MAPA 3071 Special topics in approximation theory.

2000-2002: Asymptotic estimates in complex rational approximation.

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Abstract: several ways to build rational approximations to various species of analytic functions are examined. Special emphasis is put on strong asymptotic estimates of the form $f(z) - R_n(z) \sim \sigma(z)\rho^n(z)$, when such estimates are available.

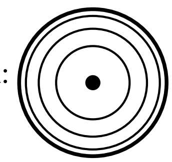
CONTENTS

1. Complex approximation theory and potential theory.

1.1. Taylor expansions.

The Taylor series expansion of a function with finite convergence domain, say

$$\frac{1}{1-z/R}=\sum_0^\infty \frac{z^k}{R^k}, \qquad \text{or} \qquad \sqrt{1-z/R}=\sum_0^\infty \left(-\frac{1}{2}\right)\frac{1}{2}\cdots\left(k-\frac{3}{2}\right)\frac{z^k}{k!R^k}$$

shows “typically” almost circular level lines of equal approximation:  , explained by a convenient representation of the error

$$f(z)-\sum_0^n c_k z^k = K_n(z)\,(z/R)^n, \tag{1}$$

where K_n is “typically” slowly variable in n. What is meant by “typical” must be estimated on particular classes of functions. The only general truth here is that K_n is bounded by a slowly variable[1] function of n when $|z|<R$:

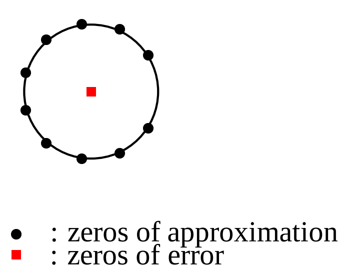
$$K_n(z)=\left(\frac{R}{z}\right)^n \frac{1}{2\pi i}\int_{|t|=r} f(t)\frac{z^{n+1}}{t^n(t-z)}\,dt,$$

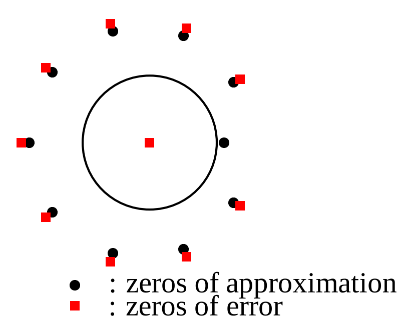
with $|z|<r<R$, and r arbitrarily close to R. For some functions f, an infinite subset $\{K_{n_i}\}_i$ may be much smaller than expected, that’s why some special classes of functions f will be described (sometimes a *single* function. . .) when accurate asymptotic estimates of (1) will be needed. For the examples above,

$$\frac{1}{1-z/R}-\sum_0^n (z/R)^k=\frac{(z/R)^{n+1}}{1-z/R}\,,\ \sqrt{1-z/R}-\sum_0^n c_k(z/R)^k=\frac{d_{n+1}^{(n)}(z/R)^{n+1}+\cdots+d_{2n}^{(n)}(z/R)^{2n}}{\sqrt{1-z/R}\,+\,\sum_0^n c_k(z/R)^k},$$

with suitable $d_{n+1}^{(n)}$, etc. A special asymptotic analysis is needed to show that $d_{n+1}^{(n)}=2c_{n+1}\sim -1/\sqrt{\pi n^3}$.

The circle $|z|=R$ is the boundary of the convergence region. It is often conveniently approached by the zeros of the truncated expansion, here with $n=10$:





[1] slowly variable = less than exponentially variable.

For $f(z) = 1/(1-z/R)$, the truncated Taylor expansion is $p_n(z) = (1-(z/R)^{n+1})/(1-z/R)$, therefore vanishes at the n nontrivial $(n+1)^{\text{th}}$ roots of unity. For $f(z) = \sqrt{1-z/R}$, the picture is not so simple, but the zeros of p_n appear to have the circle as limit set as well. It is not essential to take zeros, similar results hold for $z : p_n(z) = \xi$, for any ξ: as a "typical" function f is expected to be almost univalent, or not very many-valent, in the convergence disk, most of the roots of $p_n(z) = \xi$ will be outside the disk interior.

Darboux

1.2. **Interpolatory (Jacobi) expansions.**

..., tout entier à une idée
qui lui était venue sur les *potentiels*.

Alphonse Allais (from *Madrigal manqué*

1.3. **General polynomial interpolation.**

1.4. **Padé approximation.**

Padé approximation is closely linked to continued fractions, also to orthogonal polynomials. The latter aspect appears especially with expansion about ∞.

Just as the simplest explicit arithmetic continued fractions are the periodic ones giving square roots, we have square roots of polynomials:

$$\sqrt{z^2-1} = z - \frac{1}{2z} - \frac{1}{8z^3} - \cdots = z - \cfrac{1/2}{z - \cfrac{1/4}{z - \cfrac{1/4}{z - \cdots}}}$$

and the $[n+1/n]$ Padé approximant is the ratio of ***Chebyshev*** polynomials

$$\frac{T_{n+1}(z)}{U_n(z)} = \sqrt{z^2-1}\,\frac{(z+\sqrt{z^2-1})^{n+1} + (z-\sqrt{z^2-1})^{n+1}}{(z+\sqrt{z^2-1})^{n+1} - (z-\sqrt{z^2-1})^{n+1}}$$

Square roots of polynomials of degree > 2 will usually not yield periodic continued fractions, but a kind of quasi-periodicity which will be described further on. It will also be seen that the arrangement $f = \dfrac{X-\sqrt{Y}}{Z}$ is almost pure periodic [49]. So,

$$\frac{z^2-\alpha z-\beta\sqrt{1-\alpha^2} - \sqrt{(z^2-1)(z-\alpha-\beta)(z-\alpha+\beta)}}{z-\alpha} = \frac{\gamma}{z} + \alpha\frac{\gamma-(1-\beta^2)/2}{z^2} + \cdots$$
$$= \cfrac{\gamma}{z - \alpha\beta(\beta-\sqrt{1-\alpha^2})/\gamma - \cfrac{\gamma'}{z-\cdots}}$$

with $\gamma = (1+\beta^2)/2 - \beta\sqrt{1-\alpha^2}$, and $\gamma' = \ldots$ turns into an (apparent) mess. Special periodic cases are

- Two-periodic $\alpha = 0$:

$$\frac{z^2-\beta-\sqrt{(z^2-1)(z^2-\beta^2)}}{z} = \cfrac{(\beta-1)^2/2}{z - \cfrac{(\beta+1)^2/4}{z - \cfrac{(\beta-1)^2/4}{z-\cdots}}}$$

Approximants $q_n/p_n = [n-1/n]$:

$$p_{n+2} = (z^2 - (1+\beta^2)/2)p_n - \left(\frac{1-\beta^2}{4}\right)^2 p_{n-2}.$$

$$\frac{q_{2n}}{p_{2n}} = \frac{z\dfrac{(\beta-1)^2}{2}\dfrac{\rho^{2n}-\rho^{-2n}}{\sqrt{(z^2-1)(z^2-\beta^2)}}\left(\dfrac{1-\beta^2}{4}\right)^n}{(A\rho^{2n}+B\rho^{-2n})\left(\dfrac{1-\beta^2}{4}\right)^n},$$

$$\frac{q_{2n+1}}{p_{2n+1}} = ((\beta-1)^2/2)\frac{(C\rho^{2n+1}+D\rho^{-2n-1})\left(\dfrac{1-\beta^2}{4}\right)^n}{z(\rho^{2n}+\rho^{-2n})\left(\dfrac{1-\beta^2}{4}\right)^n}$$

with $\rho = \sqrt{\dfrac{z^2-(1+\beta^2)/2+\sqrt{(z^2-1)(z^2-\beta^2)}}{(1-\beta^2)/2}}$, and $A,B = \dfrac{1}{2} \pm \dfrac{z^2-\beta}{2\sqrt{(z^2-1)(z^1-\beta^2)}}$, $C,D = \rho^{\mp 1}\left(\dfrac{1}{2} \pm \dfrac{z^2+\beta}{2\sqrt{(z^2-1)(z^2-\beta^2)}}\right)$.

The approximants converge (check that the limit is f) outside the singular locus $S = \{z : |\rho(z)| = 1\}$. All the zeros and poles but one are on, or tend when $n \to \infty$ towards, S. There is a supplementary zero at 0 for the even approximants, a supplementary pole at 0 for the odd approximants.

The numerator and the denominator have the forms $C_n\rho^n + D_n\rho^{-n}$, and $A_n\rho^n + B_n\rho^{-n}$, with A_n, etc., D_n two-periodic in n.

- Period three: $\alpha = (1-\beta^2)/2$.

$$\frac{z^2 - \dfrac{1-\beta^2}{2}z + \beta\dfrac{1-2\beta-\beta^2}{2} - \sqrt{(z^2-1)\left(z-\dfrac{1-2\beta-\beta^2}{2}\right)\left(z-\dfrac{1+2\beta-\beta^2}{2}\right)}}{z-\dfrac{1-2\beta-\beta^2}{2}} =$$

$$\cfrac{(1-\beta)(1+\beta)^2/2}{z+\beta(1+\beta)/2-\cfrac{(1-\beta)^2/4}{z+\beta(1+\beta)/2-\cfrac{(1-\beta)(1+\beta)^2/4}{z-\beta-\cfrac{(1-\beta)(1+\beta)^2/4}{z+\beta(1+\beta)/2-\cdots}}}}$$

Approximants $[n-1/n] = q_n/p_n$:

$$p_{n+3} = \left(z^3+\beta^2z^2+\frac{\beta^4-2\beta^2-3}{4}z-\beta^2\right)p_n - \left(\frac{(1-\beta^2)^2}{8}\right)^2 p_{n-3}.$$

$$\frac{q_{3n}}{p_{3n}} = \frac{(1-\beta)(1+\beta)^2}{2}\frac{\left(z-\dfrac{\beta+1}{2}\right)\dfrac{\rho^{3n}-\rho^{-3n}}{2\sqrt{Y}}\left(\dfrac{(\beta^2-1)^2}{8}\right)^n}{(A\rho^{3n}+B\rho^{-3n})\left(\dfrac{(\beta^2-1)^2}{8}\right)^n}$$

$$\frac{q_{3n+1}}{p_{3n+1}} = \frac{(1-\beta)(1+\beta)^2}{2}\frac{(C\rho^{3n+1}+D\rho^{-3n-1})\left(\dfrac{(\beta^2-1)^2}{8}\right)^n}{(A'\rho^{3n+1}+B'\rho^{-3n-1})\left(\dfrac{(\beta^2-1)^2}{8}\right)^n}$$

$$\frac{q_{3n+2}}{p_{3n+2}} = \frac{(1-\beta)(1+\beta)^2}{2}\frac{(C'\rho^{3n+2}+D'\rho^{-3n-2})\left(\frac{(\beta^2-1)^2}{8}\right)^n}{\left(z+\frac{\beta^2+2\beta-1}{2}\right)\frac{\rho^{3n+3}-\rho^{-3n-3}}{2\sqrt{Y}}\left(\frac{(\beta^2-1)^2}{8}\right)^n}$$

with $Y=(z^2-1)\left(z-\frac{1-2\beta-\beta^2}{2}\right)\left(z-\frac{1+2\beta-\beta^2}{2}\right)$,
$\rho=\sqrt[3]{\frac{z^3+\beta^2z^2-\frac{3+2\beta^2-\beta^4}{4}z-\beta^2+\left(z+\frac{1+\beta^2}{2}\right)\sqrt{Y}}{(1-\beta^2)^2/4}}$,
$A,B=\frac{1}{2}\pm\frac{2z^2+(\beta^2-1)z+\beta(1-2\beta-\beta^2)}{4\sqrt{Y}}, C,D=\rho^{\mp 1}\left(\frac{1}{2}\pm\frac{2z^2+(\beta-1)^2z-2\beta}{4\sqrt{Y}}\right), A'\rho=C'\rho^2, B'\rho^{-1}=$ $D'\rho^{-2}=\frac{z+\beta(1+\beta)/2}{2}\pm\frac{4z^3+2(2\beta^2+\beta-1)z^2+(\beta^4+\beta^3-3\beta^2-\beta-2)z-(2\beta^3+6\beta^2+2\beta-2)}{8\sqrt{Y}}$.

- General quadratic case [49]

We consider $f(z):=f_0(z)=\cfrac{\beta_0(z)}{\alpha_0(z)-\cfrac{\beta_1(z)}{\alpha_1(z)-\cdots}}=\frac{X_0(z)-\sqrt{Y(z)}}{Z_0(z)}$, where X_0, Y, and Z_0 are given polynomials, and where the continued fraction arrangement (Jacobi, or Stieltjes, or $C-$fraction, etc.) keeps α_n and β_n to be polynomials of small degrees.

Then, $f_n(z):=\cfrac{\beta_n(z)}{\alpha_n(z)-\cfrac{\beta_{n+1}(z)}{\alpha_{n+1}(z)-\cdots}}=\frac{X_n(z)-\sqrt{Y(z)}}{Z_n(z)}$, or

$$f_n=\frac{\beta_n}{\alpha_n-f_{n+1}}, \tag{2}$$

with polynomial X_n and Z_n of ultimate degrees related to the degree of Y. For Jacobi continued fractions (Padé at ∞), the degrees are $g+1$ and g if the degree of Y is $2g+2$.

$$X_n^2-Y=\beta_nZ_nZ_{n+1}, \qquad X_{n+1}=-X_n-\alpha_nZ_{n+1}. \tag{3}$$

Also, the X_n's and Z_n's have a quasi-periodic behaviour with respect to n.

For the two and three periodic examples above,

$$X_{2n}(z)=z-\beta, X_{2n+1}(z)=z+\beta; \qquad Z_n(z)=z;$$

$$X_{3n}(z)=X_{3n+2}(z)=z^2-\frac{1-\beta^2}{2}z+\beta\frac{1-2\beta-\beta^2}{2}, X_{3n+1}(z)=z^2-\frac{1-\beta^2}{2}z-\beta,$$

$$Z_{3n}(z)=z-\frac{1-2\beta-\beta^2}{2}, Z_{3n+1}(z)=Z_{3n+2}(z)=z-\frac{1+\beta}{2}.$$

The recurrence $p_{n+1}=\alpha_np_n-\beta_np_{n-1}$ of p_n and q_n of the approximants $q_n/p_n=1/0, \beta_0/\alpha_0$, $\beta_0\alpha_1/(\alpha_0\alpha_1-\beta_1),\ldots$, has also the solution $f_0f_1\ldots f_n$ (from (2), $\alpha_nf_0\ldots f_n-f_0\ldots f_nf_{n+1}=\beta_nf_0\ldots f_{n-1}$), and we have

$$f_0\ldots f_n=q_n-fp_n. \tag{4}$$

Considering the two possible choices of the square root of Y in f and the f_n's,

$$p_n=-\frac{f_{0,\mathrm{conj.}}\cdots f_{n,\mathrm{conj.}}-f_0\cdots f_n}{f_{\mathrm{conj.}}-f}, \qquad q_n=\frac{f_{1,\mathrm{conj.}}\cdots f_{n,\mathrm{conj.}}-f_1\cdots f_n}{1/f_{\mathrm{conj.}}-1/f}. \tag{5}$$

$$(f_0\dots f_n)^2 = \frac{(X_0-\sqrt{Y})^2\cdots(X_n-\sqrt{Y})^2\, Z_{n+1}}{Z_0\ \underbrace{Z_0Z_1}_{(X_0^2-Y)/\beta_0}\ \cdots\ \underbrace{Z_nZ_{n+1}}_{(X_n^2-Y)/\beta_n}}$$
$$= \beta_0\dots\beta_n\,\frac{Z_{n+1}}{Z_0}\,\frac{X_0-\sqrt{Y}}{X_0+\sqrt{Y}}\cdots\frac{X_n-\sqrt{Y}}{X_n+\sqrt{Y}}$$

Definition. *The average complex rate of growth = exponential of the complex Green function is*

$$\rho = e^{\mathcal{G}} = \lim_{n\to\infty}\left(\frac{X_0+\sqrt{Y}}{X_0-\sqrt{Y}}\cdots\frac{X_n+\sqrt{Y}}{X_n-\sqrt{Y}}\right)^{1/(2n)},$$

with the square root of Y such that $|\rho|\geqslant 1$ (= the real Green function $G := \mathrm{Re}\,\mathcal{G}\geqslant 0$) everywhere.

1.5. **Strong asymptotics of Padé terms for functions with branch points.**

Two branch points. Szegő: denominator p_n is orthogonal to w, if $f(z) = \int_a^b \frac{w(t)}{z-t}\,dt$. Then, $p_n(z)\sim \alpha(z)\rho^n(z)$,

1.6. **General rational interpolation. Condenser capacity.**

2. Asymptotic features of rational interpolation.

2.1. **According to Gončar-Stahl (a sloppy rendering).**

Interpolation to $f_n(z) = \int_{C_f}\frac{\varphi_0(t)\varphi^n(t)}{z-t}\,dt$ at $z_0,\dots,z_{m+n}$ by p_m/q_n yields

$$f_n(z)-\frac{p_m(z)}{q_n(z)} = \frac{\prod_0^{m+n}(z-z_j)}{q_n^2(z)}\int_{C_f}\frac{q_n^2(t)}{\prod_0^{m+n}(t-z_j)}\,\frac{\varphi_0(t)\varphi^n(t)}{z-t}\,dt,$$

where q_n is (formally) orthogonal with respect to $\frac{\varphi_0(t)\varphi^n(t)}{\prod_0^{m+n}(t-z_j)}$ on C_f.

Well, we expect that most of the poles of q_n will tend to a set of arcs S, with a limit distribution μ_p, that C_f may be modified within the closure of the domain where φ_0 and φ are analytic, so that $S\subseteq C_f$. On the support of μ_p, q_n is almost a Szegő orthogonal polynomial! which means that $\pm q_n(t)\frac{\varphi^{n/2}(t)}{\sqrt{\prod_0^{m+n}(t-z_j)}}$ has slowly varying phase and absolute value there.

Sloppy asymptotic explanation with (complex) potentials $\mathcal{V}_p(z) := \int_{\mathrm{supp}\,\mu_p}\log(z-t)\,d\mu_p(t)$ and $\mathcal{V}_i(z) := \int_{\mathrm{supp}\,\mu_i}\log(z-t)\,d\mu_i(t)$ (interpolation points), so that $q_n(z)\sim\exp(n\mathcal{V}_p(z))$ when $z\notin\mathrm{supp}(\mu_p)$, $\prod_0^{m+n}(z-z_j)\sim\exp(2n\mathcal{V}_i(z))$, and $q_n(z)\approx\exp(n\mathcal{V}_{p,+}(z))+\exp(n\mathcal{V}_{p,-}(z))$ on $\mathrm{supp}(\mu_p)$. Then,

$$q_n(t)\frac{\varphi^{n/2}(t)}{\sqrt{\prod_0^{m+n}(t-z_j)}} \approx [\exp(n\mathcal{V}_{p,+}(t))+\exp(n\mathcal{V}_{p,-}(t))]\,\exp(n\log\varphi(t)/2-n\mathcal{V}_i(t))$$
$$\approx \exp\left(n\left[\frac{\log\varphi(t)}{2}-\mathcal{V}_i(t)+\frac{\mathcal{V}_{p,+}(t)+\mathcal{V}_{p,-}(t)}{2}\right]\right)\ \cos n\left(\frac{\mathcal{V}_{p,+}(t)-\mathcal{V}_{p,-}(t)}{i}\right)$$

on S, or:

$$\log\varphi(t)/2 - \mathcal{V}_i(t) + [\mathcal{V}_{p,+}(t) + \mathcal{V}_{p,-}(t)]/2 = \text{constant}, \tag{6}$$

the same real constant on all the arcs of S, has a real part smaller than this constant on $C_f \setminus S$.

For derivatives:

$$(\log\varphi(z))'/2 - \mathcal{V}_i''(z) + \fint_S \frac{d\mu_p(t)}{z-t} = 0 \text{ on } z \in S. \tag{7}$$

Remark that the (complex conjugate of) the derivative $(\log\varphi(z))'/2 - \mathcal{V}_i''(z) + \mathcal{V}_p''(z)$ on the two sides of S gives the gradient of the real potential $\mathrm{Re}[\log\varphi(z)/2 - \mathcal{V}_i(z) + \mathcal{V}_p(z)]$, and has ***opposite*** values $\pm\pi i\overline{\mu'(z)}$ on the two sides of S, from (7) and the ***Sokhotskyi-Plemelj*** formulas (see § 2.5) for $\mathcal{V}_p''$: ***symmetry property*** [4, 25, 74, 75, etc.].

2.2. According to Nuttall (in construction).

2.3. The Riemann-Hilbert way (not even started).

2.4. Conditions on a single arc.

Let the function (often associated to a distribution of poles) $\mathcal{V}_p''(z) = \int_\alpha^\beta \frac{d\mu_p(t)}{z-t}$. Suppose that we know that

$$= \fint_\alpha^\beta \frac{d\mu_p(t)}{z-t} = g(z), \tag{8}$$

with g analytic in some domain (the arc $[\alpha,\beta]$ is not yet known). The trick is to multiply $\mathcal{V}_p''$ by a function $[(z-\alpha)(z-\beta)]^{\gamma/2}$ taking *opposite* values on the two sides of $[\alpha,\beta]$. We consider only $\gamma = 1$ and $\gamma = -1$. Also, $[(z-\alpha)(z-\beta)]^{\gamma/2}$ is defined to be continuous outside the arc, and behaves like z^γ for large z. As $\mathcal{V}_p''(z)[(z-\alpha)(z-\beta)]^{\gamma/2} - \delta_{\gamma,1}$ has a Laurent expansion with only negative powers at ∞,

$$\mathcal{V}_p''[(z-\alpha)(z-\beta)]^{\gamma/2} - \delta_{\gamma,1} = \frac{1}{2\pi i}\oint \frac{\mathcal{V}_p''(t)[(t-\alpha)(t-\beta)]^{\gamma/2} - \delta_{\gamma,1}}{z-t}\,dt$$

on a big counterclockwise contour having the arc $[\alpha,\beta]$ inside and z outside. Making the contour shrink to a neighbourhood of the arc $[\alpha,\beta]$, we get $\dfrac{1}{2\pi i}\displaystyle\int_\alpha^\beta \frac{\Delta\{\mathcal{V}_p''(t)[(t-\alpha)(t-\beta)]^{\gamma/2} - \delta_{\gamma,1}\}}{z-t}\,dt$, where $\Delta\{F\}$ means the difference $F_- - F_+$ between the limit values of F on the "lower" side of the arc (from which the arc is seen at left), and the "upper" side. The difference is here $2g(t)[(t-\alpha)(t-\beta)]_-^{\gamma/2}$, whence quite explicit solutions

$$\mathcal{V}_p''[(z-\alpha)(z-\beta)]^{\gamma/2} - \delta_{\gamma,1} = \frac{1}{\pi i}\int_\alpha^\beta \frac{g(t)[(t-\alpha)(t-\beta)]_-^{\gamma/2}}{z-t}\,dt, \qquad \gamma = \pm 1. \tag{9}$$

It may help to realize that the phase of $\dfrac{\beta-\alpha}{[(t-\alpha)(t-\beta)]_-^{1/2}}$ is exactly the one of $+i$ on the rectilinear segment $[\alpha,\beta]$.

Some questions: the -1 in the left-hand side of (9) when $\gamma = 1$ is needed from $\mathcal{V}_p''(z) = 1/z + O(1/z^2)$ for large z. But the two sides of (9) when $\gamma = -1$ should be $\sim 1/z^2$ for large z, everything works only if

$$\int_\alpha^\beta \frac{g(t)\,dt}{[(t-\alpha)(t-\beta)]_-^{1/2}} = 0, \quad \int_\alpha^\beta \frac{tg(t)\,dt}{[(t-\alpha)(t-\beta)]_-^{1/2}} = \pi i. \tag{10}$$

The two forms of (9) then agree, either with $\gamma = -1$, or $\gamma = 1$. It will also be useful to check that, as (8) is a plain integral when $z = \alpha$ and $z = \beta$, one has $\mathcal{V}_p''(\alpha) = g(\alpha)$, and $\mathcal{V}_p''(\beta) = g(\beta)$.

2.4.1. *A little bit of Chebyshev polynomials calculus.* N.B. Ullman

Let us consider the Chebyshev polynomials expansion of a generic function F on $[\alpha,\beta]$:

$$F(t) = \frac{c_0}{2} + \sum_1^\infty c_n T_n\left(\frac{2t-\alpha-\beta}{\beta-\alpha}\right).$$

Then, we have the integral

$$\int_\alpha^\beta \frac{F(t)\,dt}{[(t-\alpha)(t-\beta)]_-^{1/2}} = \pi i \frac{c_0}{2}.$$

Therefore, from (9) with $\gamma = -1$, $\mathcal{V}_p''[(z-\alpha)(z-\beta)]^{-1/2}$ is the constant term of the Chebyshev expansion of $g(t)/(z-t)$.

Let $g_0/2 + \sum_1^\infty g_n T_n$ be the expansion of g. Remark that (10) becomes

$$g_0 = 0, \qquad g_1 = \frac{4}{\beta-\alpha}. \tag{11}$$

We need the expansion of $1/(z-t) = X_0/2 + \sum_1^\infty X_n T_n$, which we multiply by $\dfrac{2(z-t)}{\beta-\alpha} = \dfrac{2z-\alpha-\beta}{\beta-\alpha} - \dfrac{2t-\alpha-\beta}{\beta-\alpha}$:

$$\frac{2}{\beta-\alpha} = \frac{X_0}{2}\left(\frac{2z-\alpha-\beta}{\beta-\alpha} - T_1((2t\alpha-\beta)/(\beta-\alpha))\right) + \sum_{n=1}^\infty X_n\left(\frac{2z-\alpha-\beta}{\beta-\alpha}T_n - \frac{T_{n-1}+T_{n+1}}{2})\right)$$

whence the recurrence $X_{n+1} - 2\dfrac{2z-\alpha-\beta}{\beta-\alpha}X_n + X_{n-1} = 0$ for $n = 1,2,\dots$ solved by $X_n = X_0\rho^n$, where ρ is a root of

$$\frac{\rho+\rho^{-1}}{2} = \frac{2z-\alpha-\beta}{\beta-\alpha}\,, \tag{12}$$

normally with $|\rho| < 1$, but this will have to be discussed later. The value of X_0 comes from $n = 0$: $4/(\beta-\alpha) = X_0((\rho+\rho^{-1})/2) - X_1 = X_0(\rho^{1}-\rho)/2$, so

$$X_0 = \frac{8}{(\beta-\alpha)(\rho^{-1}-\rho)}.$$

Remark that $[(z-\alpha)(z-\beta)]^{1/2} = (\beta-\alpha)^2(1-\rho^2)^2/(16\rho^2)$, so that

$$\mathcal{V}_p''(z) = \sum_{n=1}^\infty g_n\rho^n. \tag{13}$$

The two determinations of $\mathcal{V}_p''$ on the two sides of the cut $[\alpha,\beta]$ are obtained with the *two* roots ρ and $1/\rho$ of (12). One checks that the arithmetic mean is indeed

$$(\mathcal{V}_{p,+}''(z) + \mathcal{V}_{p,-}''(z))/2 = \sum_1^\infty g_n(\rho^n+\rho^{-n})/2 = \sum_1^\infty g_n T_n = g(z).$$

As for the discontinuity along the cut,

$$\pm\pi i\mu_p'(z) = \mathcal{V}_{p,-}''(z) + \mathcal{V}_{p,+}''(z) = \sum_1^\infty g_n(\rho^{-n}-\rho^n) = \frac{4}{\beta-\alpha}[(z-\alpha)(z-\beta)]^{1/2}\sum_1^\infty g_n U_{n-1}(z), \tag{14}$$

it appears as a kind of harmonic conjugate to g.

2.5. **Conditions on several arcs.**

Let (7) be

$$\fint_S \frac{d\mu_p(t)}{z-t} = g(z)\ , \quad z \in S, \tag{15}$$

with $g(z) = \mathcal{V}_i''(z) - (\log\varphi(z))'/2$, whereas $\mathcal{V}_p''(z) = \fint_S \frac{d\mu_p(t)}{z-t} \pm \pi i \mu_p'(z)$ (***Sokhotskyi-Plemelj*** formulas [31], chap. 14, etc.).

If the (unknown) endpoints are $\alpha_1, \dots, \alpha_{2m}$, and $P(z) = (z-\alpha_1)\cdots(z-\alpha_{2m})$,
$\dfrac{\mathcal{V}_p''(z)}{\sqrt{P(z)}} = \pm \dfrac{g(z)}{\sqrt{P(z)}} +$ the same function on the two sides of S, so,

$$\frac{\mathcal{V}_p''(z)}{\sqrt{P(z)}} = \frac{1}{\pi i}\int_S \frac{g(t)}{\sqrt{P(t)}}\,\frac{dt}{z-t}$$

(Sokhotskyi-Plemelj again).

As $\mathcal{V}_p''(z) = z^{-1} + \cdots$ for large z, $m+1$ equations follow

$$\int_S t^k\,\frac{g(t)}{\sqrt{P(t)}}\,dt = 0, \quad k = 0, \dots, m-1; \quad \int_S t^m\,\frac{g(t)}{\sqrt{P(t)}}\,dt = \pi i,$$

and the $m-1$ further equalities, from (6)

$$\log\varphi(\alpha_{2k})/2 - \mathcal{V}_i'(\alpha_{2k})\mathcal{V}_p(\alpha_{2k} = \log\varphi(\alpha_{2k+1})/2 - \mathcal{V}_i'(\alpha_{2k+1})\mathcal{V}_p(\alpha_{2k+1} \ \ k = 1, \dots, m-1.$$

2.5.1. *Differential equation for $\mathcal{V}_p''$ [25]* . If there is a polynomial Q such that Qg'/g is a polynomial,

$$Q(z)\sqrt{P(z)}\ \mathcal{V}_p''(z) = \text{ polynomial } + \frac{1}{\pi i}\int_S Q(t)g(t)\sqrt{P(t)}\,\frac{dt}{z-t}$$

we derive in z and perform an integration by parts

$$[Q(z)\sqrt{P(z)}\ \mathcal{V}_p''(z)]' = \text{ another polynomial } + \left(\frac{P'(z)Q(z)}{2} + \frac{P(z)Q(z)g'(z)}{g(z)} + P(z)Q'(z)\right)\frac{1}{\pi i}\int_S \frac{g(t)}{\sqrt{P(t)}}\,\frac{dt}{z-t},$$

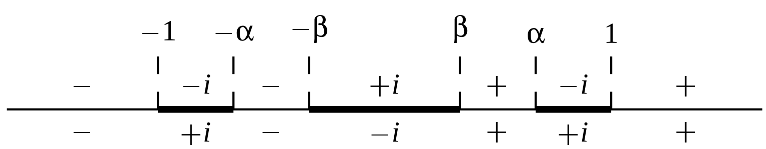
or $Q\sqrt{P}\left(\dfrac{\mathcal{V}_p''}{g}\right)' = \dfrac{\text{polynomial}}{g}$.

2.5.2. *An example from [16, p. 404] (J. Meinguet).* We have $g(x) = -2\delta x^3$ on $[-1,1]$ with $\delta > 0$ and we suspect S to be $[-1,-\alpha]\bigcup[-\beta,\beta]\bigcup[\alpha,1]$.

The formula for $\mathcal{V}_p''$ is surprisingly elementary:

$$\mathcal{V}_p''(z) = -2\delta z^3 + \delta(2z^2 + \alpha^2 + \beta^2 - 1)\sqrt{\frac{(z^2-\alpha^2)(z^2-\beta^2)}{z^2-1}}$$

we have indeed $(\mathcal{V}_{p,+}''(x) + \mathcal{V}_{p,-}''(x))/2 = -2\delta x^3$ on $S =$ the set of points where the square root is pure imaginary. Let us look at the argument of this square root just above and just below the real axis:



The coefficient of z^{-1} in the expansion of $\mathcal{V}_p''(z)$ about ∞ is 1 if $4 = \delta[3 - 2(\alpha^2+\beta^2) - (\alpha^2-\beta^2)^2 - (\alpha^2+\beta^2-1)^2]$. Finally, Re $\mathcal{V}_p(x) + \delta x^4/2$ must be the same constant on the intervals of S, which holds if $\int_\beta^\alpha (2t^2+\alpha^2+\beta^2-1)\sqrt{\dfrac{(t^2-\alpha^2)(t^2-\beta^2)}{t^2-1}}\,dt = 0$. This latter equation gives, say, α as a function of β, and the former equation gives the corresponding δ:

β^2	α^2	δ
0	0.5402057601	3.319654336
0.0001	0.5398986321	3.318351926
0.01	0.5221272482	3.237132862
0.02	0.5073410955	3.172076626
0.03	0.4936208059	3.115410709
0.04	0.4805403767	3.064945359
0.05	0.4679087488	3.019526908
0.06	0.4556185329	2.978410767
0.07	0.4436016321	2.941064425
0.08	0.4318114417	2.907083036
0.09	0.4202143377	2.876146324
0.10	0.4087851073	2.847993877
0.11	0.3975042842	2.822409762
0.12	0.3863564943	2.799212342
0.13	0.3753293774	2.778247201
0.14	0.3644128565	2.759382044
0.15	0.3535986256	2.742502906
0.16	0.3428797816	2.727511262
0.17	0.3322505543	2.714321783
0.18	0.3217061032	2.702860566
0.19	0.3112423637	2.693063714
0.20	0.3008559284	2.684876183
0.21	0.2905439552	2.678250852
0.22	0.2803040951	2.673147746
0.23	0.2701344357	2.669533405
0.24	0.2600334576	2.667380353
0.25	0.2500000000	2.666666667

Inequalities:
We want $\mu'(x) \geqslant 0$ on the whole of S, OK if $3\alpha^2+\beta^2-1 \geqslant 0$ and $\alpha^2+3\beta^2-1 \leqslant 0$. The real part of $\mathcal{V}_p(x)+\delta x^4/2$ must be smaller on (β,α) than its constant value on S, OK, as

$$\mathcal{V}_p(x)+\delta x^4/2 - \mathcal{V}_p(\beta) - \delta\beta^4/2 = \delta\int_\beta^x (2t^2+\alpha^2+\beta^2-1)\sqrt{\frac{(t^2-\alpha^2)(t^2-\beta^2)}{t^2-1}}\,dt$$

is the integral of a function which is negative on a first part of (β,α), positive on another part, but the integral remains negative, as it must vanish when $x=\alpha$ (the square root is positive in the whole interval (β,α)).

The set S is reduced to two intervals when $\delta > 3.319\ldots$, and to a single interval when $\delta < 8/3$. It is conjectured in [16] that S has at most 3 parts for any $g(x) = -\delta x^{2q-1}$.

3. The exponential function e^{Az}.

3.1. Taylor.

3.2. Padé.

for e^z,

$$[m/n] = \frac{1+\frac{m}{m+n}\frac{z}{1!}+\frac{m(m-1)}{(m+n)(m+n-1)}\frac{z^2}{2!}+\cdots+\frac{m(m-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(n+1)}\frac{z^m}{m!}}{1-\frac{n}{m+n}\frac{z}{1!}+\frac{n(n-1)}{(m+n)(m+n-1)}\frac{z^2}{2!}-\cdots+(-1)^n\frac{n(n-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(m+1)}\frac{z^n}{n!}} \tag{16}$$

$$\begin{aligned} e^z \text{ den.} - \text{num.} &= \frac{(-1)^n}{(m+1)\cdots(m+n)}\sum_{k=m+n+1}^{\infty}\frac{(k-m-1)\cdots(k-m-n)}{k!}z^k \\ &= \frac{(-1)^n}{(m+n)!}\left[\int_{-\infty}^{z}-\int_{-\infty}^{0}=\int_0^z e^t(z-t)^m t^n\,dt\right] \end{aligned} \tag{17}$$

[66]

Exponential behaviour of numerator and denominator has been much worked, especially the distribution of zeros and poles. Saff & Varga remark [67, II] that, when $m\sim n$, these distributions had already been examined by Olver [64] in a study of Bessel functions.

Integrals of the form (17) behave for large n as value at saddlepoint. With $m\sim n$, saddlepoint is a root of

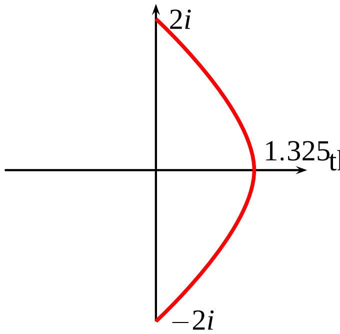
$$1+\frac{n}{t-z}+\frac{n}{t}=0,\ \text{whence } t=\frac{z}{2}-n+\sqrt{\frac{z^2}{4}+n^2},$$

with some choice for the square root (see later), and $(z-t)t = 2n^2\left(\sqrt{1+\frac{z^2}{4n^2}}-1\right)$.

Denominator behaves like n^{th} power of

$$\frac{1+\sqrt{1+\frac{z^2}{4n^2}}}{2}\exp\left(1-\frac{z}{2n}-\sqrt{1+\frac{z^2}{4n^2}}\right), \tag{18}$$

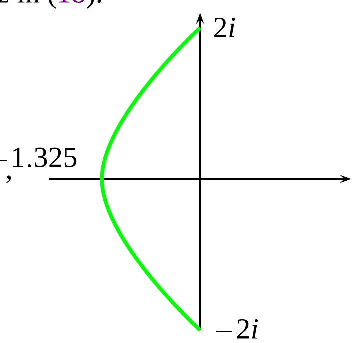
where the value of the square root is 1 at $z=0$. When z increases, the value of (18) becomes very small and must be replaced by the same formula with the other choice of the square root as soon as the new formula has an absolute value which is larger than the former one. This happens near $z/n = 1.3255$. In the

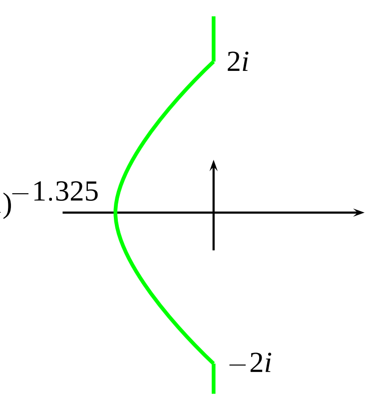
$z/n-$plane: 

the formula (18) holds with a continuous square root outside the shown arc, which is the locus where the two formulas have the same absolute value, also the limit of the ***poles*** of the approximant. The equation of the arc is $|w(z)| = 1$, where

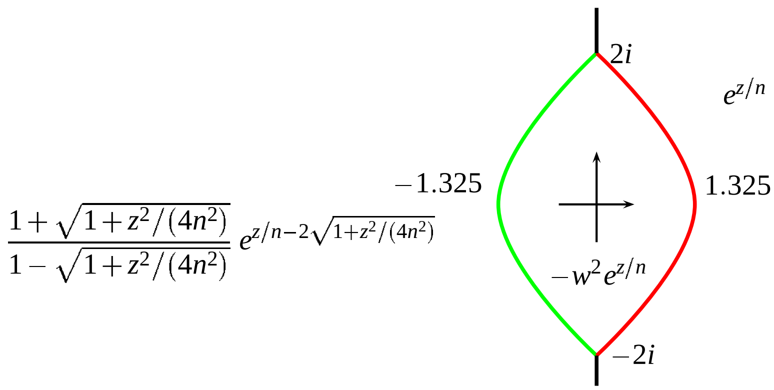
$$w(z) = \sqrt{\frac{\sqrt{1+\frac{z^2}{4n^2}}-1}{\sqrt{1+\frac{z^2}{4n^2}}+1}}\ \exp\left(\sqrt{1+\frac{z^2}{4n^2}}\right) = \frac{(z/2n)\exp\left(\sqrt{1+z^2/(4n^2)}\right)}{1+\sqrt{1+\frac{z^2}{4n^2}}}$$

This w looks like the ρ of before, but is not a plain quadratic algebraic function.
Remark the square root behaves like $-z/(2n)$ for large z in (18).

Numerator: $\dfrac{1+\sqrt{1+\dfrac{z^2}{4n^2}}}{2}\ \exp\left(1+\dfrac{z}{2n}-\sqrt{1+\dfrac{z^2}{4n^2}}\right)$, 

Here the square root behaves like $z/(2n)$ for large z. The formula holds with a continuous square root outside another arc (the limit set of the ***zeros*** of the approximant) which is another part of the locus $|w(z)| = 1$.
And the remainder (or residual) e^z den. $-$ num. behaves like the n^{th} power of
$\dfrac{1-\sqrt{1+\dfrac{z^2}{4n^2}}}{2}\ \exp\left(1+\dfrac{z}{2n}-\sqrt{1+\dfrac{z^2}{4n^2}}\right)$ to the right of the latter arc, together with the part of the imaginary axis outside $[-2ni, 2ni]$ (which is also a part of the locus $|w(z)| = 1$) 

Finally, for the error $e^z -$ approximant, the powers are



The example just seen shows the interest of a technique much used by J. Nuttall, for instance in [62, § 3.2], consisting in building asymptotic approximations χ_1, χ_2, and χ_3 to the remainder, denominator, and numerator[2]. These χ functions, which are here these n^{th} powers of exponentials and algebraic functions, are expected to be piecewise analytic,

3.3. **Padé approximation to** $\exp(nB_1z+nB_2z^2)$ **and a bit of Painlevé functions.**

Laguerre wrote quite a number of papers dealing with continued fraction expansion of solutions of linear differential equations of first order with polynomial coefficients. His theory culminates in [41], but he considered separately in [40] exponentials of polynomials $\exp(F(z))$. Laguerre starts immediately with the Padé property (here, about 0):

$$e^{F(z)}-\frac{q_n(z)}{p_n(z)}=O(z^{m+n+1})=\varepsilon_n z^{m+n+1}+\eta_n z^{m+n+2}+\cdots,$$

if the degrees of q_n and p_n are m and n ($m-n$ a small fixed integer). Divide by e^F and derivate:

$$-\frac{d}{dz}\left(\frac{q_n}{p_n e^F}\right)=e^{-F(0)}\left[(m+n+1)\varepsilon_n z^{m+n}+(m+n+2)(\eta_n-\varepsilon_n F'(0))z^{m+n+1}+\cdots\right]$$

The left-hand side must be a polynomial divided by $p_n^2 e^F$. The left-hand side numerator is $-q_np_n(q_n'/q_n-p_n'/p_n-F')=q_np_nF'-p_nq_n'+p_n'q_n$ of degree $m+n+g$ if the degree of F is $g+1$ (yes, g will be the genus of something...) And has the right-hand side has order $m+n$ at 0, we have

$$-\frac{d}{dz}\left(\frac{q_n(z)}{p_n(z)e^F(z)}\right)=\frac{z^{m+n}\varepsilon_n\Theta_n(z)}{p_n^2(z)e^{F(z)}} \tag{19}$$

with Θ_n of degree g. Remark that $\Theta_n(0)=m+n+1$ if we choose $p_n(0)=1$. As the right-hand side is the derivative of a meromorphic function, it must have vanishing residues: $\dfrac{m+n}{z}+\dfrac{\Theta_n'(z)}{\Theta_n(z)}-\dfrac{p_n''(z)}{p_n'(z)}-F'(z)=0$ at the zeros of p_n, whence

$$z\Theta_n(z)p_n''(z)+[zF'(z)\Theta_n(z)-z\Theta_n'(z)-(m+n)\Theta_n(z)]p_n'(z)+M_n(z)p_n(z)=0, \tag{20}$$

where M_n is a new polynomial of degree $2g$.

We need to know a little more on Θ_n and M_n for large n, in order to extract the asymptotic behaviour of p_n.

[2] In [62, § 3.2], χ_1, χ_2, and χ_3 are called R, χ_1, and χ_2.

Any other solution y of (20) satisfies $\left(\dfrac{m+n}{z}+\dfrac{\Theta_n'(z)}{\Theta_n(z)}-F'(z)\right)\left(\dfrac{p_n'(z)}{p_n(z}-\dfrac{y'}{y}\right)=\dfrac{p_n''(z)}{p_n(z}-\dfrac{y''}{y}$, or $p_n'y-p_ny'=\text{const.}\, z^{m+n}e^{-F}\Theta_n$, whence $\left(\dfrac{y}{p_n}\right)'=\text{const.}\,\dfrac{z^{m+n}\Theta_n}{e^F\ p_n^2}$, showing that

$$Ap_n(z)+Bq_n(z)e^{-F(z)}$$

is the general solution of (20).

Such differential equations with apparent singular points (as all solutions are entire functions) have been much worked by W. Hahn [26,27,28,29]. The coefficients must be Painlevé-like functions (see [47], containing references to R. Fuchs, D. & G. Chudnovsky, etc.).

We use $q_np_{n-1}-p_nq_{n-1}=\varepsilon_{n-1}z^{m+n-1}$ to write (19) as

$$-\frac{d}{dz}\left(\frac{q_n(z)}{p_n(z)e^{F(z)}}\right)=\frac{z\varepsilon_n[q_np_{n-1}-p_nq_{n-1}]\Theta_n(z)}{\varepsilon_{n-1}p_n^2(z)e^{F(z)}},$$

or

$$q_n[p_n'+F'p_n]-q_n'p_n=z\frac{\varepsilon_n}{\varepsilon_{n-1}}[q_np_{n-1}-p_nq_{n-1}]\Theta_n,$$

$$q_n\left[p_n'-z\frac{\varepsilon_n}{\varepsilon_{n-1}}\Theta_np_{n-1}\right]=p_n\left[q_n'-F'q_n-z\frac{\varepsilon_n}{\varepsilon_{n-1}}\Theta_nq_{n-1}\right]$$

must be a polynomial of the form $p_nq_n\Omega_n$, with Ω_n of degree g. Whence the ***differential system*** of first order

$$p_n'=\Omega_np_n+z\frac{\varepsilon_n}{\varepsilon_{n-1}}\Theta_np_{n-1}\ ,q_n'=(F'+\Omega_n)q_n+z\frac{\varepsilon_n}{\varepsilon_{n-1}}\Theta_nq_{n-1}.$$

Conditions at $n=0$: $p_0=1$, $p_{-1}=0$, $q_0=$ Taylor-Maclaurin expansion of e^F truncated to degree $m-n$. $q_np_{n-1}-p_nq_{n-1}=\varepsilon_{n-1}z^{m+n-1}\Rightarrow q_{-1}=-\varepsilon_{-1}z^{m-n-1}$. Also, $\Omega_0=0$.

The same equations for p_{n-1} and q_{n-1} involve p and q_{n-2}, which we eliminate through the three-terms recurrence relation $p_n(z)=(1-\alpha_nz)-\dfrac{\varepsilon_{n-1}}{\varepsilon_{n-2}}z^2p_{n-2}(z)$, and we arrive at

$$\boldsymbol{Y_n}'=\boldsymbol{A}_n\boldsymbol{Y}_n\ ,\ \boldsymbol{Y}_n=\begin{bmatrix}p_n & q_ne^{-F}\\ p_{n-1} & q_{n-1}e^{-F}\end{bmatrix}\ ,\ \boldsymbol{A}_n=\begin{bmatrix}\Omega_n & z\dfrac{\varepsilon_n}{\varepsilon_{n-1}}\Theta_n\\ -z^{-1}\Theta_{n-1} & \Omega_{n-1}+(z^{-1}-\alpha_n)\Theta_{n-1}\end{bmatrix}. \tag{21}$$

Remark that $\boldsymbol{Y}_0$ being upper triangular, so is $\boldsymbol{A}_0=(\boldsymbol{Y}_0')^{-1}\boldsymbol{Y}_0\Rightarrow\Theta_{-1}=0$.

From trace $\boldsymbol{A}_n=(\log\det\boldsymbol{Y})'$,

$$\Omega_n(z)+\Omega_{n-1}(z)=-\frac{(1-\alpha_nz)\Theta_{n-1}(z)+zF'(z)-m-n+1}{z}. \tag{22}$$

When we reconstruct the scalar second order differential equation for p_n using now $\boldsymbol{A}_n=\begin{bmatrix}\Omega_n & z\dfrac{\varepsilon_n}{\varepsilon_{n-1}}\Theta_n\\ -z^{-1}\Theta_{n-1} & z^{-1}(m+n-1)-F'-\Omega_n\end{bmatrix}$, we recover the two first terms of (20), and conclude that $M_n=z(\Omega_n\Theta_n'-\Omega_n'\Theta_n)+(m+n)\Omega_n\Theta_n-z\Theta_n[\Omega_n^2+F'\Omega_n-(\varepsilon_n/\varepsilon_{n-1})\Theta_n\Theta_{n-1}]$. This complicated formula makes not even clear if M_n has indeed degree $2g$ (the degree seems to be $3g+1$). We still need more relations by looking at the first order differential equation for $p_{n+1}(z)=(1-\alpha_{n+1}z)p_n(z)-(\varepsilon_n/\varepsilon_{n-1})z^2p_{n-1}(z)$ to find

$$(1-\alpha_{n+1}z)(\Omega_n-\Omega_{n+1})+(\varepsilon_n/\varepsilon_{n-1})z\Theta_{n-1}-\alpha_{n+1}=z(\varepsilon_{n+1}/\varepsilon_n)\Theta_{n+1},$$

or, at $n-1$,

$$(1-\alpha_nz)(\Omega_n-\Omega_{n-1})=\frac{\varepsilon_{n-1}}{\varepsilon_{n-2}}z\Theta_{n-2}-\alpha_n-\frac{\varepsilon_n}{\varepsilon_{n-1}}z\Theta_n, \tag{23}$$

leading to, by multiplying (22) by (23), and by summation in n:

$$\Omega_n^2 - \frac{\varepsilon_n}{\varepsilon_{n-1}}\Theta_n\Theta_{n-1} + \Omega_n\frac{zF' - m - n}{z} = \frac{\sum_0^{n-1}\Theta_k - mn}{z^2} + \frac{nF'}{z}\ . \tag{24}$$

It is then possible to advance the recurrence for the Θ's and the Ω's: if Θ_{n-1}, Ω_{n-1}, and $\varepsilon_{n-1}/\varepsilon_{n-2}$ are known, we find

$$\alpha_n = \frac{2\Omega_{n-1}(0) + \Theta_n'(0) + F'(0)}{m+n} \tag{25}$$

from the difference of (22) and (23) at $z=0$; then (22) yields Ω_n, and (23) gives Θ_n and $\varepsilon_n/\varepsilon_{n-1}$, knowing that $\Theta_n(0) = m+n+1$.

With $F(z) = Az$, one finds the explicit solution $\Theta_n = m+n+1$, $\Omega_n = -\dfrac{nA}{m+n}$, $\alpha_n = \dfrac{(m-n)A}{(m+n)(m+n-2)}$, ($\alpha_1 = \dfrac{A}{m-n+2}$), $\varepsilon_n/\varepsilon_{n-1} = -\dfrac{mnA^2}{(m+n-1)(m+n)^2(m+n+1)}$.

See later for a much more difficult example of degree 2 for F.

The recurrence equations (22) and (24) have a structure somewhat similar to the equations (3)! This is an important remark already made by Gammel and Nuttall in [20] also about the Laguerre theory.

We are very close to (3) by taking $X_n = \Omega_n + F'/2 - (m+n)/(2z)$ and $\Theta_n = Z_{n+1}$. But what is Y?

Ah, Painlevé. Let us introduce a parameter t in F. Then, the coefficients of p_n, q_n, etc. will be functions of t. The Painlevé-like equations are differential equations in t for the coefficients of the polynomials Θ_n and Ω_n, for each n.

Let $\boldsymbol{H}_n := \dot{\boldsymbol{Y}}_n\boldsymbol{Y}_n^{-1}$. From (21),

$$\boldsymbol{H}_n = \frac{-1}{\varepsilon_{n-1}z^{m+n-1}}\begin{bmatrix} \dot p_n q_{n-1} - \dot q_n p_{n-1} + \dot F q_n p_{n-1} & -\dot p_n q_n + \dot q_n p_n - \dot F q_n p_n \\ \dot p_{n-1} q_{n-1} - \dot q_{n-1} p_{n-1} + \dot F q_{n-1} p_{n-1} & -\dot p_{n-1} q_n + \dot q_{n-1} p_n - \dot F q_{n-1} p_n \end{bmatrix}$$

is a matrix of rational functions in z. Remark that trace $\boldsymbol{H}_n = \dfrac{\dot\varepsilon_{n-1}}{\varepsilon_{n-1}} - \dot F$. Is there such a big pole at the origin? No: from $q_n(z)e^{-F(z)} - p_n(z) = -\varepsilon_n z^{m+n+1} + \cdots$,

$$\begin{aligned}\dot p_{n-r}q_{n+s-1} - \dot q_{n-r}p_{n+s-1} + \dot F q_{n-r}p_{n+s-1} &= e^F\left\{\dot p_{n-r}e^{-F}q_{n+s-1} - \frac{\partial}{\partial t}\left[e^{-F}q_{n-r}\right]p_{n+s-1}\right\} \\ &= e^F\left\{p_{n+s-1}\frac{\partial}{\partial t}\left[p_{n-r} - e^{-F}q_{n-r}\right] + \dot p_{n-r}(e^{-F}q_{n+s-1} - p_{n+s-1})\right\}\end{aligned}$$

showing that the orders in $\boldsymbol{H}_n$ are $\binom{1\ 2}{0\ 0}$ ($r,s = 0,1$), so, polynomials of degrees not higher than $g+1$, much less if only coefficients of low degree in F actually depend on t: if the degree of $\partial F/\partial t$ is δ, then the degrees of the elements of $\boldsymbol{H}_n$ are $\binom{\delta\ ,\delta+1}{\delta-1,\delta}$. More precisely, the coefficients of lowest order in $\boldsymbol{H}_n$ are $\begin{bmatrix} \dot p_n'(0)z & \dfrac{\dot\varepsilon_n}{\varepsilon_{n-1}}z^2 \\ -\dfrac{\dot\varepsilon_{n-1}}{\varepsilon_{n-1}} & \dfrac{\dot\varepsilon_{n-1}}{\varepsilon_{n-1}} \end{bmatrix}$.

When $\delta = 1$ (and when $F(0)$ does not depend on t), we have

$$\boldsymbol{H}_n = \begin{bmatrix} \dot p_n'(0)z & \dfrac{\dot\varepsilon_n}{\varepsilon_{n-1}}z^2 \\ -\dfrac{\dot\varepsilon_{n-1}}{\varepsilon_{n-1}} & \dfrac{\dot\varepsilon_{n-1}}{\varepsilon_{n-1}} - \dot p_n'(0)z - \dot F \end{bmatrix}.$$

And the differential equation in t for $\boldsymbol{A}_n$ comes from

$$\frac{\partial}{\partial z}\frac{\partial \boldsymbol{Y}_n}{\partial t} = \frac{\partial \boldsymbol{H}_n \boldsymbol{Y}_n}{\partial z} = \frac{\partial \boldsymbol{H}_n}{\partial z}\boldsymbol{Y}_n + \boldsymbol{H}_n \boldsymbol{A}_n \boldsymbol{Y}_n$$
$$= \frac{\partial}{\partial t}\frac{\partial \boldsymbol{Y}_n}{\partial z} = \frac{\partial \boldsymbol{A}_n \boldsymbol{Y}_n}{\partial t} = \frac{\partial \boldsymbol{A}_n}{\partial t}\boldsymbol{Y}_n + \boldsymbol{A}_n \boldsymbol{H}_n \boldsymbol{Y}_n$$

whence

$$\frac{\partial \boldsymbol{A}_n}{\partial t} = \frac{\partial \boldsymbol{H}_n}{\partial z} + \boldsymbol{H}_n \boldsymbol{A}_n - \boldsymbol{A}_n \boldsymbol{H}_n \tag{26}$$

We apply to the Padé approximations $[n-1/n$ to $\exp(\alpha x + \beta x^2)$. It is not bad to look at low degrees, with the `GP-PARI` program [7]:

```
{
/* padee21.gp:  launch gp and type   \r padee21

Pade [n-1 / n]  de exp(ax+b x2)

*/
default(seriesprecision,10);

Yn=matrix(2,2); An=matrix(2,2) ; Hn=matrix(2,2) ;

F=a*x+b*x^2;  Fp=deriv(F,x);        eF=exp(F);

/* n=0:  */   thn1=0+0*x;  omn1=0+0*x ;   epn1=0;    thn2=thn1;
      ep=1;  pn=1; pn1=0 ;  qn=0 ; qn1=0;
 for(n=1,3,
  print(" n=",n);
  m=n-1;
  aln=(2*polcoeff(omn1,0)+polcoeff(thn1,1)+polcoeff(Fp,0))/(m+n);
  print("    alphan=",aln);
  omn=-omn1-(1-aln*x)*thn1/x-Fp+(m+n-1)/x;
  thn=((1-aln*x)*(omn1-omn)+epn1*x*thn2-aln)/x;
  epn=polcoeff(thn,0)/(m+n+1);
  pnp=(1-aln*x)*pn-epn*x^2*pn1;
  qnp=(1-aln*x)*qn-epn*x^2*qn1;
  pn1=pn; pn=pnp; qn1=qn; qn=qnp;
  if(n==1,qn=1);
  print(" qn= ",qn);
  print(" pn= ",pn);
  thn=thn/epn;
  ep=ep*epn;
  print("   epsilon n=",ep);
  print(" Theta n =",thn);
  print(" Omega n =",omn);
    Yn[1,1]=pn;Yn[1,2]=qn/eF;
    Yn[2,1]=pn1;Yn[2,2]=qn1/eF;
    An=deriv(Yn,x)*Yn^(-1);
    Hn=deriv(Yn,a)*Yn^(-1);
   print("check: ",deriv(An,a)-deriv(Hn,x)-Hn*An+An*Hn);
  omn1=omn; epn1=epn; thn2=thn1;thn1=thn;
```

```
    );
}

Script V1.1 session started Wed Oct 11 16:07:49 2000

C:\calc\pari>pari217

                    GP/PARI CALCULATOR Version 2.0.17 (beta)
                  Windows NT ix86 (ix86 kernel) 32-bit version
                (readline disabled, extended help not available)

                           Copyright (C) 1989-1999 by
          C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

? \r padee2
 n=1
    alphan=a
 qn= 1
 pn= -a*x + 1
   epsilon n=-1/2*a^2 + b
 Theta n =-4*a*b/(-a^2 + 2*b)*x + 2
 Omega n =-2*b*x - a
check: [O(x^9), O(x^9); 0, 0]
 n=2
    alphan=(a^3 - 6*a*b)/(-3*a^2 + 6*b)
 qn= ((-a^3 + 6*a*b)/(-3*a^2 + 6*b))*x + 1
```

etc. In a more readable writing:

$$\frac{q_1}{p_1}=\frac{1}{1-\alpha z}\ ,\ \alpha_1=\alpha\ ,\ \varepsilon_1=\frac{2\beta-\alpha^2}{2},$$

$$\Theta_1=2+\frac{4\alpha\beta}{\alpha^2-2\beta}z\quad ,\quad \Omega_1=-F'=-\alpha-2\beta z;$$

$$\frac{q_2}{p_2}=\frac{1+\alpha\dfrac{\alpha^2-6\beta}{3(\alpha^2-2\beta)}z}{1-\dfrac{2\alpha^3}{3(\alpha^2-2\beta)}z+\dfrac{\alpha^4+12\beta^2}{6(\alpha^2-2\beta)}z^2}\ ,$$

$$\alpha_2=-\alpha\frac{\alpha^2-6\beta}{3(\alpha^2-2\beta)}\ ,\ \varepsilon_2=\frac{\alpha^6-6\beta\alpha^4+36\beta^2\alpha^2+72\beta^3}{72(\alpha^2-2\beta)}\ ,$$

$$\Theta_2=4+8\alpha\beta\frac{(\alpha^2-6\beta)(\alpha^4+12\beta^2)}{(\alpha^2-2\beta)(\alpha^6-6\beta\alpha^4+36\beta^2\alpha^2+72\beta^3)}z\ ,\ \Omega_2=-\frac{2\alpha^3}{3(\alpha^2-2\beta)}-\frac{4\beta\alpha^2}{3}\ \frac{\alpha^2-6\beta}{(\alpha^2-2\beta)^2}\ z;$$

$$\frac{q_3}{p_3}=\frac{1+\dfrac{\dfrac{2\alpha^3}{5}(\alpha^4-12\beta\alpha^2+60\beta^2)z+\dfrac{1}{20}(\alpha^8-16\beta\alpha^6+120\beta^2\alpha^4+720\beta^4)z^2}{\alpha^6-6\beta\alpha^4+36\beta^2\alpha^2+72\beta^3}}{1+\dfrac{\dfrac{-3\alpha}{5}(\alpha^6-2\beta\alpha^4+20\beta^2\alpha^2+120\beta^3)z+\dfrac{3}{20}(\alpha^8+40\beta^2\alpha^4-240\beta^4)z^2-\dfrac{\alpha}{60}(\alpha^8+72\beta^2\alpha^4-2160\beta^4)z^3}{\alpha^6-6\beta\alpha^4+36\beta^2\alpha^2+72\beta^3}}\ ,$$

$$\alpha_3=-\frac{\alpha}{15}\ \frac{\alpha^8-24\beta\alpha^6+144\beta^2\alpha^4+2160\beta^4}{(\alpha^2-2\beta)(\alpha^6-6\beta\alpha^4+36\beta^2\alpha^2+72\beta^3)}$$

$$\varepsilon_3 = -\frac{\alpha^{12} - 12\beta\alpha^{10} + 180\beta^2\alpha^8 - 480\beta^3\alpha^6 - 3600\beta^4\alpha^4 - 43200\beta^5\alpha^2 + 43200\beta^6}{7200(\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3)}$$

$$\Theta_3 = 6 + \frac{12\alpha\beta(\alpha^8 - 16\beta\alpha^6 + 120\beta^2\alpha^4 + 720\beta^4)(\alpha^8 + 72\beta^2\alpha^4 - 2160\beta^4)}{(\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3)(\alpha^{12} - 12\beta\alpha^{10} + 180\beta^2\alpha^8 - 480\beta^3\alpha^6 - 3600\beta^4\alpha^4 - 43200\beta^5\alpha^2 + 43200\beta^6)} z,$$

$$\Omega_3 = -\frac{3\alpha}{5}\,\frac{\alpha^6 - 2\beta\alpha^4 + 20\beta^2\alpha^2 + 120\beta^3}{\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3} - \frac{6\beta}{5}\,\frac{(\alpha^4 + 12\beta^2)(\alpha^8 - 16\beta\alpha^6 + 120\beta^2\alpha^4 + 720\beta^4)}{(\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3)^2} z.$$

Hmm... these functions contain more and more complicated terms. For each n, there are basically four non obvious terms to consider: ε_n, $\Omega_n(0)$, and the coefficients of z of Θ_n and Ω_n, i.e., Θ'_n and Ω'_n. It happens that, if one looks for differential equations (in $t = \alpha$) for these four terms through (26), what comes out is a real mess (I tried).

One finds more tractable expressions by considering the even function $\exp(\widetilde{F}(Z))$, with $\widetilde{F}(Z) = F(Z^2) = \alpha Z^2 + \beta Z^4$. Nothing is lost, as the $[2n-1/2n]$ Padé approximants to $\exp(F(Z^2))$ is related to the $[n-1/n]$ Padé approximant to $\exp(F)$: with $z = Z^2$,

$$e^{F(z)} - \frac{q_n(z)}{p_n(z)} = \varepsilon_n z^{2n} + \cdots \Rightarrow e^{\widetilde{F}(Z)} - \frac{\tilde{q}_{2n}(z)}{\tilde{p}_{2n}(z)} = \tilde{\varepsilon}_{2n} Z^{4n} + \cdots,$$

so, $p_n(z) = \tilde{p}_{2n}(Z) = \tilde{p}_{2n}(\sqrt{z})$, $q_n(z) = \tilde{q}_{2n}(Z) = \tilde{q}_{2n}(\sqrt{z})$, $\tilde{\varepsilon}_{2n} = \varepsilon_n$. Remark that $\tilde{q}_{2n}$ is an even polynomial of true degree $2n-2$.

The Laguerre theory above (p. 13) builds the Θ's and the Ω's for this $\exp(\alpha Z^2 + \beta Z^4)$ function, which one writes $\widetilde{\Theta}_N$ and $\widetilde{\Omega}_N$ (and which may be interesting on their own right):

N	$\dfrac{\tilde{q}_N(Z)}{\tilde{p}_N(Z)}$	$\widetilde{\Theta}_N(Z)$	$\widetilde{\Omega}_N(Z)$	$\tilde{\varepsilon}_N$
0	$\dfrac{0}{1}$	0	0	1
1	$\dfrac{1}{1}$	$2\left(1 + 2\dfrac{\beta}{\alpha}Z^2\right)$	$-2\alpha Z - 4\beta Z^3$	α
2	$\dfrac{1}{1-\alpha Z^2}$	$4\left(1 + \dfrac{2\alpha\beta}{\alpha^2 - 2\beta}Z^2\right)$	$-\dfrac{4\beta}{\alpha}Z$	$-\dfrac{\alpha^2 - 2\beta}{2}$
3	$\dfrac{1 + \dfrac{\alpha^2 - 2\beta}{2\alpha}Z^2}{1 - \dfrac{\alpha^2 + 2\beta}{2\alpha}Z^2}$	$6\left(1 + \dfrac{2\beta(\alpha^4 - 4\beta^2)}{\alpha(\alpha^4 + 12\beta^2)}Z^2\right)$	$-\dfrac{2(\alpha^4 + 4\beta^2)}{\alpha(\alpha^2 - 2\beta)}Z - 4\beta Z^3$	$-\dfrac{\alpha^4 + 12\beta^2}{12\alpha}$

Now, $\widetilde{\Theta}_N$ is an even polynomial of degree 2, whose constant coefficient is $2N$; $\widetilde{\Omega}_N$ is an odd polynomial of degree 3, whose coefficient of Z^3 is given by a simple rule, easily deduced from (22) and $\widetilde{\Omega}_0 = 0, \widetilde{\Omega}_1 = -\widetilde{F}' = -2\alpha Z - 4\beta Z^3$: this coefficient of Z^3 is $2\beta((-1)^N - 1)$.

So, we now have three unknown terms for each N: ε_N, and, say ξ_N and χ_N in $\widetilde{\Theta}_N(Z) = 2N(1 + \xi_N Z^2)$ and $\widetilde{\Omega}_N(Z) = 2\beta((-1)^N - 1)Z^3 - \chi_N Z$.

We know apply the differential relations (26) with $t=\alpha$, remarking first that the matrix $\widetilde{\boldsymbol{H}}_N$ is made of even polynomials, therefore of degrees $\binom{2,2}{0,2}$, (as $\delta=$ degree of $\partial\widetilde{F}/\partial\alpha=2$), and orders $\binom{2,2}{0,0}$, (as seen before): we have monomials, excepting the second diagonal element,

$$\widetilde{\boldsymbol{H}}_N(Z)=\begin{bmatrix}\zeta_N Z^2 & \dfrac{\dot{\tilde{\varepsilon}}_N}{\tilde{\varepsilon}_{N-1}}Z^2\\ -\dfrac{\dot{\tilde{\varepsilon}}_{N-1}}{\tilde{\varepsilon}_{N-1}} & \dfrac{\dot{\tilde{\varepsilon}}_{N-1}}{\tilde{\varepsilon}_{N-1}}-\dot{\widetilde{F}}(Z)-\zeta_N Z^2\end{bmatrix},$$

as we don't yet know what the first diagonal could be, but we know the trace.

(26) yields

$$\frac{\partial\widetilde{\Omega}_N}{\partial\alpha}=2\zeta_N Z+Z\frac{\tilde{\varepsilon}_N\dot{\tilde{\varepsilon}}_{N-1}}{\tilde{\varepsilon}_{N-1}^2}\widetilde{\Theta}_N-Z\frac{\dot{\tilde{\varepsilon}}_N}{\tilde{\varepsilon}_{N-1}}\widetilde{\Theta}_{N-1},$$

for the two diagonal elements of $\widetilde{\boldsymbol{A}}_N$, leading to the coefficient of Z^2 of $\widetilde{\Theta}_N$

$$2N\xi_N=\widetilde{\Theta}_{N,2}=4\beta\frac{\dot{\tilde{\varepsilon}}_N}{\tilde{\varepsilon}_n}\,,\qquad \dot{\chi}_N=1-(-1)^N-2N\frac{\tilde{\varepsilon}_N\dot{\tilde{\varepsilon}}_{N-1}}{\tilde{\varepsilon}_{N-1}^2}\widetilde{\Theta}_N+2(N-1)\frac{\dot{\tilde{\varepsilon}}_N}{\tilde{\varepsilon}_{N-1}},$$

knowing that the coefficient of Z^3 in $\widetilde{\Omega}_N$ does not depend on α, and using $\widetilde{\Theta}_N(Z)=2N(1+\xi_N Z^2)$; the off-diagonal yield, after cleaning a little bit,

$$\dot{\widetilde{\Theta}}{}_N'=\frac{\dot{\tilde{\varepsilon}}_N}{\tilde{\varepsilon}_N}[2N-\widetilde{\Theta}_N-Z(\widetilde{F}'+2\widetilde{\Omega}_N)]+(1+2\zeta_N)Z^2\widetilde{\Theta}_N\,,$$

$$\dot{\widetilde{\Theta}}{}_{N-1}'=\frac{\dot{\tilde{\varepsilon}}_{N-1}}{\tilde{\varepsilon}_{N-1}}[\widetilde{\Theta}_N-(2N-2)+Z(\widetilde{F}'+2\widetilde{\Omega}_N)]-(1+2\zeta_N)Z^2\widetilde{\Theta}_{N-1}\,,$$

whence $\zeta_N=((-1)^N-1)/2$, as the coefficient of Z^3 of $\widetilde{F}'+2\widetilde{\Omega}_N$ is $4(-1)^N\beta$, and as the coefficient of Z^2 of $\widetilde{\Theta}_N$ is $4\beta\dot{\tilde{\varepsilon}}_N/\tilde{\varepsilon}_n$, as seen above;

$$2N\dot{\xi}_N=\frac{\dot{\tilde{\varepsilon}}_N}{\tilde{\varepsilon}_N}[-2N\xi_N-2\alpha+2\chi_N]+2N(-1)^N\,,$$

$$2(N-1)\dot{\xi}_{N-1}=\frac{\dot{\tilde{\varepsilon}}_{N-1}}{\tilde{\varepsilon}_{N-1}}[2(N-1)\xi_{N-1}+2\alpha-2\chi_N]-2(N-1)(-1)^N\,,$$

amounting to the differential system

$$4\beta\frac{\partial}{\partial\alpha}\begin{bmatrix}\tilde{\varepsilon}_N/\tilde{\varepsilon}_{N-1}\\ \chi_N\\ \xi_N\\ \xi_{N-1}\end{bmatrix}=\begin{bmatrix}(\tilde{\varepsilon}_N/\tilde{\varepsilon}_{N-1})(2N\xi_N-2(N-1)\xi_{N-1})\\ 4\beta[1-(-1)^N]-4N(N-1)(\tilde{\varepsilon}_N/\tilde{\varepsilon}_{N-1})(\xi_N-\xi_{N-1})\\ \xi_N[-2N\xi_N-2\alpha+2\chi_N]+4\beta(-1)^N\\ \xi_{N-1}[2(N-1)\xi_{N-1}+2\alpha-2\chi_N]+4\beta(-1)^{N-1}\end{bmatrix}\tag{27}$$

a system of 4 equations, but the coefficients of Z^4 and Z^2 of (24) yield 2 first integrals

$$(-1)^N\beta\chi_N+N(N-1)\frac{\tilde{\varepsilon}_N}{\tilde{\varepsilon}_{N-1}}\xi_N\xi_{N-1}=\alpha\beta[(-1)^N-1],$$

$$\chi_N^2-4N(N-1)\frac{\tilde{\varepsilon}_N}{\tilde{\varepsilon}_{N-1}}(\xi_N+\xi_{N-1})-2\alpha\chi_N=2\beta[(2N-1)(-1)^N+1].$$

There must remain a system of two first order differential equations, or scalar second order differential equations.

Well, let us try with ξ_N first:

from the third row of (27), $\chi_N = \alpha + N\xi_N + 2\beta[\dot{\xi}_N - (-1)^N]/\xi_N$; use the first integrals:

$$N(N-1)\frac{\tilde{\varepsilon}_N}{\tilde{\varepsilon}_{N-1}}\xi_{N-1} = -\frac{\alpha\beta}{\xi_N} - (-1)^N\beta\frac{N\xi_N + 2\beta[\dot{\xi}_N - (-1)^N]/\xi_N}{\xi_N} = -\frac{\alpha\beta}{\xi_N} - N\beta(-1)^N - 2\beta^2\frac{(-1)^N\dot{\xi}_N - 1}{\xi_N^2},$$

$$\frac{\tilde{\varepsilon}_N}{\tilde{\varepsilon}_{N-1}} = \frac{N}{4(N-1)}\xi_N + \frac{\beta}{N-1}\frac{\dot{\xi}_N}{\xi_N} - \beta\frac{(2N-1)(-1)^N+1}{2N(N-1)\xi_N} + \frac{\beta^2}{N(N-1)}\frac{1}{\xi_N}\left(\frac{\dot{\xi}_N}{\xi_N}\right)^2 - \frac{1}{4N(N-1)\xi_N}\left(\frac{2\beta}{\xi_N} - \alpha\right)^2$$

gives $\tilde{\varepsilon}_N/\tilde{\varepsilon}_{N-1}$ as a function of ξ_N and its first derivative $\dot{\xi}_N = \partial\xi_N/\partial\alpha$. We enter all that in the first equation of (27), and get a second order differential equation for ξ_N

$$\frac{\ddot{\xi}_N}{\xi_N} - \frac{3}{2}\left(\frac{\dot{\xi}_N}{\xi_N}\right)^2 + \frac{1+(-1)^N}{4\beta} + \frac{\alpha^2}{8\beta^2} + \frac{3}{2\xi_N^2} - \frac{\alpha}{\beta\xi_N} - \frac{N^2\xi_N^2}{8\beta^2} = 0 \tag{28}$$

which relates ξ_N to a solution of the Painlevé-IV equation by

$$\xi_N = \frac{\sqrt{-4\beta}}{\mathfrak{P}_{IV,N}\left(\dfrac{\alpha}{\sqrt{-4\beta}}\right)}, \tag{29}$$

$\mathfrak{P}_{IV,N}(x)$ being a solution of the Painlevé IV equation

$$p'' = \frac{p'^2}{2p} + p\frac{(3p+2x)(p+2x)}{2} - [1+(-1)^N]p - \frac{2N^2}{p} \tag{30}$$

of parameters $[1+(-1)^N]/2$ and $-2N^2$ (see [36, chap. 3] for the Painlevé-IV equation, also Ince [34, § 14.4] of course).

Remark that, as ξ_N is rational in β, (29) shows that $\mathfrak{P}_{IV,N}$ is an *odd* function. From the instances of Θ_n and $\widetilde{\Theta}_N$ seen above, we have some samples

N	1	2	3	4
$\mathfrak{P}_{IV,N}(x)$	$-2x$	$-\dfrac{2x^2+1}{x}$	$-\dfrac{2x(4x^4+3)}{4x^4-1}$	$-\dfrac{(2x^2+1)(8x^6+12x^4+18x^2-9)}{x(2x^2+3)(4x^4+3)}$

For the other functions of interest,

$$\chi_N = \sqrt{-4\beta}\left[x + \frac{N}{\mathfrak{P}_{IV,N}} + \frac{\mathfrak{P}'_{IV,N}}{2\mathfrak{P}_{IV,N}} + \frac{(-1)^N}{2\mathfrak{P}_{IV,N}}\right],$$

$$N(N-1)\frac{\tilde{\varepsilon}_N}{\tilde{\varepsilon}_{N-1}}\xi_{N-1} = \beta\left[-x\mathfrak{P}_{IV,N} - N(-1)^N - \frac{(-1)^N\mathfrak{P}'_{IV,N}}{2} + \frac{\mathfrak{P}^2_{IV,N}}{2}\right],$$

$$8N(N-1)\frac{\tilde{\varepsilon}_N}{\tilde{\varepsilon}_{N-1}} = \sqrt{-4\beta}\left[\frac{2N^2}{\mathfrak{P}_{IV,N}} + \frac{2N\mathfrak{P}'_{IV,N}}{\mathfrak{P}_{IV,N}} + [(2N-1)(-1)^N+1]\mathfrak{P}_{IV,N} + \frac{{\mathfrak{P}'_{IV,N}}^2}{2\mathfrak{P}_{IV,N}} - \frac{\mathfrak{P}_{IV,N}}{2}(\mathfrak{P}_{IV,N}+2x)^2\right]$$

but what may be the use of all this?

To be continued...

3.4. **Rational interpolation.**

3.4.1. *Equidistant points.* [35]

As far as we only need e^{Az} at $z = z_0, z_0 + h, \ldots, z_0 + (m+n)h$,

$$\begin{aligned} e^{Az} &= (\boldsymbol{I} + \boldsymbol{\Delta})^{(z-z_0)/h} e^{Az_0} \\ &= \sum_{k=0}^{m+n} \binom{(z-z_0)/h}{k} \boldsymbol{\Delta}^k e^{Az_0} \\ &= \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{1}{k!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \end{aligned}$$

which we multiply by the denominator $Q(z) = \sum_{j=0}^{n} q_j (z-z_0)\cdots(z-z_0-(j-1)h)$, using

$$\begin{aligned} &(z-z_0)(z-z_0-h)\cdots(z-z_0-(j-1)h)e^{Az} = \\ &\qquad e^{A(z_0+jh)} \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^{k-j} \frac{1}{(k-j)!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \end{aligned}$$

$$Q(z)e^{Az} = e^{Az_0} \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{C(k)}{k!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

where $C(k) = \sum_{j=0}^{n} q_j e^{Ajh} \left(\frac{e^{Ah}-1}{h}\right)^{-j} \frac{1}{(k-j)!}$ is a polynomial of degree n in k, which must vanish at $k = m+1, m+2, \ldots, m+n$,

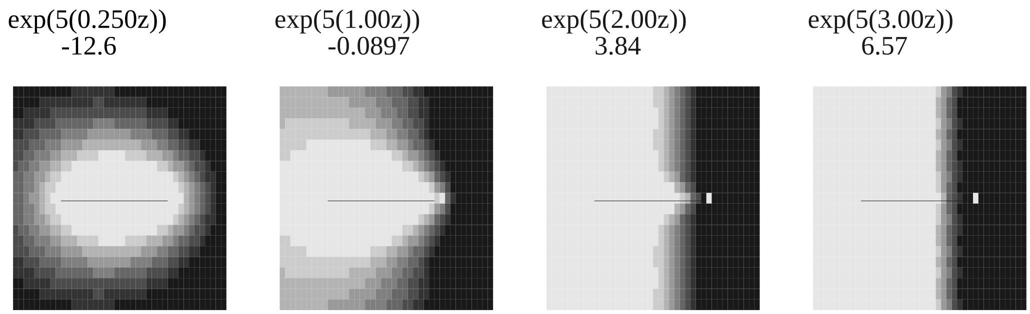
$$P(z) = e^{Az_0} \sum_{k=0}^{m} \left(\frac{e^{Ah}-1}{h}\right)^k \binom{m}{k} (m+n-k)!(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

$$Q(z) = \sum_{k=0}^{n} \left(\frac{e^{-Ah}-1}{h}\right)^k \binom{n}{k} (m+n-k)!(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

and, formally:

$$\begin{aligned} &Q(z)e^{Az} - P(z) = \\ &e^{Az_0} m!(-1)^n \sum_{k=m+n+1}^{\infty} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{(k-m-1)(k-m-2)\cdots(k-m-n)}{k!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \end{aligned} \tag{31}$$

We look at the performance of some examples of the region of good approximation in the complex plane, coloured in light gray:

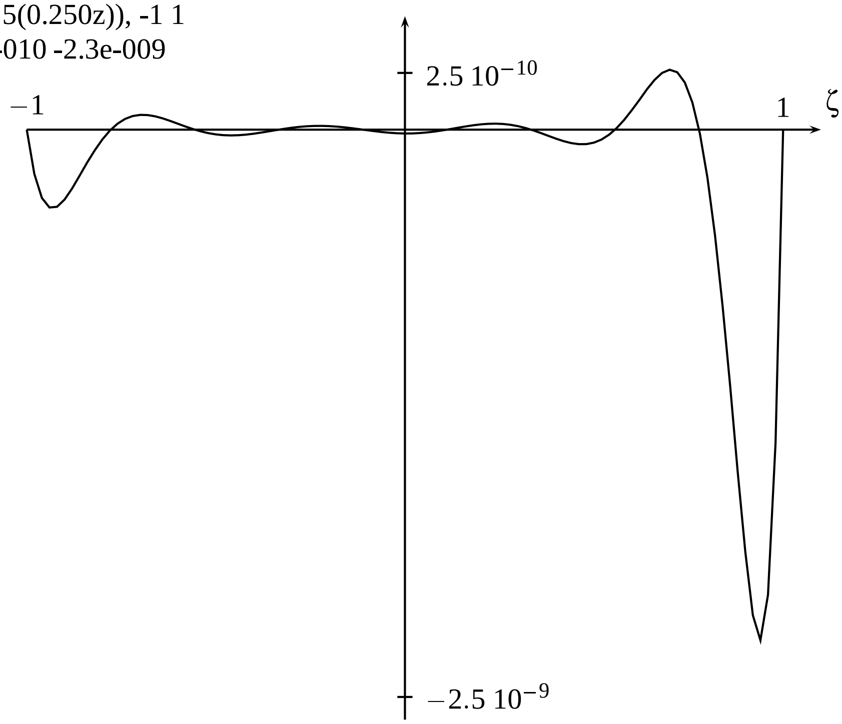


colouring is made with respect of the average of $\log|\exp(Az) - P(z)/Q(z)|$ in the square $[-2,2]\times[-2,2]$. The degrees of P and Q are here 5 and 4. When A is small, the region is an oval around the locus of the interpolation points (here, the interval $[-1,1]$ shown by a thin horizontal black line).

The interpolation points should appear as bright white dots, but they are hardly visible in somewhat big pixels, if colouring is made according to an arbitrary point of the pixel. This chosen point happens to be an actual interpolation point only for the endpoints, whence the rightmost interpolation point looking like a beacon in a dark environment.

The graph of the error function in the interpolation interval also looks like its envelope:

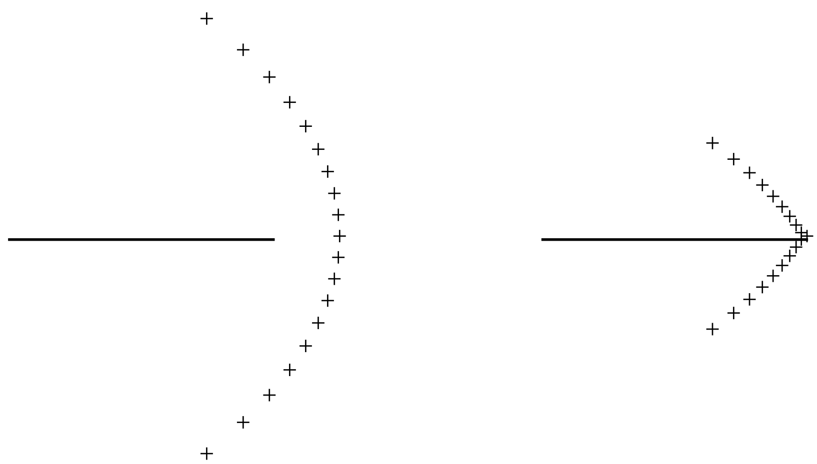
exp(5(0.250z)), -1 1
2.6e-010 -2.3e-009



This suggests that we shall have valuable asymptotic estimates containing something like a nonvanishing power ρ^n, even up to the interpolation locus.

Somewhat similarly poles may even enter the locus of interpolation points. Here, the poles of P/Q (degrees $20/19$) interpolating $\exp(20az)$, with $a=1$ and $a=2$:

n=20 A=1 B1= I B2=0. n=20 A=1 B1= 2*I B2=0.



So, the locus of poles of P/Q of degrees $m \sim n$ and n approximating $\exp(naz)$ enters the locus $[-1,1]$ of interpolation points when a becomes larger than a number slightly smaller than 2. Such features will be explained.

Integral form.

The sums and series (31) are special hypergeometric expansions formally related to Jacobi polynomials with large symbols, say $P_n^{(n\alpha+\beta,n\gamma+\delta)}(x)$. Detailed asymptotics of these polynomials are achieved in [13] and [23]. These authors used either generating functions or integral forms. We adapt the integral form here:

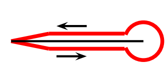
the polynomials of (31) have the same form

$$S = \sum_{k=0}^{M} X^k \binom{M}{k} (P-k)!Y(Y-1)\cdots(Y-k+1), \tag{32}$$

where P is an integer larger than $M($ in (31)), $X = e^{Ah}-1$ or $e^{-Ah}-1$, $M = m$ or n, $P = m+n$, $Y = (z-z_0)/h)$).

As a hypergeometric function, $S =$ const. ${}_2F_1(-M,-Y;-P;X)$, which is the Jacobi polynomial $P_M^{(-P-1,-Y-M+P)}(1+2X)$ with parameters $-P-1$ and $-Y-M+P$ which are never both positive, so that we cannot just quote [13, 23], but follow their methods.

We try to write $(P-k)!Y(Y-1)\cdots(Y-k+1)$ as an integral involving a k^{th} power, in order to achieve a closed form of (32) through the binomial theorem.

The special Beta integral $\int_C u^{P-k}(1-u)^{k-Y-1}du$, where C is an arc starting from the origin, turning around $u=1$, and returning to the origin 0  1 is what we need:

$\int_C u^{P-k}(1-u)^{k-Y-1}du = \cdots = \dfrac{(-1)^{P-k}(P-k)!}{(Y-k)(Y-k-1)\cdots(Y-P+1)} \int_C (1-u)^{P-Y-1}du$, or

$(P-k)!Y(Y-1)\cdots(Y-k+1) = (-1)^{P-k}Y(Y-1)\cdots(Y-P+1)\dfrac{\int_C u^{P-k}(1-u)^{k-Y-1}du}{\int_C (1-u)^{P-Y-1}du}$, then (32) becomes

$$S = (-1)^P Y(Y-1)\cdots(Y-P+1)\frac{\displaystyle\int_C u^{P-M}(1-u)^{-Y-1}[u(1+X)-X]^M\,du}{\displaystyle\int_C (1-u)^{P-Y-1}du}.$$

(Check: if $X = -1$, $S = (P-M)!(Y-P)(Y-P+1)\cdots(Y-P+M-1)$).

The formula is not intended to be of any use if $Y = P-1, P-2, \ldots$ where it yields $0/0$. The integral of the denominator is $(1-e^{-2\pi iY})/(P-Y)$ if the phase of $1-u$ is 0 in the first part of the integral. Then, another form of the formula is

$$S = (-1)^{P-1}\frac{\Gamma(Y+1)\Gamma(P-Y+1)e^{\pi iY}}{2\pi i}\int_C u^{P-M}(1-u)^{-Y-1}[u(1+X)-X]^M\,du$$

Rough asymptotics. Let $E := \exp(Ah)$ and $\zeta := \dfrac{z-z_0-(m+n)h/2}{nh} = \dfrac{Y-(m+n)/2}{n}$. Then, with $m \sim n$, we intend to follow things at constant E and ζ, i.e., a fixed exponential and z expanding linearly with n, or A increasing linearly with n, and $2n$ interpolating points filling a fixed segment $[z_0, z_0+2nh]$. Remark that this segment of interpolation points is $-1 \leqslant \zeta \leqslant 1$.

Remark also that $e^{Az} = e^{Az_0}E^Y$, so

$$e^{Az}Q(z) - P(z) = (-1)^{m+n-1}e^{Az_0}\frac{\Gamma(Y+1)\Gamma(m+n-Y+1)e^{\pi i(m+n)}}{2\pi i}$$

$$\int_C \{u^m E^Y(1-u)^{-Y-1}[uE^{-1}-E^{-1}+1]^n - u^n(1-u)^{-Y-1}[uE-E+1]^m\}\,du$$

...

Saddlepoint & Stirling:

$$S \approx e^{-P}P^P\left\{\frac{1}{4}(1+\zeta)^{1+\zeta}(1-\zeta)^{1-\zeta}u(1-u)^{-\zeta-1}[u(1+X)-X]\right\}^n,$$

where

$$\frac{1}{u} - \frac{\zeta+1}{u-1} + \frac{1}{u-\dfrac{X}{1+X}} = 0,$$

or

$$(\zeta-1)u^2 + \left(2-\frac{\zeta X}{1+X}\right)u - \frac{X}{1+X} = 0$$

so,

$$u = \frac{1+X-\zeta X/2 \pm \sqrt{1+X+\zeta^2X^2/4}}{(1-\zeta)(1+X)}. \tag{33}$$

One of these saddlepoints, or both, enter an asymptotic expression of the denominator and the numerator. It is easy to get lost in intractable, although elementary, formulas.

Strangely enough, a cruder method ends up with almost readable expressions:

At least when each term has the same phase, the sum is roughly given by the term reached when the preceding ratio has unit value:

$$\frac{X\kappa(\zeta+\kappa)}{1-\kappa^2} = 1, \tag{34}$$

where $\kappa := 1-k/n$. Remark that the ratio $\to 0$ when $\kappa \to 0$, and $\to \infty$ when $\kappa \to 1$. The roots are

$$\kappa = \frac{-\zeta X/2 \pm \sqrt{1+X+\zeta^2X^2/4}}{1+X},$$

clearly related to the saddlepoints of (33) by

$$\kappa = (1-\zeta)u - 1.$$

The dominant term (there will sometimes be two dominant terms), is, still roughly,

$$\frac{X^{n-n\kappa}n!(n+n\kappa)!\Gamma(n\zeta+n)}{(n-n\kappa)!(n\kappa)!\Gamma(n\zeta+n\kappa)},$$

where κ is one of the roots of (34). Tedious application of Stirling's formula yields, keeping only the wildest factors,

$$\begin{aligned}
& e^{-2n}n^{2n}\left[\frac{X^{1-\kappa}(1+\kappa)^{1+\kappa}(1+\zeta)^{1+\zeta}}{(1-\kappa)^{1-\kappa}\kappa^{\kappa}(\zeta+\kappa)^{\zeta+\kappa}}\right]^n \\
=& e^{-2n}n^{2n}\left[\underbrace{\left(\frac{1-\kappa^2}{X\kappa(\zeta+\kappa)}\right)^{\kappa}}_{1}\frac{X(1+\kappa)(1+\zeta)^{1+\zeta}}{(1-\kappa)(\zeta+\kappa)^{\zeta}}\right]^n \\
=& e^{-2n}n^{2n}\left[\frac{X(1+\kappa)(1+\zeta)^{1+\zeta}}{(1-\kappa)(\zeta+\kappa)^{\zeta}}\right]^n \\
=& e^{-2n}n^{2n}\left[\frac{(1+(1+X)\kappa)^{1+\zeta}(1+\kappa)^{1-\zeta}}{\kappa}\right]^n,
\end{aligned} \tag{35}$$

using $\zeta = \dfrac{\kappa^{-1}-(1+X)\kappa}{X}$, from (34).

Remark that, if κ is one of the roots of (34), for the numerator ($X = E-1$), the other root is $-1/(E\kappa)$, and that the values of κ for the denominator expansion are the inverses of the ones for the numerator:

$$E^{-1}\kappa_{\text{den}}^2 + (E^{-1}-1)\zeta\kappa_{\text{den}} - 1 = 0 \Rightarrow E + (E-1)\zeta\kappa_{\text{den}} - \kappa_{\text{den}}^2 = 0.$$

Finally, the last expansion of (31) yields the same equation (34) as for the numerator (i.e., with $X = E-1$), and we have the following possible exponential behaviours:

num.	$\dfrac{P(z)}{e^{-2n}n^{2n}}$	$\left[\dfrac{(1+E\kappa)^{1+\zeta}(1+\kappa)^{1-\zeta}}{\kappa}\right]^n$	$\left[-E^{\zeta}\dfrac{(\kappa-1)^{1+\zeta}(E\kappa-1)^{1-\zeta}}{\kappa}\right]^n$
den.	$\dfrac{Q(z)}{e^{-2n}n^{2n}}$	$\left[E^{-1-\zeta}\dfrac{(1+E\kappa)^{1+\zeta}(1+\kappa)^{1-\zeta}}{\kappa}\right]^n$	$\left[-\dfrac{(\kappa-1)^{1+\zeta}(E\kappa-1)^{1-\zeta}}{E\kappa}\right]^n$
ratio	$\dfrac{P(z)}{Q(z)}$	$\left[E^{1+\zeta}\right]^n$	$E^{n(\zeta+1)}\left[-E^{\zeta}\dfrac{(\kappa-1)^{1+\zeta}(E\kappa-1)^{1-\zeta}}{(1+\kappa)^{1-\zeta}(1+E\kappa)^{1+\zeta}}\right]^{\pm n}$
	$\dfrac{Q(z)e^{Az}-P(z)}{e^{-2n}n^{2n}}$	$\left[\dfrac{(1+E\kappa)^{1+\zeta}(1+\kappa)^{1-\zeta}}{\kappa}\right]^n$	$\left[-E^{\zeta}\dfrac{(\kappa-1)^{1+\zeta}(E\kappa-1)^{1-\zeta}}{\kappa}\right]^n$
err.	$e^{Az}-\dfrac{P(z)}{Q(z)}$	$\left[E^{1+\zeta}\right]^n$	$E^{n(\zeta+1)}\left[-E^{\zeta}\dfrac{(\kappa-1)^{1+\zeta}(E\kappa-1)^{1-\zeta}}{(1+\kappa)^{1-\zeta}(1+E\kappa)^{1+\zeta}}\right]^{\pm n}$

Of course, the integral formula above gives more weight to these asymptotic constructions, with $\kappa = (1-\zeta)u-1$.

We expect the numerator, denominator, ratio = the approximation, and the error to behave as one of the given n^{th} powers, or a combination of both. At the present level of formal manipulation, we don't yet know which power(s) to use.

At least for large z, we know that we should have $(E^{\pm 1}-1)^n n!(z/h)^n \approx (E^{\pm 1}-1)^n e^{-n} n^{2n}\zeta^n$ for the numerator and the denominator. Indeed, with the root $\kappa = (E^{-1}-1)\zeta + \cdots$ one finds, with the second power, $\left[\dfrac{P(z)}{(e^{-2n}n^{2n}}\right]^{1/n} \approx -E\kappa\left(\dfrac{\kappa-1}{\kappa-1/E}\right)^{\zeta} \approx (E-1)\zeta\left(1+\dfrac{1}{\zeta}\right)^{\zeta} \approx e(E-1)\zeta$. The first power behaves for large z as $[e(1-E)\zeta E^{1+\zeta}]^n$, such a behaviour is impossible with a polynomial. Similarly, one has for the denominator, with the first power, $\left[\dfrac{Q(z)}{(e^{-2n}n^{2n}}\right]^{1/n} \approx e(E^{-1}-1)\zeta$.

The region where the asymptotic behaviours of P and Q are such that $(P/Q)^{1/n} \approx E^{1+\zeta}$ is simply the region of good approximation! Indeed, $E^{n(1+\zeta)} = e^{Ahn(1+\zeta)} = e^{A(z-z_0)}$.

When E is close to 1, we almost have the Padé situation of the figure of p. 13

Various interesting situations occur, the wildest situation being $E=-1$: we then interpolate merely the sequence $1,-1,1,-1,\dots$ at $z_0,z_1,\dots,z_{m+n}$ by q/p of degrees m and n, without any reference to an exponential function anymore!

Moreover, the solution of the Cauchy problem (in the sense of [52]) is then immediate: $q+p$ must vanish at $z_1,z_3,\dots$, and $q-p$ vanishes at $z_0,z_2,\dots$:

$$\frac{q(z)}{p(z)} = \frac{c(z-z_1)(z-z_3)\cdots + c'(z-z_0)(z-z_2)\cdots}{c(z-z_1)(z-z_3)\cdots - c'(z-z_0)(z-z_2)\cdots},$$

where one of the two numbers c or c' may very well vanish if it is the only way to achieve degrees $\leqslant m$ and n!

The integral of the numerator is $\displaystyle\int_C F(u)e^{n\Phi(u)}\,du$ estimated, through deformation of the contour, by steepest descent contributions of the neighbourhoods of saddle points (there will be at most two of them) u_s such that $\Phi'(u_s)=0$:

$$\Phi(u) = \Phi(u_s) + \frac{(u-u_s)^2}{2}\Phi''(u_s) + \cdots, \qquad \int_C F(u)e^{n\Phi(u)}\,du \sim F(u_s)e^{n\Phi(u_s)}\sqrt{\frac{2\pi}{\Phi''(u_s)}}.$$

Here, when $m \sim n$,
$F(u) = e^{Az_0}(1-u)^{-1}[u(1+E)-E]^{m-n}$, $\Phi(u) = \log\{u(1-u)^{-\zeta-1}[u(1+E)-E]\}$ (numerator P),
$F(u) = u^{m-n}(1-u)^{-1}$, $\Phi(u) = \log\{u(1-u)^{-\zeta-1}[u(1+E^{-1})-E^{-1}]\}$ (denominator Q).

3.4.2. *Check with* § *2.4*.

From (35),

$$\mathcal{V}_p(z) = \lim_{n\to\infty}\frac{\log Q(z)}{n} = (1+\zeta)\log(1+E^{-1}\kappa) + (1-\zeta)\log(1+\kappa) - \log(\kappa),$$

where ζ is basically our z (managed so that the interpolation points are in $[-1,1]$). Then, the derivative in ζ simplifies into

$$\frac{d\mathcal{V}_p(z)}{d\zeta} = \log\frac{1+E^{-1}\kappa}{1+\kappa},$$

where κ is related to ζ through $\zeta = \dfrac{\kappa^{-1}-E^{-1}\kappa}{E^{-1}-1}$, from (34). This matches (12) provided

$$\rho = iE^{-1/2}\kappa, \alpha = -\beta = \frac{2i}{E^{1/2}-E^{-1/2}},$$

and

$$\frac{d\mathcal{V}_p(z)}{d\zeta} = \log\frac{1-iE^{-1/2}\rho}{1-iE^{1/2}\rho} = -\sum_{n=1}^{\infty}\frac{i^n(E^{-n/2}-E^{n/2})}{n}\rho^n.$$

Remark that $g_1 = -i(E^{-1/2}-E^{1/2}) = 4/(\beta-\alpha)$ as it should.

4. Rational interpolation to $\exp(nB_1 z + nB_2 z^2)$.

This very interesting rational interpolation appears in special nonlinear Schrödinger problems ([54] and remarks by J. Nuttall) . The Padé approximation (§ 3.3) already told that there may be several arcs in the discussion.

4.1. **The single arc case.**

Let the interpolation points be equidistant on $[I_1, I_2]$. Then,

$$g(z) = \int_{I_1}^{I_2}\frac{(I_2-I_1)^{-1}\,dt}{z-t} - \frac{B_1}{2} - B_2 z = \frac{\log\dfrac{z-I_1}{z-I_2}}{I_2-I_1} - \frac{B_1}{2} - B_2\left(\frac{\beta-\alpha}{2}\frac{2z-\alpha-\beta}{\beta-\alpha} + \frac{\alpha+\beta}{2}\right) \tag{36}$$

The logarithms have the expansions

$$\log(z-I_k) = \log\frac{\alpha-\beta}{4\rho_k} - 2\sum_1^{\infty}\frac{\rho_k^n}{n}T_n,$$

where ρ_k is now a root of

$$\frac{\rho_k+\rho_k^{-1}}{2} = \frac{2I_k-\alpha-\beta}{\beta-\alpha}\;, k=1,2, \tag{37}$$

where $|\rho_k| < 1$ should be the orthodox choice, but which will not be kept in the final formula. Precisely, the closed form is now

$$\mathcal{V}_p'(z) = \sum_1^{\infty} g_n\rho^n = \frac{2}{I_2-I_1}\log\frac{1-\rho_1\rho}{1-\rho_2\rho} - B_2\frac{\beta-\alpha}{2}\rho, \tag{38}$$

with the conditions (11) on g_0 and g_1

$$\frac{\log(\rho_2/\rho_1)}{I_2-I_1} - \frac{B_1}{2} - B_2\frac{\alpha+\beta}{2} = 0, \tag{39}$$

$$g_1 = 2\frac{\rho_2-\rho_1}{I_2-I_1} - B_2\frac{\beta-\alpha}{2} = \frac{4}{\beta-\alpha}, \tag{40}$$

If ρ_1 and ρ_2 are known, α and β are got by (37):

$$\alpha = \frac{\rho_1(1+\rho_2)^2 I_1 - \rho_2(1+\rho_1)^2 I_2}{(\rho_2-\rho_1)(\rho_1\rho_2-1)}\;, \beta = \frac{\rho_1(1-\rho_2)^2 I_1 - \rho_2(1-\rho_1)^2 I_2}{(\rho_2-\rho_1)(\rho_1\rho_2-1)}\;. \tag{41}$$

(As for ρ_1 and ρ_2, they are simply found to be, if $B_2 = 0$, $\rho_1 = i\exp(-B_1(I_2-I_1)/4)$ and $\rho_2 = i\exp(B_1(I_2-I_1)/4)$. Remark that $\rho_1\rho_2 = -1$: no chance to have the comfortable $|\rho_k| < 1\ldots$)

Also,

$$\frac{\beta-\alpha}{2} = \frac{2\rho_1\rho_2(I_2-I_1)}{(\rho_2-\rho_1)(\rho_1\rho_2-1)}, \frac{\alpha+\beta}{2} = \frac{I_1+I_2}{2} - \frac{(\rho_1+\rho_2)(\rho_1\rho_2+1)(I_2-I_1)}{2(\rho_2-\rho_1)(\rho_1\rho_2-1)}.$$

and (39) and (40) become

$$\frac{\log(\rho_2/\rho_1)}{I_2-I_1} - \frac{B_1}{2} - B_2\frac{I_1+I_2}{2} + \frac{B_2}{2}\frac{(\rho_2+\rho_1)(\rho_1\rho_2+1)}{(\rho_2-\rho_1)(\rho_1\rho_2-1)}(I_2-I_1) = 0, B_2(I_2-I_1)^2 = \frac{(\rho_2-\rho_1)^2(\rho_1^2\rho_2^2-1)}{2\rho_1^2\rho_2^2},$$

or this form emphasizing $\rho_1\rho_2$ and ρ_2/ρ_1:

$$2B_2(I_2-I_1)^2 = \left(\frac{\rho_2}{\rho_1}-2+\frac{\rho_1}{\rho_2}\right)\left(\rho_1\rho_2-\frac{1}{\rho_1\rho_2}\right), \tag{42}$$

$$\log\left(\frac{\rho_2}{\rho_1}\right)+\frac{1}{4}\left(\frac{\rho_2}{\rho_1}-\frac{\rho_1}{\rho_2}\right)\left(\rho_1\rho_2+2+\frac{1}{\rho_1\rho_2}\right) = \frac{B_1}{2}(I_2-I_1)+B_2\frac{I_2^2-I_1^2}{2}. \tag{43}$$

For a given B_2 and various ratios ρ_2/ρ_1, we find valid values for B_1, etc. For instance, $I_1=-Ai$, $I_2=Ai$, B_2 negative imaginary and ρ_2/ρ_1 negative real, which is of interest in [54]:

```
Script V1.1 session started Mon Jan 17 14:32:31 2000

C:\calc\pari>gp
                   GP/PARI CALCULATOR Version 2.0.12 (alpha)
                           Copyright (C) 1989-1998 by
          C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

? \r expr1r2
```

$A=1, B_1=\pi-2ix, x=0.5, B_2=-2iAt$,

At	$-\rho_2/\rho_1$	Q	ρ_1	ρ_2	$\alpha=\overline{\beta}$
0.10	2.758	0.02455	$0.6002+0.04717i$	$-1.655-0.1301i$	$0.03677+0.8865i$
0.10	0.439	3.971	$0.1284+1.502i$	$-0.05646-0.6606i$	$-4.554+10.81i$
0.20	2.888	0.09570	$0.5812+0.09100i$	$-1.679-0.2629i$	$0.07604+0.8847i$
0.20	0.429	3.883	$0.2604+1.504i$	$-0.1117-0.6454i$	$-2.307+5.374i$
0.30	3.138	0.2038	$0.5499+0.1274i$	$-1.725-0.3998i$	$0.1197+0.8788i$
0.30	0.411	3.737	$0.3997+1.507i$	$-0.1643-0.6197i$	$-1.574+3.545i$
0.40	3.553	0.3275	$0.5082+0.1518i$	$-1.806-0.5394i$	$0.1674+0.8641i$
0.40	0.384	3.533	$0.5507+1.514i$	$-0.2119-0.5830i$	$-1.221+2.622i$
0.50	4.176	0.4363	$0.4618+0.1616i$	$-1.928-0.6749i$	$0.2147+0.8365i$
0.50	0.351	3.277	$0.7171+1.527i$	$-0.2517-0.5363i$	$-1.023+2.064i$
0.60	5.006	0.5082	$0.4175+0.1593i$	$-2.090-0.7976i$	$0.2544+0.7974i$
0.60	0.311	2.988	$0.9009+1.548i$	$-0.2805-0.4824i$	$-0.9027+1.692i$
0.70	6.007	0.5430	$0.3792+0.1503i$	$-2.278-0.9031i$	$0.2832+0.7524i$
0.70	0.270	2.692	$1.099+1.578i$	$-0.2972-0.4266i$	$-0.8243+1.431i$
0.80	7.139	0.5518	$0.3474+0.1390i$	$-2.480-0.9924i$	$0.3017+0.7071i$
0.80	0.232	2.414	$1.307+1.612i$	$-0.3033-0.3742i$	$-0.7690+1.241i$
0.90	8.372	0.5452	$0.3211+0.1275i$	$-2.689-1.068i$	$0.3124+0.6643i$
0.90	0.198	2.166	$1.517+1.650i$	$-0.3019-0.3282i$	$-0.7263+1.099i$
1.0	9.692	0.5303	$0.2991+0.1169i$	$-2.899-1.133i$	$0.3178+0.6252i$
1.0	0.171	1.953	$1.727+1.687i$	$-0.2962-0.2893i$	$-0.6908+0.9883i$
1.1	11.08	0.5113	$0.2804+0.1073i$	$-3.109-1.190i$	$0.3195+0.5899i$
1.1	0.149	1.770	$1.933+1.723i$	$-0.2882-0.2568i$	$-0.6599+0.9000i$
1.2	12.55	0.4906	$0.2643+0.09885i$	$-3.318-1.240i$	$0.3187+0.5582i$
1.2	0.130	1.613	$2.136+1.757i$	$-0.2791-0.2296i$	$-0.6323+0.8278i$

```
? quit
Good bye!

C:\calc\pari>exit

Script completed Mon Jan 17 14:34:12 2000
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We integrate (38) along the lines suggested by the exercises of section 3.4.2, p. 26:

$$\mathcal{V}_p(z) = \frac{2}{I_2-I_1}\left[(z-I_1)\log(1-\rho_1\rho)-(z-I_2)\log(1-\rho_2\rho)+X(\rho)\right],$$

which yields indeed, using from (12) and (37) $z-I_k = \dfrac{\beta-\alpha}{4}\left(1-\dfrac{\rho_k}{\rho}\right)\left(\rho-\dfrac{1}{\rho_k}\right)$,

$$\frac{d\mathcal{V}_p(z)}{dz} = \frac{2}{I_2-I_1}\left\{\log\frac{1-\rho_1\rho}{1-\rho_2\rho} + \left[\frac{z-I_1}{\rho-\rho_1^{-1}} - \frac{z-I_2}{\rho-\rho_1^{-2}} + \frac{dX(\rho)}{d\rho}\right]\frac{d\rho}{dz}\right\}$$

$$= \frac{2}{I_2-I_1}\left\{\log\frac{1-\rho_1\rho}{1-\rho_2\rho} + \left[\frac{\beta-\alpha}{4}\frac{\rho_2-\rho_1}{\rho} + \frac{dX(\rho)}{d\rho}\right]\frac{4}{(\beta-\alpha)(1-\rho^{-2})}\right\}$$

One must have $\dfrac{dX}{d\rho} = -\dfrac{\beta-\alpha}{4}\dfrac{\rho_2-\rho_1}{\rho} - B_2\dfrac{(\beta-\alpha)^2}{16}(I_2-I_1)\left(\rho-\dfrac{1}{\rho}\right)$, finally:

$$\frac{dX}{d\rho} = -\frac{I_2-I_1}{2}\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\rho + \frac{1}{\rho},$$

$$\mathcal{V}_p(z) = \frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho_1\rho) - (z-I_2)\log(1-\rho_2\rho)] - \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2}{2} - \log\rho. \tag{44}$$

The two determinations of $\mathcal{V}_p$ on the two sides of the cut are found with the two roots ρ and $1/\rho$ of (12). In particular, the *arithmetic mean* of the two values of the derivative must give (8) again, with g given by (36). Indeed, one finds

$$\frac{1}{I_2-I_1}\{\log[(1-\rho_1\rho)(1-\rho_1/\rho)] - \log[(1-\rho_2\rho)(1-\rho_2/\rho)]\} - B_2\frac{\beta-\alpha}{2}\frac{\rho+\rho^{-1}}{2},$$

which is

$$\frac{1}{I_2-I_1}\left[\log\frac{z-I_1}{z-I_2} + \log\frac{\rho_1}{\rho_2}\right] - B_2 z + B_2\frac{\alpha+\beta}{2},$$

The *difference* of the two determinations of $\mathcal{V}_p''$ must be $\pm 2\pi i\mu'$:

$$\pm 2\pi i\mu'(z) = \frac{2}{I_2-I_1}\left[\log\frac{1-\rho_1\rho}{1-\rho_1/\rho} - \log\frac{1-\rho_2\rho}{1-\rho_2/\rho}\right] - B_2\frac{\beta-\alpha}{2}(\rho-\rho^{-1}), \tag{45}$$

(Nuttall's $\Delta\Psi_2$)

and the cut itself is the locus $\{z : \mu'(z)\,dz$ real $\}$, which is integrated as $\{z : \mathcal{V}_{p,+}(z) - \mathcal{V}_{p,-}(z)$ pure imaginary $\}$,

$$\frac{2}{I_2-I_1}\left[(z-I_1)\log\frac{1-\rho_1\rho}{1-\rho_1/\rho} - (z-I_2)\log\frac{1-\rho_2\rho}{1-\rho_2/\rho}\right] - \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2-\rho^{-2}}{2} - 2\log\rho \text{ pure imaginary.} \tag{46}$$

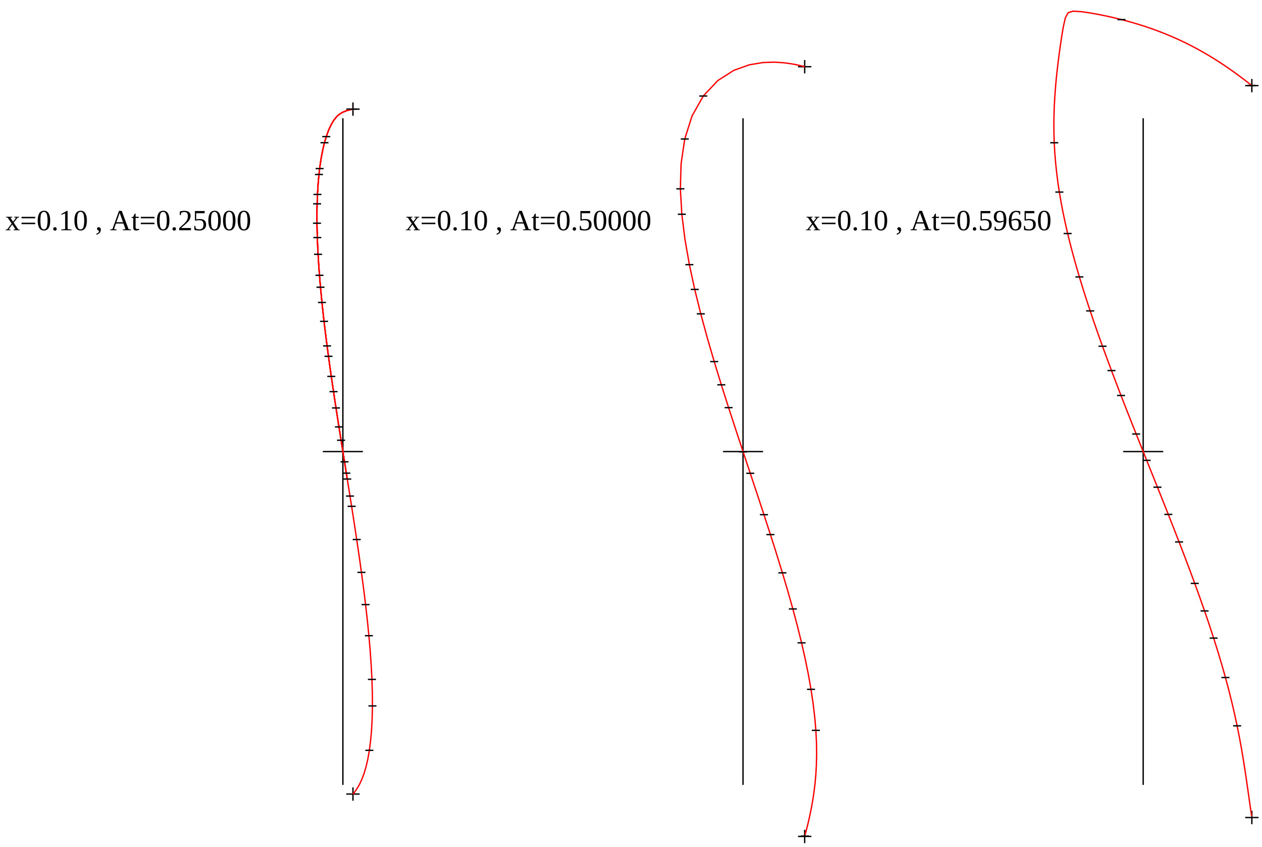
Writing (46) as a function of ρ (using (12) and (37)), we have

$$F(\rho) = \frac{2}{(\rho_2-\rho_1)(1-1/(\rho_1\rho_2))}\left[(\rho-\rho_1)\left(1-\frac{1}{\rho\rho_1}\right)L_1 - (\rho-\rho_2)\left(1-\frac{1}{\rho\rho_2}\right)L_2\right] - \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2-\rho^{-2}}{2},$$

with $L_1 = \log\dfrac{1-\rho_1\rho}{\rho-\rho_1}$, $L_2 = \log\dfrac{1-\rho_2\rho}{\rho-\rho_2}$, and where, for given B_1, B_2, I_1, I_2, one must determine ρ_1 and ρ_2 from (42) and (43).

4.2. **First caustic.**

The present setting of the limit set of poles as a single arc joining $z=\alpha$ to $z=\beta$ (or $\rho=-1$ to $\rho=1$) holds as long as $\mu_p'(z)dz$ remains positive on the cut. A critical situation occurs when μ_p' happens to vanish right on the cut, i.e., if dF/dz vanishes at a point where the real part of F vanishes too.



The locus of (x,At) with $B_1=\pi-2ix$, $B_2=-2iAt$, where this happens is called the (first) caustic in [54]. We then have $\rho_1=R^{-1/2}e^{i\theta}$, $\rho_2=-R^{1/2}e^{i\theta}$, with real R and θ. For a trial value of At, we look for R and θ such that $(R+1/R)/2=2At/\sin 2\theta\ -1$ (from (42)) and $2x=\log R+(1/R-R)\sin^2\theta$ (from (43)). Knowing ρ_1 and ρ_2, one looks for the zero of the analytic function dF/dz, or $dF/d\rho$. This yields the equation $\mu_p'=0$ in (45) as

$$L_1-L_2=\left(1+\frac{1}{\rho_1\rho_2}\right)(\rho_2-\rho_1)\frac{\rho-\rho^{-1}}{2}. \tag{47}$$

One then manages to have the real part of $F=0$ as well.

Some values:

x	At	$\sin\theta$	ρ_2	$R=-\rho_2/\rho_1$	ρ	F
0.001	0.500973	0.6990	-0.749 -0.733 i	1.098	-1.27846 -0.24352 i	6.282 i
0.010	0.509711	0.6781	-0.854 -0.788 i	1.350	-1.52506 -0.51989 i	6.261 i
0.050	0.548383	0.6288	-1.095 -0.885 i	1.982	-1.84975 -0.94099 i	6.123 i
0.100	0.596697	0.5838	-1.324 -0.952 i	2.660	-2.06555 -1.23283 i	5.922 i
0.250	0.744301	0.4821	-1.928 -1.061 i	4.842	-2.48476 -1.76294 i	5.301 i
0.500	1.009193	0.3636	-2.919 -1.139 i	9.817	-2.98988 -2.26747 i	4.371 i
0.750	1.312145	0.2794	-4.017 -1.169 i	17.504	-3.41534 -2.56407 i	3.587 i
1.000	1.672677	0.2167	-5.311 -1.179 i	29.600	-3.78112 -2.72755 i	2.923 i
1.250	2.117750	0.1686	-6.897 -1.180 i	48.954	-4.06939 -2.79596 i	2.359 i
1.500	2.682341	0.1314	-8.888 -1.178 i	80.381	-4.27061 -2.81082 i	1.884 i
1.750	3.408138	0.1024	-11.423 -1.176 i	131.867	-4.39913 -2.80582 i	1.493 i
2.000	4.344519	0.0798	-14.669 -1.174 i	216.553	-4.47798 -2.79719 i	1.175 i
2.500	7.107960	0.0484	-24.179 -1.172 i	585.997	-4.55461 -2.78484 i	0.721 i
3.000	11.684073	0.0294	-39.858 -1.171 i	1590.003	-4.58262 -2.77939 i	0.439 i

We see that $\theta\to\pi/4$ when $x\to 0$, and that $\theta\to 0$ when $x\to\infty$, but many features are still unexplained...

Here is a tentative explanation of the behaviour for large x: as it seems that $|\rho_1| << |\rho| << |\rho_2|$, the logarithms are approximated by $L_1\approx\rho_1(\rho^{-1}-\rho)-\log\rho$, $L_2\approx(\rho-\rho^{-1})/\rho_2+\log\rho$, the equation (47) becomes $\dfrac{\log\rho}{\rho-\rho^{-1}}\approx i\xi$, with $\xi=\theta\sqrt{R}/2$. Also, $F/\theta\approx(\rho-\rho^{-1})[i(\rho+\rho^{-1})/2-\xi^{-1}-2\xi]$ must be pure imaginary, making a second equation for ξ and ρ, whence fixed solutions. And

$$\frac{At}{e^x}\approx\frac{\theta R/4}{\sqrt{R}\,\exp(-\theta^2R/2)}=\frac{\xi}{2}\exp(2\xi^2).$$

```
Script V1.1 session started Thu Feb 17 11:24:23 2000

C:\calc\pari>gp

                   GP/PARI CALCULATOR Version 2.0.12 (alpha)
                           Copyright (C) 1989-1998 by
          C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

? \r expcaus2
...
    rho=-4.59885439353246016460 - 2.77599828040642492631*I

 F/theta= -1.14496588753536343500 E-40 + 14.9829543045360004158*I
? xi
  0.585318492448534646977
? (xi/2)*exp(2*xi*xi)
  0.580682668039111487078
? quit
Good bye!

C:\calc\pari>exit

Script completed Thu Feb 17 11:29:16 2000
```

Numerator and interpolation.

Remind that $\mathcal{V}_p(z)$ is the limit when $n\to\infty$ of $n^{-1}\log Q_n(z) = n^{-1}\sum \log(z-$ poles$)$. It must behave like $\log(z)+O(1/z)$ for large z, compatible with (44) if one adds a constant:

$$\mathcal{V}_p(z) = \frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho_1\rho)-(z-I_2)\log(1-\rho_2\rho)] - \frac{\rho^2}{2}\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1} - \log\rho + C_p,$$

which behaves for large z as $\log z - log\dfrac{\beta-\alpha}{4} + \dfrac{(\beta-\alpha)(\rho_2-\rho_1)}{2(I_2-I_1)} + C_p$, from $\rho\sim(\beta-\alpha)/(4z)$. Therefore,

$$C_p = log\frac{\beta-\alpha}{4} - \frac{(\beta-\alpha)(\rho_2-\rho_1)}{2(I_2-I_1)}.$$

The numerator of the interpolant to $\exp(n(B_1z+B_2z^2))$ is the denominator of the interpolant to $\exp(n(-B_1z-B_2z^2))$, so that the calculations made before apply with $(B_1,B_2)\longrightarrow(-B_1,-B_2)$. The equations (39) and (40) are now satisfied by $(\rho_1,\rho_2)\longrightarrow(1/\rho_1,1/\rho_2)$. And the values for α and β are the same as before. Let $\mathcal{V}_{\text{num}}(z)$ be the (presumed to exist) limit when $n\to\infty$ of $n^{-1}\log P_n(z)$, where P_n is the numerator. We expect a formula similar to (44), but with another constant:

$$\mathcal{V}_{\text{num}}(z) = \frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho/\rho_1)-(z-I_2)\log(1-\rho/\rho_2)] + \frac{\rho^2}{2}\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1} - \log\rho + C_{\text{num}}.$$

The remaining constant C_{num} is determined by $\mathcal{V}_{\text{num}}(z)-\mathcal{V}_p(z) = B_1z+B_2z^2$ in a neighbourhood of the set of the interpolation points. Everything works if one determination, say with ρ, is used for $\mathcal{V}_p$, while the determination with $1/\rho$ is used for $\mathcal{V}_{\text{num}}$:

$$\mathcal{V}_{\text{num}}(z)-\mathcal{V}_p(z) = \frac{2}{I_2-I_1}\left[(z-I_1)\log\frac{1-1/(\rho\rho_1)}{1-\rho\rho_1}-(z-I_2)\log\frac{1-1/(\rho/\rho_2)}{1-\rho\rho_2}\right] + \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^{-2}+\rho^2}{2} + 2\log\rho + C_{\text{num}} - C$$

$$= \frac{2}{I_2-I_1}[(z-I_2)\log(-\rho_2)-(z-I_1)\log(-\rho_1)] + \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\left[2\left(\frac{2z-\alpha-\beta}{\beta-\alpha}\right)^2-1\right] + C_{\text{num}} - C_p,$$

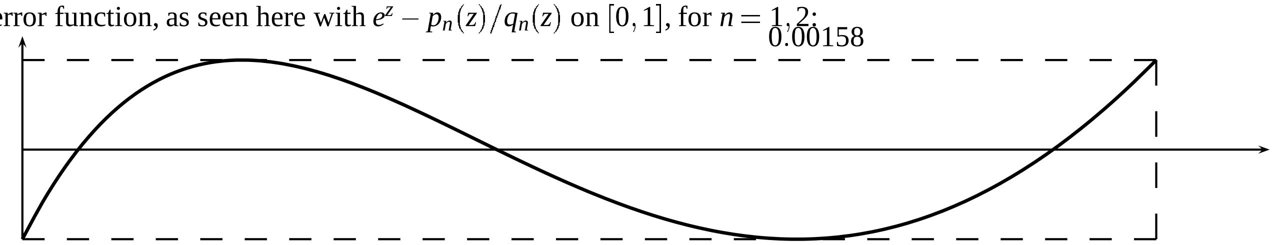
$$= [B_1+B_2(\alpha+\beta)]z + 2\frac{I_1\log(-\rho_1)-I_2\log(-\rho_2)}{I_2-I_1} + \frac{B_2}{4}(2z-\alpha-\beta)^2 - \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1} + C_{\text{num}} - C_p,$$

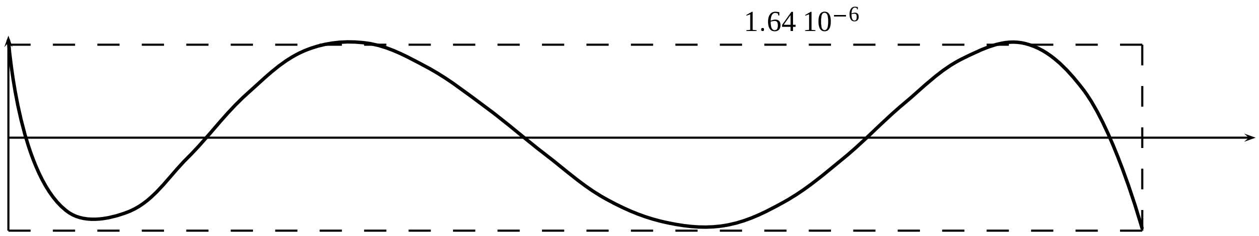
whence

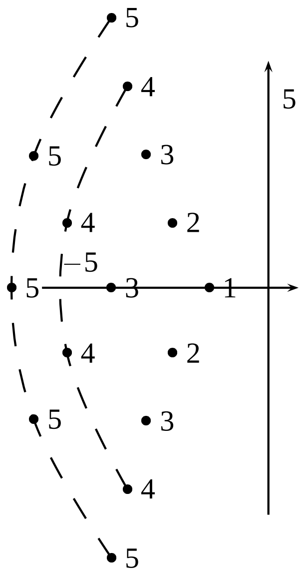
$$C_{\text{num}} - C_p = 2\frac{I_2\log(-\rho_2)-I_1\log(-\rho_1)}{I_2-I_1} - \frac{B_2}{4}(\alpha+\beta)^2 + \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}. \tag{48}$$

5. Best rational approximation to $e^{-(An+B)z}$ on a real interval

5.1. Best rational approximation to $\exp(-z)$ on a given real interval, say $[0,c]$ has a strict equioscillating error function, as seen here with $e^z - p_n(z)/q_n(z)$ on $[0,1]$, for $n=1,2$:

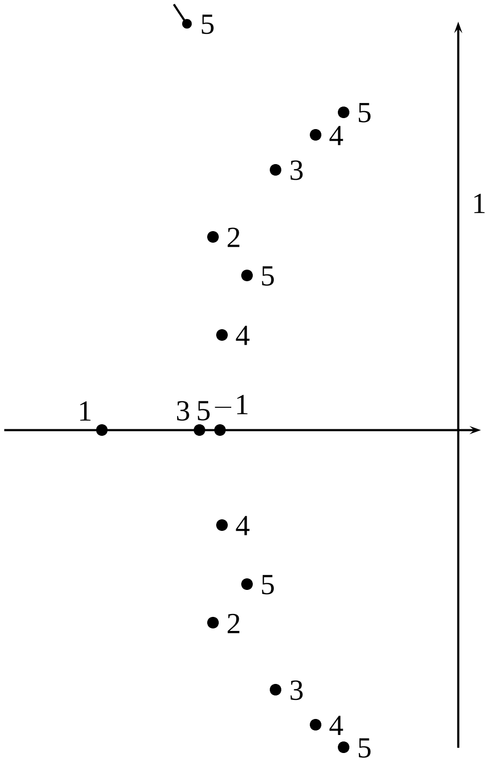






For varying degrees, we have a now familiar scaling effect best seen through the poles:

Sets of poles expand and tend to follow the Padé poles; errors decrease factorially fast with n(here, the error is about $\dfrac{e^{-1/2}n!(n-1)!}{4^{2n+1}(2n+1)!(2n-1)!}$. For an accurate asymptotic picture, see Braess' proof of Meinardus' conjecture [9]).



We find a stable picture if we look at the poles of the best approximants of degree n to $\exp(-nz)$. Moreover, the norms E_n of the errors tend to decrease in an exponential way with n:

$n=1$	2	3	4	5	6
1.58 E-3	3.197 E-5	5.921 E-7	1.068 E-8	1.91 E-10	3.383 E-12

The ratio of two successive errors seems to tend towards a limit of about $1/60$. The exact value, as it will be shown later (in (72), p. 45), is $\rho = 1/57.0699681\cdots$. Could we have $E_n \sim C\rho^n$, and what is the value of C? I can't wait: here are the products $E_n\rho^{-n}$:

$n=1$	2	3	4	5	6
0.090	0.104	0.110	0.113	0.116	0.117

Hmmm, what could it be? The numbers follow the approximate formula $0.125 - 0.05/(n+1/2)$. The limit 0.125 is reasonably close to an estimate which will be given in § 5.3.

Ah, an obscure insight (hindsight?) coming from long and painful experiments with the $'1/9'$ problem [45, 46] tells me to try $\exp(-(n+1/2)z)$ instead of $\exp(-nz)$, and to multiply the errors by $\rho^{-n+1/2}$:

$n=0$	1	2	3	4	5	6
0.197	4.161 E-3	7.610 E-5	1.356 E-6	2.406 E-8	4.244 E-10	7.470 E-12
1.488	1.794	1.872	1.904	1.928	1.941	1.950

Aha! Now, the limit seems to be 2. This phenomenon will also be explained in § 5.3.

Ah, the numerical experiments with best real rational approximation on a bounded interval have been performed with the truly remarkable MATLAB CF program of Trefethen [79]. Here is how it is adapted to the present problem of approximation to $\exp(-(An+B)z)$ on $[0,c]$. Of course, one must enter c, n, A, and B, also nfft = preferably a power of 2, say, 512, or 1024, and K = order of an auxiliary Hankel matrix, say, between 20 and 30.

```
%cfneuf3.m :
%
%  Quasi best rational approximation of degree n to  exp(- nf z)
%
%        on 0 <=  z  <=  c.
%
% enter:  c   n   nfft (a power of 2, say 1024)  K ( order of Hankel
%             matrix in CF construction, say 20 or more)
%
% Output:  plot of error function,    restart with higher nfft and/or K
%               if not nice;
%          two estimates of error norm. They should of course be
%              very close.
%          zeros &  poles.
%
%  cfneuf.m  adaptation de
% cftref.m
%  Approximation Theory V
% RCF -- REAL RATIONAL CF APPROXIMATION ON THE UNIT INTERVAL
%
% Loyd N. Trefethen, Dept. of Math., M.I.T., March 1986
% Reference: L.N.T. and M.H. Gutknecht,
%            SIAM J. Numer. Anal. 20 (1983), pp. 420-436
%   L.N.~Trefethen,  MATLAB programs for CF approximation,  pp.599-602
%     {\it in}  {\sl Approximation Theory V\ },
%     ( C.K.Chui, L.L.Schumaker, J.D.Ward, eds.),
%      Academic Press, Orlando 1986.
%
%
%     Fx(x) - function to be approximated by R(x)=P(x)/Q(x)
%      m,n  - degree of P,Q
%     nfft  - number of points in FFT (power of 2)
%        K  - degree at which Chebyshev series is truncated
%   F,P,Q,R - functions evaluated on FFT mesh (Chebyshev points)
%    Pc,Qc  - Chebyshev coefficients of P and Q
%
%  If Fx is even, take (m,n)=( odd, even).
%  If Fx is odd , take (m,n)=(even,even ).
%
  diary cfneuf3.txt
%
%  CONTROL PARAMETERS
   format long;
   c=input('c in interval 0..c ? ');
     m=input('degree n? ') ; n=m; np=n+1;
    aa=input(' A  in exp(-(An+B)z) ? ');bb=input(' B  in exp(-(An+B)z) ? ');
       nf=aa*n+bb;
     nfft=input('nfft? ');nfft2=nfft/2;
     K=input('K? '); dim=K+n-m;
%
%  CHEBYSHEV COEFFICIENTS OF Fx
     z = exp(2*pi*sqrt(-1)*(0:nfft-1)/nfft);
```

```
     x = real(z); F = exp(nf*c*(x-1)/2); Fc = real(fft(F))/nfft2;
%    disp(Fc(dim:dim+2));
%
%  SVD OF HANKEL MATRIX H
     H = toeplitz(Fc(1+rem((dim:-1:1)+nfft+m-n,nfft)));
        H = triu(H); H=H(:,(dim:-1:1));
     [u,s,v] = svd(H);
     s = s(np,np); u = u((dim:-1:1),np)';v = v(:,np)';
%
%  DENOMINATOR POLYNOMIAL Q
     zr = roots(v); qout = []; for i=1:dim-1;
         if abs(zr(i))>1 qout = [qout, zr(i)];end; end;
     qc = real(poly(qout)); qc = qc/qc(np); q = polyval(qc,z);
     Q = q.*conj(q); Qc = real(fft(Q))/nfft2;
     Qc(1)= Qc(1)/2; Q=Q/Qc(1); Qc = Qc(1:np)/Qc(1);
%
%  NUMERATOR POLYNOMIAL P
     b = fft([u zeros(1,nfft-dim)])./fft([v zeros(1,nfft-dim)]);
     Rt = F-real(s*z.^K.*b); Rtc = real(fft(Rt))/nfft2;
     gam = real(fft((1)./Q))/nfft2; gam = toeplitz(gam(1:2*m+1));
     if m==0 Pc = 2*Rtc(1)/gam;
       else Pc = 2*[Rtc(m+1:-1:2) Rtc(1:m+1)]/gam; end;
     Pc = Pc(m+1:2*m+1); Pc(1) = Pc(1)/2;
     P = real(polyval(Pc(m+1:-1:1),z)); R = P./Q;
%
% RESULTS
%    plot(x,F-R,'-',x,[s;0;-s]*ones(1,nfft),':'); pause;
%   s, err = norm(F-R,'inf'), Pc, Qc
       err = norm(F-R,'inf');
    serr=[s,err],
  disp(' zeros');
  qth=zeros(1,2*n+1);qth(1:n)=Pc(np:-1:2);
                  qth(n+2:2*n+1)=Pc(2:np);
                  qth(n+1)=2*Pc(1);
  polth=roots(qth);
  polx=[];
  ki=0;for kz=1:2*n;
        polxx=( polth(kz)+1/polth(kz) )/2;
         if kz==1, polx=[polx,polxx]; ki=1;kip=0;end;
         if kz>1 , kip=1;
             for kz2=1:ki;  if abs(polxx-polx(kz2))<0.0000001 , kip=0;end;
             end;
         end;
         if kip==1 , ki=ki+1; polx=[polx,polxx] ; end;
       end;
  polz= c*(1-polx)/2;
%  real(polz'), imag(polz')
  Pcc=real(poly(polz));
  polz'
  disp(' poles');
  qth=zeros(1,2*n+1);qth(1:n)=Qc(np:-1:2);
                  qth(n+2:2*n+1)=Qc(2:np);
                  qth(n+1)=2;
  polth=roots(qth);
```

```
  polx=zeros(1,n+1);
  ki=0;for kz=1:2*n;if abs(polth(kz))>1 , ki=ki+1;
                             polx(ki)=( polth(kz)+1/polth(kz) )/2;
                      end;
       end;
  polx=polx(1:ki);
  polz= c*(1-polx)/2;
%  real(polz'), imag(polz')
  polz'
  Qcc=real(poly(polz));
  Pcc=Pcc*R(1)*Qcc(np)/Pcc(np), Qcc
    tx=0:(c/100):c;er2=polyval(Pcc,tx)./polyval(Qcc,tx)-exp(-nf*tx);
    plot(tx,er2,'-',tx,[s;0;-s]*ones(1,101),':');  pause;
```

5.2. **Root asymptotics.**

We expect the poles to tend to be ultimately distributed on a fixed arc F with a limit distribution $d\mu_p$, and the interpolation points on $E=[0,c]$ with a limit distribution $d\mu_i$, so that the complex potential

$$\mathcal{V}(z) := \int_F \log\frac{1}{z-t}\,d\mu_p(t) - \int_E \log\frac{1}{z-t}\,d\mu_i(t) \tag{49}$$

satisfies

$$V := \mathrm{Re}\,\mathcal{V} = \text{ a constant } = \rho \text{ on } E, \tag{50}$$

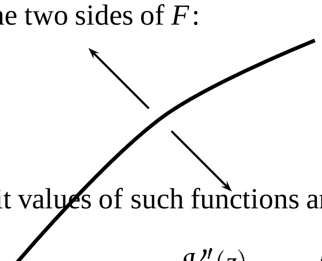
$$V(z)+\frac{A\mathrm{Re}\,z}{2} = \text{ a constant } = \sigma \text{ on } F, \tag{51}$$

$$V(z)+\frac{A\mathrm{Re}\,z}{2} \text{ has equal normal derivatives on the two sides of } F, \tag{52}$$

$$\int_E d\mu_i(t) = \int_F d\mu_p(t) = 1 \tag{53}$$

(charges on E and F), equivalent to $\mathcal{V}$ bounded at ∞, actually, $\mathcal{V}'(z)\sim$ constant z^{-2} for large z, and $\displaystyle\int_C \frac{\partial V(t)}{\partial n}|dt| = -2\pi$ on any contour containing F but not E, or also, that the imaginary part of $\mathcal{V}$ increases by π on $[0,c]$.

Conditions (51) and (52) amount to the realization that $V+A\mathrm{Re}\,z/2$ has opposite *gradients* along the normal on the two sides of F:



F

As the derivative of an analytic function has real and imaginary parts building the gradient of its real part (Cauchy-Riemann: grad $\mathrm{Re}\,F=\overline{F'}$), it follows that $\mathcal{V}'+A/2$ takes opposite values on the two sides of F.

Now, limit values of such functions are given by ***Sokhotskyi-Plemelj*** formulas [31], chap. 14, etc.

$$\mathcal{V}'(z) = -\int_{E\bigcup F}\frac{d\mu(t)}{z-t} = -⨍_{E\bigcup F}\frac{d\mu(t)}{z-t} \pm \pi i\mu'(z) \tag{54}$$

when z tends to a point of E or F, and where $⨍$ is the Cauchy principal value. We therefore have

$$⨍_{E\bigcup F}\frac{d\mu(t)}{z-t} = ⨍_F \frac{d\mu_p(t)}{z-t} - \int_E \frac{d\mu_i(t)}{z-t} = \frac{A}{2},\ z\in F, \tag{55}$$

which is an integral equation for the distribution μ_p, to be considered with (50) as another equation for μ_p and μ_i...

Now, there are various ways to go further, and to conclude with more or less neat expressions. There may be wrong turns, which may however yield a useful piece of information.

We get rid of the condition (50) by using complex Green functions [3] of E: first, let

$$\varphi(z) := \frac{2z}{c} - 1 + \sqrt{\left(\frac{2z}{c}-1\right)^2 - 1} \tag{56}$$

with the square root such that $|\varphi(z)| > 1$ for $z \notin E$: φ maps $\mathbb{C}\setminus E$ on the exterior of the unit disk, with $\varphi(\infty) = \infty$. Remark that $\varphi(z) + \dfrac{1}{\varphi(z)} = \dfrac{4z}{c} - 2$.

We now build $\varphi(z,t)$, with $\varphi(t,t) = \infty$:

$$\varphi(z,t) = \frac{\varphi(z)\overline{\varphi(t)} - 1}{\varphi(z) - \varphi(t)}, \qquad t \notin E, \tag{57}$$

and reconsider a formula for $\mathcal{V}$:

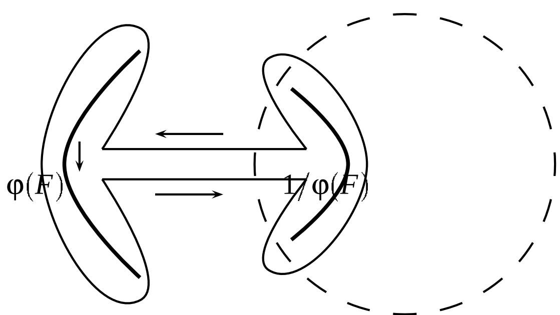
$$\mathcal{V}(z) := \int_F \log\varphi(z,t)\, d\mu(t), \tag{58}$$

which automatically satisfies (50), with $\rho = 0$, as $\operatorname{Re}\log\varphi(z,t) = \log|\varphi(z,t)| = 0$ when $z \in E$.

As $\dfrac{d}{dz}\log\varphi(z,t) = \dfrac{\varphi'(z)}{\varphi(z) - 1/\overline{\varphi(t)}} - \dfrac{\varphi'(z)}{\varphi(z)-\varphi(t)}$,

$$\frac{d\mathcal{V}(z)}{d\varphi(z)} = \int_F \frac{d\mu(t)}{\varphi(z) - 1/\overline{\varphi(t)}} - \int_F \frac{d\mu(t)}{\varphi(z)-\varphi(t)} \tag{59}$$

corresponding to charges and their images spread on $\varphi(F)$ and $1/\overline{\varphi(F)}$ in the $\varphi-$plane.



?? As an argument of validity of the form (58), let us show how to recover, at least partially, the derivative of (58):

$\mathcal{V}'(z) = -\displaystyle\int_{E\cup F} \frac{d\mu(t)}{z-t}$ is analytic in a neighbourhood of ∞, and can be written for large z as $\mathcal{V}'(z) = \alpha z^{-1} + \beta z^{-2} + \cdots$ (actually, $\alpha = 0$), with the contour integrals $\alpha = (2\pi i)^{-1}\displaystyle\int_C \mathcal{V}'(t)\,dt$, $\beta = (2\pi i)^{-1}\displaystyle\int_C t\mathcal{V}'(t)\,dt, \ldots$ so, $\mathcal{V}'(z) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{\mathcal{V}'(t)}{z-t}\,dt$ on a large contour C containing the singular loci E and F, for z outside C. We may as well consider the Laurent series in powers of $\varphi(z)$

$$\mathcal{V}'(z) = \mathcal{V}'\left(c\frac{\varphi + 1/\varphi + 2}{4}\right) = \frac{1}{2\pi i}\int_D \frac{\mathcal{V}'((c/4)(u+1/u+2))}{\varphi - u}\,du\,,$$

and we make the contour D shrink about the singular loci $\varphi(F)$ and $1/\varphi(F)$

The contributions about $\varphi(F)$ and $1/\varphi(F)$ sum as

$$\mathcal{V}'((c/4)(\varphi+1/\varphi+2)) = \frac{1}{2\pi i}\int_{\varphi(F)} \mathcal{V}'\left[\frac{1}{\varphi-u} - \frac{1/u^2}{\varphi - 1/u}\right] du$$

It seems simpler to return now to the original variable z, but we learned at least that $\mathcal{V}'$ has the same singularities than φ' at 0 and c, i.e.,

$$\mathcal{V}'(z) \sim \text{ constant } z^{-1/2}, z \to 0\,; \qquad \mathcal{V}'(z) \sim \text{ constant } (z-c)^{-1/2}, z \to c.$$

[3] used by Gonchar in several works...

We turn to a classical way to deal with the Sokhotskyi-Plemelj formulas (54)-(55) in the $z-$plane, by considering $\sqrt{z(z-c)(z-a)(z-b)}\ \mathcal{V}''(z)$ which is meromorphic outside F, even holomorphic, as the product remains bounded near 0 and c. The product has therefore a Laurent expansion about ∞, say

$$\sqrt{z(z-c)(z-a)(z-b)}\ \mathcal{V}''(z) = \delta_0 + \delta_1 z^{-1} + \cdots$$

with $\delta_k = \dfrac{1}{2\pi i}\displaystyle\int_C t^{k-1}\sqrt{t(t-c)(t-a)(t-b)}\ \mathcal{V}''(t)\, dt$, on a contour C around F, or

$$\sqrt{z(z-c)(z-a)(z-b)}\ \mathcal{V}''(z) = \delta_0 + \frac{1}{2\pi i}\int_C \frac{\sqrt{t(t-c)(t-a)(t-b)}}{z-t}\ \mathcal{V}''(t)\, dt$$

which, considering that $\mathcal{V}''(t) = -A/2 \pm \pi i \mu_p'(t)$ on the two sides of F, where the square root takes opposite values, yields

$$\sqrt{z(z-c)(z-a)(z-b)}\ \mathcal{V}''(z) = \delta_0 + \frac{A}{2\pi i}\int_a^b \frac{\sqrt{t(t-c)(t-a)(t-b)}}{z-t}\, dt.$$

Now, everything is known up to the three numbers a, b, and δ_0! The clumsy move leaving the δ_0 term comes from $\mathcal{V}''(z) = O(z^{-2})$ at ∞, so that the product of $\mathcal{V}''$ and the big square root has an unknown nonzero limit there. Multiplying by $\sqrt{z^{-1}(z-c)(z-a)(z-b)}$ instead of $\sqrt{z(z-c)(z-a)(z-b)}$ removes this problem, but introduces an unwanted residue at the origin! After several trials, I came on the– Bingo!

$$\sqrt{\frac{z(z-c)}{(z-a)(z-b)}}\ \mathcal{V}''(z) = \frac{A}{2\pi i}\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\, \frac{dt}{z-t}, \tag{60}$$

where one not only got rid of unwanted constants, but, as $\mathcal{V}''(z)$ is only $O(z^{-2})$ at ∞, leaves

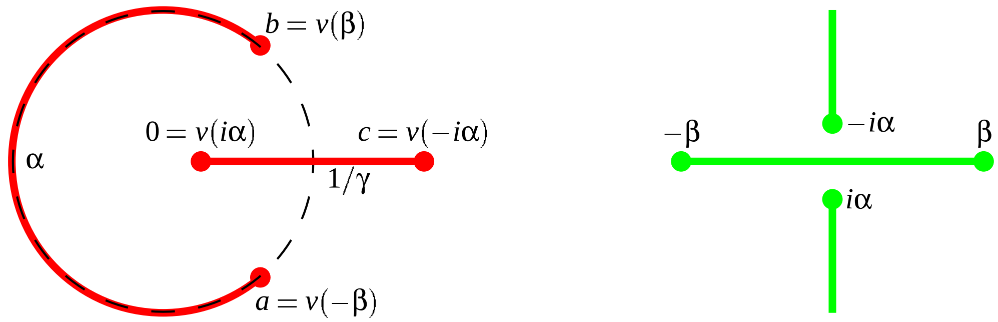
$$\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\, dt = 0 \tag{61}$$

as a bonus!! (61) gives one equation for a and b, knowing c (and another equation will be worked later on, from the unit charge condition (53)). For instance, when $c = 0$, we have indeed a vanishing integral of an odd function if $a = -b$, but, as we know (or suspect) that a and b are complex conjugates, we see that a and b must be opposite pure imaginary numbers, as they are indeed in the Padé case. To work (61) a bit further, we see that it is a complete elliptic integral of the third kind (complete because one integrates on a arc joining two branchpoints; of the third kind because the incomplete integral behaves like a logarithm somewhere [near ∞, the square root is $1 + (a+b-c)/(2t) + \cdots$]).

A convenient transformation sending the four branchpoints 0, c, a, and b on and from a symmetric set is

$$t = \frac{\alpha + iv}{1 + i\gamma v}.$$

So, $v = i\alpha$ is mapped on $t = 0$, one must have, for $v = -i\alpha$, $\dfrac{2\alpha}{1+\gamma\alpha} = c$, and $\dfrac{\alpha \pm i\beta}{1 \pm i\gamma\beta} = a, b$. As neither a nor b is known, we may as well take α and β, keeping in mind that $\gamma = \dfrac{2}{c} - \dfrac{1}{\alpha}$ (for given α, a and b are on a circle of diametral points α and $1/\gamma$).



The $z-$ plane and the $v-$plane.

Now, (61) becomes $\displaystyle\int_{-\beta}^{\beta}\frac{1}{(1+i\gamma v)^2}\sqrt{\frac{\alpha^2+v^2}{\beta^2-v^2}}\,dv=0$, or, with $v=\beta\cos\theta$,

$$\int_0^\pi \frac{\sqrt{1+(\beta^2/\alpha^2)\cos^2\theta}}{(1+i\gamma\beta\cos\theta)^2}\,d\theta=\int_0^\pi\frac{1-\gamma^2\beta^2\cos^2\theta}{(1+\gamma^2\beta^2\cos^2\theta)^2}\sqrt{1+(\beta^2/\alpha^2)\cos^2\theta}=0.$$

The best way to study this integral is to look how it may be computed efficiently. Consider first the Fourier expansion of $(1+i\gamma\beta\cos\theta)^{-2}$:

$$\frac{1}{(1+i\gamma\beta\cos\theta)^2}=\left[\frac{2}{(\xi-\xi^{-1})i\gamma\beta}\left(\frac{\xi e^{-i\theta}}{1-\xi e^{-i\theta}}+\frac{1}{1-\xi e^{i\theta}}\right)\right]^2=\frac{-4}{(\xi-\xi^{-1})^2\gamma^2\beta^2}\sum_{p=-\infty}^{\infty}\left(|p|+\frac{1+\xi^2}{1-\xi^2}\right)\xi^{|p|}e^{ip\theta},$$

where ξ is the root of $\xi^2+\dfrac{2}{i\gamma\beta}\xi+1=0$, with $|\xi|<1$. The integral is therefore $\displaystyle\frac{1}{1+\gamma^2\beta^2}\sum_{n=-\infty}^{\infty}\left(2|n|+\frac{1+\xi^2}{1-\xi^2}\right)\xi^{2|n|}\tau_n$,

where $\displaystyle\tau_n=\int_0^\pi e^{2in\theta}\sqrt{1+(\beta/\alpha)^2\cos^2\theta}\,d\theta$, computed from the recurrence relation

$$0=\int_0^\pi d\left[e^{2in\theta}\left(1+\frac{\beta^2}{\alpha^2}\cos^2\theta\right)^{3/2}\right]\Rightarrow(2n+3)\tau_{n+1}+4n(1+2\alpha^2/\beta^2)\tau_n+(2n-3)\tau_{n-1}=0.$$

The recurrence must be performed backwards (Miller's algorithm), as τ_n tends to zero exponentially fast when $n\to\infty$, see Gautschi [22] for a survey of these matters. The τ_n's are particular Legendre functions, by the way. And their generating function $F(Z)=\sum_0^\infty\tau_nZ^n$ satisfies

$$\left(Z+Z^{-1}+2+4\frac{\alpha^2}{\beta^2}\right)\frac{dF(Z)}{dZ}+\frac{Z^{-2}-1}{2}F(Z)=\frac{\tau_0+3\tau_1Z}{2Z^2}$$

i.e. $\displaystyle F(Z)=\left(Z+Z^{-1}+2+4\frac{\alpha^2}{\beta^2}\right)^{1/2}\int_0^Z\frac{\tau_0+3\tau_1Y}{2Y^2}(Y+Y^{-1}+2+4\alpha^2/\beta^2)^{-3/2}\,dY$, a way to see how complete elliptic integrals of the thitd kind turn into incomplete elliptic of first and second kind. Indeed, the integral is $G(\xi^2)=4\xi^2F'(\xi^2)+\dfrac{1+\xi^2}{1-\xi^2}[2F(\xi^2)-\tau_0]$, and it satisfies

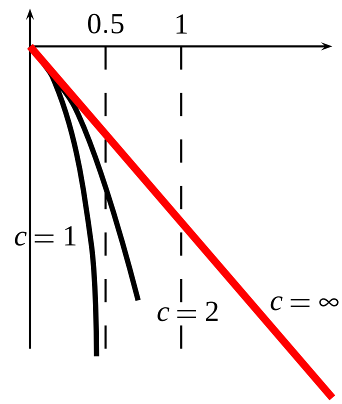
$$\frac{d}{dZ}\left[\frac{1-Z}{1+Z}(Z+Z^{-1}+2+4\alpha^2/\beta^2)G(Z)\right]+\frac{Z^{-2}-1}{2}\frac{1-Z}{1+Z}G(Z)=\frac{(1-Z)^2}{2Z(1+Z)^2}[6\tau_1-(Z+4+Z^{-1})\tau_0],$$

$$(Z+Z^{-1}+2+4\alpha^2/\beta^2)^{1/2}\frac{1-Z}{1+Z}G(Z)=\int_0^Z(t+t^{-1}+2+4\alpha^2/\beta^2)^{-1/2}\frac{(1-t)^2}{2t(1+t)^2}[6\tau_1-(t+4+t^{-1})\tau_0]\,dt$$

$$=-i\int_{+i\infty}^{(2i)^{-1}\log Z}\frac{\tan^2\theta}{\sqrt{\cos^2\theta+\alpha^2/\beta^2}}[3\tau_1-(1+2\cos^2\theta)\tau_0]\,d\theta\qquad(t=e^{2i\theta})$$

Some points of the locus:

$a/c = \overline{b}/c$	α/c	β/c	γc
$0.023671 - 0.0280i$	-0.0390	0.0809	27.614
$0.061784 - 0.0757i$	-0.1159	0.2206	10.628
$0.091190 - 0.1152i$	-0.1910	0.3385	7.2343
$0.114713 - 0.1488i$	-0.2647	0.4412	5.7773
$0.134055 - 0.1781i$	-0.3371	0.5327	4.9659
$0.150306 - 0.2040i$	-0.4084	0.6156	4.4482
$0.204616 - 0.3022i$	-0.7512	0.9493	3.3310
$0.241306 - 0.3830i$	-1.1399	1.2531	2.8772
$0.358549 - 0.8401i$	-6.3630	3.7085	2.1571
$0.406333 - 1.3084i$	-20.529	7.8098	2.0487
$0.425044 - 1.6475i$	-39.115	11.848	2.0255
$0.439587 - 2.0532i$	-73.476	17.878	2.0136
$0.445607 - 2.2839i$	-100.04	21.890	2.0099



For each c, the locus of a is a curve with vertical asymptote of abscissa $c/2$, and of tangent at the origin matching the $c = \infty$ locus, given by $\arg a = -0.860274\ldots$ (see [48, end of § 3.2]).

Constants already encountered appear as similar integrals, as

$$\gamma_0 = \lim_{z\to\infty} z^2\mathcal{V}''(z) = \frac{A}{2\pi i}\int_a^b t\sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\,dt$$

using $\dfrac{z^2}{z-t} = z + t + O(z^{-1}$ in (60). More integrals appear in a sequence of transformations needed in order to get a convenient incomplete elliptic integral form for $\mathcal{V}''$. We first multiply (60) by $z(z-c)(z-a)(z-b)$:
$\sqrt{z^3(z-c)^3(z-a)(z-b)}\;\mathcal{V}''(z) =$

$$\gamma_0 z^2 + [\gamma_1 - (a+b+c)\gamma_0]z + \gamma_2 - (a+b+c)\gamma_1 + (ab+ac+bc)\gamma_0 + \frac{A}{2\pi i}\int_a^b \sqrt{t^3(t-c)^3(t-a)(t-b)}\;\frac{dt}{z-t},$$

using $\dfrac{z(z-c)(z-a)(z-b)}{z-t} = (z-c)(z-a)(z-b) + tz^2 + (t-c-a-b)tz + t^3 - (a+b+c)t^2 + (ab+ac+bc)t + \dfrac{t(t-c)(t-a)(t-b)}{z-t}$, and where

$$\gamma_k = \frac{A}{2\pi i}\int_a^b t^{k+1}\sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\,dt \tag{62}$$

Now, we take the derivative in z, and perform an integration by parts in the integral:

$$\sqrt{P(z)}\;\mathcal{V}'''(z) + \frac{P'(z)}{2\sqrt{P(z)}}\mathcal{V} = 2\gamma_0 z + \gamma_1 - (a+b+c)\gamma_0 - \frac{A}{2\pi i}\int_a^b \frac{P'(t)}{2\sqrt{P(t)}}\,\frac{dt}{z-t},$$

where $P(z) = z^3(z-c)^3(z-a)(z-b)$. Now, $\dfrac{P'(t)}{2\sqrt{P(t)}}$ is $\sqrt{\dfrac{t(t-c)}{(t-a)(t-b)}}$ (well, who's there!) times $[4t^3 - (7a+7b+5c)t^2/2 + (3ab+2ac+2bc)t - 3abc/2]/(z-t) = -4z^2 + ((7a+7b+5c)/2 - 4t)z - 4t^2 + (7a+7b+5c)t/2 - 3ab - 2ac - 2bc + [4z^3 - (7a+7b+5c)z^2/2 + (3ab+2ac+2bc)z - 3abc/2]/(z-t)$, so that the integral is $-4\gamma_0 z - 4\gamma_1 + (7a+7b+5c)\gamma_0/2 +$ what turns out to be just $\dfrac{P'(z)}{2\sqrt{P(z)}}\mathcal{V}''(z)$, so,

$$\sqrt{P(z)}\;\mathcal{V}'''(z) = -2\gamma_0 z - \gamma_1',$$

where $\gamma_1' = 3\gamma_1 - (5a+5b+3c)\gamma_0/2$.

As remarked by J. Nuttall (22 Oct. 1999), we could have adapted Gonchar and Rakhmanov [25], and use the form

$$\mathcal{V}'''(z) = \frac{\text{constant} + \text{constant}\, z}{\sqrt{z^3(z-c)^3(z-a)(z-b)}}$$

but without knowledge of the two constants...

Check: from (60), $\left(1+\dfrac{a+b-c}{2z}+\cdots\right)\mathcal{V}'(z) = \dfrac{\gamma_0}{z^2}+\gamma_1 z^3+\cdots$, $\mathcal{V}''(z) = \dfrac{\gamma_0}{z^2}+2\gamma_1+(c-a-b)\gamma_0 2z^3+\cdots$,
$\sqrt{P(z)}\mathcal{V}'''(z) = \left(z^4-\dfrac{a+b+3c}{2}z^3+\cdots\right)\left(\dfrac{-2\gamma_0}{z^3}-3\dfrac{2\gamma_1+(c-a-b)\gamma_0}{2z^4}+\cdots\right) = -2\gamma_0 z-\gamma_1'$.

$$\begin{aligned}\mathcal{V}'(z) &= -\int_\infty^z \frac{(\gamma_1'+2\gamma_0 t)\,dt}{\sqrt{t^3(t-c)^3(t-a)(t-b)}}\,,\\ \mathcal{V}(z) &= \text{constant} - \int_\infty^z \frac{(z-t)(\gamma_1'+2\gamma_0 t)\,dt}{\sqrt{t^3(t-c)^3(t-a)(t-b)}}\end{aligned} \tag{63}$$

are elliptic integrals of first and second kind (there is no more logarithmic behaviour).

The standard forms of the elliptic integrals of 1st and 2nd kinds are [55]

$$F(x,k^2) = \int_0^{\arcsin x}(1-k^2\sin^2\theta)^{-1/2}d\theta = \int_0^x [(1-u^2)(1-k^2u^2)]^{-1/2}du,$$
$$E(x,k^2) = \int_0^{\arcsin x}(1-k^2\sin^2\theta)^{1/2}d\theta = \int_0^x (1-u^2)^{-1/2}(1-k^2u^2)^{1/2}du.$$

Elementary change of variable will not easily lead to these forms, but what is closest to our needs appears to be [55, 17.4.51]

$$\begin{aligned}F(y,k^2) &= (\alpha^2+\beta^2)^{1/2}\int_0^x \frac{dv}{[(v^2+\alpha^2)(\beta^2-v^2)]^{1/2}}\,,\\ E(y,k^2) &= \alpha^2(\alpha^2+\beta^2)^{1/2}\int_0^x \frac{1}{v^2+\alpha^2}\,\frac{dv}{[(v^2+\alpha^2)(\beta^2-v^2)]^{1/2}}\,,\end{aligned} \tag{64}$$

where $k^2 = \beta^2/(\alpha^2+\beta^2)$, $y^2 = \dfrac{x^2(\alpha^2+\beta^2)}{\beta^2(\alpha^2+x^2)}$. One may check that dF/dy and dE/dy are what they should be (using $1-y^2 = \dfrac{\alpha^2(\beta^2-x^2)}{\beta^2(\alpha^2+x^2)}$ and $1-k^2y^2 = \dfrac{\alpha^2}{\alpha^2+x^2}$).

Also, for complete elliptic integrals,

$$\begin{aligned}&0 = \int_{-\beta}^{\beta} d[v(\alpha^2+v^2)^{p+1/2}(\beta^2-v^2)^{1/2}] \Rightarrow\\ &\int_{-\beta}^{\beta}(\alpha^2+v^2)^p\frac{-(2p+3)(v^2+\alpha^2)^2+2(p+1)(\beta^2+2\alpha^2)(v^2+\alpha^2)-(2p+1)\alpha^2(\alpha^2+\beta^2)}{\sqrt{(v^2+\alpha^2)(\beta^2-v^2)}}dv = 0.\end{aligned} \tag{65}$$

In particular, with $p=-1$, we find two equivalent formulas for the complete elliptic integral of second kind:

$$\mathsf{E} = \alpha^2(\alpha^2+\beta^2)^{1/2}\int_0^\beta \frac{1}{v^2+\alpha^2}\,\frac{dv}{[(v^2+\alpha^2)(\beta^2-v^2)]^{1/2}} = (\alpha^2+\beta^2)^{-1/2}\int_0^\beta \left[\frac{v^2+\alpha^2}{\beta^2-v^2}\right]^{1/2}dv.$$

We need integrals

$$\int_\infty^z R(t)\frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}\,, \tag{66}$$

where R is a rational function with at most simple poles at 0 and/or c. One finds

$$\sqrt{(1+\gamma^2\beta^2)\frac{1+\gamma\alpha}{1-\gamma\alpha}}\int_{i/\gamma}^{v(z)} R\left(\frac{\alpha+iv}{1+i\gamma v}\right)\frac{dv}{\sqrt{(\alpha^2+v^2)(\beta^2-v^2)}},$$

where $x=v(z)=i\dfrac{\alpha-z}{1-\gamma z}$.

So, the constant and the simple fractions $1/t$ and $1/(t-c)$ lead to the (indefinite) integrals

$$R(t)=1:\sqrt{\frac{(1+\gamma^2\beta^2)(1+\gamma\alpha)}{(1-\gamma\alpha)(\alpha^2+\beta^2)}}F(y,k^2),$$

$$R(t)=\frac{1}{t}:\sqrt{\frac{(1+\gamma^2\beta^2)(1+\gamma\alpha)}{(1-\gamma\alpha)(\alpha^2+\beta^2)}}\left\{\gamma F(x,k^2)+i\frac{1-\alpha\gamma}{\sqrt{\alpha^2+\beta^2}}\sqrt{\frac{\beta^2-x^2}{x^2+\alpha^2}}+(1-\alpha\gamma)E(y,k^2)/\alpha\right\},$$

$$R(t)=\frac{1}{t-c}:\sqrt{\frac{(1+\gamma^2\beta^2)(1+\gamma\alpha)^3}{(1-\gamma\alpha)^3(\alpha^2+\beta^2)}}\left\{\gamma F(y,k^2)+i\frac{1+\alpha\gamma}{\sqrt{\alpha^2+\beta^2}}\sqrt{\frac{\beta^2-x^2}{x^2+\alpha^2}}-(1+\alpha\gamma)E(y,k^2)/\alpha\right\},$$

For $\mathcal{V}''(z), R(t)=-\dfrac{\gamma_1'+2\gamma_0 t}{t(t-c)}=\dfrac{\gamma_1'/c}{t}-\dfrac{\gamma_1''/c}{t-c}$,

$$\begin{aligned}\mathcal{V}''(z)=\text{constant }+&\sqrt{\frac{(1+\gamma^2\beta^2)(1+\gamma\alpha)}{(1-\gamma\alpha)^3(\alpha^2+\beta^2)}}\\ &\left\{\frac{\gamma}{c}\left[\gamma_1'(1-\gamma\alpha)-\gamma_1''(1+\gamma\alpha)\right]F(y,k^2)\right.\\ &+\frac{i}{c\sqrt{\alpha^2+\beta^2}}\left[\gamma_1'(1-\gamma\alpha)^2-\gamma_1''(1+\gamma\alpha)^2\right]\sqrt{\frac{\beta^2-x^2}{x^2+\alpha^2}}\\ &\left.+\left[\gamma_1'(1-\gamma\alpha)^2+\gamma_1''(1+\gamma\alpha)^2\right]\frac{E(y,k^2)}{\alpha c}\right\}.\end{aligned}\tag{67}$$

Remark that everything but the constant vanishes in (67) at $x=\pm\beta$, i.e., at $z=a$ and $z=b$, so that this constant is $\mathcal{V}''(a)=\mathcal{V}''(b)=-A/2$.

The function $\mathcal{V}''$ must be single-valued in $\mathbb{C}\setminus\{E\bigcup F\}$, i.e., have vanishing periods about the sets E and F.

$$\gamma\left[-\gamma_1'(1-\gamma\alpha)-\gamma_1''(1+\gamma\alpha)\right]\mathsf{K}+\left[\gamma_1'(1-\gamma\alpha)^2+\gamma_1''(1+\gamma\alpha)^2\right]\mathsf{E}/\alpha=0,\tag{68}$$

What a mess! Wait! The integrals γ_0 and γ_1 entering γ_1' and γ_1'' in (68) are complete elliptic of first and second kind too: (62) and (61) mean that $\gamma_{-1}=0$, which will be useful in the calculation of the subsequent γ_k's, from the recurrence relation

$$(k+2)\gamma_k-[k(a+b+c)+3(a+b)/2+c/2]\gamma_{k-1}+[k(ab+ac+bc)+ab]\gamma_{k-2}-(k-1/2)abc\,\gamma_{k-3}=0,\tag{69}$$

found by

$$0=\int_a^b d[t^{k-1/2}(t-c)^{3/2}(t-a)^{1/2}(t-b)^{1/2}\Rightarrow$$

$$\int_a^b t^{k-2}\sqrt{\frac{t(t-c)}{(t-a)(t-b)}}[(k-1/2)(t-a)(t-b)(t-c)+3t(t-a)(t-b)/2+t(t-c)(t-b)/2+t(t-c)(t-a)/2]dt=0.$$

When k is negative enough, γ_k in (62) is immediately an elliptic integral of second kind at most:

$$\gamma_k=\frac{A}{2\pi i}\int_a^b t^{k+1}\sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\,dt=\frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\int_{-\beta}^{\beta}\frac{(\alpha+iv)^{k+1}}{(1+i\gamma v)^{k+3}}\sqrt{\frac{v^2+\alpha^2}{\beta^2-v^2}}\,dv$$

So,

$$\gamma_{-4} = \frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\int_{-\beta}^{\beta}\frac{1+i\gamma v}{(\alpha+iv)^3}\sqrt{\frac{v^2+\alpha^2}{\beta^2-v^2}}\,dv$$

$$= \frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\int_{-\beta}^{\beta}\frac{(1+i\gamma v)(\alpha-iv)^3}{(v^2+\alpha^2)^2\sqrt{(v^2+\alpha^2)(\beta^2-v^2)}}\,dv$$

$$= \frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\int_{-\beta}^{\beta}\frac{-\gamma v^4+3\alpha(\alpha\gamma-1)v^2+\alpha^3}{(v^2+\alpha^2)^2\sqrt{(v^2+\alpha^2)(\beta^2-v^2)}}\,dv \qquad \text{(keep only even terms)}$$

$$= \frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\int_{-\beta}^{\beta}\frac{[(\alpha^2-3\beta^2)\gamma-4\alpha](v^2+\alpha^2)+\alpha[\alpha\gamma(7\beta^2-\alpha^2)+7\alpha^2-\beta^2]}{3(\alpha^2+\beta^2)(v^2+\alpha^2)\sqrt{(v^2+\alpha^2)(\beta^2-v^2)}}\,dv$$

(subtract a suitable multiple of (65) with $p=-2$)

$$= \frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\frac{[(\alpha^2-3\beta^2)\gamma-4\alpha]\mathsf{K}+[\gamma(7\beta^2-\alpha^2)+7\alpha-\beta^2/\alpha]\mathsf{E}}{3(\alpha^2+\beta^2)^{3/2}}$$

$$\gamma_{-3} = \frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\int_{-\beta}^{\beta}\frac{(\alpha-iv)^2}{\alpha^2+v^2}\frac{dv}{\sqrt{(v^2+\alpha^2)(\beta^2-v^2)}}$$

$$= \frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\frac{4\mathsf{E}-2\mathsf{K}}{\sqrt{\alpha^2+\beta^2}}$$

The relation (69) allows to find the next γ_k's… excepting γ_{-2}, which is precisely the first elliptic integral of third kind. However, knowing that $\gamma_{-1}=0$, (69) at $k=-1$ actually gives

$$\gamma_{-2} = c\frac{-2(a+b)\gamma_{-3}+3ab\gamma_{-4}}{a+b-c},$$

amounting to

$$\gamma_{-2} = \frac{A}{2\pi}\sqrt{\frac{(1+\gamma^2\beta^2)(\alpha\gamma-1)}{\alpha\gamma+1}}\frac{2\alpha\mathsf{K}-2(\alpha+1/\gamma)\mathsf{E}}{\sqrt{\alpha^2+\beta^2}}.$$

For γ_0, use (69) at $k=0$: $\gamma_0 = -ab(c\gamma_{-3}+2\gamma_{-2})/4$,

$$\gamma_0 = \frac{A}{2\pi}\sqrt{\frac{(\alpha\gamma-1)(\alpha^2+\beta^2)}{(\alpha\gamma+1)^3(1+\gamma^2\beta^2)}}\left[\alpha^2\gamma(\mathsf{K}-\mathsf{E})-\frac{\mathsf{E}}{\gamma}\right]$$

and finally, $\gamma_1 = \dfrac{(5(a+b)+3c)\gamma_0+abc\gamma_{-2}}{6}$, $\gamma_1' = 3\gamma_1 - \dfrac{(5(a+b)+3c)\gamma_0}{2} = abc\gamma_{-2}/2$, $\gamma_1'' = \gamma_1'+2c\gamma_0 = -abc(\gamma_{-2}+c\gamma_{-3})/2$:

$$\gamma_1' = \frac{A}{2\pi}\sqrt{\frac{(\alpha\gamma-1)(\alpha^2+\beta^2)}{(\alpha\gamma+1)^3(1+\gamma^2\beta^2)}}\,2\alpha^2\left[\mathsf{K}-\mathsf{E}-\frac{\mathsf{E}}{\alpha\gamma}\right],$$

$$\gamma_1'' = \frac{A}{2\pi}\sqrt{\frac{(\alpha\gamma-1)(\alpha^2+\beta^2)}{(\alpha\gamma+1)^3(1+\gamma^2\beta^2)}}\,\frac{1-\alpha\gamma}{1+\alpha\gamma}\,2\alpha^2\left[\mathsf{K}-\mathsf{E}+\frac{\mathsf{E}}{\alpha\gamma}\right],$$

and (67) becomes

$$\mathcal{V}'(z) = -\frac{A}{2}+\frac{A}{\pi}\left\{\mathsf{K}E(y,k^2)-\mathsf{E}F(y,k^2)-\frac{i\alpha^2}{c\sqrt{\alpha^2+\beta^2}}\left[\frac{2\alpha\gamma}{1+\alpha\gamma}\mathsf{K}+\frac{2}{\alpha\gamma(1-\alpha\gamma)}\mathsf{E}\right]\sqrt{\frac{\beta^2-x^2}{x^2+\alpha^2}}\right\}$$

which confirms (68).

For $\mathcal{V}$, use (63) with the discussion following (66): $R(t) = \dfrac{(t-z)(\gamma_1'+2\gamma_0 t)}{t(t-c)} = 2\gamma_0 + \dfrac{z\gamma_1'/c}{t} - \dfrac{(z-c)\gamma_1''/c}{t-c}$,

$$\begin{aligned}\mathcal{V}(z) = \text{constant } &+ \sqrt{\frac{(1+\gamma^2\beta^2)(1+\gamma\alpha)}{(1-\gamma\alpha)^3(\alpha^2+\beta^2)}} \\ &\Big\{2\gamma_0(1-\gamma\alpha)F(y,k^2) + \frac{\gamma}{c}\left[z\gamma_1'(1-\gamma\alpha)-(z-c)\gamma_1''(1+\gamma\alpha)\right]F(y,k^2) \\ &+ \frac{i}{c\sqrt{\alpha^2+\beta^2}}\left[z\gamma_1'(1-\gamma\alpha)^2-(z-c)\gamma_1''(1+\gamma\alpha)^2\right]\sqrt{\frac{\beta^2-x^2}{x^2+\alpha^2}} \\ &+ \left[z\gamma_1'(1-\gamma\alpha)^2+(z-c)\gamma_1''(1+\gamma\alpha)^2\right]\frac{E(y,k^2)}{\alpha c}\Big\}.\end{aligned} \tag{70}$$

Now we find, at last!, a second equation for a and b, from $\mathcal{V}(b)-\mathcal{V}(a) = i\pi$: at $z=a$ and $z=b$, $x=\pm\beta$, $F=\pm\mathsf{K}$, $E=\pm\mathsf{E}$,

$$\begin{aligned}\mathcal{V}(a) \text{ and } \mathcal{V}(b) = \text{const.} \pm &\frac{iA}{2\pi(1-\alpha^2\gamma^2)}2\alpha^2\left\{\gamma\left[\mathsf{K}-\mathsf{E}-\frac{\mathsf{E}}{\alpha^2\gamma^2}\right](1-\gamma\alpha)\mathsf{K}\right. \\ &+z(1-\gamma\alpha)2\alpha^2\left[\mathsf{K}-\mathsf{E}-\frac{\mathsf{E}}{\alpha\gamma}\right]\frac{\gamma}{c}\left[\mathsf{K}-\mathsf{E}+\frac{\mathsf{E}}{\alpha\gamma}\right] \\ &\left.-(z-c)(1+\gamma\alpha)\,2\alpha^2\frac{1-\alpha\gamma}{1+\alpha\gamma}\left[\mathsf{K}-\mathsf{E}+\frac{\mathsf{E}}{\alpha\gamma}\right]\frac{\gamma}{c}\left[\mathsf{K}-\mathsf{E}-\frac{\mathsf{E}}{\alpha\gamma}\right]\right\} \\ &= \text{const.} \pm \frac{iAc}{2\pi}\left\{\left[\alpha\gamma(\mathsf{K}-\mathsf{E})-\frac{\mathsf{E}}{\alpha\gamma}\right]\mathsf{K}-\left[\alpha\gamma(\mathsf{K}-\mathsf{E})^2-\frac{\mathsf{E}^2}{\alpha\gamma}\right]\right\},\end{aligned}$$

so that, $\mathcal{V}(b)-\mathcal{V}(a) = i\pi \Rightarrow$

$$Ac = \frac{\pi^2}{\left(\dfrac{1}{\alpha\gamma}-\alpha\gamma\right)\mathsf{E}(\mathsf{K}-\mathsf{E})} \tag{71}$$

which, together with (61), gives α, γ, K, etc., from Ac (no wonder that everything depends essentially on the product Ac: remember that we approximate $\exp(-nAz)$ on $0 \leqslant z \leqslant c$, equivalent up to a scaling to the approximation of $\exp(-nAct)$ on $[0,1]$.

Check: when $c\to\infty$, $\alpha\gamma = -1+2\alpha/c \to -1$, $1/(\alpha\gamma)-\alpha\gamma \sim -4\alpha/c$, and we should check that $-4\alpha\mathsf{E}(\mathsf{K}-\mathsf{E}) \to \pi^2$. Yes: $|a|^2 = ab = \dfrac{\alpha^2+\beta^2}{1+\gamma^2\beta^2} \to \alpha^2$, and we know that $\mathsf{E}\to\mathsf{K}/2$ when $c\to\infty$, so that the limit of (71) is $|a|\mathsf{K}^2 = \pi^2$, confirmed by $|\xi_1|\mathsf{K} = \pi$ in [46, § 2] (where $\xi_1 = \sqrt{-a}$), and by $|a| = \pi/\omega$ of [48, § 3.4, eq. (34)].

For $\mathcal{V}(0)$ and $\mathcal{V}(c)$, it is better to show $\mathcal{V}$ to be bounded at 0 and c by adapting (65) to incomplete elliptic integrals:

$$E(y,k^2) = \frac{x}{\sqrt{\alpha^2+\beta^2}}\sqrt{\frac{\beta^2-x^2}{\alpha^2+x^2}} + \frac{1}{\sqrt{\alpha^2+\beta^2}}\int_0^x\sqrt{\frac{\alpha^2+v^2}{\beta^2-v^2}}\,dv,$$

and the two last lines of (70) become

$$\frac{i(1-\gamma\alpha)}{c\alpha(1+i\gamma x)}[\gamma_1'(1-\gamma\alpha)+\gamma_1''(1+\gamma\alpha)]\frac{\sqrt{(\alpha^2+x^2)(\beta^2-x^2)}}{\sqrt{\alpha^2+\beta^2}} + \frac{z\gamma_1'(1-\gamma\alpha)^2+(z-c)\gamma_1''(1+\gamma\alpha)^2}{\alpha c\sqrt{\alpha^2+\beta^2}}\int_0^x\sqrt{\frac{\alpha^2+v^2}{\beta^2-v^2}}\,dv,$$

using $z = \dfrac{\alpha+ix}{1+i\gamma x}$ and $z-c = z-\dfrac{2\alpha}{1+\gamma\alpha} = \dfrac{(\gamma\alpha-1)(\alpha-ix)}{(1+\gamma\alpha)(1+i\gamma x)}$

$\mathcal{V}(c)-\mathcal{V}(0)=\pi i$: at $z=0$ and $z=c$, $x=i(\alpha-z)/(1-\gamma z)=\pm i\alpha$, where

$$F(y,k^2)=\sqrt{\alpha^2+\beta^2}\int_0^{\pm i\alpha}\frac{dv}{\sqrt{(v^2+\alpha^2)(\beta^2-v^2)}}=\pm i\sqrt{\alpha^2+\beta^2}\int_0^{\alpha}\frac{dw}{\sqrt{(\alpha^2-w^2)(\beta^2+w^2)}}=\pm i\mathsf{K}',$$

as α and β have been interchanged in K.

Also, $\dfrac{1}{\sqrt{\alpha^2+\beta^2}}\displaystyle\int_0^{\pm i\alpha}\sqrt{\frac{\alpha^2+v^2}{\beta^2-v^2}}\,dv=\pm i\frac{1}{\sqrt{\alpha^2+\beta^2}}\int_0^{\alpha}\frac{\alpha^2+\beta^2-(\beta^2+w^2)}{\sqrt{(\alpha^2-w^2)(\beta^2+w^2)}}\,dw=\pm i(\mathsf{K}'-\mathsf{E}')$.

What a mess again. Well, the important thing now is the rate of decrease of the error with n, which is $\rho=\exp\{-2[\ \mathrm{Re}\ (\mathcal{V}(z)+Az/2)\text{ on }F-\ \mathrm{Re}\ \mathcal{V}(z)\text{ on }E]\}$. From $\mathcal{V}'(a)+A/2=0$, one has $\mathcal{V}'(z)+A/2=-\displaystyle\int_a^z\frac{\gamma_1'+2\gamma_0 t=\gamma_1''+2\gamma_0(t-c)}{\sqrt{t^3(t-c)^3(t-a)(t-b)}}\,dt$, $\mathcal{V}(z)+Az/2=\text{const.}-\displaystyle\int_a^z(z-t)\frac{\gamma_1''+2\gamma_0(t-c)}{\sqrt{t^3(t-c)^3(t-a)(t-b)}}\,dt$, and what is needed is the complete elliptic integral

$$\log\rho=2\,\mathrm{Re}\,[\mathcal{V}(0)-\mathcal{V}(a)-Aa/2]=-2\ \mathrm{Re}\ \int_0^a\frac{\gamma_1''+2\gamma_0(t-c)}{\sqrt{t(t-c)^3(t-a)(t-b)}}\,dt,$$

which turns as

$$\log\frac{1}{\rho}=\pi\frac{\alpha\gamma(\mathsf{K}-\mathsf{E})(\mathsf{K}'-\mathsf{E}')-\mathsf{E}\mathsf{E}'}{(\alpha\gamma-1)\mathsf{E}(\mathsf{K}-\mathsf{E})}\tag{72}$$

The Legendre relation $\mathsf{E}\mathsf{E}'-(\mathsf{K}-\mathsf{E})(\mathsf{K}'-\mathsf{E}')=\pi/2$ may be useful here, but does not give a much nicer formula.

Check when $c\to\infty$: $\alpha\gamma\to-1$ and $\mathsf{K}-2\mathsf{E}\to 0$, and we recover the $\rho=\exp(-\pi\mathsf{K}'/\mathsf{K})$ of [45, 46, 48]!

In the limit case $c\to 0$, $\mathcal{V}\sim\mathcal{V}_{p,\text{Padé}}-\mathcal{V}_{[0,c]}$, where $\mathcal{V}_{p,\text{Padé}}$ is the potential of the poles of the Padé approximant, and where $\mathcal{V}_{[0,c]}$ is the equilibrium potential of $[0,c]$:

$$\mathcal{V}(z)\sim-\log\left(1+\sqrt{1+\frac{A^2z^2}{4}}\right)+\sqrt{1+\frac{A^2z^2}{4}}-\frac{Az}{2}+\log\left[\frac{2z}{c}-1+\sqrt{\left(\frac{2z}{c}-1\right)^2-1}\right]$$

where the square roots behave like $Az/2$ and $2z/c$ for large z. Then,

$$\rho\sim\exp(2\ \mathrm{Re}\ (\mathcal{V}(0)-\mathcal{V}(a)-Aa/2))\sim\exp\left(2-2\log 2-2\log\frac{8}{Ac}\right)=\left(\frac{eAc}{16}\right)^2.$$

Some instances:

Ac	$a/c = \overline{b}/c$	α/c	β/c	γc	K	E	K′	E′	$1/\rho$
∞	0	0	0	∞	2.32105 $= 2\mathrm{E}$	1.16052	1.64669	1.50011	9.2890255 $= 1/'1/9'$
10	$0.11626 - 0.15112i$	-0.27010	0.44836	5.70233	2.12833	1.22121	1.69467	1.46033	10.55420
5	$0.21759 - 0.32920i$	-0.86874	1.04722	3.15109	1.94139	1.30256	1.78286	1.39577	12.43300
2	$0.37281 - 0.94489i$	-8.58507	4.47929	2.11648	1.66674	1.48306	2.22657	1.18782	23.22870
1	$0.43707 - 1.96963i$	-65.1743	16.5290	2.01534	1.59537	1.54679	2.81732	1.07039	57.069968
0.5	$0.46869 - 3.98448i$	-514.5877	64.57005	2.00194	1.57694	1.56469	3.47937	1.02313	177.934379

The modulus of the elliptic integrals is $k = \beta/\sqrt{\alpha^2+\beta^2}$. The limit value when $Ac \to \infty$ is 0.90890856. Why don't we have a "decent" limit for k when $Ac \to \infty$, such as $k \to 1$? Well, if we extrapolate the locus of a, b, etc. "above" $Ac = \infty$, we simply get... negative values of Ac which repeat already known results for positive values, up to simple reflections: if c unchanged and $A \leftrightarrow -A$, approximating $\exp(-nAz) \leftrightarrow$ approximating $\exp(-nAc)$ times $\exp(nAz)$ with $z \leftrightarrow c - z$, so that $a, b \leftrightarrow c-a, c-b$,etc. (yes, but what about k?).

5.3. **Strong asymptotics .**

Consider rational approximants to functions $f^n g$, and suppose that the Hermite-Walsh error formula can already be written as

$$f^n(z)g(z) - \frac{p_n(z)}{q_n(z)} \sim e^{\mathcal{W}_n(z)} \frac{1}{2\pi i} \int_C f^n(t)g(t)e^{-\mathcal{W}_n(t)} \frac{dt}{z-t},$$

where $\mathcal{W}_n$ is a "smoothed" approximation of the discrete potential created by the poles and the interpolation points. The function $\exp(\mathcal{W}_n)$ (corresponding to Nuttall's χ_1 and/or χ_2 [50,61,62]) has branch points, even if f and g are entire. What is this function? The influence of f is overwhelming in the determination of the branchpoints and other main features when n is large. So, we solve first with f, and find the active part $F \subset C$ and the main behaviour $(\exp(\mathcal{W}_n))^{1/n} \to \exp(2\mathcal{V})$ (root asymptotics, also called zero order asymptotics by Nuttall).

Aptekarev [4] established in some cases a more accurate picture $\mathcal{W}_n = 2n\mathcal{V} + \widetilde{\mathcal{V}} + o(1)$ (strong asymptotics, also called first order asymptotics by Nuttall). I give here a probably very sloppy account of Aptekarev's wonderful results (to be available soon):

Also sprache Aptekarev: $\widetilde{\mathcal{V}}$ is (multivalued) analytic outside $E\bigcup F$, with a period $2\pi i$ about F, and $-2\pi i$ about E, corresponding to a positive unit charge on F, and a negative unit charge on E, with $\widetilde{\mathcal{V}}_+ + \widetilde{\mathcal{V}}_-$ constant on E, $\widetilde{\mathcal{V}}(z)_+ + \widetilde{\mathcal{V}}(z)_- + 2\log g(z) =$ another constant on F, and finally $\widetilde{\mathcal{V}}(z) =$ const. $+o(1)$ when $z\to\infty$ (if E and F are bounded).

Moreover, the error norm is $E_n \sim 2\rho^n\tilde{\rho}$, where $2\log\tilde{\rho} = \mathrm{Re}\{(\widetilde{\mathcal{V}}_+(z)+\widetilde{\mathcal{V}}_-(z))_E - [\widetilde{\mathcal{V}}_+(z)+\widetilde{\mathcal{V}}_-(z)+2\log g(z)]_F\}$.

This means also that $\widetilde{\mathcal{V}}'$ is analytic outside E and F, taking opposite values on the two sides of E, and with $\widetilde{\mathcal{V}}' + g'(z)/g(z)$ taking opposite values on the two sides of F.

Important special case: if $g=\sqrt{f}$, the conditions on $\widetilde{\mathcal{V}}$ are exactly the conditions (50)-(53) which we already saw for $\mathcal{V}$ itself! So, $\widetilde{\mathcal{V}} = \mathcal{V}$, and $\tilde{\rho} = \sqrt{\rho}$ in this case.

Remark: the real part of $\widetilde{\mathcal{V}} + \log g$ need not, and will normally not be a constant on F. However, the cut on which the boundary conditions for $\widetilde{\mathcal{V}}$ are set may be modified (keeping the endpoints as the endpoints of F), and one may dream to find the locus $\widetilde{F}$ where $\widetilde{\mathcal{V}} + \log g$ has a constant real part. The use and even the existence of $\widetilde{F}$ seem questionable (Aptekarev). It may be wiser and more useful to look for a locus F_n where the whole complex potential $\mathcal{V}_n = 2n\mathcal{V} + \widetilde{\mathcal{V}} + n\log f + \log g$ has a constant real part, as this locus may be a fair approximation to the set of poles for a given value of n (Nuttall).

Application to best approximation to $\exp(-(nA+B)z)$ on $[0,c]$:

$E_n \sim 2\rho^n\rho_B$, where $2\log\rho_B = \mathrm{Re}\{(\mathcal{V}_{B,+}+(z)+\mathcal{V}_{B,-}(z))_E - [\mathcal{V}_{B,+}(z)+\mathcal{V}_{B,-}(z)+2Bz]_F\}$, $\mathcal{V}''_B = \widetilde{\mathcal{V}}'$ being analytic outside $E\bigcup F$, taking opposite values on the two sides of $E=[0,c]$, $\mathcal{V}''_B(z)+B$ taking opposite values on the two sides of F, or any arc of endpoints a and b, and corresponding to a positive unit charge on F, and a negative unit charge on E, and finally $\mathcal{V}''_B(z) =$ const. $z^{-2}+\cdots$ when $z\to\infty$.

The problem is solved by $\mathcal{V}_{A/2} = \mathcal{V}$ if $B=A/2$.

And if $B=0$? Then, $\mathcal{V}''_0$ is the simple algebraic function $\mathcal{V}''_0(z) = \dfrac{\text{constant}}{\sqrt{z(z-c)(z-a)(z-b)}}$ associated to the potential of a plain (and plane) condenser $(E,\widetilde{F})$, although we do not need to know what $\widetilde{F}$ is. The capacity is $2\mathsf{K}/(\pi\mathsf{K}')$, and

$\rho_0 = \exp\left(-\dfrac{\pi}{2}\dfrac{\mathsf{K}'}{\mathsf{K}}\right)$.

And for any B,

$$\mathcal{V}'_B = \frac{2B}{A}\mathcal{V} + \left(1-\frac{2B}{A}\right)\mathcal{V}'_0 \tag{73}$$

does the trick, see Meinguet [53] for such relations. So,

$$\rho_B = \rho^{B/A}\rho_0^{(1-2B/A)}\,.$$

and we just have to get $\rho_0 = \exp(-1/C)$, where C is the plain condenser capacity of $(E,\widetilde{F})$.

Now, we look at some error norms $E_n = \|e^{-nz} - \dfrac{p_n(z)}{q_n(z)}\|_\infty$ on $[0,c]$, and the products $\rho^{-n}E_n/2$ which should tend towards ρ_0:

n	$c=$	0.5	$c=$	1	$c=$	2	$c=$	5	$c=$	∞
	E_n	$\rho^{-n}E_n/2$	E_n	$\rho^{-n}E_n/2$	E_n	$\rho^{-n}E_n/2$	E_n	$\rho^{-n}E_n/2$	E_n	$\rho^{-n}E_n/2$
1	$2.5352E-4$	0.02255	$1.5802E-3$	0.04509	$7.7144E-3$	0.08960	$3.13E-2$	0.1946	$6.6831E-2$	0.3104
2	$1.6454E-6$	0.02605	$3.1969E-5$	0.05206	$3.8218E-4$	0.1031	$2.7691E-3$	0.2140	$7.3587E-3$	0.3175
3	$9.7750E-9$	0.02753	$5.9205E-7$	0.05502	$1.7358E-5$	0.1088	$2.301E-4$	0.2211	$7.9938E-4$	0.3204
4	$5.658E-11$	0.02836	$1.0684E-8$	0.05667	$7.6871E-7$	0.1119	$1.8815E-5$	0.2248	$8.6522E-5$	0.3221
5	$3.240E-13$	0.02888	$1.907E-10$	0.05771	$3.3678E-8$	0.1139	$1.5284E-6$	0.2270	$9.3457E-6$	0.3232
lim		0.03123		0.06241		0.1227		0.236		0.328
$e^{-\frac{\pi K'}{2K}}$		0.03126		0.06240		0.1226		0.2362		0.328

The last rows are: the limit when $n \to \infty$ estimated through a simple step of Thiele interpolatory continued fraction, i.e., λ from $\lambda + \mu/(n+\nu)$ interpolation three values, = first nontrivial step of $\rho-$algorithm [10];
and the formula $\exp(-\pi K'/(2K))$.

6. Best rational approximation to other exponential functions

Not even started! (things from [45])

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