MAPA 3071 Special topics in approximation theory.

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Abstract: several ways to build rational approximations to various species of analytic functions are examined. Special emphasis is put on strong asymptotic estimates of the form

\[ f(z) - R_n(z) \sim \sigma(z) \rho^n(z), \]

when such estimates are available.

CONTENTS

1. Complex approximation theory and potential theory. ..................................... 2
   1.1. Taylor expansions ......................................................... 2
   1.2. Interpolatory (Jacobi) expansions ......................................... 3
   1.3. General polynomial interpolation ......................................... 3
   1.4. Padé approximation ...................................................... 3
   1.5. Strong asymptotics of Padé terms for functions with branch points .......... 6
   1.6. General rational interpolation. Condenser capacity .......................... 6

2. Asymptotic features of rational interpolation. .............................................. 6
   2.1. According to Gončar-Stahl (a sloppy rendering) ................................ 6
   2.2. According to Nuttall (in construction) ....................................... 7
   2.3. The Riemann-Hilbert way (not even started) .................................... 7
   2.4. Conditions on a single arc ................................................ 7
   2.5. Conditions on several arcs ................................................. 9

3. The exponential function \( e^{Az} \). .................................................. 11
   3.1. Taylor ................................................................. 11
   3.2. Padé ................................................................. 11
   3.3. Padé approximation to \( \exp(nB_1z + nB_2z^2) \) and a bit of Painlevé functions 13
   3.4. Rational interpolation .................................................. 20
   4. Rational interpolation to \( \exp(nB_1z + nB_2z^2) \). ............................ 27
   4.1. The single arc case .................................................... 27
   4.2. First caustic ........................................................ 30

5. Best rational approximation to \( e^{-(A_0+B)x} \) on a real interval ............. 32
   5.1. ................................................................. 32
   5.2. Root asymptotics ..................................................... 36
   5.3. Strong asymptotics .................................................... 46

6. Best rational approximation to other exponential functions ..................... 48

7. References. .................................................................................. 48
1. Complex approximation theory and potential theory.

1.1. Taylor expansions.

The Taylor series expansion of a function with finite convergence domain, say
\[ \frac{1}{1 - z/R} = \sum_{k=0}^{\infty} \frac{z^k}{R^k}, \quad \text{or} \quad \sqrt{1 - z/R} = \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k \cdot \left( k - \frac{3}{2} \right) \frac{z^k}{k!R^k} \]

shows “typically” almost circular level lines of equal approximation, explained by a convenient representation of the error
\[ f(z) - \sum_{k=0}^{n} c_k z^k = K_n(z)(z/R)^n, \tag{1} \]

where \( K_n \) is “typically” slowly variable in \( n \). What is meant by “typical” must be estimated on particular classes of functions. The only general truth here is that \( K_n \) is bounded by a slowly variable function of \( n \) when \( |z| < R \):
\[ K_n(z) = \left( \frac{R}{z} \right)^n \frac{1}{2\pi i} \int_{|z|=r} f(t) \frac{z^{n+1}}{t^n(t-z)} dt, \]

with \( |z| < r < R \), and \( r \) arbitrarily close to \( R \). For some functions \( f \), an infinite subset \( \{K_n\} \) may be much smaller than expected, that’s why some special classes of functions \( f \) will be described (sometimes a single function…) when accurate asymptotic estimates of (1) will be needed. For the examples above,
\[ \frac{1}{1 - z/R} - \sum_{k=0}^{n} (z/R)^k = \frac{(z/R)^{n+1}}{1 - z/R} \cdot \sqrt{1 - z/R} - \sum_{k=0}^{n} c_k(z/R)^k = \frac{d_{n+1}^{(n)}(z/R)^{n+1} + \cdots + d_{2n}^{(n)}(z/R)^{2n}}{\sqrt{1 - z/R} + \sum_{k=0}^{n} c_k(z/R)^k}, \]

with suitable \( d_{n+1}^{(n)} \), etc. A special asymptotic analysis is needed to show that \( d_{n+1}^{(n)} = 2c_{n+1} \sim -1/\sqrt{n^3} \).

The circle \( |z| = R \) is the boundary of the convergence region. It is often conveniently approached by the zeros of the truncated expansion, here with \( n = 10 \):

---

1. slowly variable = less than exponentially variable.
For $f(z) = 1/(1 - z/R)$, the truncated Taylor expansion is $p_n(z) = (1 - (z/R)^{n+1})/(1 - z/R)$, therefore vanishes at the $n$ nontrivial $(n+1)^{th}$ roots of unity. For $f(z) = \sqrt{1 - z/R}$, the picture is not so simple, but the zeros of $p_n$ appear to have the circle as limit set as well. It is not essential to take zeros, similar results hold for $z$: $p_n(z) = \xi$, for any $\xi$: as a “typical” function $f$ is expected to be almost univalent, or not very many-valent, in the convergence disk, most of the roots of $p_n(z) = \xi$ will be outside the disk interior.

Darboux

1.2. Interpolatory (Jacobi) expansions.

... tout entier à une idée qui lui était venue sur les potentiels.

Alphonse Allais (from Madrigal manqué

1.3. General polynomial interpolation.

1.4. Padé approximation.

Padé approximation is closely linked to continued fractions, also to orthogonal polynomials. The latter aspect appears especially with expansion about $\infty$.

Just as the simplest explicit arithmetic continued fractions are the periodic ones giving square roots, we have square roots of polynomials:

$$\sqrt{z^2 - 1} = z - \frac{1}{2z} - \frac{1}{8z^3} - \cdots = z - \frac{1/2}{z - \frac{1/4}{z - \cdots}}$$

and the $[n+1/n]$ Padé approximant is the ratio of Chebyshev polynomials

$$\frac{T_{n+1}(z)}{U_n(z)} = \sqrt{z^2 - 1} = \frac{(z + \sqrt{z^2 - 1})^{n+1} + (z - \sqrt{z^2 - 1})^{n+1}}{(z + \sqrt{z^2 - 1})^{n+1} - (z - \sqrt{z^2 - 1})^{n+1}}$$

Square roots of polynomials of degree $> 2$ will usually not yield periodic continued fractions, but a kind of quasi-periodicity which will be described further on. It will also be seen that the arrangement $f = \frac{X - \sqrt{Y}}{Z}$ is almost pure periodic [49]. So,

$$\frac{z^2 - \alpha z - \beta \sqrt{1 - \alpha^2} - \sqrt{(z^2 - 1)(z - \alpha - \beta)(z - \alpha + \beta)}}{z - \alpha} = \frac{\gamma}{z} + \frac{\gamma (1 - \beta^2)/2}{z^2} + \cdots$$

with $\gamma = (1 + \beta^2)/2 - \beta \sqrt{1 - \alpha^2}$, and $\gamma = \ldots$ turns into an (apparent) mess. Special periodic cases are

- Two-periodic $\alpha = 0$:

$$\frac{z^2 - \beta - \sqrt{(z^2 - 1)(z^2 - \beta^2)}}{z} = \frac{(\beta - 1)^2/2}{z - \frac{(\beta + 1)^2/4}{z - \frac{(\beta - 1)^2/4}{z - \cdots}}}$$
Approximants $q_n/p_n = [n-1/n]$:

$$p_{n+2} = (z^2 - (1 + \beta^2)/2)p_n - \left(\frac{1 - \beta^2}{4}\right)^2 p_{n-2},$$

$$q_{2n} = \frac{z(\beta - 1)^2}{2} \frac{\rho^{2n} - \rho^{-2n}}{\sqrt{(z^2 - 1)(z^2 - \beta^2)}} \left(\frac{1 - \beta^2}{4}\right)^n$$

$$(Ap^{2n} + Bp^{-2n}) \left(\frac{1 - \beta^2}{4}\right)^n,$$

$$q_{2n+1} = \frac{(\beta - 1)^2/2}{z(\rho^{2n} + \rho^{-2n})} \left(\frac{1 - \beta^2}{4}\right)^n$$

with $\rho = \sqrt{z^2 - (1 + \beta^2)/2 + \sqrt{(z^2 - 1)(z^2 - \beta^2)}}/(1 - \beta^2)/2$, and $A, B = 1/2 \pm z^2 - \beta/2\sqrt{(z^2 - 1)(z^1 - \beta^2)}$, $C, D = \rho^{\pm1} \left(\frac{1}{2} \pm \frac{z^2 + \beta}{2\sqrt{(z^2 - 1)(z^2 - \beta^2)}}\right)$.

The approximants converge (check that the limit is $f$) outside the singular locus $S = \{z : |\rho(z)| = 1\}$. All the zeros and poles but one are on, or tend when $n \to \infty$ towards, $S$. There is a supplementary zero at 0 for the even approximants, a supplementary pole at 0 for the odd approximants.

The numerator and the denominator have the forms $C_n\rho^n + D_n\rho^{-n}$, and $A_n\rho^n + B_n\rho^{-n}$, with $A_n$, etc., $D_n$ two-periodic in $n$.

- Period three: $\alpha = (1 - \beta^2)/2$.

$$z^2 - \frac{1 - \beta^2}{2}z + \beta \frac{1 - 2\beta - \beta^2}{2} = \frac{(1 - \beta)(1 + \beta)^2/2}{z + \beta(1 + \beta)/2 - \frac{(1 - \beta)^2/4}{z + \beta(1 + \beta)/2} - \frac{(1 - \beta)(1 + \beta)^2/4}{z - \beta} - \frac{(1 - \beta)(1 + \beta)^2/4}{z + \beta(1 + \beta)/2} - \cdots$$

Approximants $[n-1/n] = q_n/p_n$:

$$p_{n+3} = (z^3 + \beta^2z^2 + \frac{\beta^4 - 2\beta^2 - 3}{4}z - \beta^2)p_n - \left(\frac{(1 - \beta^2)^2}{8}\right)^2 p_{n-3},$$

$$q_{3n} = \frac{(1 - \beta)(1 + \beta)^2}{2} \frac{2\sqrt{Y}}{z - \frac{\beta + 1}{2}} \rho^{3n} - \rho^{-3n} \left(\frac{\beta^3 - 1}{8}\right)^n$$

$$(Ap^{3n} + Bp^{-3n}) \left(\frac{\beta^3 - 1}{8}\right)^n,$$

$$q_{3n+1} = \frac{(1 - \beta)(1 + \beta)^2}{2} \frac{Cp^{3n+1} + Dp^{-3n-1}}{z + \beta(1 + \beta)/2} \left(\frac{(1 - \beta)^2}{8}\right)^n$$

$$(A'p^{3n+1} + B'p^{-3n-1}) \left(\frac{(1 - \beta)^2}{8}\right)^n.$$
\[
\frac{q_{3n+2}}{p_{3n+2}} = \frac{(1 - \beta)(1 + \beta)^2}{2} \left( C' p_{3n+2} + D' p_{3n-2} \right) \left( \frac{(\beta^2 - 1)^2}{8} \right)^n \left( \frac{z + \beta^2 + 2\beta - 1}{2} \right)^{3n+3} - \rho^{-3n-3} \left( \frac{(\beta^2 - 1)^2}{8} \right)^n \left( z - \frac{1 + 2\beta - \beta^2}{2} \right),
\]

with \( Y = (z^2 - 1) \left( \frac{z - \frac{1 + 2\beta - \beta^2}{2}}{2} \right) \left( z - \frac{1 - 2\beta - \beta^2}{2} \right). \)

\[
\rho = \sqrt{\frac{(1 - \beta^2)^2/4}{4}}
\]

\[
A, B = \frac{1}{2} \pm \frac{2z^2 + (\beta^2 - 1)z + \beta(1 - 2\beta - \beta^2)}{4\sqrt{Y}}, C, D = \rho + 1 \left( \frac{1}{2} \pm \frac{2z^2 + (\beta^2 - 1)z - 2\beta}{4\sqrt{Y}} \right), A' \rho = C' \rho^2, B' \rho^{-1} =
\]

\[
D' \rho^{-2} = \frac{z + \beta(1 + \beta)/2}{2} \pm \frac{4z^2 + 2(2\beta^2 + \beta - 1)z^2 + (\beta^4 + 2\beta^2 - 3\beta^2 - \beta - 2)z - (2\beta^3 + 6\beta^2 + 2\beta - 2)}{8\sqrt{Y}}.
\]

- General quadratic case [49]

We consider \( f(z) := f_0(z) = \frac{\beta_0(z)}{\alpha_0(z)} = \frac{X_0(z) - \sqrt{Y(z)}}{Z_0(z)}, \) where \( X_0, Y, \) and \( Z_0 \) are given polynomials, and where the continued fraction arrangement (Jacobi, or Stieltjes, or \( C - \) fraction, etc.) keeps \( \alpha_n \) and \( \beta_n \) to be polynomials of small degrees.

Then, \( f_n(z) := \frac{\beta_n(z)}{\alpha_n(z)} = \frac{X_n(z) - \sqrt{Y(z)}}{Z_n(z)}, \) or

\[
f_n = \frac{\beta_n}{\alpha_n - f_{n+1}}.
\]

with polynomial \( X_n \) and \( Z_n \) of ultimate degrees related to the degree of \( Y \). For Jacobi continued fractions (Padé at \( \infty \)), the degrees are \( g + 1 \) and \( g \) if the degree of \( Y \) is \( 2g + 2 \).

\[X_n^2 - Y = \beta_n Z_n Z_{n+1}, \quad X_{n+1} = -X_n - \alpha_n Z_{n+1}.\] (3)

Also, the \( X_n \)'s and \( Z_n \)'s have a quasi-periodic behaviour with respect to \( n \).

For the two and three periodic examples above,

\[
X_{2n}(z) = z - \beta, X_{2n+1}(z) = z + \beta; \quad Z_n(z) = z;
\]

\[
X_{3n}(z) = X_{3n+2}(z) = z^2 - \frac{1 - \beta^2}{2} z + \beta \frac{1 - 2\beta - \beta^2}{2}, X_{3n+1}(z) = z^2 - \frac{1 - \beta^2}{2} z - \beta, \]

\[
Z_{3n}(z) = z - \frac{1 - 2\beta - \beta^2}{2}, Z_{3n+1}(z) = Z_{3n+2}(z) = z - \frac{1 + \beta}{2}.
\]

The recurrence \( p_{n+1} = \alpha_n p_n - \beta_n p_{n-1} \) of \( p_n \) and \( q_n \) of the approximants \( q_n/p_n = 1/0, \beta_0/\alpha_0, \beta_0 \alpha_1/(\alpha_0 \alpha_1 - \beta_1), \ldots \), has also the solution \( f_0 f_1 \ldots f_n \) (from (2)). \( \alpha_n f_0 \ldots f_n - f_0 \ldots f_n f_{n+1} = \beta_n f_0 \ldots f_{n-1} \), and we have

\[
f_0 \ldots f_n = q_n - f p_n. \quad \ldots \quad (4)
\]

Considering the two possible choices of the square root of \( Y \) in \( f \) and the \( f_n \)'s,

\[
p_n = \frac{f_{0,\text{conj}} \ldots f_{n,\text{conj}} - f_0 \ldots f_n}{f_{\text{conj}} \ldots f_{\text{conj}}}, \quad q_n = \frac{f_{1,\text{conj}} \ldots f_{n,\text{conj}} - f_1 \ldots f_n}{1/f_{\text{conj}} \ldots 1/f}. \quad \ldots \quad (5)
\]
\[(f_0 \cdots f_n)^2 = \frac{(X_0 - \sqrt{Y})^2 \cdots (X_n - \sqrt{Y})^2 Z_{n+1}}{Z_0 \cdots Z_n Z_{n+1}} = \beta_0 \cdots \beta_n \frac{Z_{n+1}}{Z_0} \frac{X_0 - \sqrt{Y}}{X_0 + \sqrt{Y}} \cdots \frac{X_n - \sqrt{Y}}{X_n + \sqrt{Y}}.\]

**Definition.** The average complex rate of growth = exponential of the complex Green function is

\[\rho = e^G = \lim_{n \to \infty} \left( \frac{X_0 + \sqrt{Y}}{X_0 - \sqrt{Y}} \cdots \frac{X_n + \sqrt{Y}}{X_n - \sqrt{Y}} \right)^{1/(2n)},\]

with the square root of \(Y\) such that \(|p| > 1\) (= the real Green function \(G := \text{Re } G \geq 0\)) everywhere.

### 1.5. Strong asymptotics of Padé terms for functions with branch points.

Two branch points. Szegő: denominator \(p_n\) is orthogonal to \(w\), if \(f(z) = \int_a^b w(t) \frac{dt}{z-t}\). Then, \(p_n(z) \sim \alpha(z)\rho^n(z)\).


### 2. Asymptotic features of rational interpolation.

#### 2.1. According to Gončar-Stahl (a sloppy rendering).

Interpolation to \(f_n(z) = \int_{C_f} \frac{\varphi_0(t) \varphi^n(t)}{z-t} dt\) at \(z_0, \ldots, z_{m+n}\) by \(p_m/q_n\) yields

\[f_n(z) - \frac{p_m(z)}{q_n(z)} = \frac{\prod_{i=0}^{m+n}(z-z_i)}{q_n(z)} \int_{C_f} \frac{q_n(t)}{\prod_{i=0}^{m+n}(t-z_i)} \frac{\varphi_0(t) \varphi^n(t)}{z-t} \frac{dt}{z-t},\]

where \(q_n\) is (formally) orthogonal with respect to \(\frac{\varphi_0(t) \varphi^n(t)}{\prod_{i=0}^{m+n}(t-z_i)}\) on \(C_f\).

Well, we expect that most of the poles of \(q_n\) will tend to a set of arcs \(S\), with a limit distribution \(\mu_p\), that \(C_f\) may be modified within the closure of the domain where \(\varphi_0\) and \(\varphi\) are analytic, so that \(S \subseteq C_f\). On the support of \(\mu_p\), \(q_n\) is almost a Szegő orthogonal polynomial! which means that \(\pm q_n(t)\frac{\varphi^{n/2}(t)}{\sqrt{\prod_{i=0}^{m+n}(t-z_i)}}\) has slowly varying phase and absolute value there.

Sloppy asymptotic explanation with (complex) potentials \(\psi'_p(z) := \int_{\text{supp } \mu_p} \log(z-t) \, d\mu_p(t)\) and \(\psi'_l(z) := \int_{\text{supp } \mu_p} \log(t-z) \, d\mu_l(t)\) (interpolation points), so that \(q_n(z) \sim \exp(n \psi'_p(z))\) when \(z \notin \text{supp } (\mu_p)\), \(\prod_{i=0}^{m+n}(z-z_i) \sim \exp(2n \psi'_l(z))\), and \(q_n(z) \approx \exp(n \psi'_{p,+}(z)) + \exp(n \psi'_{p,-}(z))\) on \(\text{supp } (\mu_p)\). Then,

\[q_n(t) \frac{\varphi^{n/2}(t)}{\sqrt{\prod_{i=0}^{m+n}(t-z_i)}} \approx \left[ \exp(n \psi'_{p,+}(z)) + \exp(n \psi'_{p,-}(z)) \right] \exp(n \log \varphi(t)/2 - n \psi'_l(t)) \approx \exp \left( n \left[ \frac{\log \varphi(t)}{2} - \psi'_l(t) + \frac{\psi'_{p,+}(t) + \psi'_{p,-}(t)}{2} \right] \right) \cos n \left( \frac{\psi'_{p,+}(t) - \psi'_{p,-}(t)}{i} \right).\]
on $S$, or:
\[ \log \varphi(t)/2 - \psi'(t) + \left[ \psi_{p,+}(t) + \psi_{p,-}(t) \right]/2 = \text{constant}, \tag{6} \]
the same real constant on all the arcs of $S$, has a real part smaller than this constant on $C_f \setminus S$.

For derivatives:
\[ (\log \varphi(z))'/2 - \psi'(z) + \int_{S} \frac{d\mu_p(t)}{z-t} = 0 \text{ on } z \in S. \tag{7} \]

Remark that the (complex conjugate of) the derivative $(\log \varphi(z))'/2 - \psi'(z) + \psi_{p}(z)$ on the two sides of $S$ gives the gradient of the real potential $\text{Re}[\log \varphi(z)/2 - \psi'(z) + \psi_{p}(z)]$, and has \textit{opposite} values $\pm \pi i \mu'(z)$ on the two sides of $S$, from (7) and the \textit{Sokhotskyi-Plemelj} formulas (see § 2.5) for $\psi_{p}$: \textit{symmetry property} [4, 25, 74, 75, etc.].

2.2. According to Nuttall (in construction).

2.3. The Riemann-Hilbert way (not even started).

2.4. Conditions on a single arc.

Let the function (often associated to a distribution of poles) $\psi_{p}^x(z) = \int_{\alpha}^{\beta} \frac{d\mu_p(t)}{z-t}$. Suppose that we know that
\[ = \int_{\alpha}^{\beta} \frac{d\mu_p(t)}{z-t} = g(z), \tag{8} \]
with $g$ analytic in some domain (the arc $[\alpha, \beta]$ is not yet known). The trick is to multiply $\psi_{p}^x$ by a function $[(z - \alpha)(z - \beta)]^{\gamma/2}$ taking \textit{opposite} values on the two sides of $[\alpha, \beta]$. We consider only $\gamma = 1$ and $\gamma = -1$.

Also, $[(z - \alpha)(z - \beta)]^{\gamma/2}$ is defined to be continuous outside the arc, and behaves like $z'$ for large $z$. As $\psi_{p}^x[(z - \alpha)(z - \beta)]^{\gamma/2} - \delta_{\gamma,1}$ has a Laurent expansion with only negative powers at $\infty$,
\[
\psi_{p}^x[(z - \alpha)(z - \beta)]^{\gamma/2} - \delta_{\gamma,1} = \frac{1}{2\pi i} \oint_{C} \frac{\psi_{p}^x[(t - \alpha)(t - \beta)]^{\gamma/2} - \delta_{\gamma,1}}{z-t} dt \\
\text{on a big counterclockwise contour having the arc } [\alpha, \beta] \text{ inside and } z \text{ outside. Making the contour shrink to a neighbourhood of the arc } [\alpha, \beta], \text{ we get} \\
\frac{1}{2\pi i} \int_{\alpha}^{\beta} \frac{\Delta[\psi_{p}^x[(t - \alpha)(t - \beta)]^{\gamma/2} - \delta_{\gamma,1}]}{z-t} dt, \text{ where } \Delta\{F\} \text{ means the difference } F_{-} - F_{+} \text{ between the limit values of } F \text{ on the “lower” side of the arc (from which the arc is seen at left), and the “upper” side. The difference is here } 2g(i)[(t - \alpha)(t - \beta)]^{\gamma/2}, \text{ whence quite explicit solutions } \\
\psi_{p}^x[(z - \alpha)(z - \beta)]^{\gamma/2} - \delta_{\gamma,1} = \frac{1}{\pi i} \int_{\alpha}^{\beta} \frac{g(t)[(t - \alpha)(t - \beta)]^{\gamma/2}}{z-t} dt, \quad \gamma = \pm 1. \tag{9} \]

It may help to realize that the phase of $\frac{\beta - \alpha}{[(t - \alpha)(t - \beta)]^{\gamma/2}}$ is exactly the one of $+i$ on the rectilinear segment $[\alpha, \beta]$.

Some questions: the $-1$ in the left-hand side of (9) when $\gamma = 1$ is needed from $\psi_{p}^x(z) = 1/z + O(1/z^2)$ for large $z$. But the two sides of (9) when $\gamma = -1$ should be $\sim 1/z^2$ for large $z$, everything works only if
\[ \int_{\alpha}^{\beta} \frac{g(t) dt}{[(t - \alpha)(t - \beta)]^{\gamma/2}} = 0, \quad \int_{\alpha}^{\beta} \frac{tg(t) dt}{[(t - \alpha)(t - \beta)]^{\gamma/2}} = \pi i. \tag{10} \]
The two forms of (9) then agree, either with $\gamma = -1$, or $\gamma = 1$. It will also be useful to check that, as (8) is a plain integral when $z = \alpha$ and $z = \beta$, one has $\mathcal{V}_\rho^\gamma(\alpha) = g(\alpha)$, and $\mathcal{V}_\rho^\gamma(\beta) = g(\beta)$.

2.4.1. A little bit of Chebyshev polynomials calculus. N.B. Ullman

Let us consider the Chebyshev polynomials expansion of a generic function $F$ on $[\alpha, \beta]$:

$$F(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n T_n \left( \frac{2t - \alpha - \beta}{\beta - \alpha} \right).$$

Then, we have the integral

$$\int_{\alpha}^{\beta} \frac{F(t) \, dt}{[\,(t - \alpha)(t - \beta)\,]^{1/2}} = \pi i \frac{c_0}{2}.$$ 

Therefore, from (9) with $\gamma = -1$, $\mathcal{V}_\rho^\gamma([z - \alpha](z - \beta)]^{-1/2}$ is the constant term of the Chebyshev expansion of $g(t)/(z - t)$.

Let $g_0/2 + \sum_{n=1}^{\infty} g_n T_n$ be the expansion of $g$. Remark that (10) becomes

$$g_0 = 0, \quad g_1 = \frac{4}{\beta - \alpha}.$$ (11)

We need the expansion of $1/(z - t) = X_0/2 + \sum_{n=1}^{\infty} X_n T_n$, which we multiply by $\frac{2(z - t)}{\beta - \alpha} = \frac{2z - \alpha - \beta}{\beta - \alpha} - \frac{2t - \alpha - \beta}{\beta - \alpha}$.

$$\frac{2}{\beta - \alpha} = \frac{X_0}{2} \left( \frac{2z - \alpha - \beta}{\beta - \alpha} - T_1((2\alpha - \beta)/(\beta - \alpha)) \right) + \sum_{n=1}^{\infty} X_n \left( \frac{2z - \alpha - \beta}{\beta - \alpha} - T_n - \frac{T_{n-1} + T_{n+1}}{2} \right)$$

whence the recurrence $X_{n+1} - 2\frac{2z - \alpha - \beta}{\beta - \alpha} X_n + X_{n-1} = 0$ for $n = 1, 2, \ldots$ solved by $X_n = X_0 \rho^n$, where $\rho$ is a root of

$$\frac{\rho + \rho^{-1}}{2} = \frac{2z - \alpha - \beta}{\beta - \alpha},$$ (12)

normally with $|\rho| < 1$, but this will have to be discussed later. The value of $X_0$ comes from $n = 0$: $4/(\beta - \alpha) = X_0((\rho + \rho^{-1})/2) - X_1 = X_0(\rho^i - \rho)/2$, so

$$X_0 = \frac{8}{(\beta - \alpha)(\rho^{-1} - \rho)}.$$ 

Remark that $[\,(z - \alpha)(z - \beta)\,]^{1/2} = (\beta - \alpha)^2(1 - \rho^2)^2/(16\rho^2)$, so that

$$\mathcal{V}_\rho^\gamma(z) = \sum_{n=1}^{\infty} g_n \rho^n.$$ (13)

The two determinations of $\mathcal{V}_\rho^\gamma$ on the two sides of the cut $[\alpha, \beta]$ are obtained with the two roots $\rho$ and $1/\rho$ of (12). One checks that the arithmetic mean is indeed

$$\left( \mathcal{V}_\rho^\gamma(z) + \mathcal{V}_{\rho^{-1}}^\gamma(z) \right)/2 = \sum_{n=1}^{\infty} g_n (\rho^n + \rho^{-n})/2 = \sum_{n=1}^{\infty} g_n T_n = g(z).$$

As for the discontinuity along the cut,

$$\pm \pi i \mu_\rho(z) = \mathcal{V}_{\rho^{-1}}^\gamma(z) + \mathcal{V}_{\rho}^\gamma(z) = \sum_{n=1}^{\infty} g_n (\rho^{-n} - \rho^n) = \frac{4}{\beta - \alpha} [(z - \alpha)(z - \beta)]^{1/2} \sum_{n=1}^{\infty} g_n U_{n-1}(z),$$ (14)

it appears as a kind of harmonic conjugate to $g$. 

2.5. Conditions on several arcs.

Let (7) be
\[ \int_S \frac{d\mu_p(t)}{z-t} = g(z), \quad z \in S, \]  
(15)

with \( g(z) = \Psi_p(z) - (\log \varphi(z))'/2 \), whereas \( \Psi_p(z) = \int_S \frac{d\mu_p(t)}{z-t} \pm \pi i \mu_p(z) \) (Sokhotskyi-Plemelj formulas [31], chap. 14, etc.).

If the (unknown) endpoints are \( \alpha_1, \ldots, \alpha_{2m} \), and \( P(z) = (z - \alpha_1) \cdots (z - \alpha_{2m}) \),
\[ \frac{\Psi_p(z)}{\sqrt{P(z)}} = \pm \frac{g(z)}{\sqrt{P(z)}} \]  
the same function on the two sides of \( S \), so,
\[ \frac{\Psi_p(z)}{\sqrt{P(z)}} = \frac{1}{\pi i} \int_S \frac{g(t)}{\sqrt{P(t)}} \frac{dt}{z-t} \]
(Sokhotskyi-Plemelj again).

As \( \Psi_p(z) = z^{-1} + \cdots \) for large \( z \), \( m + 1 \) equations follow
\[ \int_S \frac{g(t)}{\sqrt{P(t)}} \frac{dt}{t} = 0, \quad k = 0, \ldots, m - 1; \quad \int_S \frac{g(t)}{\sqrt{P(t)}} \frac{dt}{t} = \pi i, \]
and the \( m - 1 \) further equalities, from (6)
\[ \log \varphi(\alpha_{2k})/2 - \varphi'(\alpha_{2k}) \varphi'(\alpha_{2k}) = \log \varphi(\alpha_{2k+1})/2 - \varphi'(\alpha_{2k+1}) \varphi'(\alpha_{2k+1}) \quad k = 1, \ldots, m - 1. \]

2.5.1. Differential equation for \( \Psi_p \) [25]. If there is a polynomial \( Q \) such that \( Qg'/g \) is a polynomial,
\[ Q(z) \sqrt{P(z)} \Psi_p(z) = \text{polynomial} + \frac{1}{\pi i} \int_S Q(t) g(t) \sqrt{P(t)} \frac{dt}{z-t} \]
we derive in \( z \) and perform an integration by parts
\[ [Q(z) \sqrt{P(z)} \Psi_p(z)]' = \text{another polynomial} + \left( \frac{P'(z) Q(z)}{2} + \frac{P(z) Q(z) g'(z)}{g(z)} + P(z) Q'(z) \right) \frac{1}{\pi i} \int_S \frac{g(t)}{\sqrt{P(t)}} \frac{dt}{z-t}, \]
or \[ Q \sqrt{P} \left( \frac{\Psi_p}{g} \right)' = \frac{\text{polynomial}}{g}. \]

2.5.2. An example from [16, p. 404] (J. Meinguet). We have \( g(x) = -2\delta x^3 \) on \([-1, 1]\) with \( \delta > 0 \) and we suspect \( S \) to be \([-1, -\alpha] \cup [-\beta, \beta] \cup [\alpha, 1]\).

The formula for \( \Psi_p \) is surprisingly elementary:
\[ \Psi_p(z) = -2\delta z^3 + 3\delta(2z^2 + \alpha^2 + \beta^2 - 1) \sqrt{\frac{(z^2 - \alpha^2)(z^2 - \beta^2)}{z^2 - 1}} \]
we have indeed \( (\Psi_{p+}^*(x) + \Psi_{p-}^*(x))/2 = -2\delta x^3 \) on \( S \) = the set of points where the square root is pure imaginary. Let us look at the argument of this square root just above and just below the real axis:

\[
\begin{array}{cccccccc}
-1 & -\alpha & -\beta & \beta & \alpha & 1 \\
- & -i & - & +i & + & -i & + & +i & +
\end{array}
\]
The coefficient of \( z^{-1} \) in the expansion of \( \mathcal{V}_p'(\zeta) \) about \( \infty \) is 1 if \( 4 = \delta \left[ 3 - 2(\alpha^2 + \beta^2) - (\alpha^2 - \beta^2)^2 - (\alpha^2 + \beta^2 - 1)^2 \right] \). Finally, \( \text{Re} \mathcal{V}_p'(x) + \delta x^4/2 \) must be the same constant on the intervals of \( S \), which holds if \( \int_{\beta}^{\alpha} \left( 2t^2 + \alpha^2 + \beta^2 - 1 \right) \sqrt{\frac{(t^2 - \alpha^2)(t^2 - \beta^2)}{t^2 - 1}} \, dt = 0 \). This latter equation gives, say, \( \alpha \) as a function of \( \beta \), and the former equation gives the corresponding \( \delta \):

\[
\begin{array}{ccc}
\beta^2 & \alpha^2 & \delta \\
0 & 0.5402057601 & 3.319654336 \\
0.0001 & 0.5398986321 & 3.318351926 \\
0.01 & 0.5221727482 & 3.237132862 \\
0.02 & 0.5073410955 & 3.172076626 \\
0.03 & 0.4936208059 & 3.115410709 \\
0.04 & 0.4805403767 & 3.064945359 \\
0.05 & 0.4679087488 & 3.019526908 \\
0.06 & 0.4556185329 & 2.978410767 \\
0.07 & 0.4436016321 & 2.941064425 \\
0.08 & 0.4318114417 & 2.907083036 \\
0.09 & 0.4202143377 & 2.876146324 \\
0.10 & 0.4087851073 & 2.847993877 \\
0.11 & 0.3975042842 & 2.822409762 \\
0.12 & 0.3863564943 & 2.799212342 \\
0.13 & 0.3753293774 & 2.778247201 \\
0.14 & 0.3644128565 & 2.759382044 \\
0.15 & 0.3535986256 & 2.742502906 \\
0.16 & 0.3428797816 & 2.727511262 \\
0.17 & 0.3322505543 & 2.714321783 \\
0.18 & 0.3217061032 & 2.702860566 \\
0.19 & 0.3112423637 & 2.693063714 \\
0.20 & 0.3008592824 & 2.684876183 \\
0.21 & 0.2905439552 & 2.678250852 \\
0.22 & 0.2803040951 & 2.673147746 \\
0.23 & 0.2701344357 & 2.669533405 \\
0.24 & 0.2600334576 & 2.667380353 \\
0.25 & 0.2500000000 & 2.666666667 \\
\end{array}
\]

Inequalities:
We want \( \mu'(x) \geq 0 \) on the whole of \( S \), OK if \( 3\alpha^2 + \beta^2 - 1 \geq 0 \) and \( \alpha^2 + 3\beta^2 - 1 \leq 0 \). The real part of \( \mathcal{V}_p'(x) + \delta x^4/2 \) must be smaller on \( (\beta, \alpha) \) than its constant value on \( S \), OK, as

\[
\mathcal{V}_p'(x) + \delta x^4/2 - \mathcal{V}_p'(\beta) - \delta \beta^4/2 = \delta \int_{\beta}^{x} \left( 2t^2 + \alpha^2 + \beta^2 - 1 \right) \sqrt{\frac{(t^2 - \alpha^2)(t^2 - \beta^2)}{t^2 - 1}} \, dt
\]

is the integral of a function which is negative on a first part of \( (\beta, \alpha) \), positive on another part, but the integral remains negative, as it must vanish when \( x = \alpha \) (the square root is positive in the whole interval \( (\beta, \alpha) \)).

The set \( S \) is reduced to two intervals when \( \delta > 3.319\ldots \), and to a single interval when \( \delta < 8/3 \). It is conjectured in [16] that \( S \) has at most 3 parts for any \( g(x) = -\delta x^3 e^{-1} \).
3. The exponential function $e^{Az}$.

3.1. Taylor.

for $e^z$, 

$$
\left[ \frac{m}{n} \right] = \frac{1 + \frac{m}{m+n} \frac{z}{1!} + \frac{m(m-1)}{(m+n)(m+n-1)} \frac{z^2}{2!} + \cdots + \frac{m(m-1)\cdots 2,1}{(m+n)(m+n-1)\cdots (n+1) \frac{z^n}{m!}}}{1 - \frac{n}{m+n} \frac{z}{1!} + \frac{n(n-1)}{(m+n)(m+n-1)} \frac{z^2}{2!} - \cdots + (-1)^n \frac{n(n-1)\cdots 2,1}{(m+n)(m+n-1)\cdots (m+1) \frac{n!}}}
$$

(16)

$$
e^z \text{ den. - num.} = \frac{(-1)^n}{(m+1)\cdots (m+n)} \sum_{k=m+n+1}^{\infty} \frac{(k-m-1)\cdots (k-m-n)}{k!} z^k
$$

$$
= \frac{(-1)^n}{(m+n)!} \left[ \int_{-\infty}^{z} - \int_{-\infty}^{0} = \int_{0}^{z} e^{(z-t)^m} n \ dt \right]
$$

(17)

[66]

Exponential behaviour of numerator and denominator has been much worked, especially the distribution of zeros and poles. Saff & Varga remark [67, II] that, when $m \sim n$, these distributions had already been examined by Olver [64] in a study of Bessel functions.

Integrals of the form (17) behave for large $n$ as value at saddlepoint. With $m \sim n$, saddlepoint is a root of 

$$
1 + \frac{n}{t-z} + \frac{n}{t} = 0, \text{ whence } t = \frac{z}{2} - n + \sqrt{\frac{z^2}{4} + n^2},
$$

with some choice for the square root (see later), and $(z-t)^t = 2n^2 \left( \sqrt{1 + \frac{z^2}{4n^2}} - 1 \right)$.

Denominator behaves like $n^{th}$ power of 

$$
1 + \frac{1 + \frac{z^2}{4n^2}}{2n^2} \exp \left( 1 - \frac{z}{2n} - \sqrt{1 + \frac{z^2}{4n^2}} \right),
$$

(18)

where the value of the square root is 1 at $z = 0$. When $z$ increases, the value of (18) becomes very small and must be replaced by the same formula with the other choice of the square root as soon as the new
formula has an absolute value which is larger than the former one. This happens near \( z/n = 1.3255 \). In the

\[
\frac{2i}{z/n - \text{plane}}: \quad 1.325
\]

the formula (18) holds with a continuous square root outside

the shown arc, which is the locus where the two formulas have the same absolute value, also the limit of the **poles** of the approximant. The equation of the arc is \( |w(z)| = 1 \), where

\[
w(z) = \frac{\sqrt{1 + \frac{z^2}{4n^2} - 1}}{\sqrt{1 + \frac{z^2}{4n^2} + 1}} \exp \left( \sqrt{1 + \frac{z^2}{4n^2}} \right) = \frac{(z/2n) \exp \left( \sqrt{1 + \frac{z^2}{4n^2}} \right)}{1 + \sqrt{1 + \frac{z^2}{4n^2}}}
\]

This \( w \) looks like the \( p \) of before, but is not a plain quadratic algebraic function. Remark the square root behaves like \(-z/(2n)\) for large \( z \) in (18).

\[
\text{Numerator: } \frac{1 + \sqrt{1 + \frac{z^2}{4n^2}}}{2} \exp \left( \frac{1 + \frac{z}{2n} - \sqrt{1 + \frac{z^2}{4n^2}}}{1 + \sqrt{1 + \frac{z^2}{4n^2}}} \right), \quad 1.325
\]

\[
\text{Here the square root behaves like } z/(2n) \text{ for large } z. \text{ The formula holds with a continuous square root outside another arc (the limit set of the zeros of the approximant) which is another part of the locus } |w(z)| = 1.
\]

And the remainder (or residual) \( e^z \) den. – num. behaves like the \( n^{\text{th}} \) power of

\[
1 - \frac{1 + \sqrt{1 + \frac{z^2}{4n^2}}}{2} \exp \left( 1 + \frac{z}{2n} - \sqrt{1 + \frac{z^2}{4n^2}} \right) \text{ to the right of the latter arc, together with the part of the imaginary axis outside } [-2ni, 2ni] \text{ (which is also a part of the locus } |w(z)| = 1)^{-1.325}
\]

Finally, for the error \( e^z \) – approximant, the powers are
The example just seen shows the interest of a technique much used by J. Nuttall, for instance in [62, § 3.2], consisting in building asymptotic approximations $\chi_1, \chi_2, \text{and } \chi_3$ to the remainder, denominator, and numerator. These $\chi$ functions, which are here these $n^{th}$ powers of exponentials and algebraic functions, are expected to be piecewise analytic.

3.3. **Padé approximation to $\exp(nB_1z + nB_2z^2)$ and a bit of Painlevé functions.**

Laguerre wrote quite a number of papers dealing with continued fraction expansion of solutions of linear differential equations of first order with polynomial coefficients. His theory culminates in [41], but he considered separately in [40] exponentials of polynomials $\exp(F(z))$. Laguerre starts immediately with the Padé property (here, about 0):

$$e^{F(z)} \frac{q_n(z)}{p_n(z)} = O(z^{m+n+1}) = \epsilon_n z^{m+n+1} + \eta_n z^{m+n+2} + \cdots,$$

if the degrees of $q_n$ and $p_n$ are $m$ and $n$ ($m - n$ a small fixed integer). Divide by $e^F$ and derive:

$$\frac{d}{dz} \left( \frac{q_n}{p_ne^F} \right) = e^{-F(0)} \left[ (m+n+1) \epsilon_n z^{m+n} + (m+n+2) (\eta_n - \epsilon_n F'(0)) z^{m+n+1} + \cdots \right]$$

The left-hand side must be a polynomial divided by $p_n^2 e^F$. The left-hand side numerator is $-q_n p_n (q_n'/p_n - p_n'/p_n - F') = q_n p_n F' - p_n q_n' + p_n' q_n$ of degree $m + n + g$ if the degree of $F$ is $g + 1$ (yes, $g$ will be the genus of something, ...). And has the right-hand side has order $m + n$ at 0, we have

$$\frac{d}{dz} \left( \frac{q_n(z)}{p_n(z) e^F(z)} \right) = \frac{z^{m+n} \epsilon_n \Theta_n(z)}{p_n^2(z) e^F(z)}$$

with $\Theta_n$ of degree $g$. Remark that $\Theta_n(0) = m + n + 1$ if we choose $p_n(0) = 1$. As the right-hand side is the derivative of a meromorphic function, it must have vanishing residues: $\frac{m + n + \frac{\Theta_n'(z)}{\Theta_n(z)} - \frac{p_n''(z)}{p_n'(z)} - F'(z)}{z} = 0$ at the zeros of $p_n$, whence

$$z \Theta_n(z) p_n'(z) + [z F'(z) \Theta_n(z) - z \Theta_n'(z) - (m + n) \Theta_n(z)] p_n'(z) + M_n(z) p_n(z) = 0, \quad (20)$$

where $M_n$ is a new polynomial of degree $2g$.

We need to know a little more on $\Theta_n$ and $M_n$ for large $n$, in order to extract the asymptotic behaviour of $p_n$.

---

2In [62, § 3.2], $\chi_1, \chi_2, \text{and } \chi_3$ are called $R, \chi_1, \text{and } \chi_2$. 

3A slightly more general result can be obtained: if $F(z)$ is meromorphic and has no pole of order greater than 1, then $\Theta_n(z)$ is a polynomial of degree $m + n + 1$.
Any other solution $y$ of (20) satisfies
\[
\left(\frac{m+n}{z} + \Theta_n(z) - F'(z) \right) \left( \frac{p_n(z)}{p_n(z)} - \frac{y'}{y} \right) = \frac{p_n(z)}{p_n(z)} - \frac{y''}{y},
\]
or
\[
p_n y - p_n y' = \text{const.} \, z^{m+n} e^{-F} \Theta_n, \text{ whence } \left( \frac{y}{p_n} \right)' = \text{const.} \, z^{m+n} \Theta_n e^F p_n^2,
\]
showing that
\[
A p_n(z) + B q_n(z) e^{-F(z)}
\]
is the general solution of (20).

Such differential equations with apparent singular points (as all solutions are entire functions) have been much worked by W. Hahn \[26, 27, 28, 29\]. The coefficients must be Painlevé-like functions (see \[47\], containing references to R. Fuchs, D. & G. Chudnovsky, etc.).

We use $q_n p_n - p_n q_n = \epsilon_n^{-1} z^{m+n-1}$ to write (19) as
\[
-d \left( \frac{q_n(z)}{p_n(z) e^F(z)} \right) = \frac{\epsilon_n [q_n p_n - p_n q_n]}{\epsilon_n^{-1} p_n(z) e^F(z)},
\]
or
\[
q_n [p_n' + F' p_n] - q_n' p_n = \epsilon_n z \left[ q_n p_n - p_n q_n \right] \Theta_n,
\]

\[
q_n \left[ p_n' - \frac{\epsilon_n}{\epsilon_n^{-1}} \Theta_n p_n \right] = p_n \left[ q_n' - F' q_n - \frac{\epsilon_n}{\epsilon_n^{-1}} \Theta_n q_n \right]
\]
must be a polynomial of the form $p_n q_n \Theta_n$, with $\Theta_n$ of degree $g$. Whence the differential system of first order
\[
p_n' = \Omega_n z + \frac{\epsilon_n}{\epsilon_n^{-1}} \Theta_n p_n, \quad q_n' = (F' + \Omega_n) q_n + \frac{\epsilon_n}{\epsilon_n^{-1}} \Theta_n q_n.
\]

Conditions at $n = 0$: $p_0 = 1$, $p_1 = 0$, $q_0 = \text{Taylor-Maclaurin expansion of } e^F$ truncated to degree $n$. $q_n p_n - p_n q_n = \epsilon_n^{-1} z^{m+n-1} \Rightarrow q_n = -\epsilon_n^{-1} z^{m+n-1}.$ Also, $\Omega_0 = 0.$

The same equations for $q_{n-1}$ and $q_{n-2}$ involve $p$ and $q_{n-2}$, which we eliminate through the three-terms recurrence relation $p_n(z) = (1 - \alpha_n z) \frac{\epsilon_n}{\epsilon_n^{-1}} z^2 p_{n-2}(z)$, and we arrive at

\[
Y_n' = A_n Y_n, \quad Y_n = \left[ \begin{array}{c} p_n q_n e^{-F} \\ p_{n-1} q_{n-1} e^{-F} \end{array} \right], \quad A_n = \left[ \begin{array}{c} \Omega_n \\ -z^{-1} \Theta_{n-1} \\ \Omega_{n-1} + (z^{-1} - \alpha_n) \Theta_{n-1} \end{array} \right].
\]

Remark that $Y_0$ being upper triangular, so is $A_0 = (Y_0)^{-1} Y_0 \Rightarrow \Theta_{-1} = 0.$

From trace $A_n = (\log \text{det} Y)'$,
\[
\Omega_n(z) + \Omega_{n-1}(z) = -\frac{(1 - \alpha_n z) \Theta_{n-1}(z) + z F'(z) - m - n + 1}{z}.
\]

When we reconstruct the scalar second order differential equation for $p_n$ using now
\[
A_n = \left[ \begin{array}{c} \Omega_n \\ -z^{-1} \Theta_{n-1} \\ z^{-1} (m - n - 1) - F' - \Omega_n \end{array} \right],
\]
we recover the two first terms of (20), and conclude that $M_n = \zeta(\Omega_n \Theta_n - \Omega_n' \Theta_n) + (m - n) \Omega_n \Theta_n - z \Theta_n \Omega_n + F' \Omega_n - (\epsilon_n/\epsilon_{n-1}) \Theta_n \Theta_{n-1}].$ This complicated formula makes not even clear if $M_n$ has indeed degree $2g$ (the degree seems to be $3g + 1$). We still need more relations by looking at the first order differential equation for $p_{n+1}(z) = (1 - \alpha_{n+1} z) p_n(z) - (\epsilon_n/\epsilon_{n-1}) z^2 p_{n-1}(z)$ to find
\[
(1 - \alpha_{n+1} z) (\Omega_n - \Omega_{n+1}) + (\epsilon_n/\epsilon_{n-1}) z \Theta_{n-1} - \alpha_{n+1} = z (\epsilon_n/\epsilon_{n-1}) \Theta_{n+1},
\]
or, at $n = 1$,
\[
(1 - \alpha_{n+1} z) (\Omega_n - \Omega_{n+1}) = \frac{\epsilon_{n-1}}{\epsilon_{n-2}} z \Theta_{n-2} - \alpha_n - \frac{\epsilon_n}{\epsilon_{n-1}} z \Theta_n.
\]
leading to, by multiplying (22) by (23), and by summation in $n$:

$$\Omega_n^2 - \frac{\varepsilon_n}{\varepsilon_{n-1}} \Theta_n\Theta_{n-1} + \Omega_n \frac{\zeta F' - m - n}{z} = \frac{1}{z^2} + \frac{nF'}{z}.$$  \hfill (24)

It is then possible to advance the recurrence for the $\Theta$'s and the $\Omega$'s: if $\Theta_{n-1}$, $\Omega_{n-1}$, and $\varepsilon_{n-1}/\varepsilon_{n-2}$ are known, we find

$$\alpha_n = \frac{2\Omega_{n-1}(0) + \Theta'_n(0) + F'(0)}{m + n}$$  \hfill (25)

from the difference of (22) and (23) at $z = 0$; then (22) yields $\Omega_n$, and (23) gives $\Theta_n$ and $\varepsilon_n/\varepsilon_{n-1}$, knowing that $\Theta_n(0) = m + n + 1$.

With $F(z) = Az$, one finds the explicit solution $\Theta_n = m + n + 1$, $\Omega_n = -\frac{nA}{m + n}$, $\alpha_n = \frac{(m - n)A}{(m + n)(m + n - 2)}$.

(\alpha_1 = \frac{A}{m - n + 2}, \varepsilon_n/\varepsilon_{n-1} = -\frac{mnA^2}{(m + n - 1)(m + n)^2(m + n + 1)}.

See later for a much more difficult example of degree 2 for $F$.

The recurrence equations (22) and (24) have a structure somewhat similar to the equations (3)! This is an important remark already made by Gammel and Nuttall in [20] also about the Laguerre theory.

We are very close to (3) by taking $X_n = \Omega_n + F'/2 - (m + n)/2z$ and $\Theta_n = Z_{n+1}$. But what is $Y$?

Ah, Painlevé. Let us introduce a parameter $t$ in $F$. Then, the coefficients of $p_n$, $q_n$, etc. will be functions of $t$. The Painlevé-like equations are differential equations in $t$ for the coefficients of the polynomials $\Theta_n$ and $\Omega_n$, for each $n$.

Let $H_n := \hat{Y}_nY_n^{-1}$. From (21),

$$H_n = \frac{-1}{\varepsilon_{n-1}z^{m+n-1}} \begin{bmatrix} p_nq_{n+1} - \hat{q}_nq_{n+1} + \hat{F}q_{n+1}p_{n+1} & -p_nq_{n+1} + \hat{q}_nq_{n+1} - \hat{F}q_{n+1}p_{n+1} \\ \hat{p}_nq_{n+1} - \hat{q}_np_{n+1} + \hat{F}q_{n+1}p_{n+1} & -\hat{p}_nq_{n+1} + \hat{q}_np_{n+1} - \hat{F}q_{n+1}p_{n+1} \end{bmatrix}$$

is a matrix of rational functions in $z$. Remark that trace $H_n = \frac{\hat{\varepsilon}_{n-1}}{\varepsilon_{n-1}} - \hat{F}$. Is there such a big pole at the origin? No: from $q_n(z)e^{-F(z)} = p_n(z) = -\varepsilon_nz^{m+n+1} + \cdots$,

$$\hat{p}_nq_{n+s-1} - \hat{q}_nq_{n+s-1} + Fq_{n+s-1}p_{n+s-1} = e^F \left\{ \hat{p}_n - e^{-F}q_{n+s-1} - \frac{\partial}{\partial t} [e^{-F}q_{n+s-1}], p_{n+s-1} \right\}$$

shows that the orders in $H_n$ are $\binom{1}{0}$ ($r, s = 0, 1$), so, polynomials of degrees not higher than $g + 1$, much less if only coefficients of low degree in $F$ actually depend on $t$: if the degree of $\partial F/\partial t$ is $\delta$, then the degrees of the elements of $H_n$ are $\binom{\delta}{\delta}$.

More precisely, the coefficients of lowest order in $H_n$ are

$$H_n = \begin{bmatrix} \hat{p}_n(0)z & \frac{\hat{\varepsilon}_{n-1}z^2}{\varepsilon_{n-1}} \\ \frac{\hat{\varepsilon}_{n-1}}{\varepsilon_{n-1}} & \frac{\hat{\varepsilon}_{n-1}}{\varepsilon_{n-1}} - \hat{p}_n(0)z - \hat{F} \end{bmatrix}.$$
And the differential equation in \( t \) for \( A_n \) comes from

\[
\frac{\partial}{\partial z} \frac{\partial Y_n}{\partial t} = \frac{\partial H_n Y_n}{\partial z} = \frac{\partial H_n}{\partial z} Y_n + H_n A_n Y_n
\]

\[
= \frac{\partial}{\partial t} \frac{\partial Y_n}{\partial z} = \frac{\partial A_n Y_n}{\partial t} = \frac{\partial A_n}{\partial t} Y_n + A_n H_n Y_n
\]

whence

\[
\frac{\partial A_n}{\partial t} = \frac{\partial H_n}{\partial z} + H_n A_n - A_n H_n
\]  \hspace{1cm} (26)

We apply to the Padé approximations \([n - 1 / n] \to \exp(\alpha x + \beta x^2)\). It is not bad to look at low degrees, with the GP-PARI program [7]:

```plaintext
/*padee21.gp: launch gp and type \r padee21
Pade [n-1 / n] de exp(ax+b x2)
*/
default(seriesprecision,10);

Yn=matrix(2,2); An=matrix(2,2) ; Hn=matrix(2,2) ;
F=a*x+b*xˆ2; Fp=deriv(F,x); eF=exp(F);
/* n=0: */ thn1=0+0*x; omn1=0+0*x ; epn1=0; thn2=thn1;
ep=1; pn=1; pn1=0 ; qn=0 ; qn1=0;
for(n=1,3,
print(" n=",n);
if(n==1,qn=1);
m=n-1;
aln=(2*polcoeff(omn1,0)+polcoeff(thn1,1)+polcoeff(Fp,0))/(m+n);
print(" alphan=",aln);
omn=-omn1-(1-aln*x)*thn1/x-Fp+(m+n-1)/x;

\[
\begin{align*}
\text{thn} & = \left( (1-aln*x) \cdot (omn1-omn) + epn1*x \cdot thn2-aln \right) / x; \\
\text{epn} & = \text{polcoeff}((\text{thn},0)) / (m+n+1); \\
\text{pnp} & = (1-aln*x) \cdot \text{pn} - \text{epn} \cdot x^2 \cdot \text{pn1}; \\
\text{qnp} & = (1-aln*x) \cdot \text{qn} - \text{epn} \cdot x^2 \cdot \text{qn1}; \\
\text{pn1} & = \text{pn}; \quad \text{pn} = \text{pnp}; \quad \text{qn} = \text{qn}; \quad \text{qnp} = \text{qnp}; \\
\end{align*}
\]
if(n==1,qn=1);
print(" qn= ",qn);
print(" pn= ",pn);
\text{thn} = \text{thn}/\text{epn};
ep=ep\cdot epn;
print(" epsilon n=",ep);
print(" Theta n =",thn);
print(" Omega n =",omn);
Yn[1,1]=pn;Yn[1,2]=qn/eF;
Yn[2,1]=pn1;Yn[2,2]=qn1/eF;
An=deriv(Yn,x)*Yn^(-1);
Hn=deriv(Yn,a)*Yn^(-1);
print("check: ",deriv(An,a)-deriv(Hn,x)-Hn*An+An*Hn);

omn1=omn1; epn1=epn; thn2=thn1;thn1=thn;
```

```
\( \text{Script V1.1 session started Wed Oct 11 16:07:49 2000} \)

\[ \text{C:\calc\pari>pari217} \]

\[ \text{GP/PARI CALCULATOR Version 2.0.17 (beta)} \]
\[ \text{Windows NT ix86 (ix86 kernel) 32-bit version} \]
\[ \text{(readline disabled, extended help not available)} \]

\[ \text{Copyright (C) 1989-1999 by} \]
\[ \text{C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.} \]

? \r padee2
n=1
  alphan=a
  qn= 1
  pn= -a*x + 1
  epsilon n=-1/2*a^2 + b
  Theta n =-4*a*b/(-a^2 + 2*b)*x + 2
  Omega n =-2*b*x - a
  check: [O(x^9), O(x^9); 0, 0]

n=2
  alphan=(a^3 - 6*a*b)/(-3*a^2 + 6*b)
  qn= ((-a^3 + 6*a*b)/(-3*a^2 + 6*b))*x + 1
  etc. In a more readable writing:

\[ \frac{q_1}{p_1} = \frac{1}{1 - \alpha z}, \quad \alpha_1 = \alpha, \quad \epsilon_1 = \frac{2\beta - \alpha^2}{2}, \]
\[ \Theta_1 = 2 + \frac{4\alpha\beta}{\alpha^2 - 2\beta} z, \quad \Omega_1 = -\epsilon' = -\alpha - 2\beta z; \]
\[ \frac{q_2}{p_2} = \frac{1 + \alpha \frac{\alpha^2 - 6\beta}{3(\alpha^2 - 2\beta)} z}{1 - \frac{2\alpha^3}{3(\alpha^2 - 2\beta)} z + \frac{\alpha^3 + 12\beta^2}{6(\alpha^2 - 2\beta)} z^2}, \]
\[ \alpha_2 = -\alpha \frac{\alpha^2 - 6\beta}{3(\alpha^2 - 2\beta)}, \quad \epsilon_2 = \frac{\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3}{72(\alpha^2 - 2\beta)}, \]
\[ \Theta_2 = 4 + 8\alpha \beta \frac{(\alpha^2 - 6\beta)(\alpha^4 + 12\beta^2)}{(\alpha^2 - 2\beta)(\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3)} z, \quad \Omega_2 = -\frac{2\alpha^3}{3(\alpha^2 - 2\beta)} - \frac{4\beta\alpha^2}{3} \frac{\alpha^2 - 6\beta}{(\alpha^2 - 2\beta)^2} z; \]
\[ \frac{q_3}{p_3} = \frac{\frac{2\alpha^3}{5}(\alpha^4 - 12\beta\alpha^2 + 60\beta^2) z + \frac{1}{20}(\alpha^8 - 16\beta\alpha^6 + 120\beta^2\alpha^4 + 720\beta^4) z^2}{1 + \frac{3\alpha^6}{5}(\alpha^6 - 2\beta\alpha^4 + 20\beta^2\alpha^2 + 120\beta^3) z + \frac{3}{20}(\alpha^8 + 40\beta^2\alpha^4 - 240\beta^4) z^2 + \frac{\alpha^6}{60}(\alpha^8 + 72\beta^2\alpha^4 - 2160\beta^4) z^3,} \]
\[ \alpha_3 = -\frac{\alpha^8 - 24\beta\alpha^6 + 144\beta^2\alpha^4 + 2160\beta^4}{15 (\alpha^2 - 2\beta)(\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3)} \]
\[
\varepsilon_3 = \frac{\alpha_{12} - 12\beta\alpha_{10} + 180\beta^2\alpha_8 - 480\beta_3\alpha_6 - 3600\beta_4\alpha_4 - 43200\beta_5\alpha_2 + 43200\beta^6}{7200(\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3)}
\]

\[
\Theta_3 = 6 + \frac{12\alpha\beta(\alpha^8 - 16\alpha^6 + 120\beta^2\alpha^4 + 720\beta^4)(\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3)}{\alpha^6 - 6\beta\alpha^4 + 36\beta^2\alpha^2 + 72\beta^3}
\]

Hmm... these functions contain more and more complicated terms. For each \(n\), there are basically four non-obvious terms to consider: \(\varepsilon_n\), \(\Omega_n(0)\), and the coefficients of \(z\) of \(\Theta_n\) and \(\Omega_n\), i.e., \(\Theta_n\) and \(\Omega_n^\prime\). It happens that, if one looks for differential equations (in \(t = \alpha\)) for these four terms through (26), what comes out is a real mess (I tried).

One finds more tractable expressions by considering the even function \(\exp(\tilde{F}(Z))\), with \(\tilde{F}(Z) = F(Z^2) = \alpha Z^2 + \beta Z^4\). Nothing is lost, as the \([2n - 1/2n] \) Padé approximants to \(\exp(F(Z^2))\) is related to the \([n - 1/n] \) Padé approximant to \(\exp(F)\) with \(z = Z^2\),

\[
e^{\tilde{F}(z)} - \frac{\tilde{q}_n(z)}{\tilde{p}_n(z)} = \varepsilon_n Z^{2n} + \ldots \Rightarrow e^{\tilde{F}(z)} - \frac{\tilde{q}_{2n}(z)}{\tilde{p}_{2n}(z)} = \varepsilon_{2n} Z^{4n} + \ldots,
\]

so, \(p_n(z) = \tilde{p}_{2n}(z) = \tilde{p}_{2n}(\sqrt{\alpha})\), \(q_n(z) = \tilde{q}_{2n}(z) = \tilde{q}_{2n}(\sqrt{\alpha})\), \(\varepsilon_n = \varepsilon_{2n}\). Remark that \(\tilde{q}_{2n}\) is an even polynomial of true degree \(2n - 2\).

The Laguerre theory above (p. 13) builds the \(\Theta\)’s and the \(\Omega\)’s for this \(\exp(\alpha Z^2 + \beta Z^4)\) function, which one writes \(\tilde{\Theta}_N\) and \(\tilde{\Omega}_N\) (and which may be interesting on their own right):

<table>
<thead>
<tr>
<th>(N)</th>
<th>(\tilde{\Theta}_N(\alpha))</th>
<th>(\tilde{\Omega}_N(\alpha))</th>
<th>(\varepsilon_N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(2 + \frac{\beta}{\alpha})</td>
<td>(-2\alpha Z - 4\beta Z^3)</td>
<td>(\alpha)</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{1}{1 - \alpha Z^2})</td>
<td>(4 + \frac{2\beta}{\alpha^2 - 2\beta Z^2})</td>
<td>(-\frac{4\beta}{\alpha Z})</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{\alpha^2 - 2\beta Z^2}{2\alpha(\alpha^3 + 12\beta Z^2)})</td>
<td>(6 + \frac{2\beta(\alpha^4 - 4\beta^3) Z}{(\alpha(\alpha^3 + 12\beta Z^2)})</td>
<td>(-\frac{2(\alpha^4 + 4\beta^3)}{(\alpha\alpha^2 - 2\beta)Z - 4\beta Z^3})</td>
</tr>
</tbody>
</table>

Now, \(\tilde{\Theta}_N\) is an even polynomial of degree 2, whose constant coefficient is \(2N\); \(\tilde{\Omega}_N\) is an odd polynomial of degree 3, whose coefficient of \(Z^3\) is given by a simple rule, easily deduced from (22) and \(\tilde{\Omega}_0 = 0\). \(\tilde{\Omega}_1 = \tilde{F}' = -2\alpha Z - 4\beta Z^3\) this coefficient of \(Z^3\) is \(2\beta((1 - N)^2 - 1)\).

So, we now have three unknown terms for each \(N\): \(\varepsilon_N\), and, say \(\xi_N\) and \(\chi_N\) in \(\tilde{\Theta}_N(\alpha) = \tilde{N}(1 + \xi_N Z^2)\) and \(\tilde{\Omega}_N(\alpha) = 2\beta((1 - N)^2 - 1)Z^3 - \chi_N Z\).
We know apply the differential relations (26) with \( t = \alpha \), remarking first that the matrix \( \mathbf{H}_N \) is made of even polynomials, therefore of degrees \( \frac{2 \cdot 2}{0 \cdot 2} \), (as \( \delta = \) degree of \( \partial \mathbf{F}/\partial \alpha = 2 \), and orders \( \frac{2 \cdot 2}{0 \cdot 0} \), (as seen before): we have monomials, excepting the second diagonal element,

\[
\mathbf{H}_N(Z) = \begin{bmatrix}
\frac{\xi N^2}{\xi N-1} \\
\frac{\xi N}{\xi N-1} - \frac{\xi N}{\xi N} - \frac{\dot{F}(Z)}{\xi N Z^2}
\end{bmatrix},
\]

as we don’t yet know what the first diagonal could be, but we know the trace, (26) yields

\[
\frac{\partial \Omega_N}{\partial \alpha} = 2\xi N Z + Z \frac{\xi N^2}{\xi N} \Omega_N - Z \frac{\xi N}{\xi N-1} \Omega_N - 1,
\]

for the two diagonal elements of \( \mathbf{A}_N \), leading to the coefficient of \( Z^2 \) of \( \hat{\Omega}_N \)

\[
2\xi N = \hat{\Omega}_{N,2} = 4\beta \frac{\xi N}{\xi N}, \quad \chi N = 1 - (1)^N - 2N \frac{\xi N^2}{\xi N} \Omega N + 2(N-1) \frac{\xi N}{\xi N-1},
\]

knowing that the coefficient of \( Z^3 \) in \( \hat{\Omega}_N \) does not depend on \( \alpha \), and using \( \hat{\Omega}_N(Z) = 2N(1 + \xi N^2) \); the off-diagonal yield, after cleaning a little bit,

\[
\hat{\Omega}_N' = \frac{\xi N}{\xi N} [2N - \hat{\Omega}_N - Z(\hat{F}' + 2\hat{\Omega}_N)] + (1 + 2\xi N^2) Z^2 \hat{\Omega}_N,
\]

\[
\hat{\Omega}_{N-1}' = \frac{\xi N-1}{\xi N-1} [\hat{\Omega}_N - (2N - 2) + Z(\hat{F}' + 2\hat{\Omega}_N)] - (1 + 2\xi N^2) Z^2 \hat{\Omega}_{N-1},
\]

whence \( \xi N = ((-1)^N - 1)/2 \), as the coefficient of \( Z^3 \) of \( \hat{F}' + 2\hat{\Omega}_N \) is \( 4(-1)^N \beta \), and as the coefficient of \( Z^2 \) of \( \hat{\Omega}_N \) is \( 4\beta \xi N^2/\xi N \), as seen above;

\[
2N \xi N = \frac{\xi N}{\xi N} [-2N \xi N - 2\alpha + 2\chi N] + 2N(-1)^N,
\]

\[
2(N-1) \xi N_{N-1} = \frac{\xi N-1}{\xi N-1} [2(N-1) \xi N_{N-1} + 2\alpha - 2\chi N] - 2(N-1)(-1)^N,
\]

amounting to the differential system

\[
4\beta \frac{\partial}{\partial \alpha} \begin{bmatrix}
\xi N/\xi N-1 \\
\chi N/\xi N \\
\xi N_{N-1}
\end{bmatrix} = 4\beta [1 - ((-1)^N)] - 4N(-1)^N \frac{\xi N/\xi N-1}{\xi N-1} \xi N [2N \xi N - 2\alpha + 2\chi N] + 4\beta (-1)^N
\]

\[
\frac{\xi N}{\xi N-1} [2(N-1) \xi N_{N-1} + 2\alpha - 2\chi N] + 4\beta (-1)^{N+1},
\]

a system of 4 equations, but the coefficients of \( Z^4 \) and \( Z^2 \) of (24) yield 2 first integrals

\[
(-1)^N \beta \chi N + N(-1)^N \frac{\xi N}{\xi N-1} \xi N_{N-1} = \alpha \beta [(-1)^N - 1],
\]

\[
\frac{\xi N}{\xi N-1} [2(N-1) \xi N_{N-1} + 2\alpha - 2\chi N] = 2\beta [2(N-1)(-1)^N + 1].
\]

There must remain a system of two first order differential equations, or scalar second order differential equations.

Well, let us try with \( \xi N \) first:
from the third row of (27), $\chi_N = \alpha + N\xi_N + 2\beta [\xi_N - (-1)^N]/\xi_N$; use the first integrals:

$$N(N-1) \frac{\ddot{\xi}_N}{\xi_{N-1}} \xi_{N-1} = -\frac{\alpha \beta}{\xi_N} - \frac{(1)^N \beta N \xi_N + 2\beta [\xi_N - (-1)^N]/\xi_N}{\xi_N} = -\frac{\alpha \beta}{\xi_N} - \frac{(1)^N \beta N - 2\beta^2 (1)^N \xi_N - 1}{\xi_N},$$

$$\frac{\ddot{\xi}_N}{\xi_{N-1}} = \frac{N}{4(N-1)} \xi_N + \frac{\beta}{N-1} \frac{\ddot{\xi}_N}{\xi_N} - \frac{\beta}{N-1} \frac{2(N-1)(1)^N + 1}{2N(N-1) \xi_N} + \frac{\beta^2}{N(N-1)} \frac{1}{\xi_N} \left( \frac{\ddot{\xi}_N}{\xi_N} \right)^2 - \frac{1}{4N(N-1)} \xi_N \left( \frac{2\beta}{\xi_N} - \alpha^2 \right)^2$$

gives $\ddot{\xi}_N/\xi_{N-1}$ as a function of $\xi_N$ and its first derivative $\dot{\xi}_N = \partial \xi_N/\partial \alpha$. We enter all that in the first equation of (27), and get a second order differential equation for $\xi_N$

$$\frac{\ddot{\xi}_N}{\xi_N} = \frac{3}{2} \left( \frac{\ddot{\xi}_N}{\xi_N} \right)^2 + \frac{1 + (1)^N}{4\beta} + \frac{\alpha^2}{8\beta^2} + \frac{3}{2\xi_N} - \frac{\alpha}{\beta \xi_N} - \frac{N^2 \xi_N^2}{8\beta^2} = 0 \quad (28)$$

which relates $\xi_N$ to a solution of the Painlevé-IV equation by

$$\xi_N = \frac{\sqrt{-4\beta}}{\Psi_{IV,N} \left( \alpha \frac{\beta}{\sqrt{-4\beta}} \right)} \quad (29)$$

$\Psi_{IV,N}(x)$ being a solution of the Painlevé IV equation

$$p'' = \frac{p'^2}{2p} + p \frac{(3p + 2x)(p + 2x)}{2} - [1 + (1)^N]p - \frac{2N^2}{p} \quad (30)$$

of parameters $[1 + (1)^N]/2$ and $-2N^2$ (see [36, chap. 3] for the Painlevé-IV equation, also Ince [34, § 14.4] of course).

Remark that, as $\ddot{\xi}_N$ is rational in $\beta$, (29) shows that $\Psi_{IV,N}$ is an odd function. From the instances of $\Theta_n$ and $\widetilde{\Theta}_n$ seen above, we have some samples

<table>
<thead>
<tr>
<th>$N$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_{IV,N}(x)$</td>
<td>$-2x$</td>
<td>$-\frac{2x^2 + 1}{x}$</td>
<td>$-\frac{2x(4x^4 + 3)}{4x^4 - 1}$</td>
<td>$\frac{(2x^2 + 1)(8x^6 + 12x^4 + 18x^2 - 9)}{x(2x^2 + 3)(4x^4 + 3)}$</td>
</tr>
</tbody>
</table>

For the other functions of interest,

$$\chi_N = \sqrt{-4\beta} \left[ x + \frac{N}{\Psi_{IV,N}} + \frac{\Psi_{IV,N}}{2\Psi_{IV,N}} + \frac{(-1)^N}{2} \right],$$

$$N(N-1) \frac{\ddot{\xi}_N}{\xi_{N-1}} \xi_{N-1} = \beta \left[ -x \Psi_{IV,N} - N(-1)^N - \frac{(-1)^N \Psi_{IV,N}}{2} + \frac{\Psi_{IV,N}^2}{2} \right],$$

$$8N(N-1) \frac{\ddot{\xi}_N}{\xi_{N-1}} = \sqrt{-4\beta} \left[ \frac{2N^2}{\Psi_{IV,N}} + \frac{2N\Psi_{IV,N}}{2\Psi_{IV,N}} + [(2N-1)(1)^N + 1] \Psi_{IV,N} + \frac{\Psi_{IV,N}^2}{2\Psi_{IV,N}} - \frac{\Psi_{IV,N}^2}{2} (\Psi_{IV,N} + 2x)^2 \right]$$

but what may be the use of all this?

To be continued...

3.4. **Rational interpolation.**
3.4.1. **Equidistant points.** [35]

As far as we only need \( e^{Az} \) at \( z = z_0, z_0 + h, \ldots, z_0 + (m+n)h \),

\[
e^{Az} = (I + \Delta)^{(z-z_0)/h} e^{Az_0}
= \sum_{k=0}^{m+n} \binom{(z-z_0)/h}{k} \Delta^k e^{Az_0}
= \sum_{k=0}^{m+n} \left( \frac{e^{Ah} - 1}{h} \right)^k \frac{1}{k!} (z-z_0)(z-z_0-h) \cdots (z-z_0-(k-1)h),
\]

which we multiply by the denominator \( Q(z) = \sum_{j=0}^{n} q_j (z-z_0) \cdots (z-z_0-(j-1)h) \), using

\[
(z-z_0)(z-z_0-h) \cdots (z-z_0-(j-1)h) e^{Az} =
\]

\[
e^{A(z_0+jh)} \sum_{k=0}^{m+n} \left( \frac{e^{Ah} - 1}{h} \right)^{k-j} \frac{1}{(k-j)!} (z-z_0)(z-z_0-h) \cdots (z-z_0-(k-1)h),
\]

\[
Q(z) e^{Az} = e^{Az_0} \sum_{k=0}^{m+n} \left( \frac{e^{Ah} - 1}{h} \right)^k \frac{C(k)}{k!} (z-z_0)(z-z_0-h) \cdots (z-z_0-(k-1)h),
\]

where \( C(k) = \sum_{j=0}^{n} q_j e^{Ahj} \left( \frac{e^{Ah} - 1}{h} \right)^{-j} \frac{1}{(k-j)!} \) is a polynomial of degree \( n \) in \( k \), which must vanish at \( k = m+1, m+2, \ldots, m+n \),

\[
P(z) = e^{Az_0} \sum_{k=0}^{m} \left( \frac{e^{Ah} - 1}{h} \right)^k \frac{m+n-k}{k!} (z-z_0)(z-z_0-h) \cdots (z-z_0-(k-1)h),
\]

\[
Q(z) = \sum_{k=0}^{n} \left( \frac{e^{-Ah} - 1}{h} \right)^k \frac{n}{k!} (m+n-k)! (z-z_0)(z-z_0-h) \cdots (z-z_0-(k-1)h),
\]

and, formally:

\[
Q(z) e^{Az} - P(z) =
\]

\[
e^{Az_0} m! \sum_{k=m+n+1}^{\infty} \left( \frac{e^{Ah} - 1}{h} \right)^k \frac{(k-m-1)(k-m-2) \cdots (k-m-n)}{k!} (z-z_0)(z-z_0-h) \cdots (z-z_0-(k-1)h),
\]

We look at the performance of some examples of the region of good approximation in the complex plane, coloured in light gray:
colouring is made with respect of the average of \( \log |\exp(Az) - P(z)/Q(z)| \) in the square \([-2,2] \times [-2,2]\). The degrees of \( P \) and \( Q \) are here 5 and 4. When \( A \) is small, the region is an oval around the locus of the interpolation points (here, the interval \([-1,1]\) shown by a thin horizontal black line).

The interpolation points should appear as bright white dots, but they are hardly visible in somewhat big pixels, if colouring is made according to an arbitrary point of the pixel. This chosen point happens to be an actual interpolation point only for the endpoints, whence the rightmost interpolation point looking like a beacon in a dark environment.

The graph of the error function in the interpolation interval also looks like its envelope:

\[
\exp(5(0.250z)) = -1.26 \\
\exp(5(1.000z)) = -0.0897 \\
\exp(5(2.000z)) = 3.84 \\
\exp(5(3.000z)) = 6.57
\]

This suggests that we shall have valuable asymptotic estimates containing something like a nonvanishing power \( \rho^n \), even up to the interpolation locus.

Somewhat similarly poles may even enter the locus of interpolation points. Here, the poles of \( P/Q \) (degrees 20/19) interpolating \( \exp(20az) \), with \( a = 1 \) and \( a = 2 \):
n = 20 A = 1 B1 = I B2 = 0.

\[ + ++ \]

n = 20 A = 1 B1 = 2*I B2 = 0.

\[ + ++ \]

So, the locus of poles of \( P/Q \) of degrees \( m \sim n \) and \( n \) approximating \( \exp(naZ) \) enters the locus \([-1, 1]\) of interpolation points when \( a \) becomes larger than a number slightly smaller than 2. Such features will be explained.

**Integral form.**

The sums and series (31) are special hypergeometric expansions formally related to Jacobi polynomials with large symbols, say \( \mathcal{P}_n^{(\alpha+\beta,\gamma+\delta)}(x) \). Detailed asymptotics of these polynomials are achieved in [13] and [23]. These authors used either generating functions or integral forms. We adapt the integral form here:

the polynomials of (31) have the same form

\[
S = \sum_{k=0}^{M} X^k \binom{M}{k} (P-k)!Y(Y-1)\cdots(Y-k+1),
\]

where \( P \) is an integer larger than \( M \) (in (31), \( X = e^{Ah-1} \) or \( e^{-Ah-1} \), \( M = m \) or \( n \), \( P = m+n \), \( Y = (z-z_0)/h) \).

As a hypergeometric function, \( S = \text{const.} \ \mathcal{Z}{F}_1(-M,-Y;(-P);X) \), which is the Jacobi polynomial \( \mathcal{P}_M^{(-P+1,-Y-M+P)}(1+2X) \) with parameters \(-P+1\) and \(-Y-M+P\) which are never both positive, so that we cannot just quote [13, 23], but follow their methods.

We try to write \( (P-k)!Y(Y-1)\cdots(Y-k+1) \) as an integral involving a \( k \)th power, in order to achieve a closed form of (32) through the binomial theorem.

The special Beta integral

\[
\int_C u^{P-k}(1-u)^{k-Y-1}du,
\]

where \( C \) is an arc starting from the origin, turning around \( u = 1 \), and returning to the origin is what we need:

\[
\int_C u^{P-k}(1-u)^{k-Y-1}du = \cdots = \frac{(-1)^{P-k}(P-k)!}{(Y-k)(Y-k-1)\cdots(Y-P+1)} \int_C (1-u)^{P-Y-1}du,
\]

or

\[ (P-k)!Y(Y-1)\cdots(Y-k+1) = (-1)^{P-k}Y(Y-1)\cdots(Y-P+1) \frac{\int_C u^{P-k}(1-u)^{k-Y-1}du}{\int_C (1-u)^{P-Y-1}du} \]

then (32) becomes

\[
S = (-1)^P Y(Y-1)\cdots(Y-P+1) \frac{\int_C u^{P-M}(1-u)^{-Y-1}[u(1+X)-X]Mdu}{\int_C (1-u)^{P-Y-1}du}.
\]
Remark that this segment of interpolation points is of the denominator is clearly related to the saddlepoints of \((1 - e^{-2\pi i y})/(P - Y)\) if the phase of \(1 - u\) is 0 in the first part of the integral. Then, another form of the formula is

\[
S = (-1)^{p-1} \frac{\Gamma(Y + 1) \Gamma(P - Y + 1)}{2\pi i} \int_c u^{P-M} (1-u)^{-Y-1} [u(1+X) - X]^M du
\]

**Rough asymptotics.** Let \(E := \exp(Ah)\) and \(\zeta := \frac{z - z_0 - (m+n)h/2}{nh} = \frac{Y - (m+n)/2}{n}\). Then, with \(m \sim n\), we intend to follow things at constant \(E\) and \(\zeta\), i.e., a fixed exponential and \(z\) expanding linearly with \(n\), or \(A\) increasing linearly with \(n\), and \(2n\) interpolating points filling a fixed segment \([z_0, z_0 + 2nh]\).

Remark that this segment of interpolation points is \(-1 \leq \zeta \leq 1\).

Remark also that \(e^{Az} = e^{Az_0} E^Y\), so

\[
e^{Az} Q(z) - P(z) = (-1)^{m+n} e^{Az_0} \frac{\Gamma(Y + 1) \Gamma(m + n - Y + 1)}{2\pi i} \int_C \{u^m E^Y (1-u)^{-Y-1} [u E^{-1} - E^{-1} + 1]^n - u^n (1-u)^{-Y-1} [u E - E + 1]^n\} du
\]

... Saddlepoint & Stirling:

\[
S \approx e^{-P} P^p \left\{ \frac{1}{4} (1 + \zeta)^{1+\zeta} (1 - \zeta)^{-1-\zeta} u (1-u)^{-1-\zeta} [u(1+X) - X] \right\}^n
\]

where

\[
\frac{1}{u} - \frac{\zeta + 1}{u-1} + \frac{1}{X} \cdot \frac{1}{1+X} = 0,
\]

or

\[
(\zeta - 1) u^2 + \left(2 - \frac{\zeta X}{1+X}\right) u - \frac{X}{1+X} = 0
\]

so,

\[
u = \frac{1 + X - \zeta X/2 \pm \sqrt{1 + X + \zeta^2 X^2/4}}{(1 - \zeta)(1 + X)}, \quad (33)
\]

One of these saddlepoints, or both, enter an asymptotic expression of the denominator and the numerator. It is easy to get lost in intractable, although elementary, formulas.

Strangely enough, a cruder method ends up with almost readable expressions:

At least when each term has the same phase, the sum is roughly given by the term reached when the preceding ratio has unit value:

\[
\frac{X \kappa (\zeta + \kappa)}{1 - \kappa^2} = 1, \quad (34)
\]

where \(\kappa := 1 - k/n\). Remark that the ratio \(\to 0\) when \(\kappa \to 0\), and \(\to \infty\) when \(\kappa \to 1\). The roots are

\[
\kappa = -\frac{X \zeta / 2 \pm \sqrt{1 + X + \zeta^2 X^2 / 4}}{1 + X},
\]

clearly related to the saddlepoints of (33) by

\[
\kappa = (1 - \zeta) u - 1.
\]

The dominant term (there will sometimes be two dominant terms), is, still roughly,

\[
\frac{X^{n-k} n! (n + n \kappa) \Gamma(n \zeta + n)}{(n - n \kappa)! (n \kappa)! \Gamma(n \zeta + n \kappa)},
\]
where $\kappa$ is one of the roots of (34). Tedious application of Stirling’s formula yields, keeping only the wildest factors,

$$
e^{-2n^2} e^{2n} \left[ \frac{X^{1-e^{-\kappa}}(1+e^{-\kappa})^{1+e^{-\kappa}}(1+\zeta)^{1+\zeta}}{(1-\kappa)^{1-e^{-\kappa}}(\zeta+\kappa)^{\zeta+\kappa}} \right]^n
$$

$$= e^{-2n^2} e^{2n} \left[ \frac{(1-\kappa^2)}{X\kappa(\zeta+\kappa)^{\zeta}} \right]^{\kappa} \frac{X(1+\kappa)(1+\zeta)^{1+\zeta}}{(1-\kappa)(\zeta+\kappa)^{\zeta}}
$$

$$= e^{-2n^2} e^{2n} \left[ \frac{(1+X+1+X)^{1+\zeta}(1+\kappa)^{1-\zeta}}{\kappa} \right]^n
$$

(35)

using $\zeta = \frac{\kappa^{-1} - (1+X)\kappa}{X}$, from (34).

Remark that, if $\kappa$ is one of the roots of (34), for the numerator ($X = E - 1$), the other root is $-1/(E\kappa)$, and that the values of $\kappa$ for the denominator expansion are the inverses of the ones for the numerator:

$$E^{-1} \kappa_{\text{den}}^2 + (E^{-1} - 1) \kappa_{\text{den}} - 1 = 0 \Rightarrow E + (E - 1) \kappa_{\text{den}} - \kappa_{\text{den}}^2 = 0.$$

Finally, the last expansion of (31) yields the same equation (34) as for the numerator (i.e., with $X = E - 1$), and we have the following possible exponential behaviours:

<table>
<thead>
<tr>
<th></th>
<th>$P(z)\over e^{-2n^2} e^{2n}$</th>
<th>$\left[ \frac{(1+X+1+X)^{1+\zeta}(1+\kappa)^{1-\zeta}}{\kappa} \right]^n$</th>
<th>$\left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</th>
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</thead>
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<tr>
<td>num.</td>
<td>$\left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</td>
<td>$E^{n(\zeta+1)} \left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</td>
<td></td>
</tr>
<tr>
<td>den.</td>
<td>$\left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</td>
<td>$\left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</td>
<td></td>
</tr>
<tr>
<td>ratio</td>
<td>$\left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</td>
<td>$\left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</td>
<td></td>
</tr>
<tr>
<td>err.</td>
<td>$\left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</td>
<td>$\left[ -E^\zeta (\kappa - 1)^{1+\zeta}(E\kappa - 1)^{1-\zeta} \right]^n$</td>
<td></td>
</tr>
</tbody>
</table>

Of course, the integral formula above gives more weight to these asymptotic constructions, with $\kappa = (1 - \zeta)\mu - 1$. 
We expect the numerator, denominator, ratio = the approximation, and the error to behave as one of the given \( n \)th powers, or a combination of both. At the present level of formal manipulation, we don’t yet know which power(s) to use.

At least for large \( z \), we know that we should have \((E^{\pm 1} - 1)^n(z/h)^n \approx (E^{\pm 1} - 1)^ne^{-n^2n\zeta^n}\) for the numerator and the denominator. Indeed, with the root \( \kappa = (E^{-1} - 1)^\zeta + \cdots \) one finds, with the second power, \[
\left( \frac{P(z)}{e^{-2n^2n^2}} \right)^{1/n} \approx -E\kappa \left( \frac{\kappa - 1}{\kappa - 1/E} \right)^{\zeta} \approx (E - 1)^{\zeta} \left( 1 + \frac{1}{\zeta} \right)^{\zeta} \approx \varepsilon(E - 1)^{\zeta}.
\]
The first power behaves for large \( z \) as \([e(1 - E)^{\zeta E^{1+\zeta}}]^n\), such a behaviour is impossible with a polynomial. Similarly, one has for the denominator, with the first power, \[
\left( \frac{Q(z)}{e^{-2n^2n^2}} \right)^{1/n} \approx \varepsilon(E - 1)^{-1}\zeta.
\]

The region where the asymptotic behaviours of \( P \) and \( Q \) are such that \((P/Q)^{1/n} \approx E^{1+\zeta}\) is simply the region of good approximation! Indeed, \( E^{n(1+\zeta)} = e^{\ln(1+\zeta)} = e^{h(z-z_0)}\).

When \( E \) is close to 1, we almost have the Padé situation of the figure of p. 13.

Various interesting situations occur, the wildest situation being \( E = -1 \): we then interpolate merely the sequence \( 1, -1, 1, -1, \ldots \) at \( z_0, z_1, \ldots, z_{m+n} \) by \( q/p \) of degrees \( m \) and \( n \), without any reference to an exponential function anymore!

Moreover, the solution of the Cauchy problem (in the sense of [52]) is then immediate: \( q + p \) must vanish at \( z_1, z_3, \ldots \), and \( q - p \) vanishes at \( z_0, z_2, \ldots \):

\[
\frac{q(z)}{p(z)} = \frac{c(z-z_1)(z-z_3)\cdots + c'(z-z_0)(z-z_2)\cdots}{c(z-z_1)(z-z_3)\cdots - c'(z-z_0)(z-z_2)\cdots},
\]

where one of the two numbers \( c \) or \( c' \) may very well vanish if it is the only way to achieve degrees \( \leq m \) and \( n! \).

The integral of the numerator is \( \int_{C} F(u) e^{n\Phi(u)} du \) estimated, through deformation of the contour, by steepest descent contributions of the neighbourhoods of saddle points (there will be at most two of them) \( u \), such that \( \Phi'(u) = 0 \):

\[
\Phi(u) = \Phi(u_0) + \frac{(u-u_0)^2}{2}\Phi''(u_0) + \cdots, \quad \int_{C} F(u) e^{n\Phi(u)} du \sim F(u_0) e^{n\Phi(u_0)} \sqrt{\frac{2\pi}{\Phi''(u_0)}}.
\]

Here, when \( m \sim n \),

\[
F(u) = e^{\ln(1-u)^{-1}}[u(1+E)-E]^{m-n}, \quad \Phi(u) = \log\{u(1-u)^{-\zeta^{-1}}[u(1+E)-E]\} \quad \text{(numerator } P),
\]

\[
F(u) = u^{\zeta^{-1}}[u(1-E)-E]^{-1}, \quad \Phi(u) = \log\{u(1-u)^{-\zeta^{-1}}[u(1-E)-E]\} \quad \text{(denominator } Q).
\]

3.4.2. Check with § 2.4.

From (35),

\[
\Psi'(z) = \lim_{n \to \infty} \frac{\log Q(z)}{n} = (1 + \zeta) \log(1 + E^{-1}\kappa) + (1 - \zeta) \log(1 + \kappa) - \log(\kappa),
\]

where \( \zeta \) is basically our \( z \) (managed so that the interpolation points are in \([-1, 1]\)). Then, the derivative in \( \zeta \) simplifies into

\[
\frac{d\Psi'(z)}{d\zeta} = \log \frac{1 + E^{-1}\kappa}{1 + \kappa},
\]

where \( \kappa \) is related to \( \zeta \) through \( \zeta = \frac{\kappa^{-1} - E^{-1}\kappa}{E^{-1} - 1} \), from (34). This matches (12) provided

\[
\rho = iE^{-1/2}\kappa, \quad \alpha = -\beta = \frac{2i}{E^{1/2} - E^{-1/2}},
\]
and
\[ \frac{dV'_\rho(z)}{dz} = \log\frac{1-iE^{-1/2}\rho}{1-iE^{1/2}\rho} = -\sum_{n=1}^{\infty} \frac{\rho^n(E^{-n/2}-E^{n/2})}{n}\rho^n. \]
Remark that \( g_1 = -i(E^{-1/2}-E^{1/2}) = 4/(\beta-\alpha) \) as it should.

4. Rational interpolation to \( \exp(nB_1z + nB_2z^2) \).

This very interesting rational interpolation appears in special nonlinear Schrödinger problems (\cite{54} and remarks by J. Nuttall). The Padé approximation (§ 3.3) already told that there may be several arcs in the discussion.

4.1. The single arc case.

Let the interpolation points be equidistant on \([I_1, I_2]\). Then,
\[ g(z) = \int_{I_1}^{I_2} \frac{(I_2-I_1)^{-1}}{z-t} - \frac{B_1}{2} - B_2 z = \frac{\log(z-I_1)}{I_2-I_1} - \frac{B_1}{2} - B_2 \left( \frac{\beta-\alpha}{2} - \frac{\alpha+\beta}{2} \right) \]  \hspace{1cm} (36)

The logarithms have the expansions
\[ \log(z-I_k) = \log \frac{\alpha-\beta}{4\rho_k} - 2\sum_{n=1}^{\infty} \frac{\rho_k^n}{n} T_n, \]
where \( \rho_k \) is now a root of
\[ \frac{\rho_k + \rho_k^{-1}}{2} = \frac{2I_k - \alpha - \beta}{\beta - \alpha}, \hspace{1cm} k=1,2, \]  \hspace{1cm} (37)
where \( |\rho_k| < 1 \) should be the orthodox choice, but which will not be kept in the final formula. Precisely, the closed form is now
\[ V'_\rho(z) = \sum_{n=1}^{\infty} g_0 n^p \rho^n = \frac{2}{I_2-I_1} \log \frac{1-\rho_1 \rho}{1-\rho_2 \rho} - B_2 \frac{\beta-\alpha}{2} \rho, \]  \hspace{1cm} (38)
with the conditions (11) on \( g_0 \) and \( g_1 \)
\[ \frac{\log(\rho_2/\rho_1)}{I_2-I_1} - \frac{B_1}{2} - B_2 \frac{\alpha+\beta}{2} = 0, \]  \hspace{1cm} (39)
\[ g_1 = 2 \rho_2 - \rho_1 \frac{B_1}{I_2-I_1} - B_2 \frac{\beta-\alpha}{2} = \frac{4}{\beta-\alpha}. \]  \hspace{1cm} (40)
If \( \rho_1 \) and \( \rho_2 \) are known, \( \alpha \) and \( \beta \) are got by (37):
\[ \alpha = \frac{\rho_1 (1+\rho_2)^2 I_1 - \rho_2 (1+\rho_1)^2 I_2}{(\rho_2 - \rho_1)(\rho_1 \rho_2 - 1)}, \beta = \frac{\rho_1 (1-\rho_2)^2 I_1 - \rho_2 (1-\rho_1)^2 I_2}{(\rho_2 - \rho_1)(\rho_1 \rho_2 - 1)}. \]  \hspace{1cm} (41)
(As for \( \rho_1 \) and \( \rho_2 \), they are simply found to be, if \( B_2 = 0, \rho_1 = i \exp(-B_1(I_2-I_1)/4) \) and \( \rho_2 = i \exp(B_1(I_2-I_1)/4) \). Remark that \( \rho_1 \rho_2 = -1 \): no chance to have the comfortable \( |\rho_k| < 1 \ldots \)
Also,
\[ \frac{\beta-\alpha}{2} = \frac{2 \rho_1 \rho_2 (I_2-I_1)}{(\rho_2 - \rho_1)(\rho_1 \rho_2 - 1)}, \frac{\alpha+\beta}{2} = \frac{I_1 + I_2}{2} - \frac{(\rho_1 + \rho_2)(\rho_1 \rho_2 + 1)(I_2-I_1)}{2(\rho_2 - \rho_1)(\rho_1 \rho_2 - 1)}. \]
and (39) and (40) become
\[ \frac{\log(\rho_2/\rho_1)}{I_2-I_1} = \frac{B_1}{2} - \frac{B_2 (I_1+I_2)}{2} + \frac{B_2 (\rho_2 + \rho_1)(\rho_1 \rho_2 + 1)}{2(\rho_2 - \rho_1)(\rho_1 \rho_2 - 1)} (I_2-I_1) = 0, B_2(I_2-I_1)^2 = \frac{(\rho_2-\rho_1)^2(\rho_1 \rho_2 - 1)}{2\rho_1^2 \rho_2^2}, \]
or this form emphasizing $p_1 p_2$ and $p_2 / p_1$:

$$2 B_2 (I_2 - I_1)^2 = \left( \frac{p_2}{p_1} - 2 + \frac{p_1}{p_2} \right) \left( p_1 p_2 - \frac{1}{p_1 p_2} \right), \quad (42)$$

$$\log \left( \frac{p_2}{p_1} \right) + \frac{1}{4} \left( \frac{p_2}{p_1} - \frac{p_1}{p_2} \right) \left( p_1 p_2 + 2 + \frac{1}{p_1 p_2} \right) = \frac{B_1}{2} (I_2 - I_1) + B_2 \frac{I_2^2 - I_1^2}{2}. \quad (43)$$

For a given $B_2$ and various ratios $p_2 / p_1$, we find valid values for $B_1$, etc. For instance, $I_1 = -A i$, $I_2 = A i$, $B_2$ negative imaginary and $p_2 / p_1$ negative real, which is of interest in [54]:

Script VI.1 session started Mon Jan 17 14:32:31 2000

C:\calc\pari>gp

GP/PARI CALCULATOR Version 2.0.12 (alpha)
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? \r expr1r2

A = 1, B_1 = \pi - 2i, x = 0.5, B_2 = -2i A t, 

<table>
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<tr>
<th>A t</th>
<th>-p_2 / p_1</th>
<th>Q</th>
<th>p_1</th>
<th>p_2</th>
<th>\alpha</th>
<th>\beta</th>
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? quit
Good bye!

C:\calc\pari>exit

Script completed Mon Jan 17 14:34:12 2000

We integrate (38) along the lines suggested by the exercises of section 3.4.2, p. 26:

$$\mathcal{V}_p(z) = \frac{2}{I_2 - I_1} \left[ (z - I_1) \log(1 - \rho_1 p) - (z - I_2) \log(1 - \rho_2 p) + X(p) \right],$$
which yields indeed, using from (12) and (37) \[ z - I_k = \frac{\beta - \alpha}{4} \left(1 - \frac{\rho_k}{\rho}\right) \left(\rho - \frac{1}{\rho_k}\right). \]

\[
\frac{d \mathcal{V}_\rho(z)}{dz} = \frac{2}{I_2 - I_1} \left\{ \log \frac{1 - \rho_1 \rho}{1 - \rho_2 \rho} + \left[ \frac{z - I_1}{\rho - \rho_1^{-1}} - \frac{z - I_2}{\rho - \rho_2^{-1}} + \frac{dX(\rho)}{d\rho} \right] \frac{d\rho}{dz} \right\}
\]

\[= \frac{2}{I_2 - I_1} \left\{ \log \frac{1 - \rho_1 \rho}{1 - \rho_2 \rho} + \frac{\beta - \alpha \rho_2 - \rho_1}{\rho} + \frac{dX(\rho)}{d\rho} \right\} \frac{4}{(\beta - \alpha)(1 - \rho^{-2})} \]

One must have

\[
\frac{dX}{d\rho} = - \frac{\beta - \alpha \rho_2 - \rho_1}{\rho} - B_2 \left(\frac{\beta - \alpha}{16}\right) (I_2 - I_1) \left(\frac{\rho - 1}{\rho}\right),
\]

finally:

\[
\frac{dX}{d\rho} = - \frac{I_2 - I_1}{2} \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} \rho + \frac{1}{\rho},
\]

\[
\mathcal{V}_\rho(z) = \frac{2}{I_2 - I_1} \left[ (z - I_1) \log(1 - \rho_1 \rho) - (z - I_2) \log(1 - \rho_2 \rho) \right] - \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} \log \rho. \tag{44}
\]

The two determinations of \( \mathcal{V}_\rho \) on the two sides of the cut are found with the two roots \( \rho \) and \( 1/\rho \) of (12). In particular, the arithmetic mean of the two values of the derivative must give (8) again, with \( g \) given by (36). Indeed, one finds

\[ \frac{1}{I_2 - I_1} \left[ \log((1 - \rho_1 \rho)(1 - \rho_1/\rho)) - \log((1 - \rho_2 \rho)(1 - \rho_2/\rho)) \right] - B_2 \frac{\beta - \alpha + \rho^{-1}}{2}, \]

which is

\[ \frac{1}{I_2 - I_1} \left[ \log \frac{z - I_1}{z - I_2} + \log \frac{\rho_1}{\rho_2} \right] - B_2 z + B_2 \frac{\alpha + \beta}{2}, \]

The difference of the two determinations of \( \mathcal{V}_\rho'' \) must be \( \pm 2\pi i \mu' \):

\[ \pm 2\pi i \mu'(z) = \frac{2}{I_2 - I_1} \left[ \log \frac{1 - \rho_1 \rho}{1 - \rho_1/\rho} - \log \frac{1 - \rho_2 \rho}{1 - \rho_2/\rho} \right] - B_2 \frac{\beta - \alpha}{2} (\rho - \rho^{-1}), \tag{45} \]

(Nuttall’s \( \Delta \Psi_2 \))

and the cut itself is the locus \( \{z : \mu'(z) \, dz \text{ real}\} \), which is integrated as \( \{z : \mathcal{V}_{\rho,+}(z) - \mathcal{V}_{\rho,-}(z) \text{ pure imaginary}\} \).

\[ \frac{2}{I_2 - I_1} \left[ (z - I_1) \log \frac{1 - \rho_1 \rho}{1 - \rho_1/\rho} - (z - I_2) \log \frac{1 - \rho_2 \rho}{1 - \rho_2/\rho} \right] - \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} \frac{\rho^2 - \rho^{-2}}{2} - 2 \log \rho \text{ pure imaginary}. \tag{46} \]

Writing (46) as a function of \( \rho \) (using (12) and (37)), we have

\[ F(\rho) = \frac{2}{(\rho_2 - \rho_1) (1 - 1/\rho_1 \rho_2)} \left[ (\rho - \rho_1) \left(1 - \frac{1}{\rho \rho_1}\right) L_1 - (\rho - \rho_2) \left(1 - \frac{1}{\rho \rho_2}\right) L_2 \right] - \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} \frac{\rho^2 - \rho^{-2}}{2}, \]

with \( L_1 = \log \frac{1 - \rho_1 \rho}{\rho - \rho_1}, L_2 = \log \frac{1 - \rho_2 \rho}{\rho - \rho_2} \), and where, for given \( B_1, B_2, I_1, I_2, \) one must determine \( \rho_1 \) and \( \rho_2 \) from (42) and (43).
4.2. **First caustic.**

The present setting of the limit set of poles as a single arc joining \( z = \alpha \) to \( z = \beta \) (or \( \rho = -1 \) to \( \rho = 1 \)) holds as long as \( \mu'_{\rho}(z) dz \) remains positive on the cut. A critical situation occurs when \( \mu'_{\rho} \) happens to vanish right on the cut, i.e., if \( dF/dz \) vanishes at a point where the real part of \( F \) vanishes too.

The locus of \( (x, At) \) with \( B_1 = \pi - 2ix, B_2 = -2iAt \), where this happens is called the (first) caustic in [54]. We then have \( \rho_1 = R^{-1/2}e^{i\theta}, \rho_2 = -R^{1/2}e^{i\theta} \), with real \( R \) and \( \theta \). For a trial value of \( At \), we look for \( R \) and \( \theta \) such that \( (R + 1/R)/2 = 2At/\sin2\theta - 1 \) (from (42)) and \( 2x = \log R + (1/R - R)\sin^2\theta \) (from (43)). Knowing \( \rho_1 \) and \( \rho_2 \), one looks for the zero of the analytic function \( dF/dz \), or \( dF/d\rho \). This yields the equation \( \mu'_{\rho} = 0 \) in (45) as

\[
L_1 - L_2 = \left(1 + \frac{1}{\rho_1 \rho_2}\right) (\rho_2 - \rho_1) \frac{\rho - \rho^{-1}}{2}.
\]  

One then manages to have the real part of \( F = 0 \) as well.

Some values:
We see that \( \theta \to \pi/4 \) when \( x \to 0 \), and that \( \theta \to 0 \) when \( x \to \infty \), but many features are still unexplained... Here is a tentative explanation of the behaviour for large \( x \): as it seems that \( |\rho_1| << |\rho| << |\rho_2| \), the logarithms are approximated by \( L_1 \approx \rho_1(\rho^{-1} - \rho) - \log \rho \), \( L_2 \approx (\rho - \rho^{-1})/\rho_2 + \log \rho \). The equation (47) becomes
\[
\log \frac{\rho}{\rho - \rho^{-1}} \approx i\xi, \quad \text{with} \quad \xi = \theta \sqrt{R}/2.
\]
Also, \( F/\theta \approx (\rho - \rho^{-1})[i(\rho + \rho^{-1})/2 - \xi^{-1} - 2\xi] \) must be pure imaginary, making a second equation for \( \xi \) and \( \rho \), whence fixed solutions. And
\[
\frac{At}{e^x} \approx \frac{\theta R/4}{\sqrt{R} \exp(-\theta^2 R/2)} = \frac{\xi}{2} \exp(2\xi^2).
\]

Numerator and interpolation.
Remind that $\mathcal{Q}_n'(z)$ is the limit when $n \to \infty$ of $n^{-1} \log Q_n(z) = n^{-1} \sum \log(z-\text{poles})$. It must behave like $\log(z) + O(1/z)$ for large $z$, compatible with (44) if one adds a constant:

$$\mathcal{Q}_n'(z) = \frac{2}{I_2 - I_1} [(z - I_1) \log(1 - \rho_1 p) - (z - I_2) \log(1 - \rho_2 p)] - \frac{\rho^2}{2} \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} - \log \rho + C_p,$$

which behaves for large $z$ as $\log z - \log \frac{\beta - \alpha}{4} + \frac{(\beta - \alpha)(\rho_2 - \rho_1)}{2(I_2 - I_1)} + C_P$, from $\rho \sim (\beta - \alpha)/(4z)$. Therefore,

$$C_P = \log \frac{\beta - \alpha}{4} - \frac{(\beta - \alpha)(\rho_2 - \rho_1)}{2(I_2 - I_1)}.$$

The numerator of the interpolant to $\exp(n(B_1z + B_2z^2))$ is the denominator of the interpolant to $\exp(n(-B_1z - B_2z^2))$, so that the calculations made before apply with $(B_1, B_2) \to (-B_1, -B_2)$. The equations (39) and (40) are now satisfied by $(\rho_1, \rho_2) \to (1/\rho_1, 1/\rho_2)$. And the values for $\alpha$ and $\beta$ are the same as before. Let $\mathcal{Q}_n'(z)$ be the (presumed to exist) limit when $n \to \infty$ of $n^{-1} \log P_n(z)$, where $P_n$ is the numerator. We expect a formula similar to (44), but with another constant:

$$\mathcal{Q}_n'(z) = \frac{2}{I_2 - I_1} [(z - I_1) \log(1 - \rho/p_1) - (z - I_2) \log(1 - \rho/p_2)] + \frac{\rho^2}{2} \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} - \log \rho + C_{\text{num}}.$$

The remaining constant $C_{\text{num}}$ is determined by $\mathcal{Q}_n'(z) - \mathcal{Q}_n'(z) = B_1z + B_2z^2$ in a neighborhood of the set of the interpolation points. Everything works if one determination, say with $\rho$, is used for $\mathcal{Q}_n'$, while the determination with $1/\rho$ is used for $\mathcal{Q}_n'$:

$$\mathcal{Q}_n'(z) - \mathcal{Q}_n'(z) = \frac{2}{I_2 - I_1} [(z - I_1) \log \frac{1}{1 - \rho_1 p} - (z - I_2) \log \frac{1}{1 - \rho_2 p}] + \frac{\rho^2}{2} \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} + 2 \log \rho + C_{\text{num}} - C_{\text{num}} - C_{\rho},$$

$$= \frac{2}{I_2 - I_1} [(z - I_2) \log(-\rho_2) - (z - I_1) \log(-\rho_1)] + \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} \left[ \frac{2(z - \alpha - \beta)}{\beta - \alpha} - 1 \right] + C_{\text{num}} - C_{\rho},$$

whence

$$C_{\text{num}} - C_{\rho} = \frac{2}{I_2 - I_1} \frac{I_1 \log(-\rho_1) - I_2 \log(-\rho_2)}{I_2 - I_1} + \frac{B_2}{4} \frac{(\alpha + \beta)^2}{\beta - \alpha} - \frac{\rho_1 \rho_2 + 1}{\rho_1 \rho_2 - 1} + C_{\text{num}} - C_{\rho},$$

(48)

5. **Best rational approximation to $e^{-(An+B)}z$ on a real interval**

5.1. **Best rational approximation to $\exp(-z)$ on a given real interval**, say $[0, c]$ has a strict equioscillating error function, as seen here with $e^z - p_n(z)/q_n(z)$ on $[0, 1]$, for $n = 1, 2$:

![Graph of exponential function](image)
For varying degrees, we have a now familiar scaling effect best seen through the poles:

Sets of poles expand and tend to follow the Padé poles; errors decrease factorially fast with \( n \) (here, the error is about \( \frac{e^{-1/2}n!(n-1)!}{4(2n+1)!(2n-1)!} \)). For an accurate asymptotic picture, see Braess’ proof of Meinardus’ conjecture \([9]\)).

We find a stable picture if we look at the poles of the best approximants of degree \( n \) to \( \exp(-nz) \). Moreover, the norms \( E_n \) of the errors tend to decrease in an exponential way with \( n \):

\[
\begin{array}{cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 \\
 E_n & 0.197 & 1.488 & 4.161 & 7.610 & 13.56 & 24.06 \\
 & 6 & 4 & 3 & 5 & 10 & \ 
\end{array}
\]

The ratio of two successive errors seems to tend towards a limit of about 1/60. The exact value, as it will be shown later (in \((72), p. 45\)), is \( \rho = 1/57.0699681 \cdots \). Could we have \( E_n \sim C \rho^n \), and what is the value of \( C \)? I can’t wait: here are the products \( E_n \rho^n \):

\[
\begin{array}{cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 E_n \rho^n & 0.090 & 0.104 & 0.110 & 0.113 & 0.116 & 0.117 & \ 
\end{array}
\]

Hmmm, what could it be? The numbers follow the approximate formula 0.125 – 0.05/(\( n + 1/2 \)). The limit 0.125 is reasonably close to an estimate which will be given in \( § 5.3 \).

Ah, an obscure insight (hindsight?) coming from long and painful experiments with the \(^11/9\) problem \([45, 46]\) tells me to try \( \exp(-(n + 1/2)z) \) instead of \( \exp(-nz) \), and to multiply the errors by \( \rho^{-n+1/2} \):

\[
\begin{array}{cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 E_n \rho^{-n+1/2} & 0.197 & 1.488 & 4.161 & 7.610 & 13.56 & 24.06 & \ 
\end{array}
\]

Ah! Now, the limit seems to be 2. This phenomenon will also be explained in \( § 5.3 \).
and \( B \), also \( nfft = \) preferably a power of 2, say, 512, or 1024, and \( K = \) order of an auxiliary Hankel matrix, say, between 20 and 30.

%cfneuf3.m:

% Quasi best rational approximation of degree \( n \) to \( \exp(-nfz) \)
% on \( 0 \leq z \leq c \).
% enter: \( c \ n \ nfft \) (a power of 2, say 1024) \( K \) (order of Hankel matrix in CF construction, say 20 or more)
% Output: plot of error function, restart with higher \( nfft \) and/or \( K \)
% if not nice;
% two estimates of error norm. They should of course be very close.
% zeros & poles.
% cfneuf.m adaptation de
% cftref.m
% Approximation Theory V
% RCF -- REAL RATIONAL CF APPROXIMATION ON THE UNIT INTERVAL
% Loyd N. Trefethen, Dept. of Math., M.I.T., March 1986
% Reference: L.N.T. and M.H. Gutknecht,
% L.N. Trefethen, MATLAB programs for CF approximation, pp.599-602
% \{it in} \{sl Approximation Theory V\},
% (C.K.Chui, L.L.Schumaker, J.D.Ward, eds.),

Fx(x) - function to be approximated by \( R(x) = P(x)/Q(x) \)
\( m, n \) - degree of \( P, Q \)
\( nfft \) - number of points in FFT (power of 2)
\( K \) - degree at which Chebyshev series is truncated
\( F, P, Q, R \) - functions evaluated on FFT mesh (Chebyshev points)
\( Pc, Qc \) - Chebyshev coefficients of \( P \) and \( Q \)

If \( Fx \) is even, take \((m,n) = (\text{odd, even}).\)
If \( Fx \) is odd, take \((m,n) = (\text{even, even}).\)

diary cfneuf3.txt

CONTROL PARAMETERS
format long;
c=input(‘c in interval 0..c ? ’);
m=input(‘degree n ? ’); n=m; np=n+1;
bb=input(‘ A in \exp(-(An+B)z) ? ’); bb=exp(-(An+B)*z);
af=aa*n+bb;
nfft=input(‘nfft ? ’); nf=nfft/2;
K=input(‘K ? ’); dim=K+m-n;

CHEBYSHEV COEFFICIENTS OF \( Fx \)
\( z = \exp(2pi*sqrt(-1)*(0:nfft-1)/nfft) \);
x = real(z); F = exp(nf*c*(x-1)/2); Fc = real(fft(F))/nfft2;

% disp(Fc(dim:dim+2));
%
% SVD OF HANKEL MATRIX H
H = toeplitz(Fc(1+rem((dim:-1:1)+nfft+m-n,nfft)));
H = triu(H); H=H(:,(dim:-1:1));
[u,s,v] = svd(H);
s = s(np,np); u = u((dim:-1:1),np)';v = v(:,np)';
%
% DENOMINATOR POLYNOMIAL Q
zr = roots(v); qout = []; for i=1:dim-1;
    if abs(zr(i))>1 qout = [qout, zr(i)];end; end;
qc = real(poly(qout)); QC = qc/qc(np); q = polyval(qc,z);
Q = q.*conj(q); QC = real(fft(Q))/nfft2;
QC(1)= QC(1)/2; Q=Q/QC(1); Qc = QC(1:np)/QC(1);
%
% NUMERATOR POLYNOMIAL P
b = fft([u zeros(1,nfft-dim)])./fft([v zeros(1,nfft-dim)]);
Rt = F-real(s.*K.*b); Rtc = real(fft(Rt))/nfft2;
gam = real(fft((1./Q))/nfft2; gam = toeplitz(gam(1:2*m+1));
if m==0 Pc = 2*Rtc(1)/gam;
else Pc = 2*[Rtc(m+1:-1:2) Rtc(1:m+1)]/gam; end;
Pc = Pc(m+1:2*m+1); Pc(1) = Pc(1)/2;
P = real(polyval(Pc(m+1:-1:1),z)); R = P./Q;
%
% RESULTS
s, err = norm(F-R,'inf'), Pc, QC
serr=[s,err],
disp(‘ zeros’);
qth=zeros(1,2*n+1); qth(1:n)=Pc(np:-1:2);
qth(n+2:2*n+1)=Pc(2:np);
qth(n+1)=2*Pc(1);
polth=roots(qth);
polx=[];
ki=0;for kz=1:2*n;
polxx=( polth(kz)+1/polth(kz) )/2;
    if kz==1, polx=[polx,polxx]; ki=1;kip=0;end;
    if kz>1, kip=1;
        for kz2=1:ki; if abs(polxx-polx(kz2))<0.0000001 , kip=0;end;
    end;
    if kip==1 , ki=ki+1; polx=[polx,polxx] ; end;
end;
polz= c*(1-polx)/2;
% real(polz'), imag(polz')
Pcc=real(poly(polz));
polz'
disp(‘ poles’);
qth=zeros(1,2*n+1);qth(1:n)=QC(np:-1:2);
qth(n+2:2*n+1)=QC(2:np);
qth(n+1)=2;
polth=roots(qth);
5.2. Root asymptotics.

We expect the poles to tend to be ultimately distributed on a fixed arc $F$ with a limit distribution $d\mu_p$, and the interpolation points on $E = [0, c]$ with a limit distribution $d\mu_i$, so that the complex potential

$$V(z) := \int_F \log \frac{1}{z-t} \ d\mu_p(t) - \int_E \log \frac{1}{z-t} \ d\mu_i(t)$$

satisfies

$$V := \text{Re} V = \text{a constant} = \rho \text{ on } E,$$

$$V(z) + \frac{A \text{Re } z}{2} = \text{a constant} = \sigma \text{ on } F,$$

$$V(z) + \frac{A \text{Re } z}{2} \text{ has equal normal derivatives on the two sides of } F,$$

$$\int_E d\mu_i(t) = \int_F d\mu_p(t) = 1$$

(conditions on $E$ and $F$), equivalent to $V$ bounded at $\infty$, actually, $V(z) \sim \text{constant } z^{-2}$ for large $z$, and $\int_C \frac{\partial V(t)}{\partial n} \ | dt| = -2\pi$ on any contour containing $F$ but not $E$, or also, that the imaginary part of $V$ increases by $\pi$ on $[0, c]$.

Conditions (51) and (52) amount to the realization that $V + A \text{Re } z/2$ has opposite gradients along the normal on the two sides of $F$:

As the derivative of an analytic function has real and imaginary parts building the gradient of its real part (Cauchy-Riemann: $\text{grad Re } F = \overline{F}$), it follows that $V + A/2$ takes opposite values on the two sides of $F$.

Now, limit values of such functions are given by Sokhotskyi-Plemelj formulas [31], chap. 14, etc.

$$V'(z) = -\int_{E \cup F} \frac{d\mu(t)}{z-t} = -\oint_{E \cup F} \frac{d\mu(t)}{z-t} \pm \pi \mu'(z)$$

when $z$ tends to a point of $E$ or $F$, and where $\oint$ is the Cauchy principal value. We therefore have

$$\int_{E \cup F} \frac{d\mu(t)}{z-t} = \oint_F \frac{d\mu_p(t)}{z-t} - \int_E \frac{d\mu_i(t)}{z-t} = \frac{A}{2}, \ z \in F,$$

which is an integral equation for the distribution $\mu_p$, to be considered with (50) as another equation for $\mu_p$ and $\mu_i$...
Now, there are various ways to go further, and to conclude with more or less neat expressions. There may be wrong turns, which may however yield a useful piece of information.

We get rid of the condition (50) by using complex Green functions \(^3\) of \(E\): first, let

\[
\varphi(z) := \frac{2z}{c} - 1 + \sqrt{\left(\frac{2z}{c} - 1\right)^2 - 1}
\]

with the square root such that \(|\varphi(z)| > 1\) for \(z \notin E\): \(\varphi\) maps \(\mathbb{C} \setminus E\) on the exterior of the unit disk, with \(\varphi(\infty) = \infty\). Remark that \(\varphi(z) + \frac{1}{\varphi(z)} = \frac{4z}{c} - 2\).

We now build \(\varphi(z,t)\), with \(\varphi(t,t) = \infty\):

\[
\varphi(z,t) = \frac{\varphi(z)\varphi(t) - 1}{\varphi(z) - \varphi(t)}, \quad t \notin E,
\]

and reconsider a formula for \(\mathcal{V}\):

\[
\mathcal{V}(z) := \int_F \log \varphi(z,t) \, d\mu(t),
\]

which automatically satisfies (50), with \(\rho = 0\), as \(\text{Re} \log \varphi(z,t) = \log |\varphi(z,t)| = 0\) when \(z \in E\).

As \(\frac{d}{dz} \log \varphi(z,t) = \frac{\varphi(t)}{\varphi(z) - \varphi(t)} = \frac{\varphi'(z)}{\varphi(z) - \varphi(t)}\),

\[
\frac{d\mathcal{V}(z)}{d\varphi(z)} = \int_C \frac{d\mu(t)}{\varphi(z) - 1/\varphi(t)} - \int_C \frac{d\mu(t)}{\varphi(z) - \varphi(t)}
\]

corresponding to charges and their images spread on \(\varphi(F)\) and \(1/\varphi(F)\) in the \(\varphi\)-plane.

?? As an argument of validity of the form (58), let us show how to recover, at least partially, the derivative of (58):

\[
\mathcal{V}'(z) = -\int_{E \cup C} \frac{d\mu(t)}{z - t}
\]

is analytic in a neighbourhood of \(\infty\), and can be written for large \(z\) as \(\mathcal{V}'(z) = \alpha z^{-1} + \beta z^{-2} + \cdots\) (actually, \(\alpha = 0\)), with the contour integrals

\[
\alpha = (2\pi i)^{-1} \int_C \varphi'(t) \, dt, \quad \beta = (2\pi i)^{-1} \int_C t \varphi'(t) \, dt, \cdots
\]

\[
\mathcal{V}'(z) = \frac{1}{2\pi i} \int_C \frac{\varphi'(t)}{z - t} \, dt
\]

d on a large contour \(C\) containing the singular loci \(E\) and \(F\), for \(z\) outside \(C\). We may as well consider the Laurent series in powers of \(\varphi(z)\)

\[
\mathcal{V}'(z) = \mathcal{V}' \left( c \frac{\varphi + 1/\varphi + 2}{4} \right) = \frac{1}{2\pi i} \int_D \frac{\mathcal{V}'(c/4)(u + 1/\varphi + 2)}{\varphi - u} \, du
\]

and we make the contour \(D\) shrink about the singular loci \(\varphi(F)\) and \(1/\varphi(F)\).

The contributions about \(\varphi(F)\) and \(1/\varphi(F)\) sum as

\[
\mathcal{V}'((c/4)(\varphi + 1/\varphi + 2)) = \frac{1}{2\pi i} \int_{\varphi(F)} \varphi' \left[ \frac{1}{\varphi - u} - \frac{1/u^2}{\varphi - 1/\varphi} \right] \, du
\]

It seems simpler to return now to the original variable \(z\), but we learned at least that \(\mathcal{V}'\) has the same singularities than \(\varphi'\) at 0 and \(c\), i.e.,

\[
\mathcal{V}'(z) \sim \text{constant } z^{-1/2}, z \to 0; \quad \varphi'(z) \sim \text{constant } (z - c)^{-1/2}, z \to c.
\]

\(^3\) used by Gonchar in several works...
We turn to a classical way to deal with the Sokhotskyi-Plemelj formulas (54)-(55) in the $z-$plane, by considering $\sqrt{z(c-c)(z-a)(z-b)}\ V'(z)$ which is meromorphic outside $F$, even holomorphic, as the product remains bounded near 0 and $c$. The product has therefore a Laurent expansion about $\infty$, say

$$\sqrt{z(z-c)(z-a)(z-b)}\ V'(z) = \delta_0 + \delta_1 z^{-1} + \cdots$$

with $\delta_k = \frac{1}{2\pi i} \int_C t^{k-1} \sqrt{t(t-c)(t-a)(t-b)}\ V'(t)\ dt$, on a contour $C$ around $F$, or

$$\sqrt{z(z-c)(z-a)(z-b)}\ V'(z) = \delta_0 + \frac{1}{2\pi i} \int_C \frac{\sqrt{t(t-c)(t-a)(t-b)}}{z-t} V'(t)\ dt$$

which, considering that $V'(t) = -A/2 \pm \pi i \mu/(t)$ on the two sides of $F$, where the square root takes opposite values, yields

$$\sqrt{z(z-c)(z-a)(z-b)}\ V'(z) = \delta_0 + \frac{A}{2\pi i} \int_a^b \frac{\sqrt{t(t-c)(t-a)(t-b)}}{z-t} dt.$$

Now, everything is known up to the three numbers $a$, $b$, and $\delta_0$! The clumsy move leaving the $\delta_0$ term comes from $V'(z) = O(z^{-2})$ at $\infty$, so that the product of $V'$ and the big square root has an unknown nonzero limit there. Multiplying by $\sqrt{z^{-1}(z-c)(z-a)(z-b)}$ instead of $\sqrt{z(z-c)(z-a)(z-b)}$ removes this problem, but introduces an unwanted residue at the origin! After several trials, I came on the– Bingo!

$$\sqrt{\frac{z(z-c)}{(z-a)(z-b)}} V'(z) = \frac{A}{2\pi i} \int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\ dt,$$

where one not only got rid of unwanted constants, but, as $V'(z)$ is only $O(z^{-2})$ at $\infty$, leaves

$$\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\ dt = 0$$

as a bonus!! (61) gives one equation for $a$ and $b$, knowing $c$ (and another equation will be worked later on, from the unit charge condition (53)). For instance, when $c = 0$, we have indeed a vanishing integral of an odd function if $a = -b$, but, as we know (or suspect) that $a$ and $b$ are complex conjugates, we see that $a$ and $b$ must be opposite pure imaginary numbers, as they are indeed in the Padé case. To work (61) a bit further, we see that it is a complete elliptic integral of the third kind (complete because one integrates on a arc joining two branchpoints; of the third kind because the incomplete integral behaves like a logarithm somewhere [near $\infty$, the square root is $1 + (a+b-c)/(2t) + \cdots$]).

A convenient transformation sending the four branchpoints 0, $c$, $a$, and $b$ on and from a symmetric set is

$$t = \frac{\alpha + iv}{1 + i\gamma v}.$$ 

So, $v = i\alpha$ is mapped on $t = 0$, one must have, for $v = -i\alpha$, $\frac{2\alpha}{1 + \gamma \alpha} = c$, and $\frac{\alpha \pm i\beta}{1 \pm i\gamma \beta} = a, b$. As neither $a$ nor $b$ is known, we may as well take $\alpha$ and $\beta$, keeping in mind that $\gamma = \frac{2}{c} - \frac{1}{\alpha}$ (for given $\alpha$, $a$ and $b$ are on a circle of diametral points $\alpha$ and $1/\gamma$).
The $z-$plane and the $v-$plane.

Now, (61) becomes
\[
\int_{-\beta}^{\beta} \frac{1}{(1 + \gamma \beta \cos \theta)^2} \sqrt{\alpha^2 + \nu^2} \, dv = 0, \text{ or, with } \nu = \beta \cos \theta,
\]
\[
\int_0^\pi \frac{\sqrt{1 + (\beta^2/\alpha^2) \cos^2 \theta}}{(1 + \gamma \beta \cos \theta)^2} \, d\theta = \int_0^\pi \frac{1 - \gamma^2 \beta^2 \cos^2 \theta}{(1 + \gamma \beta \cos \theta)^2} \sqrt{1 + (\beta^2/\alpha^2) \cos^2 \theta} = 0.
\]

The best way to study this integral is to look how it may be computed efficiently. Consider first the Fourier expansion of $(1 + \gamma \beta \cos \theta)^{-2}$:
\[
\frac{1}{(1 + \gamma \beta \cos \theta)^2} = \left[ \frac{2}{(\xi - \xi^{-1}) \gamma \beta} \left( \frac{\xi e^{-i\theta}}{1 - \xi e^{-i\theta}} + \frac{1}{1 - \xi e^{i\theta}} \right) \right]^2 = \frac{2}{(\xi - \xi^{-1})^2 \gamma \beta^2} \sum_{n=-\infty}^{\infty} \left( |p| + \frac{1 + \xi^2}{1 - \xi^2} \right) \xi^{|p|} e^{i\theta},
\]
where $\xi$ is the root of $\xi^2 + \frac{2}{\gamma \beta} \xi + 1 = 0$, with $|\xi| < 1$. The integral is therefore
\[
\frac{1}{1 + \gamma \beta^2} \sum_{n=-\infty}^{\infty} \left( 2|n| + \frac{1 + \xi^2}{1 - \xi^2} \right) \xi^{|n|} \tau_n,
\]
where $\tau_n = \int_0^\pi e^{2in\theta} \sqrt{1 + (\beta^2/\alpha^2) \cos^2 \theta} \, d\theta$, computed from the recurrence relation
\[
0 = \int_0^\pi d \left[ e^{2in\theta} \left( 1 + \frac{\beta^2}{\alpha^2} \cos^2 \theta \right)^{3/2} \right] \Rightarrow (2n + 3) \tau_{n+1} + 4n(1 + 2\alpha^2/\beta^2) \tau_n + (2n - 3) \tau_{n-1} = 0.
\]

The recurrence must be performed backwards (Miller’s algorithm), as $\tau_n$ tends to zero exponentially fast when $n \to \infty$, see Gautschi [22] for a survey of these matters. The $\tau_n$’s are particular Legendre functions, by the way. And their generating function $F(Z) = \sum_0^\infty \tau_n Z^n$ satisfies
\[
\left( Z + Z^{-1} + 2 + 4 \frac{\alpha^2}{\beta^2} \right) \frac{dF(Z)}{dZ} + \frac{Z^2 - 1}{2} F(Z) = \tau_0 + \sum_0^\infty \tau_n Z^n,
\]
i.e. $F(Z) = \left( Z + Z^{-1} + 2 + 4 \frac{\alpha^2}{\beta^2} \right)^{1/2} \int_0^Z \tau_0 + 3 \tau_1 Y \frac{2Y}{2Y^2} (Y + Y^{-1} + 2 + 4\alpha^2/\beta^2)^{-3/2} \, dY$, a way to see how complete elliptic integrals of the third kind turn into incomplete elliptic of first and second kind. Indeed, the integral is $G(\xi^2) = 4\xi^2 F'(\xi^2) + \frac{1 + \xi^2}{1 - \xi^2} [2F(\xi^2) - \tau_0]$, and it satisfies
\[
\frac{d}{dZ} \left[ \frac{1-Z}{1+Z} \left( Z + Z^{-1} + 2 + 4\alpha^2/\beta^2 \right) G(Z) \right] + \frac{Z^2 - 1}{2} \frac{1-Z}{1+Z} G(Z) = \frac{(1-Z)^2}{2Z(1+Z)^2} [6\tau_1 - (Z + 4Z^{-1}) \tau_0],
\]
\[
(Z + Z^{-1} + 2 + 4\alpha^2/\beta^2)^{1/2} [1-Z] \frac{d}{dZ} G(Z) = \int_0^{Z} (t + t^{-1} + 2 + 4\alpha^2/\beta^2)^{-1/2} \frac{(1-t)^2}{2t(1+t)^2} [6\tau_1 - (t + 4t^{-1}) \tau_0] \, dt
\]
\[
= -i \int_{-\infty}^{(2i)^{-1} \log Z} \frac{\tan^2 \theta}{\sqrt{\cos^2 \theta + \alpha^2/\beta^2}} [3\tau_1 - (1 + 2\cos^2 \theta) \tau_0] \, d\theta \quad (t = e^{2\theta})
\]
For each $c$, the locus of $a$ is a curve with vertical asymptote of abscissa $c/2$, and of tangent at the origin matching the $c = \infty$ locus, given by $\arg a = -0.860274\ldots$ (see [48, end of § 3.2]).

Constants already encountered appear as similar integrals, as

$$
y_0 = \lim_{z \to \infty} z^2 P'(z) = \frac{A}{2 \pi i} \int_a^b \frac{t(t-c)}{(t-a)(t-b)} \, dt
$$

using \(z^2 = z + t + O(z^{-1})\) in (60). More integrals appear in a sequence of transformations needed in order to get a convenient incomplete elliptic integral form for $P'$. We first multiply (60) by $z(z-c)(z-a)(z-b)$:

$$
\sqrt{z^2(z-c)^3(z-a)(z-b)} \frac{P'(z)}{z-t} = \frac{A}{2 \pi i} \int_a^b \frac{t(t-c)(t-a)(t-b)}{z-t} \, dt
$$

using \(\frac{z(z-c)(z-a)(z-b)}{z-t} = (z-c)(z-a)(z-b) + t(z-t) + (t-a)(t-b) + (t-c) + (a+b+c)\), and where

$$
y_k = \frac{A}{2 \pi i} \int_a^b t^{k+1} \frac{t(t-c)}{(t-a)(t-b)} \, dt
$$

Now, we take the derivative in $z$, and perform an integration by parts in the integral:

$$
\sqrt{P(z)} \frac{P'(z)}{2 \sqrt{P(z)}} P''(z) = 2\gamma_0 z + \gamma_1 - (a+b+c) \gamma_0 - \frac{A}{2 \pi i} \int_a^b \frac{P'(t)}{2 \sqrt{P(t)}} \frac{dt}{z-t}
$$

where $P(z) = z^3(z-c)^3(z-a)(z-b)$. Now, \(\frac{P'(t)}{2 \sqrt{P(t)}}\) is \(\frac{t(t-c)}{(t-a)(t-b)}\) (well, who’s there!) times \(4t^3 - (7a + 7b + 5c)t^2/2 + (3ab + 3ac + 2bc)t - 3abc/2)/z-t\), so that the integral is $-4\gamma_0 z - 4\gamma_1 + (7a + 7b + 5c)\gamma_0/2$ what turns out to be just $\frac{P'(z)}{2 \sqrt{P(z)}} \frac{q''(z)}{z-t}$, so,

$$
\sqrt{P(z)} \frac{P'(z)}{2 \sqrt{P(z)}} P''(z) = -2\gamma_0 z - \gamma_1,
$$

where $\gamma_1 = 3\gamma_1 - (5a + 5b + 3c)\gamma_0/2$. 

Some points of the locus:

<table>
<thead>
<tr>
<th>$a/c = \overline{b}/c$</th>
<th>$\alpha/c$</th>
<th>$\beta/c$</th>
<th>$\gamma/c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.023671 − 0.0280i</td>
<td>−0.0390</td>
<td>0.0809</td>
<td>27.614</td>
</tr>
<tr>
<td>0.061784 − 0.0757i</td>
<td>−0.1159</td>
<td>0.2206</td>
<td>10.628</td>
</tr>
<tr>
<td>0.091190 − 0.1152i</td>
<td>−0.1910</td>
<td>0.3385</td>
<td>7.2343</td>
</tr>
<tr>
<td>0.114713 − 0.1488i</td>
<td>−0.2647</td>
<td>0.4412</td>
<td>5.7773</td>
</tr>
<tr>
<td>0.134055 − 0.1781i</td>
<td>−0.3371</td>
<td>0.5327</td>
<td>4.9659</td>
</tr>
<tr>
<td>0.150306 − 0.2040i</td>
<td>−0.4084</td>
<td>0.6156</td>
<td>4.4482</td>
</tr>
<tr>
<td>0.204616 − 0.3022i</td>
<td>−0.7512</td>
<td>0.9493</td>
<td>3.3310</td>
</tr>
<tr>
<td>0.241306 − 0.3830i</td>
<td>−1.1399</td>
<td>1.2531</td>
<td>2.8772</td>
</tr>
<tr>
<td>0.358549 − 0.8401i</td>
<td>−6.3630</td>
<td>3.7085</td>
<td>2.1571</td>
</tr>
<tr>
<td>0.406333 − 1.3084i</td>
<td>−20.292</td>
<td>7.8098</td>
<td>2.0487</td>
</tr>
<tr>
<td>0.425044 − 1.6475i</td>
<td>−39.115</td>
<td>11.848</td>
<td>2.0255</td>
</tr>
<tr>
<td>0.439587 − 2.0532i</td>
<td>−73.476</td>
<td>17.878</td>
<td>2.0136</td>
</tr>
<tr>
<td>0.445607 − 2.2839i</td>
<td>−100.04</td>
<td>21.890</td>
<td>2.0099</td>
</tr>
</tbody>
</table>
As remarked by J. Nuttall (22 Oct. 1999), we could have adapted Gonchar and Rakhmanov [25], and use the form

\[ \eta''(z) = \frac{\text{constant} + \text{constant} \cdot z}{\sqrt{z^2(z-c)^3(z-a)(z-b)}} \]

but without knowledge of the two constants…

Check: from (60). \( \left(1 + \frac{a+b-c}{2z} + \cdots\right) \eta''(z) = \frac{\gamma_0}{z^2} + \gamma_1 z^3 + \cdots, \eta''(z) = \frac{\gamma_0}{z^2} + 2\gamma_1 + (c-a-b)\gamma_0 2z^3 + \cdots, \)

\[ \sqrt{P(z)}\eta''(z) = \left(z^4 - \frac{a+b+3c}{2}z^3 + \cdots\right) \left(-2\frac{\gamma_0}{z^2} - 3\frac{2\gamma_1 + (c-a-b)\gamma_0}{2z^4} + \cdots\right) = -2\gamma_0 z - \gamma_1. \]

\[ \eta''(z) = -\int_{\infty}^{z} \frac{(\gamma_1 + 2\gamma_0 t)dt}{\sqrt{t^3(t-c)^3(t-a)(t-b)}}, \]

\[ \eta'(z) = \text{constant} - \int_{\infty}^{z} \frac{(z-t)(\gamma_1 + 2\gamma_0 t)dt}{\sqrt{t^3(t-c)^3(t-a)(t-b)}} \]

(63)

are elliptic integrals of first and second kind (there is no more logarithmic behaviour). The standard forms of the elliptic integrals of 1\textsuperscript{st} and 2\textsuperscript{nd} kinds are [55]

\[ F(x,k^2) = \int_0^{\text{arcsin} x} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta = \int_0^x \left(1 - u^2\right)\left(1 - k^2 u^2\right)^{-1/2} du, \]

\[ E(x,k^2) = \int_0^{\text{arcsin} x} (1 - k^2 \sin^2 \theta)^{1/2} d\theta = \int_0^x \left(1 - u^2\right)^{1/2} (1 - k^2 u^2)^{1/2} du. \]

Elementary change of variable will not easily lead to these forms, but what is closest to our needs appears to be [55, 17.4.51]

\[ F(y,k^2) = (\alpha^2 + \beta^2)^{1/2} \int_0^x \frac{dv}{\left[(v^2 + \alpha^2\beta^2 - v^2)^{1/2}\right]}, \]

\[ E(y,k^2) = \alpha^2 (\alpha^2 + \beta^2)^{1/2} \int_0^x \frac{1}{v^2 + \alpha^2 \beta^2 \left[(v^2 + \alpha^2\beta^2 - v^2)^{1/2}\right]}, \]

where \( k^2 = \beta^2/(\alpha^2 + \beta^2), y = \frac{x^2 (\alpha^2 + \beta^2)}{\beta^2(\alpha^2 + x^2)}. \) One may check that \( dF/dy \) and \( dE/dy \) are what they should be (using \( 1 - y^2 = \frac{\alpha^2 (\beta^2 - x^2)}{\beta^2(\alpha^2 + x^2)} \) and \( 1 - k^2 y^2 = \frac{\alpha^2}{\alpha^2 + x^2} \)).

Also, for complete elliptic integrals,

\[ 0 = \int_{-\beta}^{\beta} dv \left[ (\alpha^2 + v^2)^{p+1/2} (\beta^2 - v^2)^{1/2} \right] \Rightarrow \]

\[ \int_{-\beta}^{\beta} \left(\alpha^2 + v^2\right)^p \frac{-2p+3}{(v^2 + \alpha^2)^2 + 2(p+1)(\beta^2 + 2\alpha^2)(v^2 + \alpha^2) - (2p+1)\alpha^2(\alpha^2 + \beta^2)} \sqrt{(v^2 + \alpha^2)(\beta^2 - v^2)} dv = 0. \]

(65)

In particular, with \( p = -1 \), we find two equivalent formulas for the complete elliptic integral of second kind:

\[ E = \alpha^2 (\alpha^2 + \beta^2)^{1/2} \int_0^\beta \frac{1}{v^2 + \alpha^2 \beta^2} \left[\left(v^2 + \alpha^2\right)(\beta^2 - v^2)\right]^{1/2} dv = (\alpha^2 + \beta^2)^{-1/2} \int_0^\beta \frac{v^2 + \alpha^2}{\beta^2 - v^2}^{1/2} dv. \]

We need integrals

\[ \int_{\infty}^z R(t) \frac{dt}{\sqrt{t(t-c)(t-a)(t-b)}}, \]

(66)
where $R$ is a rational function with at most simple poles at 0 and/or $c$. One finds

$$
\sqrt{(1 + \gamma^2 \beta^2)} \frac{1 + \gamma \alpha}{1 - \gamma \alpha} \int_{1/\gamma}^{\nu(z)} \frac{R}{1 + iv} \frac{dv}{\sqrt{\alpha^2 + \nu^2} (\beta^2 - \nu^2)},
$$

where $x = v(z) = \frac{\alpha - z}{1 - \gamma z}$.

So, the constant and the simple fractions $1/t$ and $1/(t - c)$ lead to the (indefinite) integrals

$$
R(t) = 1: \frac{1}{\sqrt{(1 + \gamma^2 \beta^2)} (1 + \gamma \alpha)} \left\{ \begin{array}{l}
\gamma F(x, k^2) + i \frac{1 - \alpha \gamma}{\sqrt{\alpha^2 + \beta^2}} \sqrt{\frac{\beta^2 - x^2}{\alpha^2 + \beta^2}} \sqrt{\frac{1 - (1 + \alpha \gamma) E(y, k^2)}{\alpha}}
\end{array} \right,
$$

$$
R(t) = \frac{1}{t}: \frac{1}{\sqrt{(1 + \gamma^2 \beta^2)} (1 + \gamma \alpha)} \left\{ \begin{array}{l}
\gamma F(x, k^2) + i \frac{1 + \alpha \gamma}{\sqrt{\alpha^2 + \beta^2}} \sqrt{\frac{\beta^2 - x^2}{\alpha^2 + \beta^2}} \sqrt{\frac{1 + (1 + \alpha \gamma) E(y, k^2)}{\alpha}}
\end{array} \right,
$$

$$
R(t) = \frac{1}{t - c}: \frac{1}{\sqrt{(1 + \gamma^2 \beta^2)} (1 + \gamma \alpha)} \left\{ \begin{array}{l}
\gamma F(x, k^2) + i \frac{1 - \alpha \gamma}{\sqrt{\alpha^2 + \beta^2}} \sqrt{\frac{\beta^2 - x^2}{\alpha^2 + \beta^2}} \sqrt{\frac{1 - (1 + \alpha \gamma) E(y, k^2)}{\alpha}}
\end{array} \right,
$$

For $V^\alpha(z)$, $R(t) = \frac{\gamma_1 + 2 \gamma_0 t}{t(t - c)} = \frac{\gamma_1 / c}{t} - \frac{\gamma_1 / c}{t - c}$,

$$
V^\alpha(z) = \text{constant} + \frac{1}{\sqrt{(1 + \gamma^2 \beta^2)} (1 + \gamma \alpha)} \left\{ \begin{array}{l}
\gamma \left[ \gamma_1 (1 - \gamma \alpha) - \gamma_1 (1 + \gamma \alpha) \right] F(y, k^2)
\end{array} \right.
$$

$$
+ \frac{i}{c \sqrt{\alpha^2 + \beta^2}} \left[ \gamma_1 (1 - \gamma \alpha)^2 - \gamma_1 (1 + \gamma \alpha)^2 \right] \sqrt{\frac{\beta^2 - x^2}{\alpha^2 + \beta^2}}
$$

$$
+ \left[ \gamma_1 (1 - \gamma \alpha)^2 + \gamma_1 (1 + \gamma \alpha)^2 \right] \frac{E(y, k^2)}{\alpha c} \right).
$$

Remark that everything but the constant vanishes in (67) at $x = \pm \beta$, i.e., at $z = a$ and $z = b$, so that this constant is $V^\alpha(a) = V^\alpha(b) = -A / 2$.

The function $V^\alpha$ must be single-valued in $C \setminus \{ E \cup F \}$, i.e., have vanishing periods about the sets $E$ and $F$.

$$
\gamma \left[ -\gamma_1 (1 - \gamma \alpha) - \gamma_1 (1 + \gamma \alpha) \right] K + \left[ \gamma_1 (1 - \gamma \alpha)^2 + \gamma_1 (1 + \gamma \alpha)^2 \right] E / \alpha = 0,
$$

What a mess! Wait! The integrals $\gamma_0$ and $\gamma_1$ entering $\gamma_1$ and $\gamma_1'$ in (68) are complete elliptic of first and second kind too: (62) and (61) mean that $\gamma_{-1} = 0$, which will be useful in the calculation of the subsequent $\gamma_k$'s, from the recurrence relation

$$(k + 2) \gamma_k - [k(a + b + c) + 3(a + b)/2 + c/2] \gamma_{k-1} + [k(ab + ac + bc) + ab] \gamma_{k-2} - (k - 1/2) abc \gamma_{k-3} = 0, \quad (69)$$

found by

$$
0 = \int_a^b d[t^{k - 1/2}(t - c)^{3/2}(t - a)^{1/2}(t - b)^{1/2}]
$$

$$
\int_a^b \frac{t(t - c)}{t - a} (t - b) [(k - 1/2) (t - a)(t - b) + 3t(a - t)(t - b)/2 + t(t - c)(t - b)/2 + t(t - c)(t - a)/2]dt = 0.
$$

When $k$ is negative enough, $\gamma_k$ in (62) is immediately an elliptic integral of second kind at most:

$$
\gamma_k = \frac{A}{2 \pi i} \int_a^b \frac{t(t - c)}{(t - a)(t - b)} dt = \frac{A}{2 \pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \int_{-\beta}^{\beta} \frac{(\alpha + iv)^{k+1}}{(\beta^2 - v^2)^{k+3/2}} dv
$$
So,
\begin{align*}
\gamma_4 &= \frac{A}{2\pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \int_{-\beta}^{\beta} \frac{1 + i\gamma v}{\sqrt{(\alpha + iv)^2 - \beta^2}} \sqrt{\frac{1}{v^2 + \alpha^2}} d\nu \\
&= \frac{A}{2\pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \int_{-\beta}^{\beta} \frac{(1 + i\gamma v)(\alpha - iv)^3}{(v^2 + \alpha^2)^2(\beta^2 - v^2)} d\nu \\
&= \frac{A}{2\pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \int_{-\beta}^{\beta} -\gamma^4 + 3\alpha(\alpha \gamma - 1)v^2 + \alpha^3 d\nu \\
&= \frac{A}{2\pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \int_{-\beta}^{\beta} \frac{[(\alpha^2 - 3\beta^2)\gamma - 4\alpha][v^2 + \alpha^2] + \alpha[\alpha(\gamma(7\beta^2 - \alpha^2) + 7\gamma^2 - \beta^2)]}{3(\alpha^2 + \beta^2)^{3/2}} d\nu \\
&= \frac{A}{2\pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \frac{3(\alpha^2 + \beta^2)^{3/2}}{\sqrt{\alpha^2 + \beta^2}} \\
\gamma_3 &= \frac{A}{2\pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \int_{-\beta}^{\beta} \frac{(\alpha - iv)^2}{\alpha^2 + v^2} \frac{d\nu}{(v^2 + \alpha^2)(\beta^2 - v^2)} \\
&= \frac{A}{2\pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \sqrt{\frac{4E - 2K}{\alpha^2 + \beta^2}} \\
The relation (69) allows to find the next $\gamma_k$’s… excepting $\gamma_2$, which is precisely the first elliptic integral of third kind. However, knowing that $\gamma_{-1} = 0$, (69) at $k = -1$ actually gives
\begin{align*}
\gamma_2 &= \frac{\gamma_{-3} + 3ab\gamma_{-4}}{a + b - c},
\end{align*}
amounting to
\begin{align*}
\gamma_2 &= \frac{A}{2\pi} \sqrt{\frac{(1 + \gamma^2 \beta^2)(\alpha \gamma - 1)}{\alpha \gamma + 1}} \frac{2\alpha K - 2(\alpha + 1/\gamma)E}{\sqrt{\alpha^2 + \beta^2}}.
\end{align*}
For $\gamma_0$, use (69) at $k = 0$: $\gamma_0 = -ab(c\gamma_3 + 2\gamma_{-2})/4$.
\begin{align*}
\gamma_0 &= \frac{A}{2\pi} \sqrt{\frac{(\alpha \gamma - 1)(\alpha^2 + \beta^2)}{(\alpha \gamma + 1)^3(1 + \gamma^2 \beta^2)}} \left[ \frac{\alpha^2(\gamma(1 - E)/\gamma)}{\alpha \gamma} \right] \\
\gamma_1 &= \frac{(5(a + b) + 3c)\gamma_0}{6} + abc\gamma_{-2}/2, \gamma_1' = \gamma_1 + 2c\gamma_0 = -abc(\gamma_{-2} + c\gamma_{-3})/2.
\end{align*}
and finally, $\gamma_1 = \frac{(5(a + b) + 3c)\gamma_0}{6} + abc\gamma_{-2}/2, \gamma_1' = \gamma_1 + 2c\gamma_0 = -abc(\gamma_{-2} + c\gamma_{-3})/2$.
\begin{align*}
\gamma_1 &= \frac{A}{2\pi} \sqrt{\frac{(\alpha \gamma - 1)(\alpha^2 + \beta^2)}{(\alpha \gamma + 1)^3(1 + \gamma^2 \beta^2)}} 2\alpha^2 \left[ K - E - \frac{E}{\alpha \gamma} \right] \\
\gamma_1' &= \frac{A}{2\pi} \sqrt{\frac{(\alpha \gamma - 1)(\alpha^2 + \beta^2)}{(\alpha \gamma + 1)^3(1 + \gamma^2 \beta^2)}} \frac{1 - \alpha \gamma}{1 + \alpha \gamma} 2\alpha^2 \left[ K - E + \frac{E}{\alpha \gamma} \right],
\end{align*}
and (67) becomes
\begin{align*}
\varphi(z) &= -\frac{A}{2} + \frac{A}{\pi} \left\{ \frac{KE(y, k^2) - EF(y, k^2)}{K - E} - \frac{i\alpha^2 c}{\sqrt{\alpha^2 + \beta^2}} \left[ \frac{2\alpha \gamma}{\alpha \gamma(1 - \alpha \gamma)} E \right] \sqrt{\frac{\beta^2 - x^2}{x^2 + \alpha^2}} \right\}
\end{align*}
which confirms (68).
For \( \Psi \), use (63) with the discussion following (66): \( R(t) = \frac{(t-z)(\gamma'_1 + 2\gamma_0)}{t(t-c)} = 2\gamma_0 + \frac{\gamma'_1}{c} - \frac{(z-c)\gamma'_1}{t-c}, \)

\[
\Psi'(z) = \text{constant} + \sqrt{\frac{(1 + \gamma^2 \beta^2)(1 + \gamma \alpha)}{(1 - \gamma \alpha)^3(\alpha^2 + \beta^2)}} \left\{ 2\gamma_0(1 - \gamma \alpha)F(y,k^2) + \frac{\gamma}{c} \left[ \gamma'_1(1 - \gamma \alpha) - (z-c)\gamma'_1(1 + \gamma \alpha) \right] F(y,k^2) \right.
\]
\[
+ \frac{i}{c\sqrt{\alpha^2 + \beta^2}} \left[ \gamma'_1(1 - \gamma \alpha)^2 - (z-c)\gamma'_1(1 + \gamma \alpha)^2 \right] \frac{\sqrt{\beta^2 - x^2}}{x^2 + \alpha^2} \]
\[
+ \left[ \gamma'_1(1 - \gamma \alpha)^2 + (z-c)\gamma'_1(1 + \gamma \alpha)^2 \right] \frac{E(y,k^2)}{\alpha^2} \right\}. \tag{70}
\]

Now we find, at last, a second equation for \( a \) and \( b \), from \( \Psi'(b) - \Psi'(a) = i\pi \): at \( z = a \) and \( z = b \), \( x = \pm \beta, F = \pm K, E = \pm E, \)

\[
\Psi'(a) \text{ and } \Psi'(b) = \text{const.} \pm \frac{iA}{2\pi} \left\{ \gamma \left[ K - E - \frac{E}{\alpha^2 \gamma} \right] (1 - \gamma \alpha)K \right.
\]
\[
+ z(1 - \gamma \alpha)2\alpha^2 \left[ K - E - \frac{E}{\alpha^2 \gamma} \right] \frac{\gamma}{c} \left[ K - E + \frac{E}{\alpha^2 \gamma} \right] \]
\[
- (z-c)(1 + \gamma \alpha)2\alpha^2 \left[ \frac{1 - \alpha \gamma}{1 + \alpha \gamma} \left[ K - E + \frac{E}{\alpha^2 \gamma} \right] \frac{\gamma}{c} \left[ K - E - \frac{E}{\alpha^2 \gamma} \right] \right) \}
\]
\[
= \text{const.} \pm \frac{iA\gamma}{2\pi} \left\{ \left[ \alpha \gamma(K - E) - \frac{E}{\alpha^2 \gamma} \right] K - \left[ \alpha \gamma(K - E)^2 - \frac{E^2}{\alpha^2 \gamma} \right] \right\},
\]
so that, \( \Psi'(b) - \Psi'(a) = i\pi \Rightarrow \)

\[
Ac = \frac{\pi^2}{\left( \frac{1}{\alpha \gamma} - \alpha \gamma \right) E(K - E)} \tag{71}
\]

which, together with (61), gives \( \alpha, \gamma, K, \) etc., from \( Ac \) (no wonder that everything depends essentially on the product \( Ac \): remember that we approximate \( \exp(-nAc) \) on \( 0 \leq z \leq c \), equivalent up to a scaling to the approximation of \( \exp(-nAc) \) on \( [0, 1] \).

Check: when \( c \to \infty, \alpha \gamma = -1 + 2\alpha/c \to -1, 1/(\alpha \gamma) - \alpha \gamma \sim -4\alpha/c, \) and we should check that \( -4\alpha \gamma \left[ K - E \right] \to \pi^2. \) Yes: \( |a|^2 = ab = \frac{\alpha^2 + \beta^2}{1 + \gamma \beta^2} \to \alpha^2, \) and we know that \( E \to K/2 \) when \( c \to \infty, \) so that the limit of (71) is \( |a|K^2 = \pi^2, \) confirmed by \( |\xi|K = \pi \) in [46, § 2] (where \( \xi = \sqrt{-\alpha} \), and by \( |a| = \pi/\alpha \) of [48, § 3.4, eq. (34)].

For \( \Psi'(0) \) and \( \Psi'(c) \), it is better to show \( \Psi' \) to be bounded at 0 and \( c \) by adapting (65) to incomplete elliptic integrals:

\[
E(y,k^2) = \frac{x}{\sqrt{\alpha^2 + \beta^2}} \sqrt{\frac{\beta^2 - x^2}{\alpha^2 + x^2}} + \frac{1}{\sqrt{\alpha^2 + \beta^2}} \int_0^x \sqrt{\frac{\alpha^2 + v^2}{\beta^2 - v^2}} dv,
\]

and the two last lines of (70) become

\[
\frac{i}{c\alpha(1 + i\gamma \alpha)} \left[ \gamma'_1(1 - \gamma \alpha) + \gamma'_1(1 + \gamma \alpha) \right] \sqrt{\frac{(\alpha^2 + x^2)(\beta^2 - x^2)}{\alpha^2 + \beta^2}} + \frac{\gamma'_1(1 - \gamma \alpha)^2 + (z-c)\gamma'_1(1 + \gamma \alpha)^2}{\alpha c \sqrt{\alpha^2 + \beta^2}} \int_0^x \sqrt{\frac{\alpha^2 + v^2}{\beta^2 - v^2}} dv,
\]

using \( z = \frac{\alpha + ix}{1 + i\gamma \alpha} \) and \( z - c = z - \frac{2\alpha}{1 + \gamma \alpha} = \frac{(\gamma \alpha - 1)(\alpha - ix)}{(1 + \gamma \alpha)(1 + i\gamma \alpha)} \)
$\mathcal{V}'(c) - \mathcal{V}'(0) = \pi i$: at $z = 0$ and $z = c$, $x = i(\alpha - z)/(1 - \gamma z) = \pm i\alpha$, where

$$F(y, k^2) = \sqrt{\alpha^2 + \beta^2} \int_0^{\pm i\alpha} \frac{dv}{(v^2 + \alpha^2)(\beta^2 - v^2)} = \pm i\sqrt{\alpha^2 + \beta^2} \int_0^{\alpha} \frac{dw}{(\alpha^2 - w^2)(\beta^2 + w^2)} = \pm iK',$$

as $\alpha$ and $\beta$ have been interchanged in $K$.

Also,

$$- \frac{1}{\sqrt{\alpha^2 + \beta^2}} \int_0^{\pm i\alpha} \frac{\alpha^2 + \beta^2 - (\beta^2 - v^2)}{\sqrt{\beta^2 - v^2}} dv = - \frac{1}{\sqrt{\alpha^2 + \beta^2}} \int_0^{\alpha} \frac{\alpha^2 + \beta^2}{\sqrt{(\alpha^2 - w^2)(\beta^2 + w^2)}} dw = \pm i(K' - E').$$

What a mess again. Well, the important thing now is the rate of decrease of the error with $n$, which is $\rho = \exp\{-2[\text{Re}(\mathcal{V}'(z) + Az/2) - F - \text{Re} \mathcal{V}'(z) on E]\}$. From $\mathcal{V}'(a) + A/2 = 0$, one has $\mathcal{V}'(z) + Az/2 = - \int_a^z 2\gamma_0 dt = \frac{\gamma_0}{1 - \gamma} - 2\gamma_0(t - c) dt$, $\mathcal{V}'(z) + Az/2 = \text{const} - \int_a^z (z - t) \frac{\gamma_0}{1 - \gamma} - 2\gamma_0(t - c) dt$, and what is needed is the complete elliptic integral

$$\log \rho = 2 \text{Re} [\mathcal{V}'(0) - \mathcal{V}'(a) - Aa/2] = -2 \text{Re} \int_0^{\alpha} \frac{\gamma_0}{1 - \gamma} - 2\gamma_0(t - c) \sqrt{t(1 - t)} dt,$$

which turns as

$$\log \frac{1}{\rho} = \pi \frac{\alpha \gamma(K - E)(K' - E') - EE'}{(\alpha \gamma - 1)E(K - E)}$$  \hspace{1cm} (72)$$

The Legendre relation $EE' - (K - E)(K' - E') = \pi/2$ may be useful here, but does not give a much nicer formula.

Check when $c \to \infty$: $\alpha \gamma \to 1$ and $K - 2E \to 0$, and we recover the $\rho = \exp(-\pi K'/K)$ of [45, 46, 48]!

In the limit case $c \to 0$, $\mathcal{V}' \sim \mathcal{V}'_{p,Padé} - \mathcal{V}'_{[0,c]}$, where $\mathcal{V}'_{p,Padé}$ is the potential of the poles of the Padé approximant, and where $\mathcal{V}'_{[0,c]}$ is the equilibrium potential of $[0,c]$:

$$\mathcal{V}'(z) \sim - \log \left(1 + \sqrt{1 + \frac{A^2 z^2}{4}} + \sqrt{1 + \frac{A^2 z^2}{4}} - \frac{A z}{2} + \log \left[\frac{2z}{c} - 1 + \sqrt{\left(\frac{2z}{c} - 1\right)^2 - 1}\right]\right)$$

where the square roots behave like $Az/2$ and $2z/c$ for large $z$. Then,

$$\rho \sim \exp(2 \text{Re} [\mathcal{V}'(0) - \mathcal{V}'(a) - Aa/2]) \sim \exp \left(2 - 2\log 2 - 2\log \frac{8}{Ac} \right) = \left(\frac{eAc}{16}\right)^2.$$

Some instances:
The modulus of the elliptic integrals is \( k = \beta / \sqrt{\alpha^2 + \beta^2} \). The limit value when \( Ac \to \infty \) is 0.90890856. Why don’t we have a “decent” limit for \( e^{-nAc} \) times \( nAz \) with \( z \leftrightarrow c - z \), so that \( a, b \leftrightarrow c - a, c - b, \) etc. (yes, but what about \( k \)?)

5.3. **Strong asymptotics**.

Consider rational approximants to functions \( f^n g \), and suppose that the Hermite-Walsh error formula can already be written as

\[
f^n(z) g(z) = \frac{p_n(z)}{q_n(z)} \sim e^{W_n(z)} \frac{1}{2\pi i} \int_C f^n(t) g(t) e^{-W_n(t)} \frac{dt}{z-t},
\]

where \( W_n \) is a “smoothed” approximation of the discrete potential created by the poles and the interpolation points. The function \( \exp(W_n) \) (corresponding to Nuttall’s \( \chi_1 \) and/or \( \chi_2 \) [50, 61, 62]) has branch points, even if \( f \) and \( g \) are entire. What is this function? The influence of \( f \) is overwhelming in the determination of the branchpoints and other main features when \( n \) is large. So, we solve first with \( f \), and find the active part \( F \subset C \) and the main behaviour (\( \exp(W_n) \)) \( \rightarrow \exp(2\psi_f) \) (root asymptotics, also called zero order asymptotics by Nuttall).

Aptekarev [4] established in some cases a more accurate picture \( W_n = 2n\psi + \tilde{\psi} + o(1) \) (strong asymptotics, also called first order asymptotics by Nuttall). I give here a probably very sloppy account of Aptekarev’s wonderful results (to be available soon):

<table>
<thead>
<tr>
<th>( Ac )</th>
<th>( a/c = \bar{b}/c )</th>
<th>( \alpha/c )</th>
<th>( \beta/c )</th>
<th>( \gamma/c )</th>
<th>( K )</th>
<th>( E )</th>
<th>( K' )</th>
<th>( E' )</th>
<th>( 1/\rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \infty )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \infty )</td>
<td>2.32105</td>
<td>1.16052</td>
<td>1.64669</td>
<td>1.50011</td>
<td>9.2890255</td>
</tr>
<tr>
<td>0.11626 – 0.15112i</td>
<td>–0.27010</td>
<td>0.44836</td>
<td>5.70233</td>
<td>2.12833</td>
<td>1.22121</td>
<td>1.69467</td>
<td>1.46033</td>
<td>( 1/1/9i )</td>
<td></td>
</tr>
<tr>
<td>0.21759 – 0.32920i</td>
<td>–0.86874</td>
<td>1.04722</td>
<td>3.15109</td>
<td>1.94139</td>
<td>1.30256</td>
<td>1.78286</td>
<td>1.39577</td>
<td>12.43300</td>
<td></td>
</tr>
<tr>
<td>0.37281 – 0.94489</td>
<td>–8.58507</td>
<td>4.47929</td>
<td>2.11648</td>
<td>1.66674</td>
<td>1.48306</td>
<td>2.22657</td>
<td>1.18782</td>
<td>23.22870</td>
<td></td>
</tr>
<tr>
<td>0.43707 – 1.96963i</td>
<td>–65.1743</td>
<td>16.5290</td>
<td>2.01534</td>
<td>1.59537</td>
<td>1.54679</td>
<td>2.81732</td>
<td>1.07039</td>
<td>57.069968</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.46869 – 3.98448i</td>
<td>–514.5877</td>
<td>64.57005</td>
<td>2.00194</td>
<td>1.57694</td>
<td>1.56469</td>
<td>3.47937</td>
<td>1.02313</td>
<td>177.934379</td>
</tr>
</tbody>
</table>
Also sprach Aptekarev: \( \tilde{\psi} \) is (multivalued) analytic outside \( E \cup F \), with a period \( 2\pi i \) about \( F \), and \(-2\pi i\) about \( E \), corresponding to a positive unit charge on \( F \), and a negative unit charge on \( E \), with \( \tilde{\psi}_+ + \tilde{\psi}_- \) constant on \( E \), \( \tilde{\psi}(z)_+ + \tilde{\psi}(z)_- + 210g(z) = \) another constant on \( F \), and finally \( \tilde{\psi}(z) = \) const. + \( o(1) \) when \( z \to \infty \) (if \( E \) and \( F \) are bounded).

Moreover, the error norm is \( E_n \sim 2p^n \tilde{\rho} \), where \( 210g(z) = \) Re \( \{ (\tilde{\psi}_+(z) + \tilde{\psi}_-(z))_E - [\tilde{\psi}_+(z) + \tilde{\psi}_-(z) + 210g(z)]_F \} \).

This means also that \( \tilde{\psi}^h \) is analytic outside \( E \) and \( F \), taking opposite values on the two sides of \( E \), and with \( \tilde{\psi}^h + \tilde{g}(z)/\tilde{g}(z) \) taking opposite values on the two sides of \( F \).

**Important special case**: if \( g = \sqrt{f} \), the conditions on \( \tilde{\psi} \) are exactly the conditions (50)-(53) which we already saw for \( \psi \) itself! So, \( \tilde{\psi} = \psi \), and \( \tilde{\rho} = \sqrt{\rho} \) in this case.

Remark: the real part of \( \tilde{\psi}' + \log g \) need not, and will normally not be a constant on \( F \). However, the cut on which the boundary conditions for \( \tilde{\psi}' \) are set may be modified (keeping the endpoints as the endpoints of \( F \), and one may dream to find the locus \( \tilde{\tilde{F}} \) where \( \tilde{\psi}' + \log g \) has a constant real part. The use and even the existence of \( \tilde{\tilde{F}} \) seem questionable (Aptekarev). It may be wiser and more useful to look for a locus \( \tilde{F}_n \) where the whole complex potential \( \psi_n = 2n\psi' + \tilde{\psi}' + n\log f + \log g \) has a constant real part, as this locus may be a fair approximation to the set of poles for a given value of \( n \) (Nuttall).

**Application to best approximation to** \( \exp(-(nA + B)z) \) on \([0, c]\):

\( E_n \sim 2p^n \rho_B \), where \( 210g_B = \) Re \( \{ (\psi_{R,+}(z) + \psi_{R,-}(z))_E - [\psi_{R,+}(z) + \psi_{R,-}(z) + 2B_2]_F \} \). \( \psi_{R}^e = \tilde{\psi}^e \)

being analytic outside \( E \cup F \), taking opposite values on the two sides of \( E \) = \([0, c]\), \( \psi_n^e(z) + B \) taking opposite values on the two sides of \( F \), or any arc of endpoints \( a \) and \( b \), and corresponding to a positive unit charge on \( F \), and a negative unit charge on \( E \), and finally \( \psi_n^e(z) = \) const. \( z^{-2} + \cdots \) when \( z \to \infty \).

The problem is solved by \( \psi_{A/2} = \psi' \) if \( B = A/2 \).

And if \( B = 0 \)? Then, \( \psi_0' \) is the simple algebraic function \( \psi_0'(z) = \frac{\text{constant}}{\sqrt{(z-c)(z-a)(z-b)}} \) associated to the potential of a plain (and plane) condenser \((E, \tilde{\tilde{F}})\), although we do not need to know what \( \tilde{\tilde{F}} \) is. The capacity is \( 2K/(\pi K') \), and \( \rho_0 = \exp \left(-\frac{\pi K}{2} \right) \).

And for any \( B \),

\[
\psi_B' = \frac{2B}{A} \psi' + \left(1 - \frac{2B}{A}\right) \psi_0'
\]

(73)
does the trick, see Meinguet [53] for such relations. So,

\[
\rho_B = \rho_B^{(1-2B/A)}
\]

and we just have to get \( \rho_0 = \exp(-1/C) \), where \( C \) is the plain condenser capacity of \((E, \tilde{\tilde{F}})\).

Now, we look at some error norms \( E_n = \| e^{-nc} - \frac{p_n(z)}{q_n(z)} \|_\infty \) on \([0, c]\), and the products \( \rho^{-n} E_n/2 \) which should tend towards \( \rho_0 \):
<table>
<thead>
<tr>
<th>$n$</th>
<th>$c = 0.5$</th>
<th>$c = 1$</th>
<th>$c = 2$</th>
<th>$c = 5$</th>
<th>$c = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_n$</td>
<td>$\rho^{-n}E_n/2$</td>
<td>$E_n$</td>
<td>$\rho^{-n}E_n/2$</td>
<td>$E_n$</td>
</tr>
<tr>
<td>1</td>
<td>2.5352E−4</td>
<td>0.02255</td>
<td>1.5802E−3</td>
<td>0.04509</td>
<td>7.7144E−3</td>
</tr>
<tr>
<td>2</td>
<td>1.6454E−6</td>
<td>0.02605</td>
<td>3.1969E−5</td>
<td>0.05206</td>
<td>3.8218E−4</td>
</tr>
<tr>
<td>3</td>
<td>9.7750E−9</td>
<td>0.02753</td>
<td>5.9205E−7</td>
<td>0.05502</td>
<td>1.7358E−5</td>
</tr>
<tr>
<td>4</td>
<td>5.658E−11</td>
<td>0.02836</td>
<td>1.0684E−8</td>
<td>0.05667</td>
<td>7.6871E−7</td>
</tr>
<tr>
<td>5</td>
<td>3.240E−13</td>
<td>0.02888</td>
<td>1.907E−10</td>
<td>0.05771</td>
<td>3.3678E−8</td>
</tr>
<tr>
<td>lim</td>
<td>0.03123</td>
<td>0.06241</td>
<td>0.1227</td>
<td>0.236</td>
<td>0.328</td>
</tr>
<tr>
<td>$e^{\frac{\pi i}{4}}$</td>
<td>0.03126</td>
<td>0.06240</td>
<td>0.1226</td>
<td>0.2362</td>
<td>0.328</td>
</tr>
</tbody>
</table>

The last rows are: the limit when $n \to \infty$ estimated through a simple step of Thiele interpolatory continued fraction, i.e., $\lambda$ from $\lambda + \mu/(n + \nu)$ interpolation three values, $= \text{ first nontrivial step of } \rho-\text{algorithm [10]}$; and the formula $\exp(-\pi K'/(2K))$.

6. **Best rational approximation to other exponential functions**

Not even started! (things from [45])

7. **References.**


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