_1)(x-y_2)}{\cfrac{\ddots}{1+\beta_{n-2}(x-y_{2n-3})+\cfrac{\alpha_{n-1}(x-y_{2n-3})(x-y_{2n-2})}{1+\beta_{n-1}(x-y_{2n-1})+\cdots}}}}$$

```
  k   a(k-1)      b(k-1)
  1  0.02342657  0.3064279
  2  0.02442157 -0.5755228
  3  0.2808405   0.5023201
  4  0.02875770  0.3488815
  5 -0.002774129 0.6898141
  6 -2.635035    3.609245
  7  0.05390936  0.7862154
  8  0.001055843 0.2904550
  9  0.01638600 -0.8421168
 10  0.4162262   0.5008208
...
 13 -2.667423    1.473576
 14  3.576408    0.6620672
 16  0.01282034 -5.278312
 17  3.286337    0.6235315
 24 -1.406951    1.290914
 31  2.713400   -4.291244
 41  0.01600274  2.476500
 42 -2.045183    0.8222617
 49 -3.308682    4.687681
 56 -1.128588    0.1804773
 57  1.869775    0.6528640
 59  0.01285615 -8.095352
 60  5.162988    0.6383966
```

Hope to connect to the current theory of elliptic hypergeometric expansions [53–59,69–71] some day...

Remarks on this biquadratic calculus:

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Résumé / Abstract We show that non-trivial symmetries, and structures, originating from lattice statistical mechanics, provide a quite efficient way to get interesting results on elliptic curves, algebraic varieties, and even arithmetic problems in algebraic geometry. In particular, the lattice statistical mechanics approach underlines the role played by the modular $j-$invariant, and by a reduction of elliptic curves to a symmetric biquadratic curve (instead of the well-known reduction to a canonical Weierstrass form). This representation of elliptic curves in terms of a very simple symmetric biquadratic is such that the action of a (generically infinite) discrete set of birational transformations, which corresponds to important non-trivial symmetries of lattice models, can be seen clearly. This biquadratic representation also makes a particular symmetry group of elliptic curves (group of permutation of three elements related to the modular $j-$invariant) crystal clear. Using this biquadratic representation of elliptic curves, we exhibit a remarkable polynomial representation of the multiplication of the shift of elliptic curves (associated with the group of rational points of the curve). The two expressions g_2 and g_3, occurring in the Weierstrass canonical form $y^2 = 4x^3 - g_2x - g_3$, are seen to present remarkable covariance properties with respect to this infinite set of commuting polynomial transformations (homogeneous polynomial transformations of three homogeneous variables, or rational transformations of two variables).
Revue / Journal Title *Annales de la Fondation Louis de Broglie (Ann. Fond. Louis Broglie)* ISSN 0182-4295 CODEN AFLBDU Source / Source 2001, vol. **26**, NS, no2, pp. 287-328 (73 ref.) Langue / Language Anglais Editeur / Publisher Fondation Louis de Broglie, Paris, FRANCE (1975) (Revue) Mots-clés anglais / English Keywords Theoretical study ; Elliptic curve ; Algebraic number ; Integer ; Dynamical systems ; Discrete systems ; Algebraic geometry ; Statistical mechanics ; Invariance ; Mots-clés français / French Keywords Etude théorique ; Courbe elliptique ; Nombre algébrique ; Nombre entier ; Système dynamique ; Système discret ; Géométrie algébrique ; Mécanique statistique ; Invariance ; 001b00e20 ; 001b00e30 ; 0520 ; 0530 ; Mots-clés espagnols / Spanish Keywords Curva eliptica ; Número algebraico ; Entero ; Geometra algebraica ; Localisation / Location INIST-CNRS, Cote INIST : 18321, 35400010080944.0070

12. **References**

[1] M. Abramowitz, I.A. Stegun, ed., *Handbook of Mathematical Functions*, Nat. Bureau of Standards, Washington, 1964 = Dover, New York, 1965 etc. `http://members.fortunecity.com/aands/`, `http://www.convertit.com/Go/ConvertIt/Reference/AMS55.ASP`
`http://www.math.sfu.ca/~cbm/aands/` Abramowitz and Stegun: Handbook of Mathematical Functions An electronic copy of the tenth printing of this famous reference.

[2] N.I. Akhiezer, *Elements of the Theory of Elliptic Functions*, translated from the 2nd Russian edition (Nauka, Moscow, 1970), *Transl. Math. Monographs* **79**, A.M.S., Providence, 1990.

[3] G.E. Andrews, R. Askey, Classical orthogonal polynomials, *in Orthogonal polynomials and applications (Bar-le-Duc, 1984)*, 36-62, (C.Brezinski *et al.*, eds.) *Lecture Notes in Math.*, **1171**, Springer, Berlin, 1985,

[4] P. Appell and E. Goursat, *Théorie des fonctions algébriques et de leurs intégrales: étude des fonctions analytiques sur une surface de Riemann*, Gauthier-Villars 1895, `http://www.archive.org/details/theoriefonctions00apperich`

[5] R. Askey, J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. A. M. S.* **54** no. 319, 1985 1–55.

[6] A.I. Aptekarev, Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda lattices, *Mat. Sb.* **125** (**167**) (1984) 231-258 (Russian) = *Math. USSR Sb.* **53** (1986) 233-260.

[7] A.I. Aptekarev, Multiple orthogonal polynomials, *Journal Comp. Applied Mathematics*, **99** (1998), 423-447.

[8] A.I. Aptekarev, H. Stahl, Asymptotics of Hermite-Padé polynomials, *in*: A.A. Gonchar, E.B. Saff (Eds.), *Progress in Approximation Theory*, Springer, Berlin, 1992, pp. 127-167.

[9] G.A. Baker, Jr., *Essentials of Padé Approximants*, Ac. Press, new York, 1975.

[10] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982. While out of print, available as `http://tpsrv.anu.edu.au/Members/baxter/book/Exactly.pdf` or `http://tpsrv.anu.edu.au/Members/baxter/book/Exactly.djvu`

[11] J. Bewersdorff, *Galois Theory for Beginners A Historical Perspective*, American Mathematical Society, 2006.

[12] B.M. Brown, W.D. Evans, M.E.H. Ismail, The Askey-Wilson polynomials and q-Sturm-Liouville problems, *Math. Proc. Camb. Philos. Soc.* **119** (1996) 1-16.

[13] C. Brezinski, *History of Continued Fractions and Padé Approximants*, Springer, Berlin, 1991.

[14] V. P. Burskii, A. S. Zhedanov, The Dirichlet and the Poncelet problems, Reports of RIAM Symposium No.16ME-S1 *Physics and Mathematical Structures of Nonlinear Waves* Proceedings of a symposium held at Chikushi Campus, Kyushu Universiy, Kasuga, Fukuoka, Japan, November 15 - 17, 2004 `http://www.riam.kyushu-u.ac.jp/fluid/meeting/16ME-S1/papers/Article_No_24.pdf`

[15] Vladimir P. BURSKII and Alexei S. ZHEDANOV, Dirichlet and Neumann Problems for String Equation, Poncelet Problem and PellAbel Equation, *Symmetry, Integrability and Geometry: Methods and Applications* Vol. **2** (2006), Paper 041, 5 pages arXiv:math.AP/0604278 v1 12 April 2006

[16] V.P. Burskii, A.S. Zhedanov, On Dirichlet, Poncelet and Abel problems, Mar 2009, 47 pages, `arXiv:0903.2531`.

[17] J.S.R. Chisholm, A.C. Genz, M. Pusterla, A method for computing Feynman amplitudes with branch cuts, *J. Comp. Appl. Math.* **2** (1976) 73-76.

[18] J.S. Geronimo, W. Van Assche, Orthogonal polynomials with asymptotically periodic recurrence coefficients, *J. Approx. Theory* **46** (1986), 251-283.

[19] J.S. Geronimo, W. Van Assche, Orthogonal polynomials on several intervals via a polynomial mapping, *Trans. Amer. Math. Soc.* **308** (1988), 559-581.

[20] J. Gilewicz, *Approximants de Padé*, Lect. Notes Math. **667**, Springer, Berlin, 1978.

[21] Gruenbaum, F.Alberto; Haine, Luc: On a q-analogue of Gauss equation and some q-Riccati equations, *in* Ismail, Mourad E. H. (ed.) et al., *Special functions, q-series and related topics.* Providence, RI: American Mathematical Society, *Fields Inst. Commun.* **14**, 77-81 (1997).

[22] R. Haydock, C.M.M. Nex, A general terminator for the recursion method, *J. Phys. C: Solid State Phys.* **18** (1985) 2235-2248.

[23] R. Haydock, C.M.M. Nex, Densities of states, moments, and maximally broken time-reversal symmetry, *Phys. Review* **B 74** (2006) (205121-1)–(205121-8).

[24] Ismail, Mourad E. H., *Classical and quantum orthogonal polynomials in one variable, with two chapters by Walter Van Assche*, Cambridge university press, 2005

[25] Ismail, Mourad E.H.; Masson, David R., Generalized orthogonality and continued fractions. *J. Approximation Theory* **83**, No.1, 1-40 (1995).

[26] S. Klarsfeld, An improved Padé approximant method for calculating Feynman amplitudes, *J. Phys. G: Nucl. Phys.* **7** (1981) 1-7.

[27] S. Klarsfeld, Padé approximants and related methods for computing boundary values on cuts, pp. 255–262 *in* M.G. de Bruin, H. van Rossum, eds: *Padé Approximation and its Applications, Amsterdam 1980. Lect. Notes Math.* **888**, Springer-Verlag, Berlin, 1981.

[28] R. Koekoek, R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its $q-$analogue*, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17, 1998.
See also the online version in `http://aw.twi.tudelft.nl/~koekoek/askey.html`

[29] T.H. Koornwinder, *Compact quantum groups and q-special functions*, in *Representations of Lie groups and quantum groups*, V. Baldoni & M. A. Picardello (eds.), Pitman Research Notes in

Mathematics Series 311, Longman Scientific & Technical, 1994, pp. 46–128.
See also the part "q-Special functions, a tutorial", in `ftp://unvie6.un.or.at/siam/opsf_new/koornwinder/koornwinder3/koornwinder3.tex` and *ibid.*/`wilsonqhahn.ps`

[30] L. Lorentzen, H. Waadeland, *Continued Fractions with Applications*, North-Holland, 1992.

[31] M. Maillard, S. Boukraa, Modular invariance in lattice statistical mechanics, *ANNALES DE LA FONDATION LOUIS DE BROGLIE*, Volume **26** numro spcial, 2001, 287-328. `http://www.ensmp.fr/aflb/AFLB-26j/aflb26jp287.pdf`

[32] A.P.MAGNUS: Recurrence coefficients for orthogonal polynomials on connected and non connected sets, pp. 150-171 *in* L.WUYTACK,editor: *Padé Approximation and its Applications*, Lecture Notes in Mathematics **765**, Springer Verlag Berlin 1979.

[33] A.P. Magnus, Rate of convergence of sequences of Padé-type approximants and pole detection in the complex plane, pp. 300–308 *in* M.G. de Bruin, H. van Rossum, eds: *Padé Approximation and its Applications, Amsterdam 1980. Lect. Notes Math.* **888**, Springer-Verlag, Berlin, 1981.

[34] A.P.MAGNUS: Riccati acceleration of Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials, pp 213-230 *in* H. WERNER , H.J. BÜNGER, editors: *Padé Approximation and its Applications, Bad Honnef 1983*, Lecture Notes in Mathematics **1071**, Springer Verlag , Berlin , 1984.

[35] A.P.MAGNUS: Asymptotic behaviour of continued fraction coefficients related to singularities of the weight function, pp.22-45 *in* D.G.PETTIFOR, D.L.WEAIRE, editors: *The Recursion Method and its Applications*, Springer Series in Solid-State Sciences **58**, Springer, Berlin 1985.

[36] A.P. Magnus, Associated Askey-Wilson polynomials as Laguerre-Hahn orthogonal polynomials, pp. 261-278 *in* M. Alfaro *et al.*, editors: *Orthogonal Polynomials and their Applications, Proceedings, Segovia 1986. Springer Lecture Notes Math.* **1329**, Springer, Berlin, 1988.

[37] A.P. Magnus, Special non uniform lattice (*snul*) orthogonal polynomials on discrete dense sets of points, *J. Comp. Appl. Math.* **65** (1995) 253-265.

[38] A.P. Magnus, Elliptic Hypergeometric Solutions to Elliptic Difference Equations, *SIGMA* **5** (March 2009), 038, 12 pages, `http://www.emis.de/journals/SIGMA/2009/038/sigma09-038.pdf`, contribution to the Proceedings of the Workshop Elliptic Integrable Systems, Isomonodromy Problems, and Hypergeometric Functions (July 2125, 2008, MPIM, Bonn, Germany). The full collection is available at `http://www.emis.de/journals/SIGMA/Elliptic-Integrable-Systems.html`

[39] A.P. Magnus, Rational interpolation to solutions of Riccati difference equations on elliptic lattices, *Journal of Computational and Applied Mathematics*, **233** (Issue 3, 1 Dec. 2009, 9th OPSFA Conference) 793-801.
preprint `http://perso.uclouvain.be/alphonse.magnus/num3/MagnusLuminy2007.pdf`

[40] P. Maroni, An introduction to second degree forms, *Adv. Comput. Math.* **3** (1995) 59-88.

[41] L. M. Milne-Thomson, *The Calculus Of Finite Differences*, Macmillan And Company., Limited, 1933. `http://www.archive.org/details/calculusoffinite032017mbp`

[42] A.F. Nikiforov, V.B. Uvarov, *Classical orthogonal polynomials of a discrete variable on nonuniform lattices* (in Russian), Keldysh Institute of Applied Mathematics, Preprint # 17, Moscow, 1983.

[43] A.F. Nikiforov, S.K. Suslov, Classical orthogonal polynomials of a discrete variable on nonuniform lattices, *Letters in Math. Phys.* **11** (1986) 27-34.

[44] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer, Berlin, 1991.

[45] J. Nuttall, S.R. Singh, Orthogonal polynomials and Padé approximants associated to a sustem of arcs, *J. Approx. Theory* **21** (1977), 1-42.

[46] J. Nuttall, Sets of minimum capacity, Padé approximants and the bubble problem, pp. 185–201 *in* C. Bardos, D. Bessis, eds: *Bifurcation Phenomena in Mathematical Physics and Related Topics, Proceedings NATO ASI Cargèse, 1979*, NATO ASI Ser. C **54**, D. Reidel, Dordrecht, 1980.

[47] J. Nuttall, Asymptotics of diagonal Hermite-Padé polynomials, *J. Approx. Theory* **42** (1984), 299-386.

[48] F. Peherstorfer, On Bernstein-Szegö orthogonal polynomials on several intervals, *SIAM J. Math. Analysis*, **21** (1990), 461-482.

[49] F. Peherstorfer, On Bernstein-Szegö orthogonal polynomials on several intervals II: Orthogonal polynomials with periodic recurrence coefficients, *J. Approx. Theory* **64** (1991), 123-161.

[50] F. Peherstorfer, Orthogonal and extremal polynomials on several intervals, *J. Comp. Appl. Math.* **48** (1993), 187-205.

[51] O.Perron, *Die Lehre von den Kettenbrüchen*, 2$^{\text{nd}}$ edition, Teubner, Leipzig, 1929 = Chelsea, edition of 1913 in `http://www.archive.org/details/dielehrevondenk00perrgoog`, book digitized by Google from the library of the University of Michigan and uploaded to the Internet Archive by user tpb.

[52] A. Ronveaux, private communication, October 2006.

[53] V. P. Spiridonov: Classical elliptic hypergeometric functions and their applications, Lectures delivered at the workshop "Elliptic integrable systems" (RIMS, Kyoto, November 8-11, 2004). This work is supported in part by the Russian Foundation for Basic Research (RFBR) grant no. 03-01-00780. 253-287 `http://www.math.kobe-u.ac.jp/publications/rlm18/17.pdf`

[54] V. P. Spiridonov, Theta hypergeometric integrals, Russian original version: *Algebra i Analiz*, tom 15 (2003), vypusk 6. English version: *St. Petersburg Math. J.* **15** 929-967 (2004).

[55] V.P. Spiridonov, Elliptic hypergeometric functions, Abstract: This is a brief overview of the status of the theory of elliptic hypergeometric functions to the end of 2006 written as a complement to a Russian edition (to be published by the Independent University press, Moscow, 2007) of the book by G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encycl. of Math. Appl. **71**, Cambridge Univ. Press, 1999. Report number: RIMS-1589 Cite as: arXiv:0704.3099v1 [math.CA]

[56] V.P. Spiridonov, Elliptic hypergeometric functions and models of Calogero-Sutherland type. (Russian) *Teoret. Mat. Fiz.* **150** (2007), no. 2, 311–324; English translation in *Theoret. and Math. Phys.* **150** (2007), no. 2, 266–277.

[57] Spiridonov, V. P.; Zhedanov, A. S., Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, *in* Bustoz, Joaquin (ed.) et *al.*, *Special functions 2000: current perspective and future directions. Proceedings of the NATO Advanced Study Institute, Tempe, AZ, USA, May 29–June 9, 2000*, Dordrecht: Kluwer Academic Publishers. NATO Sci. Ser. II, Math. Phys. Chem. 30, 365-388 (2001).

[58] Spiridonov, Vyacheslav; Zhedanov, Alexei, Classical biorthogonal rational functions on elliptic grids. *C. R. Math. Acad. Sci., Soc. R. Can.* **22**, No.2, 70-76 (2000).

[59] V. P. Spiridonov and A. S. Zhedanov: Elliptic grids, rational functions, and the Padé interpolation *The Ramanujan Journal* **13**, Numbers 1-3, June, 2007, p. 285–310.

[60] H. Stahl, Three different approaches to a proof of convergence for Padé approximants, pp. 79-124 *in* J. Gilewicz, M. Pindor, W. Siemaszko, eds: *Rational Approximation and its Applications in Mathematics and Physics, Proceedings, Lancut 1985, Lect. Notes Math.* **1237**, Springer, Berlin, 1987.

[61] H. Stahl, Asymptotics of Hermite-Padé polynomials and related approximants A summary of results, *in*: *Nonlinear Numerical Methods and Rational Approximation*, ed. A. Cuyt (Reidel, Dordrecht, 1988) pp. 2353.

[62] H. Stahl, The convergence of Padé approximants to functions with branch points, *J. Approx. Th.* **91** (1997) 139-204.

[63] J.P. Tignol, *Galois' theory of algebraic equations*, World Scientific Publishing Co., Inc., 2001.

[64] P. Turchi, F. Ducastelle, G. Treglia, Band gaps and asymptotic behaviour of continued fraction coefficients, *J. Phys. C: Solid State Phys.* **15** (1982) 2891-2924.

[65] W. Van Assche, Christoffel functions and Turán determinants on several intervals, *J. Comp. Appl. Math.* **48** (1993), 207-223.

[66] Alfred J. van der Poorten, Elliptic Curves and Continued Fractions, *Journal of Integer Sequences*, `http://www.univie.ac.at/EMIS/journals/JIS/index.html` Volume **8**, 2005 Issue 2 Article 05.2.5, 19 pages.

[67] A. J. Van Der Poorten and C. S. Swart, Recurrence Relations for Elliptic Sequences: Every Somos 4 is a Somos k, *Bull. Lond. Math. Soc.* **38** (2006) 546-554.

[68] H. Widom, Extremal polynomials associated with a system of curves in the complex plane, *Adv. Math.* **3** (1969) 127–232.

[69] A.S. Zhedanov, Biorthogonal Rational Functions and the Generalized Eigenvalue Problem, *J. Approx. Theory*, Volume **101**, Issue 2, December 1999, Pages 303-329.

[70] A.S. Zhedanov, Padé interpolation table and biorthogonal rational functions. In: *Proceedings of RIMS Workshop on Elliptic Integrable Systems. Kyoto, November 8-11* (2004) to be published `http://www.math.kobe-u.ac.jp/publications/rlm18/20.pdf`

[71] A S Zhedanov, V P Spiridonov, "Hypergeometric biorthogonal rational functions", *Russ. Math Surv.*, 1999, **54** (2), 461-463.