MAPA 3xxxA Special topics in approximation theory.

1998-1999: Padé approximation to functions with branch points.

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Abstract: the locus of convergence of Padé approximation (or continued fraction expansion) to functions with branchpoints is examined, theoretically (recent papers by H. Stahl), and numerically (computation of the spectrum of large bandmatrices).

CONTENTS

1. Continued fractions expansions of numbers and functions.

1.1. Let $f = \cfrac{\alpha_0}{\beta_0 + \cfrac{\alpha_1}{\beta_1 + \cdots}}$ be the usual continued fraction operation. It may be a single number, or a function. If I choose to call it f, you guess that I shall mainly deal with functions.

With $f_0 = f$, $f_n = \cfrac{\alpha_n}{\beta_n + \cfrac{\alpha_{n+1}}{\beta_{n+1} + \cdots}}$, one has

$$f_n = \frac{\alpha_n}{\beta_n + f_{n+1}}\,, \qquad n = 0, 1, \ldots \tag{1}$$

or $f_n f_{n+1} + \beta_n f_n = \alpha_n$,

$$F_n = (-1)^n f_0 f_1 \ldots f_n \quad \Rightarrow \quad F_{n+1} = \beta_n F_n + \alpha_n F_{n-1}, \tag{2}$$

the second order linear ***recurrence***, whose well known fundamental system of solutions is

$$\begin{aligned} A_{n+1} &= \beta_n A_n + \alpha_n A_{n-1}, \quad n = 1, 2, \ldots, \quad A_0 = 0, A_1 = \alpha_0, \\ B_{n+1} &= \beta_n B_n + \alpha_n B_{n-1}, \quad n = 1, 2, \ldots, \quad B_0 = 1, B_1 = \beta_0, \end{aligned} \tag{3}$$

or $A_{-1} = 1$, $B_{-1} = 0$. Any solution of (2) is a combination of A_n and B_n, easily found to be

$$F_n = F_{-1}A_n + F_0 B_n$$

Proof with 2×2 matrices:

$$\begin{bmatrix} A_n & B_n \\ A_{n+1} & B_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha_n & \beta_n \end{bmatrix} \begin{bmatrix} A_{n-1} & B_{n-1} \\ A_n & B_n \end{bmatrix} : \begin{bmatrix} A_{n-1} & B_{n-1} \\ A_n & B_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha_{n-1} & \beta_{n-1} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ \alpha_0 & \beta_0 \end{bmatrix}$$

N.B. Determinants: $A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1} \alpha_0 \alpha_1 \ldots \alpha_{n-1}$, $n = 0, 1, \ldots$

$$\begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} A_{n-1} & B_{n-1} \\ A_n & B_n \end{bmatrix} \begin{bmatrix} F_{-1} \\ F_0 \end{bmatrix}.$$

So,

$$f_n = -\frac{F_n}{F_{n-1}} = -\frac{B_n f - A_n}{B_{n-1} f - A_{n-1}},$$

or $B_n f - A_n = (-1)^n f_0 f_1 \ldots f_n$, and

$$f = \frac{A_n + A_{n-1} f_n}{B_n + B_{n-1} f_n}.$$

If one starts with f_m instead of $f = f_0$:

$$f_m = \frac{A_n^{(m)} + A_{n-1}^{(m)} f_n}{B_n^{(m)} + B_{n-1}^{(m)} f_n} \qquad m \leqslant n,$$

with the same recurrences for $A_n^{(m)}$ and $B_n^{(m)}$, but the starting values $A_m^{(m)} = B_{m-1}^{(m)} = 0$, $A_{m+1}^{(m)} = \alpha_m$, $B_m^{(m)} = 1$ ($\Rightarrow A_{m-1}^{(m)} = 1$).

$$B_n^{(m)} = \begin{vmatrix} \beta_m & -1 & & & \\ \alpha_{m+1} & \beta_{m+1} & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-2} & \beta_{n-2} & -1 \\ & & & \alpha_{n-1} & \beta_{n-1} \end{vmatrix}, \; A_n^{(m)} = \alpha_m \begin{vmatrix} \beta_{m+1} & -1 & & & \\ \alpha_{m+2} & \beta_{m+2} & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-2} & \beta_{n-2} & -1 \\ & & & \alpha_{n-1} & \beta_{n-1} \end{vmatrix} = \alpha_m B_n^{(m+1)}.$$

We only encounter now a ***convergence*** question:
$f - \dfrac{A_n}{B_n} = \dfrac{(-1)^n f_0 f_1 \ldots f_n}{B_n} \to 0$
if $|B_n|$ increases infinitely faster than the product of the f_n's.

This certainly happens with the ***regular*** continued fraction expansion of a real irrational number: $0 < f_n < 1$, $\alpha_n = 1$, $\beta_n = \lfloor 1/f_n \rfloor \geqslant 1$, $B_n \geqslant B_{n-1} + B_{n-2} \geqslant \cdots$ the n^{th} Fibonacci number[1].

For another example, the approximants A_n/B_n to $\pi = \cfrac{4}{1 + \cfrac{1/3}{1 + \cfrac{4/15}{1 + \cfrac{\cdots}{\ddots + \cfrac{n^2/(4n^2-1)}{1 + \cdots}}}}}$ (which is *not* the regular continued fraction to π) do converge, as $\limsup f_n \leqslant 1/4$, and $B_n > 1$.

[1] As I already used the notation F_n elsewhere, and as Fibonacci numbers will not be used in the future, I don't write F_n for Fibonacci numbers.

: But why should the limit be π? The secret is in establishing transformations of the form (1) for ***functions***: let $f(x) = f_0(x) = 4\arctan(1/x)$. Remark that f_0 is the solution of the differential equation $(1+x^2)f_0'(x) = -4$ satisfying $f_0(\infty) = 0$. For $f_1(x) := \dfrac{1}{\arctan(1/x)} - x$, one finds the Riccati equation $(1+x^2)f_1' = 2xf_1 + f_1^2 - 1$, and $xf_1(x) \to 1/3$ when $x \to \infty$. By induction on $n = 1, 2, \ldots$, one can show that, if f_n satisfies $(1+x^2)f_n'(x) = 2nxf_n(x) + (2n-1)f_n^2(x) - \dfrac{n^2}{2n-1}$ with $xf_n(x) \to n^2/(4n^2-1)$ when $x \to \infty$, then f_{n+1} defined by $f_n(x) = \dfrac{n^2/(4n^2-1)}{x + f_{n+1}(x)}$ satisfies the same conditions, with n replaced by $n+1$.

This was an instance of the ***Jacobi continued fraction*** expansion of a ***function***

$$f(x) = \frac{\alpha_0}{x} + \frac{\alpha_0'}{x^2} + \frac{\alpha_0''}{x^3} + \cdots$$

which we may construct as $f_0(x) \equiv f(x)$,

$$f_n(x) = \frac{\alpha_n}{x} + \frac{\alpha_n'}{x^2} + \frac{\alpha_n''}{x^3} + \cdots = \frac{\alpha_n}{x - \alpha_n'/\alpha_n + f_{n+1}(x)}$$

with $f_{n+1}(x) = [(\alpha_n'/\alpha_n)^2 - \alpha_n''/\alpha_n]/x + \cdots$, so that β_n is the polynomial $x - \alpha_n'/\alpha_n''$ of degree 1 (and $\alpha_{n+1} = (\alpha_n'/\alpha_n)^2 - \alpha_n''/\alpha_n$).

A_n and B_n are polynomials, of degrees $n-1$ and n (from (3)).

A_n/B_n is the $[n-1/n]$ ***Padé*** approximant to f, as $f - \dfrac{A_n}{B_n} = \dfrac{(-1)^n f_0 f_1 \ldots f_n}{B_n} = O(x^{-2n-1})$.

1.2. **Periodicity and quadraticity.**

A very deep relation:

1. **Periodicity $\Rightarrow$ quadraticity.**
   Simplest case: $f = \cfrac{\alpha}{\beta + \cfrac{\alpha}{\beta + \cdots}}$, so, each $f_n = f$ in (1), and f is a root of $f^2 + \beta f - \alpha = 0$ (which one? this is a matter of convergence).
   In general, if $\alpha_{n+p} = \alpha_n$, $\beta_{n+p} = \beta_n$ holds for $n \geqslant n_0$, $f_{n_0} = f_{n_0+p}$ in $f_{n_0} = f_{n_0+p} = \dfrac{A_{n_0+p}^{(n_0)} + A_{n_0+p-1}^{(n_0)} f_{n_0}}{B_{n_0+p}^{(n_0)} + B_{n_0+p-1}^{(n_0)} f_{n_0}}$, whence still a quadratic equation for f_{n_0}, and for f.
2. **Quadraticity $\Rightarrow$ periodicity?**
   Much more subtle.
   Let $f = \dfrac{X - \sqrt{Y}}{Z}$, where X, Y, and Z are whole numbers (X and Z, and, why not, Y, may be negative); or polynomials. Let us show that a formula

$$f_n = \frac{X_n - \sqrt{Y}}{Z_n} \tag{4}$$

holds, with X_n and Z_n integers in case of regular expansion of a real number, or polynomials of fixed degree in case of expansion of a function. Apply (1):

$$\begin{aligned} f_n &= \frac{X_n - \sqrt{Y}}{Z_n} \\ &= \frac{1}{\dfrac{Z_n X_n + Z_n\sqrt{Y}}{X_n^2 - Y}} \\ &= \frac{\alpha_n}{\beta_n + \left[\dfrac{\alpha_n Z_n X_n + \alpha_n Z_n\sqrt{Y}}{X_n^2 - Y} - \beta_n\right]} \end{aligned}$$

Trick: suppose we managed to have $\alpha_0 Z = \alpha_0 Z_0$ to be an exact factor of $X^2 - Y$ (by multiplying X and Z by some common factor, and Y by the square of this factor, if needed), and that the property still holds at index n. Of course, we call the oher factor $-Z_{n+1}$:

$$X_n^2 - Y = -\alpha_n Z_n Z_{n+1}, \tag{5}$$

and we have

$$f_{n+1} = \frac{-X_n - \sqrt{Y}}{Z_{n+1}} - \beta_n,$$

i.e., $X_{n+1} = -X_n - \beta_n Z_{n+1}$.
Check $X_{n+1}^2 - Y = X_n^2 - Y + 2\beta_n X_n Z_{n+1} + \beta_n^2 Z_{n+1}^2 = Z_{n+1}[-\alpha_n Z_n + 2\beta_n X_n + \beta_n^2 Z_{n+1}]$, OK if $\alpha_{n+1} = 1$ (integer case), or a constant (polynomials):

$$\alpha_{n+1} Z_{n+2} = \alpha_n Z_n - 2\beta_n X_n - \beta_n^2 Z_{n+1}. \tag{6}$$

The sensible ways to construct continued fractions expansions result in keeping the Z_n's small in some sense: from (1),

$$\alpha_n - \beta_n f_n = f_n f_{n+1}$$

is small:

- for numbers, $\alpha_n = 1$, and $\beta_n =$ integer part of $1/f_n$, so that $|\alpha_n - \beta_n f_n| < 1$, and is even smaller than f_n,
- for functions, let us choose the Jacobi continued fraction, $f_n = \alpha_n/z + \alpha_n'/z^2 + \alpha_n''/z^3 + \cdots$, $\beta_n = z - \alpha_n'/\alpha_n$, so that $\alpha_n - \beta_n f_n = O(z^{-2})$.

Now, (4) yields $Z_n f_n^2 - 2X_n f_n - \alpha_n Z_{n+1} = 0$, where (5) has been used. Well, well … Take this latter identity times β_n^2, and subtract α_n times (6):

$$\begin{aligned} \alpha_n\alpha_{n+1} Z_{n+2} &= (\alpha_n^2 - \beta_n^2 f_n^2) Z_n - 2\beta_n(\alpha_n - \beta_n f_n) X_n \\ &= (\alpha_n - \beta_n f_n)\,[(\alpha_n + \beta_n f_n) Z_n - 2\beta_n X_n] \\ &= f_n f_{n+1}\left[(\alpha_n - \beta_n f_n) Z_n - 2\beta_n\sqrt{Y}\right] \qquad \text{(from (4))} \\ &= (f_n f_{n+1})^2 Z_n - 2\alpha_n f_{n+1}\sqrt{Y} + 2 f_n f_{n+1}^2\sqrt{Y} \qquad \text{(from (1))} \end{aligned} \tag{7}$$

showing that Z_n remains bounded (numbers[2]), or of bounded degree (polynomials; if $2m$ is the degree of Y, the ultimate degree of Z_n is $\leqslant m-1$).

Now, a bounded sequence $\{(X_n, Z_n)\}$ of integers ***must*** repeat footsteps: for some n_0 and p, $X_{n_0+p} = X_{n_0}$, $Z_{n_0+p} = Z_{n_0}$, so, $f_{n_0+p} = f_{n_0} \Rightarrow$ periodicity.
Example, from

[2] The full proof of boundedness is longer [24].

```
Logiciels de Christian RADOUX
URL: http://users.skynet.be/radoux/logiciel.htm
ftp://umhsp02.umh.ac.be/pub/ftp_nombres/frcont.exe
```

with $f_0 = 1/\sqrt{1789}$:

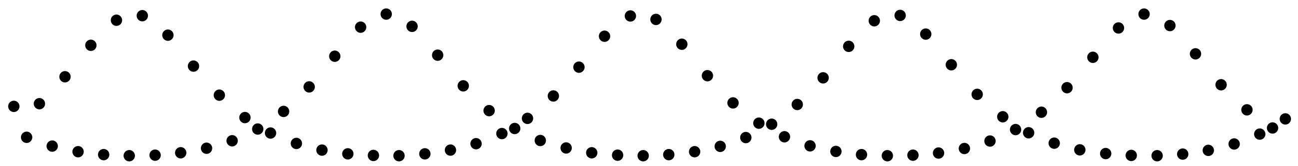
$$\begin{array}{rl}
\text{Antépériode} & X_0 = 0, Z_0 = -1789, \beta_1 = 42; \\
\text{Période} & X_1 = 42, Z_1 = -1, \beta_2 = 3, \\
 & X_2 = 33, Z_2 = -25, \beta_2 = 2, \\
 & X_2 = 23, Z_2 = -28, \beta_2 = 1, \\
 & \ldots \\
 & X_{33} = 38, Z_{33} = -69, \beta_{34} = 16, \\
 & X_{34} = 42, Z_{34} = -5, \beta_{34} = 16, \\
 & X_{35} = 38, Z_{35} = -5, \beta_{34} = 1, \\
 & X_{36} = 31, Z_{35} = -69, \beta_{34} = 6, \\
 & \ldots \\
 & X_{66} = 33, Z_{66} = -28, \beta_{66} = 3, \\
 & X_{67} = 42, Z_{67} = -25, \beta_{67} = 84,
\end{array}$$

$f_{68} = f_1$.

Whenever polynomials are known to keep their degrees under a constant d, they are in the ***compact*** set $\overline{\mathbb{C}}^{d+1}$, and some f_{n_0+p} is as close as we wish to f_{n_0}: quasi-periodicity.

See nice example ($\sqrt{-\alpha_n}$) from [27, Fig. 2], with $Y(z) = (z+2)(z+1.71)(z-1.81)(z-2)$:

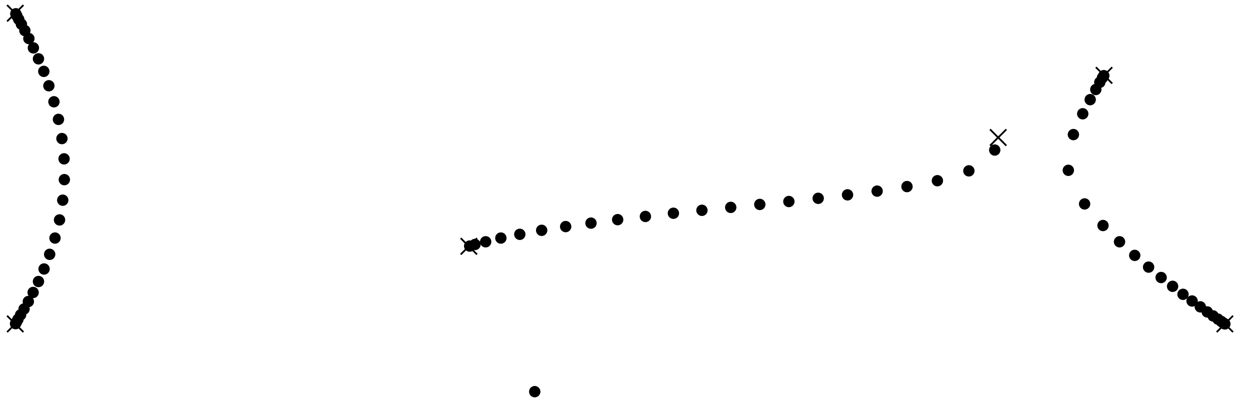
```
Script V1.1 session started Fri Oct 09 13:47:59 1998
Microsoft(R) Windows 95
   (C)Copyright Microsoft Corp 1981-1995.
C:\dos\djgpp>g77 -s -o m3xxx98.exe m3xxx98.f
C:\dos\djgpp>m3xxx98

2 100
-2 0
-1.71 0
1.81 0
2 0
```



An instance of the zeros of B_{75}, with $Y(z) = z(z-i)(z-1.5-0.25i)(z-3.25-0.6i)(z-4)(z-3.6-0.8i)$:

```
3 75
0 0
0 1
1.5 0.25
3.25 0.6
4 0
```

3.6 0.8



1.3. Terminators and (generalized) Turán determinants.

2. (Hyper)elliptic functions and all that.

2.1. The Green function .

For any n_0 and p, from (4):

$$f_{n_0}f_{n_0+1}\cdots f_{n_0+p-1}=\frac{X_{n_0}-\sqrt{Y}}{Z_{n_0}}\frac{X_{n_0+1}-\sqrt{Y}}{Z_{n_0+1}}\cdots\frac{X_{n_0+p-1}-\sqrt{Y}}{Z_{n_0+p-1}},$$

$$[f_{n_0}f_{n_0+1}\cdots f_{n_0+p-1}]^2=\frac{[X_{n_0}-\sqrt{Y}]^2[X_{n_0+1}-\sqrt{Y}]^2\cdots[X_{n_0+p-1}-\sqrt{Y}]^2}{Z_{n_0}\ (Z_{n_0}Z_{n_0+1})\ (Z_{n_0+1}Z_{n_0+2})\ \cdots(Z_{n_0+p-2}Z_{n_0+p-1})\ Z_{n_0+p-1}}$$

$$=\frac{(-1)^p\alpha_{n_0}\cdots\alpha_{n_0+p-1}[X_{n_0}-\sqrt{Y}]^2[X_{n_0+1}-\sqrt{Y}]^2\cdots[X_{n_0+p-1}-\sqrt{Y}]^2Z_{n_0+p}}{Z_{n_0}[X_{n_0}^2-Y][X_{n_0+1}^2-Y]\cdots[X_{n_0+p-1}^2-Y]}$$

(from (5))

$$=\frac{Z_{n_0+p}}{Z_{n_0}}(-1)^p\alpha_{n_0}\cdots\alpha_{n_0+p-1}\frac{[X_{n_0}-\sqrt{Y}][X_{n_0+1}-\sqrt{Y}]\cdots[X_{n_0+p-1}-\sqrt{Y}]}{[X_{n_0}+\sqrt{Y}][X_{n_0+1}+\sqrt{Y}]\cdots[X_{n_0+p-1}+\sqrt{Y}]}$$

3. The general theory.

3.1. Stahl's theorems [26].

From many theorems, problems, and conjectures by Gonchar, Nuttall, etc. a picture emerged:

- *Assumption 1.1* The function f is analytic at infinity and has all its singularities in a compact set $E\subseteq\overline{\mathbb{C}}$ with $\mathrm{cap}(E)=0$, i.e., f has analytical continuations along any path in $\overline{\mathbb{C}}\setminus E$ starting at infinity; the continuation may be multiple-valued.
- *Theorem 1.1* Let f satisfy Assumption 1.1. Then there exists a domain $D_f\subseteq\overline{\mathbb{C}}$, $\infty\in D_f$, which is unique up to a set of capacity zero, and
  - the sequence $\{A_n/B_n\}$ converges in capacity to f in D_f,
  - if $\widetilde{D}$ is a domain with $\widetilde{D}\supseteq D$ and $\mathrm{cap}(\widetilde{D}\setminus D)>0$, then $\{A_n/B_n\}$ does not converge in capacity to f in the whole domain $\widetilde{D}$.
- *Theorem 1.2*
  - for every $\varepsilon>0$ and every compact set $V\subseteq D_f\setminus\{\infty\}$ we have

$$\lim_{n\to\infty}\mathrm{cap}\,\{z\in V|\ |f(z)-A_n(z)/B_n(z)|>(G_{D_f}(z)+\varepsilon)^{2n}\}=0,$$

– and if the function f has branch points, i.e. if $G_{D_f}(z) \not\equiv 0$, then for every compact set $V \subseteq D_f \setminus \{\infty\}$ and every $0 < \varepsilon \leqslant \inf_{z \in V} G_{D_f}(z)$ we have

$$\lim_{n\to\infty} \operatorname{cap} \{z \in V \mid |f(z) - A_n(z)/B_n(z)| < (G_{D_f}(z) - \varepsilon)^{2n}\} = 0,$$

where $G_{D_f}(z) = \exp[-g_{D_f}(z,\infty)]$, g_{D_f} being the Green function of D_f with singularity at ∞.

4. The program.

```
c  m3xxx98.f    sept 1998
c
c          poles of Pade of function involving sqrt(P(z))
c
      PROGRAM M3xxx98
      parameter(ndim=200)
      INTEGER twom,m,i,nmax
      complex X(20),A(ndim),B2(ndim),b(ndim),root(ndim)
      double precision hr(ndim,ndim),hi(ndim,ndim),wr(ndim),wi(ndim)
      READ *,m,nmax
      twom=2*m
c  2m zeros of P:
      do i=1,twom
       read *,xr,yr
       x(i)=cmplx(xr,yr)
      enddo
       CALL recsqr(m,X,A,B2,NMAX)
       amaxb=0
       do i=1,nmax
        b(i)=sqrt(b2(i))
        amaxb=max(amaxb, abs(b(i)) )
       enddo
       print '(a1,''psset{unit=1cm}'')',char(92)
       print '(a1,''begin{pspicture}(17,2.5)'')',char(92)
       do i=1,nmax
       print '(a1,''psdots[dotsize=0.15](',f7.3,'','',f7.3,'')'')',
     &  char(92),i*17.0/nmax,0.2+2*real(b(i))/amaxb
       enddo
       print '(a1,''end{pspicture}'')',char(92)
       PRINT '(10F8.4)',(A(I),I=1,5)
       PRINT '(10F8.4)',(B(I),I=1,5)
c     call trilr(nmax,ntrue,b,a,root)
c     print *,ntrue,' found'
c      PRINT '(10F8.4)',(root(I),I=1,ntrue)
      do i=1,nmax
      do j=1,nmax
       hr(i,j)=0
       hi(i,j)=0
      enddo
      hr(i,i)=a(i)
      hi(i,i)=aimag( a(i) )
      enddo
      hr(1,2)=b(1)
      hi(1,2)=aimag( b(1) )
      do i=2,nmax-1
       hr(i,i-1)=b(i-1)
       hi(i,i-1)=aimag( b(i-1) )
       hr(i,i+1)=b(i)
       hi(i,i+1)=aimag( b(i) )
      enddo
```

```
      hr(nmax,nmax-1)=b(nmax-1)
      hi(nmax,nmax-1)=aimag( b(nmax-1) )
      low=1
      igh=nmax
      call comlr(ndim,nmax,low,igh,hr,hi,wr,wi,ierr)
      if(ierr.ne.0)print *,' err: ',ierr
      print '(10f8.4)',(wr(i),i=1,nmax)
      print '(10f8.4)',(wi(i),i=1,nmax)
      amaxx=wr(1)
      aminx=wr(1)
      amaxy=wi(1)
      aminy=wi(1)
      do i=2,nmax
        amaxx=max(amaxx, wr(i) )
        aminx=min(aminx, wr(i) )
        amaxy=max(amaxy, wi(i) )
        aminy=min(aminy, wi(i) )
      enddo
       print '(a1,''psset{unit=1cm}'')',char(92)
       print '(a1,''begin{pspicture}(17,5)'')',char(92)
       do i=1,twom
       print '(a1,''psdots[dotstyle=x,dotsize=0.25]
     & ('',f7.3,'','',f7.3,'')'')',
     &  char(92),16.0*(real(x(i))-aminx)/(amaxx-aminx),
     &           5.0* (aimag(x(i))-aminy)/(amaxy-aminy)
       enddo
       do i=1,nmax
       print '(a1,''psdots[dotsize=0.15]('',f7.3,'','',f7.3,'')'')',
     &  char(92),16.0*(wr(i)-aminx)/(amaxx-aminx),
     &           5.0* (wi(i)-aminy)/(amaxy-aminy)
       enddo
       print '(a1,''end{pspicture}'')',char(92)
      STOP
      SUBROUTINE recsqr(P,X,A,B,NMAX)
C  recurrence coeff  for
C                  [x ,x ] [x ,x ] ... [x    ,x  ]
C                    1  2    3  4        2p-1  2p
C
C  A partir de p, x (vecteur des 2p xi) et nmax, la routine
C  donne A et B, vecteurs des coefficients de recurrence
      INTEGER P,N,NMAX
      complex X(1),A(1),B(1),
     &       R(300),S(300),PTS(300),R0(300),S0(300),
     &       LP1(300),LP2(300),XPTS(300),LPP(20),LPP1,
     &       SOMX,X0,EPS,EPS1,PB,R10,B0,A0,RL10,PI
      SOMX=0
      DO 11 I=1,P
   11 PTS(I)=(X(2*I-1)+X(2*I))/2
      A0=0
      SOMX=0
      DO 115 I=1,P
       SOMX=SOMX+X(2*I-1)+X(2*I)
       IF(I.LT.P)A0=A0-X(2*I)
       R(I)=1
       XPTS(I)=1
       LP1(I)=1
       LP2(I)=0
       DO 1151 J=1,P
        IF(J.LT.P)R(I)=R(I)*( PTS(I)-X(2*J) )
        XPTS(I)=XPTS(I)*(PTS(I)-X(2*J-1))*(PTS(I)-X(2*J))
```

```
        IF(J.EQ.I)GOTO 1151
        LP1(I)=LP1(I)/(PTS(I)-PTS(J))
        LP2(I)=LP2(I)-PTS(J)
 1151  CONTINUE
 115   LP2(I)=LP2(I)*LP1(I)
      A0=A0+SOMX/2
      DO 14 I=1,P
      S(I)=R(I)*(PTS(I)-A0)
      R0(I)=R(I)
  14  S0(I)=S(I)
      A(1)=A0
C
C  Iterations
C
      DO 2 IT=1,NMAX
       ID=4
       IF(IT.EQ.1)ID=2
       DO 21 I=1,P
   21  R(I)=(S(I)**2-XPTS(I))/(ID*R(I))
       B0=0
       A0=0
       DO 22 I=1,P
       A0=A0+R(I)*LP2(I)
   22  B0=B0+R(I)*LP1(I)
       A0=A0/B0+SOMX/2
       DO 23 I=1,P
       R(I)=R(I)/B0
   23  S(I)=2*(PTS(I)-A0)*R(I)-S(I)
       A(IT+1)=A0
       B(IT)=B0
 2    CONTINUE
      RETURN
      END
      subroutine comlr(nm,n,low,igh,hr,hi,wr,wi,ierr)
c
cReceived: from CUNYVM.BITNET by Pythag.ucl.ac.be (Mailer R2.04) with BSMTP id
c 2883; Mon, 13 Nov 89 09:32:44 +0100
cReceived: from CUNYVM by CUNYVM.BITNET (Mailer R2.03B) with BSMTP id 2335; Mon,
c 13 Nov 89 02:56:23 EST
cReceived: from research.att.com by CUNYVM.CUNY.EDU (IBM VM SMTP R1.2.1MX) with
c TCP; Mon, 13 Nov 89 02:56:16 EST
cFrom: netlibd@research.att.com
cDate: Mon, 13 Nov 89 02:57 EST
cTo: annu76%buclln11.bitnet@cunyvm.cuny.edu
cSubject: send comlr from eispack
c
cCareful! Anything free comes with no guarantee.
c*** from netlib, Mon Nov 13 02:57:26 EST 1989 ***
      integer i,j,l,m,n,en,ll,mm,nm,igh,im1,itn,its,low,mp1,enm1,ierr
      double precision hr(nm,n),hi(nm,n),wr(n),wi(n)
      double precision si,sr,ti,tr,xi,xr,yi,yr,zzi,zzr,tst1,tst2
c
c     this subroutine is a translation of the algol procedure comlr,
c     num. math. 12, 369-376(1968) by martin and wilkinson.
c     handbook for auto. comp., vol.ii-linear algebra, 396-403(1971).
c
c     this subroutine finds the eigenvalues of a complex
c     upper hessenberg matrix by the modified lr method.
c
c     on input
```

```
c
c       nm must be set to the row dimension of two-dimensional
c         array parameters as declared in the calling program
c         dimension statement.
c
c       n is the order of the matrix.
c
c       low and igh are integers determined by the balancing
c         subroutine  cbal.  if  cbal  has not been used,
c         set low=1, igh=n.
c
c       hr and hi contain the real and imaginary parts,
c         respectively, of the complex upper hessenberg matrix.
c         their lower triangles below the subdiagonal contain the
c         multipliers which were used in the reduction by  comhes,
c         if performed.
c
c     on output
c
c       the upper hessenberg portions of hr and hi have been
c         destroyed.  therefore, they must be saved before
c         calling  comlr  if subsequent calculation of
c         eigenvectors is to be performed.
c
c       wr and wi contain the real and imaginary parts,
c         respectively, of the eigenvalues.  if an error
c         exit is made, the eigenvalues should be correct
c         for indices ierr+1,...,n.
c
c       ierr is set to
c         zero       for normal return,
c         j          if the limit of 30*n iterations is exhausted
c                    while the j-th eigenvalue is being sought.
c
c     calls cdiv for complex division.
c     calls csroot for complex square root.
c
c     questions and comments should be directed to burton s. garbow,
c     mathematics and computer science div, argonne national laboratory
c
c     this version dated august 1983.
c
c     ------------------------------------------------------------------
c
      ierr = 0
...
      return
      end
```

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Appendix. Some literature.

Demo mode: there is a limit of max. 3 allowed answers for unregistered users
Your query: (AU is (are) "MARONI, P*") and (TI contains the words "SECOND")
Answers 1-3 (out of 3)
980.21748
Beghdadi, Driss; Maroni, Pascal
Second degree classical forms. (English)
[J] Indag. Math., New Ser. 8, No.4, 439-452 (1997). [ISSN 0019-3577]

A linear functional (form) u over the space of polynomials is regular if it admits a sequence of polynomials, orthogonal with respect to u. Additionally, u is a second degree form if its Stieltjes transform

$$S(z) = \sum_{n=0}^{\infty} \frac{u(z^n)}{z^{n+1}}$$

is an algebraic function of degree 2 (in other words, S is a solution of a quadratic equation with polynomial coefficients in z). The authors show that among the classical forms (corresponding to the case when differentiation preserves orthogonality with respect to u) the only second degree form is the Jacobi one, with a suitable selection of parameters. [A.Martinez Finkelshtein (Almeria)] Keywords: Orthogonal polynomials; classical forms; regular linear functionals

Classification:

*42C05 General theory of orthogonal functions and polynomials 33C25 Orthogonal polynomials and functions

837.42009

Maroni, P.

An introduction to second degree forms. (English)

[J] Adv. Comput. Math. 3, No.1-2, 59-88 (1995). [ISSN 1019-7168]

The author studies linear functionals u on the space of polynomials with complex coefficients in a real variable x, for which the formal Stieltjes function defined from the moments

$$S(u)(z) := -\sum_{n=0}^{\infty} \frac{\langle u, x^n \rangle}{z^{n+1}}$$

satisfies a quadratic equation of the form

$$B(z)S^2(u)(z) + C(z)S(u)(z) + D(z) = 0, \qquad 1$$

for certain polynomials (B,C), where D depends on B, C and

$$B \neq 0;\ C^2 - 4BD \neq 0;\ D \neq 0. \qquad 2$$

The form u is then called a regular (2) second degree form (1).

Through a chain of lemmas, the author derives a classification of these forms in case the pair (B,C) is 'primitive', i.e., it satisfies either B and C have no common root, or for any common root c we have $\langle u, \theta_c C\rangle - \langle u^2, \theta_0\theta_c B\rangle \neq 0$. (Here $(\theta_c h)(x) = \frac{h(x)-h(c)}{x-c}$ for complex c and h a polynomial; operations on forms are defined in the usual way reminiscent of 'partial integration'.)

The classification uses the degrees of the polynomials in the unique (!) primitive pair associated with the form. For $\deg B = r$, $\deg C = m$ there appear to be 5 classes: $m \geq 1$, $r = 0$; $m \geq r+1$, $r \geq 1$; $m = r$, $r \geq 1$; $m = r-1$, $r \geq 2$; $0 \leq m < r-1$, $r \geq 2$.

An example is given for certain instances of a Tchebycheff form (not all cases treated in full because of the increasing difficulties for growing degree) and the paper concludes with an equivalent classification in terms of the so-called 'associated forms' $u^{(n+1)}$.

[M.G.de Bruin (Delft)]

Keywords: orthogonal polynomials; Chebyshev form; second degree form; Tchebycheff form

Classification:

*42C05 General theory of orthogonal functions and polynomials

837.42010

Maroni, P.

Tchebychev forms and their perturbed forms as second degree forms. (English)

[J] Ann. Numer. Math. 2, No.1-4, 123-143 (1995). [ISSN 1021-2655]

Let $\{P_n\}_{n\geq 0}$ be a sequence of orthogonal polynomials satisfying a three term recurrence relation of the form

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0.$$

Then the sequence satisfying the above recurrence relation with β_n and γ_n perturbed for $1 \leq n \leq r$ and unchanged for $n \geq r+1$ is called an r-order perturbed sequence. A linear functional u is called a second degree form if its formal Stieltjes function satisfies a certain second-order polynomial equation.

The author gives a characterization of all second-order degree forms and studies their 1-order perturbed forms in general. It is shown that every perturbed sequence of this kind is semiclassical. The structure relation is given and also the second-order differential equation. Finally, the case of the Chebyshev form of the second kind is treated in greater detail. [R.Koekoek (Delft)] Keywords: orthogonal polynomials; three term recurrence relation; second-order degree forms; perturbed forms; Chebyshev form

Classification:

*42C05 General theory of orthogonal functions and polynomials

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688.42020

Peherstorfer, Franz

On Bernstein-Szegő orthogonal polynomials on several intervals. (English)

[J] SIAM J. Math. Anal. 21, No.2, 461-482 (1990). [ISSN 0036-1410]

Let $\ell \in \mathbb{N}$, $a_1 < a_2 < ... < a_{2\ell}$, $E_\ell = \cup_{k=1}^{\ell}[a_{2k-1}, a_{2k}]$, $H(x) = \prod_{k=1}^{2\ell}(x-a_k)$ and let $\rho_\nu(x) = c\prod_{k=1}^{\nu^*}(x-w_k)^{\nu_k}$ be a real polynomial with $w_k \notin int(E_\ell)$ for $k = 1, ..., \nu^*$ and $\nu_k = 1$ if w_k is a boundary point of E_ℓ. For given ρ_ν and $\varepsilon = (\varepsilon_1, ..., \varepsilon_{\nu^*})$, $\varepsilon_k \in \{-1, 1\}$, the following linear functional on $\mathbb{P}$, $\mathbb{P}$ denoting the space of real polynomials, is defined:

$$\Psi_{H,\rho_\nu,\varepsilon}(p) = \int_{E_\ell} p(x)\frac{\sqrt{-H(x)}}{\rho_\nu(x)} sgn(-\prod_{k=1}^{\ell}(x-a_{2k-1}))dx+$$

$$+\sum_{k=1}^{\nu^*}(1-\varepsilon_k)\sum_{j=1}^{\nu_k}\mu_{j,k}p^{(j-1)}(w_k),$$

where $\mu'_{j,k}s$ are certain numbers. Polynomials orthogonal with respect to the (not necessarily positive definite) linear functional $\Psi_{H,\rho_\nu,\varepsilon}$ are characterized. Those polynomials are given the name Bernstein-Szegő orthogonal polynomials on several intervals. Special attention is given to the most interesting case $\varepsilon = (1, ..., 1)$. Keywords: Bernstein-Szegő orthogonal polynomials

Classification:

*42C05 General theory of orthogonal functions and polynomials

33C25 Orthogonal polynomials and functions

721.42017

Peherstorfer, Franz

On Bernstein-Szegő orthogonal polynomials on several intervals. II: Orthogonal polynomials with periodic recurrence coefficients. (English)

[J] J. Approximation Theory 64, No.2, 123-161 (1991). [ISSN 0021-9045]

Geronimus has shown that a sequence of orthogonal polynomials (p_n) with periodic recurrence coefficients for $n \geq n_0$ is orthogonal on a set of disjoint intervals $E_\ell = \cup_{j=1}^{\ell}[a_{2j-1}, a_{2j}]$ with respect to a distribution of the form

$$d\psi(x) = \sqrt{-\prod_{j=1}^{2\ell}(x-a_j)}/|\rho_\nu(x)|dx + d\mu(x),$$

where $\rho_\nu(x) = \prod_{j=1}^{\nu}(x-w_j)$ with sgn $\rho_{\ \nu}(x) = (-1)^{\ell+1-j}$ on (a_{2j-1}, a_{2j}) for $j = 1, ..., \ell$, $\nu \geq \ell - 1$, and where μ is a certain point measure with $\sup p(\mu) \subset \{w_1, ..., w_\nu\}$. In this paper we show (in fact a more general result is presented) that a sequence of polynomials (p_n) orthogonal with respect to $d\psi$ has recurrence coefficients of period N, $N \geq \ell$, for $n \geq n_0$, if and only if there exists a so-called Chebyshev polynomial $\mathcal{T}_N$ of degree N on E_ℓ, where a polynomial $\mathcal{T}_N$ is called a Chebyshev polynomial on E_ℓ if $|\mathcal{T}_N|$ attains its maximum value on E_ℓ at $N+\ell$ points from E_ℓ. Furthermore it is demonstrated how to get in a simple way a (nonlinear) recurrence relation for the recurrence coefficients of the orthogonal polynomials. Results on Chebyshev polynomials on several intervals are also given.

[For Part I see SIAM J. Math. Anal. 21, No.2, 461-482 (1990); Zbl. 688.42020).]

Citations: Zbl.688.42020

Keywords: Bernstein-Szegő orthogonal polynomials; periodic recurrence coefficients; Chebyshev polynomials on several intervals

Classification:

*42C05 General theory of orthogonal functions and polynomials

33C25 Orthogonal polynomials and functions

These pages: December 18, 1998.

About A_n and B_n of a Jacobi continued fraction considered as formal orthogonal polynomials.

So, $f(z)=\dfrac{\alpha_0}{z}+\dfrac{\alpha_0'}{z^2}+\dfrac{\alpha_0''}{z^3}+\cdots=\cfrac{\alpha_0}{z-b_0-\cfrac{a_1^2}{z-b_1-\cdots}}$ has approximants A_n and B_n of degrees $n-1$ and n, with the Padé property $f(z)-\dfrac{A_n(z)}{B_n(z)}=\dfrac{(-1)^n f_0(z)f_1(z)\dots f_n(z)}{B_n(z)}=O(z^{-2n-1})$ (see p. 3). This is enough to establish a property of formal orthogonality:

suppose that the Laurent expansion $f(z)=\dfrac{\alpha_0}{z}+\dfrac{\alpha_0'}{z^2}+\dfrac{\alpha_0''}{z^3}+\cdots$ converges in some region $|z|>R$, then we have the formal moments

$$\alpha_0=\frac{1}{2\pi i}\int_C f(t)\,dt,\alpha_0'=\frac{1}{2\pi i}\int_C t f(t)\,dt,\dots\alpha_0^{(n)}=\frac{1}{2\pi i}\int_C t^n f(t)\,dt,$$

where C is a big contour in $|z|>R$. If z in the exterior of C:

$$f(z)=\frac{1}{2\pi i}\int_C\frac{f(t)}{z-t}\,dt. \qquad z \text{ exterior to } C,$$

Exercise: let $u=1/z$, $g(u)=zf(z)=\alpha_0+\alpha_0'u+\alpha_0''u^2+\cdots=(2\pi i)^{-1}\int_D g(v)(v-u)^{-1}\,dv$ from the usual Cauchy formula. Take $t=v^{-1}$, $C=D^{-1}$, etc.

Now,

$$\begin{aligned}B_n(z)f(z)&=\frac{1}{2\pi i}\int_C\frac{B_n(z)}{z-t}\,f(t)\,dt \qquad \text{behaves like } z^{n-1}\\ &=\underbrace{\frac{1}{2\pi i}\int_C\frac{B_n(z)-B_n(t)}{z-t}\,f(t)\,dt}_{\text{polynomial}=A_n(z)}+\underbrace{\frac{1}{2\pi i}\int_C\frac{B_n(t)}{z-t}\,f(t)\,dt}_{\text{must be }O(z^{-n-1})}\end{aligned}$$

therefore

$$\int_C B_n(t)\,f(t)\,dt=0,\int_C B_n(t)\,t\,f(t)\,dt=0,\dots\int_C B_n(t)\,t^{n-1}f(t)\,dt=0: \qquad \int_C B_n(t)\,p(t)\,f(t)\,dt=0$$

for any polynomial p of degree $<n$.

That's for the orthogonality of $B_n=p_n=p_n^{(0)}$. And A_n?

Easy: we have to find a function such that $A_n=\alpha_0 p_{n-1}^{(1)}$ times this function $-$ a polynomial of degree $n-2$ is $O(z^{-n})$ for large z.

$$B_n f-A_n=p_n\frac{\alpha_0}{\beta_0+f_1}-\alpha_0 p_{n-1}^{(1)}=\alpha_0\frac{p_n-\beta_0 p_{n-1}^{(1)}-p_{n-1}^{(1)}f_1}{\beta_0+f_1}.$$

So, just replace $f=f_0$ by f_1!

Remark that, with the notations of p. 2,

$$p_n=B_n=B_n^{(0)}=\beta_0 B_n^{(1)}-\alpha_1 B_n^{(2)},$$

so,

$$p_n=B_n^{(0)}=\begin{vmatrix}\beta_0 & -1 & & & \\ \alpha_1 & \beta_1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-2} & \beta_{n-2} & -1\\ & & & \alpha_{n-1} & \beta_{n-1}\end{vmatrix},\ p_{n-1}^{(1)}=B_n^{(1)}=\begin{vmatrix}\beta_1 & -1 & & & \\ \alpha_2 & \beta_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-2} & \beta_{n-2} & -1\\ & & & \alpha_{n-1} & \beta_{n-1}\end{vmatrix},$$

$$p_{n-2}^{(2)} = B_n^{(2)} = \begin{vmatrix} \beta_2 & -1 & & & & \\ \alpha_3 & \beta_3 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \alpha_{n-2} & \beta_{n-2} & -1 \\ & & & \alpha_{n-1} & \beta_{n-1} \end{vmatrix}$$

which shows that $p_n - \beta_0 p_{n-1}^{(1)} = \alpha_1 p_{n-2}^{(2)}$, indeed of degree $n-2$.

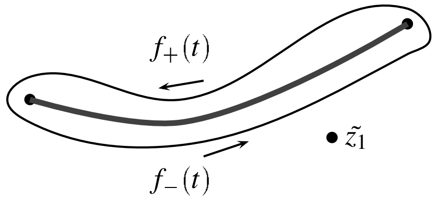
Now, all the f_n's have the same form (4), so that all the associated orthogonal polynomials will have similar complexity.

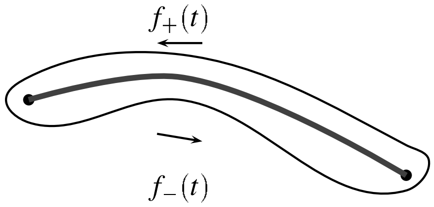
Let us look more accurately at the formal scalar product defining the orthogonality:

first, with $Y(z) = z^{2m} + \cdots$, let $\sqrt{Y(z)}$ be defined for large z as the determination of the square root behaving like z^m, whence a well-defined meaning of $f(z) = \dfrac{X(z) - \sqrt{Y(z)}}{Z(z)}$ for large z, and also a well-defined meaning of f in

$$(p,q) = \frac{1}{2\pi i}\int_C p(t)q(t)\,f(t)\,dt.$$

Now, we shrink C, *keeping* $\sqrt{Y(t)}$ *continuous*, as far as we don't encounter singularities of f. When a zero of Z is encountered, the residue is taken into account (as Z is a factor of $X^2 - Y$, the residue is either 0 or $2p(\tilde z)q(\tilde z)X(\tilde z)/Z'(\tilde z)$). We must stop at the zeros of Y, C has shrinked into a set of several thin contours around a system S of arcs joining the $2m$ zeros of Y:





and what remains is

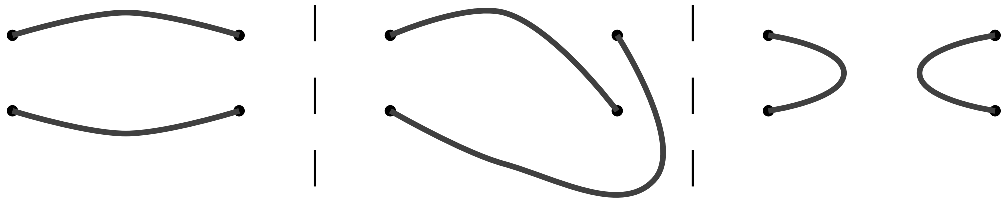
$$(p,q) = \sum_{\text{residues}} \text{res. } p(\tilde z_j)q(\tilde z_j) + \int_S p(t)q(t)\,\frac{f_-(t) - f_+(t)}{2\pi i}\,dt.$$

The "weight" function is therefore $\dfrac{\sqrt{Y(t)}_+}{\pi i Z(t)}$.

Examples.

- $f(z) = z - \sqrt{z^2-1}$. As $\sqrt{z^2-1} = z - 1/(2z) - 1/(8z^3)\cdots$ for large z, $f(z)$ is indeed $O(z^{-1}$ for large z. And, as $\sqrt{z^2-1}$ is positive imaginary on the positive imaginary axis, $\sqrt{t^2-1}_+ = +i\sqrt{1-t^2}$ on $S = [-1,1]$, whence the weight $\sqrt{1-t^2}/\pi$ on $[-1,1]$. The orthogonal polynomials are the Chebyshev polynomials of second kind U_n.
- $f(z) = 1/\sqrt{z^2-1}$: weight $1/(\pi\sqrt{1-t^2})$, 1st kind Chebyshev pol. T_n. Remark that $f(z) = \dfrac{1}{z - \text{ the } f \text{ of above}}$.
- $f(z) = \dfrac{z + c - \sqrt{z^2-1}}{z-d}$, with $c = -d \pm \sqrt{d^2-1}$, so that $Z(z) = z - d$ is a factor of $X^2(z) - Y(z) = (z+c)^2 - z^2 + 1 = 2cz + c^2 + 1$. According to the choice of c, a possible masspoint d, and the weight $\sqrt{1-t^2}/(\pi(t-d))$ on $[-1,1]$. If $d = \pm 1$: Chebyshev pol. of 3rd or 4th kind.
- Generalization of all the preceding cases: $Y(z) = (z^2-1)$ times the square of a polynomial, Z is any polynomial: Bernstein-Szegő orthogonal polynomials (on the real line). Some f_n must be $z - \sqrt{z^2-1}$. After a finite number of steps, recurrence coefficients must be exactly $\alpha_n = -a_n^2 = -1/4$, $\beta_n = z - b_n = z$ [18, 20–23, etc.].

- $Y(z) = (z-a_1)(z-a_2)(z-a_3)(z-a_4)$, the much worked "elliptic case" [1, etc.]. The three different ways to write a product $Y = Y_1Y_2$ with Y_1 and Y_2 of degree 2, make three possible ways to build S as a set of two arcs:



Of course, if $a_1,\ldots,a_4$ are real, with $a_1 < \cdots < a_4$, the only sensible way is to take the two intervals $S = [a_1,a_2]\bigcup[a_3,a_4]$. Remark that $\sqrt{Y(z)}_+$ is positive on $(-\infty,a_1)$, negative imaginary on (a_1,a_2), negative (real!) on (a_2,a_3), positive imaginary on (a_3,a_4), positive on $(a_4,+\infty)$ (use $z=x+i\varepsilon,\varepsilon>0$). For a positive weight function, the simplest f is $f(z) = \dfrac{X(z)-\sqrt{Y(z)}}{z-c}$, with $a_2 \leqslant c \leqslant a_3$, and $X(z) = z^2-(a_1+\cdots+a_4)z/2+$ a constant such that $z-c$ is a factor of X^2-Y.

More on section 2.1: the Green function.
When p is (almost) a period, $Z_{n_0+p} \approx Z_{n_0}$, we have the limit

$$\exp(\mathcal{G}(z)) = \exp(g(z)+ih(z)) = \lim_{n\to\infty}\left(\frac{a_1\cdots a_n}{f_0(z)\cdots f_{n-1}(z)}\right)^{1/n} \approx \left(\frac{(X_{n_0}+\sqrt{Y})\cdots(X_{n_0+p-1}+\sqrt{Y})}{(X_{n_0}-\sqrt{Y})\cdots(X_{n_0+p-1}-\sqrt{Y})}\right)^{1/(2p)}$$

When Y has real roots, $\exp(\mathcal{G}(z))$ is the ratio of complex conjugate numbers if $z\in S$, the locus where $Y(z)\leqslant 0$. $g(z)$ is the harmonic function vanishing on S, positive outside S, behaving like $\log|z|+$ constant for large z: the ***Green function*** of S. The constant is the ***Robin constant***[3] = -log of the ***logarithmic capacity*** of S.

What if Y has general complex zeros $a_1,\ldots,a_{2m}$? $\exp\mathcal{G}(z)$ is already on the unit circle, i.e., $g(z)=0$ at the zeros of Y. We still can think of a path S where $g(z)=0$, i.e., $\mathcal{G}(z)$ is pure imaginary, and this path is characterized by the differential condition

$$\mathcal{G}'(z)\,dz \text{ pure imaginary}$$

(or the *quadratic differential* $(\mathcal{G}')^2(z)\,dz^2\leqslant 0$). We must know more on $\mathcal{G}'$:

$$\mathcal{G}' \approx \frac{1}{2p}\sum\left[\frac{X'+\dfrac{Y'}{2\sqrt{Y}}}{X+\sqrt{Y}} - \frac{X'-\dfrac{Y'}{2\sqrt{Y}}}{X-\sqrt{Y}}\right]$$

$$\approx \frac{1}{2p}\sum\frac{-2X'\sqrt{Y}+2\dfrac{Y'X}{2\sqrt{Y}}}{X^2-Y=a_m^2Z_mZ_{m+1}}$$

is therefore a rational function divided by $\sqrt{Y}$, with possible poles at the zeros of the Z polynomials. But let us look again at $\exp(\mathcal{G}(z)) \approx \left(\dfrac{(X_{n_0}+\sqrt{Y})\cdots(X_{n_0+p-1}+\sqrt{Y})}{(X_{n_0}-\sqrt{Y})\cdots(X_{n_0+p-1}-\sqrt{Y})}\right)^{1/(2p)}$ if some $X_m\pm\sqrt{Y}$ vanishes, so does X_m^2-Y, so, either Z_m or Z_{m+1}. If it is Z_m, look at $X_m = -X_{m-1}-\beta_{m-1}Z_m$, which shows that $X_{m-1}\mp\sqrt{Y}$

[3] About Robin, see K. Gustafson, T. Abe: "The third boundary condition- was it Robin's?", *Math. Intelligencer* vol. **20** (1998), nr. 1, pp. 63-71; "(Victor) Gustave Robin: 1855-1897", *ibid.*, vol. **20**, nr. 2, pp. 47-53.

vanishes as well; and if it is Z_{m+1}, $X_{m+1} = -X_m - \beta_m Z_{m+1} = -X_m$ at a zero of Z_{m+1}. So, $\exp \mathcal{G}$ has no pole nor zero in the finite complex plane, and we must have

$$\mathcal{G}'(z) = \frac{Q(z)}{\sqrt{Y(z)}}, \tag{8}$$

where $Q(z) = z^{m-1} + \cdots$ is a monic polynomial of degree $m-1$.

When $m > 1$, Q is completely determined by the conditions

$$\text{periods } = 2\int_{a_1}^{a_k} \frac{Q(z)}{\sqrt{Y(z)}}\,dz \text{ pure imaginary.}$$

Simplest examples. They correspond to exactly periodic continued fractions:

- $f(z) = z - \sqrt{z^2-1}$, $p = 1$: $\exp(\mathcal{G}(z)) = \left(\dfrac{(X_0+\sqrt{Y}}{X_0-\sqrt{Y}}\right)^{1/2} = z + \sqrt{z^2-1}$, $\mathcal{G}'(z) = \dfrac{1}{\sqrt{z^2-1}}$.
- A case with $p = 2$: let $b_1 = -b_0$, $a_1^2 = a_0^2 + \alpha$. $f(z) = \dfrac{a_0^2}{z - b_0 - \dfrac{a_0^2+\alpha}{z+b_0-f(z)}}$:

$$f(z) = f_0(z) = \frac{z^2 - b_0^2 - \alpha - \sqrt{(z^2-b_0^2-\alpha)^2 - 4a_0^2(z^2-b_0^2}}{2(z-b_0)},$$

$$f_1(z) = \frac{z^2 - b_0^2 + \alpha - \sqrt{(z^2-b_0^2-\alpha)^2 - 4a_0^2(z^2-b_0^2)}}{2(z+b_0)},$$

$$\exp(4\mathcal{G}(z)) = \frac{\left[\Xi - \alpha + \sqrt{(\Xi-\alpha)^2 - 4a_0^2\Xi}\right]\left[\Xi + \alpha + \sqrt{(\Xi-\alpha)^2 - 4a_0^2\Xi}\right]}{\left[\Xi - \alpha - \sqrt{(\Xi-\alpha)^2 - 4a_0^2\Xi}\right]\left[\Xi + \alpha - \sqrt{(\Xi-\alpha)^2 - 4a_0^2\Xi}\right]}$$

$$(\Xi = z^2 - b_0^2)$$

$$= \frac{(\Xi-\alpha)^2 - 2a_0^2(2\Xi-\alpha) + 2a_0^4 + (\Xi - \alpha - 2a_0^2)\sqrt{(\Xi-\alpha)^2 - 4a_0^2\Xi}}{2a_0^2a_1^2}.$$

$$\mathcal{G}'(z) = \frac{z}{\sqrt{(\Xi-\alpha)^2 - 4a_0^2\Xi}}$$

$$= \frac{z}{\sqrt{(z^2-b_0^2-\alpha)^2 - 4a_0^2(z^2-b_0^2)}}$$

These pages: December 18, 1998.

Importance of the Green function $\mathcal{G}$ on the behaviour of the denominator B_n: let n be a quasi-period, then

$$B_{n-1}f - A_{n-1} = \frac{B_{n-1}X - A_{n-1}Z - B_{n-1}\sqrt{Y}}{Z} = (-1)^n f_0 \cdots f_{n-1} \approx (-1)^n a_1 \cdots a_n e^{-n\mathcal{G}}.$$

Now, take the other determination of $\sqrt{Y}$. $\mathcal{G}$ becomes $-\mathcal{G}$ (as seen form the definition of $\exp(\mathcal{G})$ through a product of $\dfrac{X_k+\sqrt{Y}}{X_k-\sqrt{Y}}$, or frm the quadratic differential), and

$$B_{n-1}f_{\text{conjugate}} - A_{n-1} = \frac{B_{n-1}X - A_{n-1}Z + B_{n-1}\sqrt{Y}}{Z} \approx (-1)^n a_1 \cdots a_n e^{n\mathcal{G}},$$

whence

$$B_{n-1} \approx (-1)^n a_1 \cdots a_n Z \frac{e^{n\mathcal{G}} - e^{-n\mathcal{G}}}{2\sqrt{Y}}.$$

showing that most of the zeros of B_{n-1} are on the locus $S = \{z:\ \mathrm{Re}\,\mathcal{G}(z) = g(z) = 0\}$.

Some instances:

```
Script V1.1 session started Tue Nov 17 16:16:24 1998


Microsoft(R) Windows 95
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C:\dos\djgpp>m3xxx98

2 100
0 0
100 50
100 -50
50 0
```

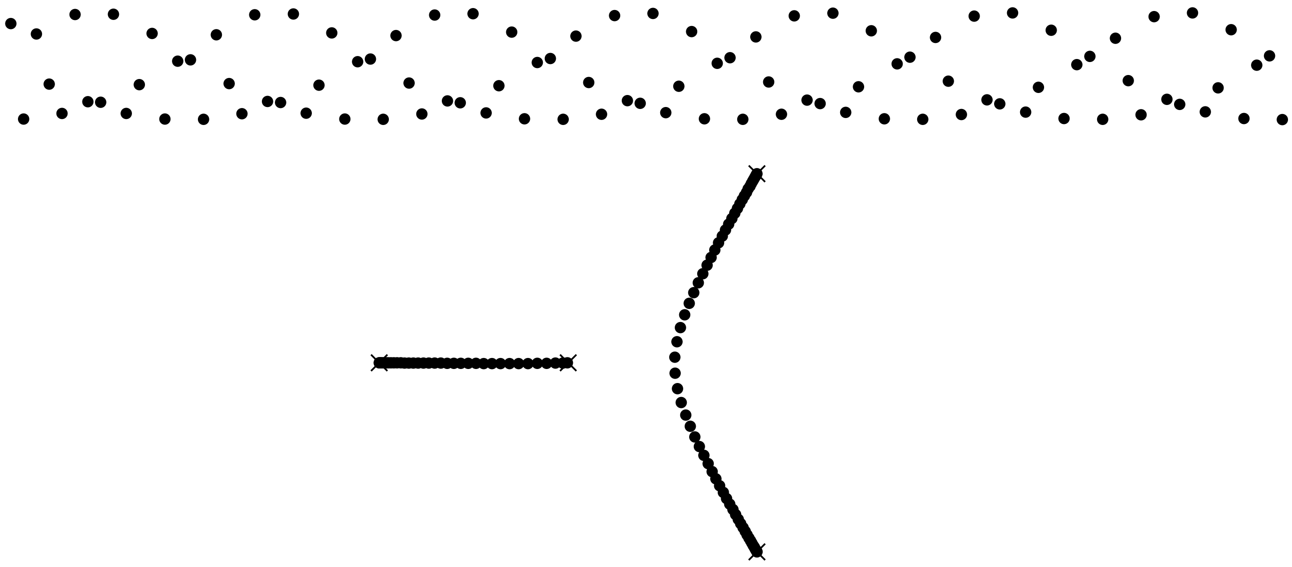

```
C:\dos\djgpp>exit
```

```
Script completed Tue Nov 17 16:17:06 1998


Script V1.1 session started Tue Nov 17 16:26:39 1998


Microsoft(R) Windows 95
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C:\dos\djgpp>m3xxx98

2 100
0 0
100 50
100 -50
75 0
```
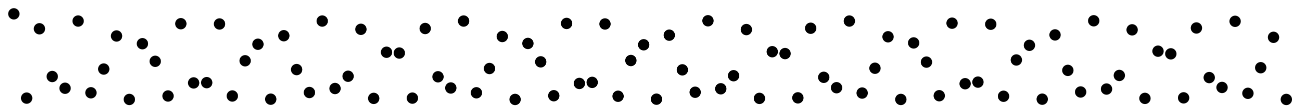
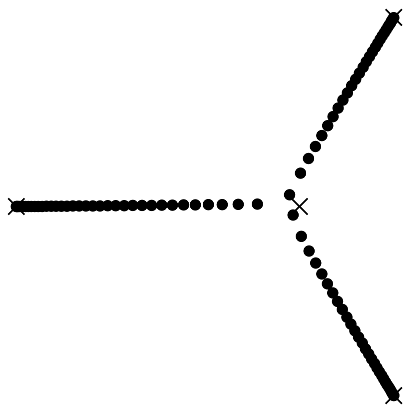
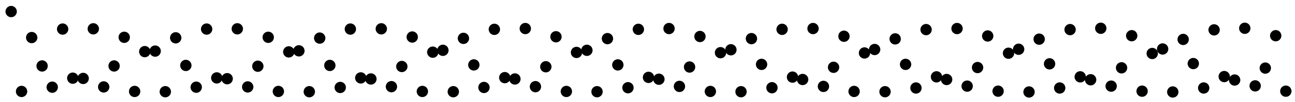
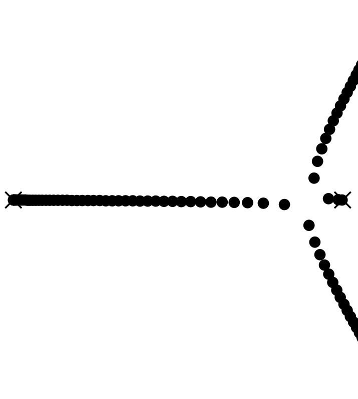





```
C:\dos\djgpp>exit

Script completed Tue Nov 17 16:27:14 1998

Script V1.1 session started Tue Nov 17 20:59:08 1998


Microsoft(R) Windows 95
   (C)Copyright Microsoft Corp 1981-1996.

C:\lang\djgpp>m3xxx98

2 125
0 0
100 50
```

```
100 -50
87 0
```





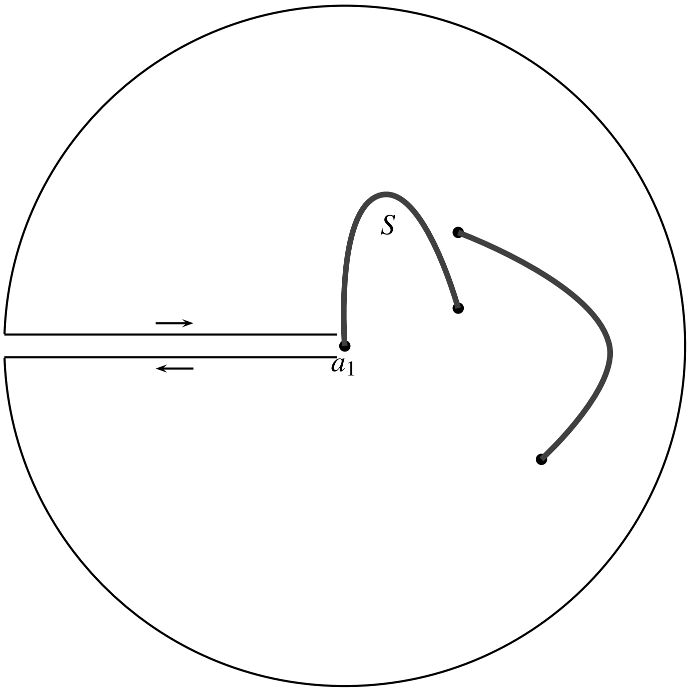
```
C:\lang\djgpp>exit

Script completed Tue Nov 17 20:59:52 1998
```

Remark: "Pell's equation". Make the product

$$\begin{aligned}
[B_{n-1}X - A_{n-1}Z]^2 - B_{n-1}^2 Y &= Z^2 f_0 \dots f_{n-1} f_{0,\text{conj.}} \dots f_{n-1,\text{conj.}} \\
&= Z^2 \frac{X_0 - \sqrt{Y}}{Z_0} \dots \frac{X_{n-1} - \sqrt{Y}}{Z_{n-1}} \frac{X_0 + \sqrt{Y}}{Z_0} \dots \frac{X_{n-1} + \sqrt{Y}}{Z_{n-1}} \\
&= a_0^2 \dots a_{n-1}^2 Z_0 Z_n.
\end{aligned}$$

2.2. What is (are) the period(s)?

If $m > 1$, let us consider the integrals of
$$\frac{t^k}{\pi i \sqrt{Y(t)}} \log \left[\frac{B_{n-1}(t)X(t) - A_{n-1}(t)Z(t) + B_{n-1}(t)\sqrt{Y(t)}}{B_{n-1}(t)X(t) - A_{n-1}(t)Z(t) - B_{n-1}(t)\sqrt{Y(t)}} \right] \, dt$$



on the big contour made of the big circle and the two horizontal sections nearby, for $k = 0, 1, \ldots, m-2$. The part on the big circle $\to 0$; the logarithms differ by a constant integer multiple of $2\pi i$ on the horizontal parts. Considering large t, we have $\log[U(t)/V(t)]$, with $U(t)$ behaving like t^{n+m-1}, and $V(t)$ like t^{-n+m-1} (remember that $UV =$ const. $Z_0 Z_n$), we see that the difference is $2n\pi i$: the value of the integrals are

$$2n\int_\infty^{a_1} \frac{t^k}{\sqrt{Y(t)}}\,dt.$$

On the other hand, making the contour shrink about S and the zeros of U and V (which are the zeros of Z_0 and Z_n), one finds

$$2n\int_\infty^{a_1} \frac{t^k}{\sqrt{Y(t)}}\,dt = \sum_{\text{zeros of } U,V} \pm \int_{a_1}^{\text{zero}} \frac{t^k}{\sqrt{Y(t)}}\,dt + 2\sum_j N_j \int_{a_j}^{a_{j+1}} \frac{t^k}{\sqrt{Y(t)}}\,dt, \qquad k = 0, \ldots, m-2, \tag{9}$$

where the $\pm$ signs tell if the zero is a zero of U or V; and where N_j are (unknown) integers. There are only $m-1$ integrals $\displaystyle\int_{a_j}^{a_{j+1}} \frac{t^k}{\sqrt{Y(t)}}\,dt$ to consider, in the "gaps" of S. Integrals on the two sides of S vanish, as $\sqrt{Y}_+ = -\sqrt{Y}_-$, so that the logarithm takes opposite values on the sides of S.

It happens that, knowing n, Y, and Z_0, (9) allows to find the remaining unknowns, including the $\pm$ signs (***Jacobi problem***, [1,2,13,18,20,21]).

There is absolutely no need for n to be an integer in the description (9) of the Jacobi problem. Let $z_1, \ldots, z_{m-1}$ be the unknown zeros of Z_n. To see how these z_ℓ's are functions of n, we … take the derivative of (9) with respect to n (!!):

$$y_k = \sum_{\ell=1}^{m-1} s_\ell \frac{z_\ell^k}{\sqrt{Y(z_\ell)}} \frac{dz_\ell}{dn} \qquad k = 0, \ldots, m-2,$$

where $y_k = 2\displaystyle\int_\infty^{a_1} \frac{t^k}{\sqrt{Y(t)}}\,dt$. We have a system of differential equations for the z_ℓ's. A single derivative is isolated through Lagrange interpolation polynomials:

$$\frac{dz_\ell}{dn} = 2s_\ell\sqrt{Y(z_\ell)} \int_\infty^{a_1} \prod_{\substack{q=1\\ q\neq\ell}} \frac{t-z_q}{z_\ell - z_q} \frac{dt}{\sqrt{Y(t)}}.$$

An initial condition consists of a set $z_1, \ldots, z_{m-1}$ ***and*** the signs $s_1, \ldots, s_{m-1}$ (= a set of places on the ***Riemann surface*** of $\sqrt{Y}$).

The vector $[z_1, \ldots, z_{m-1}]$ is a well defined function (***Jacobi-Abel funsction***) of the left-hand side $[ny_0, \ldots, ny_{m-2}]$.

Periodicity: $z_1, \ldots, z_{m-1}$ are kept unchanged if the left-hand side of (9) is augmented by integers times the integrals $2\displaystyle\int_{a_j}^{a_{j+1}} \frac{t^k}{\sqrt{Y(t)}}\,dt$ (***periods***)

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Consequences of (8) on the support S, locus $\mathcal{G}$ pure imaginary:

- Generic case: z_k is a simple zero of Y, $Q(z_k) \neq 0$. Then,

$$d\mathcal{G}(z) = idt \approx \text{ const. } (z - z_k)^{-1/2} \Rightarrow z \approx z_k + \text{ const. } t^2,$$

  z_k is an ***endpoint***.

- Y *and* Q happen to have a common simple zero at z_k. Then,

$$d\mathcal{G}(z) = idt \approx \text{ const. } (z - z_k)^{1/2} \Rightarrow z \approx z_k + \text{ const. } t^{2/3},$$

  three directions according to the three complex cubic roots of $t^2 > 0$.

- Q has a simple zero at a point z^*, with $Y(z^*) \neq 0$. Then, *if* $z^* \in S$,

$$d\mathcal{G}(z) = idt \approx \text{ const. } (z - z^*) \Rightarrow z \approx z^* + \text{ const. } t^{1/2},$$

  four directions according to the square roots of $t \in \mathbb{R}$!

The three cases happen in the examples given above. With $Y(z) = z(z-1-i/2)(z-1+i/2)(z-\alpha)$, the zero of Q happens to be

α	zero of Q
0.5	0.594
0.75	0.723
0.87	0.779

```
C
C  Iterations
C
       print '(i4,20f8.4)',0,(r(ii),ii=1,p),(s(ii),ii=1,p)
       errx=1111111
       somz0=0
      DO 2 IT=1,NMAX
       ID=4
       somz0=somz0+r(1)-pts(1)
       IF(IT.EQ.1)ID=2
       DO 21 I=1,P
   21  R(I)=(S(I)**2-XPTS(I))/(ID*R(I))
       B0=0
       A0=0
       DO 22 I=1,P
       A0=A0+R(I)*LP2(I)
   22  B0=B0+R(I)*LP1(I)
       A0=A0/B0+SOMX/2
       DO 23 I=1,P
       R(I)=R(I)/B0
   23  S(I)=2*(PTS(I)-A0)*R(I)-S(I)
       err=0
       do i=1,p
        err=err+abs(r(i)-r0(i))
        err=err+abs(s(i)-s0(i))
       enddo
       if(err.lt.errx)then
        print '(i4,20f8.4)',it,(r(ii),ii=1,p),(s(ii),ii=1,p)
        print *,' z: ',somz0/it
        errx=err
       endif
       A(IT+1)=A0
       B(IT)=B0
 2    CONTINUE

Script V1.1 session started Wed Dec 02 20:57:46 1998
```

```
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C:\lang\djgpp>m3xxx98

  ? m  nmax
2 5000
  2m times:  real  imag  of zero of X
0 0
1 0.5
1 -0.5
0.5 0

   0  -.5000  -.2500  -.2500  -.7500   .0625  -.4375   .0625  -.4375
   1   .1391   .4837   .3891  -.0163  -.1864   .0656  -.1864   .0656
  z:  (-1.,-.5)
   2   .0453   .6167   .2953   .1167   .3036  -.4404   .3036  -.4404
  z:  (-.680473387,-.133136094)
   3  -.4501  -.2639  -.2001  -.7639  -.0805  -.3531  -.0805  -.3531
  z:  (-.605227411,.033463497)
  28  -.4999  -.2500  -.2499  -.7500   .0728  -.4408   .0728  -.4408
  z:  (-.593973756,-.000180994)
 507  -.4999  -.2500  -.2499  -.7500   .0527  -.4354   .0527  -.4354
  z:  (-.593980789,8.87650731E-06)
 535  -.5000  -.2500  -.2500  -.7500   .0628  -.4390   .0628  -.4390
  z:  (-.593980134,-1.21554478E-06)

C:\lang\djgpp>m3xxx98

  ? m  nmax
2 5000
  2m times:  real  imag  of zero of X
0 0
1 -0.5
1 0.5
0.75 0
   0  -.5000   .2500  -.1250   .7500   .1250   .4062   .1250   .4062
   1  -.0506  -.4691   .3244   .0309  -.1212  -.0987  -.1212  -.0987
  z:  (-1.,.5)
   2  -.1678  -.6237   .2072  -.1237   .3450   .3157   .3450   .3157
  z:  (-.775280893,.140449435)
   3  -.4358   .2459  -.0608   .7459  -.0313   .3644  -.0313   .3644
  z:  (-.739448965,-.030925432)
  19  -.4931   .2500  -.1181   .7500   .1943   .3985   .1943   .3985
  z:  (-.722522259,.001715698)
  41  -.4915   .2499  -.1165   .7499   .0579   .3990   .0579   .3990
  z:  (-.722831786,-.000880254)
  60  -.4999   .2500  -.1249   .7500   .1177   .4062   .1177   .4062
  z:  (-.722605884,-6.06487192E-05)
 499  -.4999   .2500  -.1249   .7500   .1321   .4074   .1321   .4074
  z:  (-.722596109,7.94312564E-06)
 559  -.5000   .2500  -.1250   .7500   .1247   .4078   .1247   .4078
  z:  (-.722597241,7.39033624E-07)

C:\lang\djgpp>m3xxx98

  ? m  nmax
2 5000
  2m times:  real  imag  of zero of X
0 0
1 0.5
```

```
1 -0.5
0.87 0
   0  -.5000  -.2500  -.0650  -.7500   .1550  -.3913   .1550  -.3913
   1  -.1478   .4762   .2872  -.0238  -.0927   .1107  -.0927   .1107
  z:  (-1.,-.5)
   2  -.2159   .5800   .2191   .0800   .2802  -.3303   .2802  -.3303
  z:  (-.823914468,-.136887014)
   3  -.4738  -.2437  -.0388  -.7437   .0455  -.3933   .0455  -.3933
  z:  (-.78790164,.01875641)
  28  -.5000  -.2500  -.0650  -.7500   .1597  -.3902   .1597  -.3902
  z:  (-.778847158,-8.43278031E-05)
 703  -.5000  -.2500  -.0650  -.7500   .1509  -.3924   .1509  -.3924
  z:  (-.778865635,3.16523551E-06)
 731  -.5000  -.2500  -.0650  -.7500   .1557  -.3915   .1557  -.3915
  z:  (-.778864682,-2.14283176E-07)

C:\lang\djgpp>exit

Script completed Wed Dec 02 21:00:42 1998
```

Q is estimated from the average

$$Q(z) = \lim_{n\to\infty} \frac{Z_1 + Z_2 + \cdots + Z_n}{n}. \tag{10}$$

Proof: from “Pell’s equation” (p. 21), we have to estimate the average of $C_n D_n$, where $a_0 \ldots a_n C_n$ and $a_0 \ldots a_n D_n$ are combinations of A_n and B_n:

$$a_n C_n(z) = (z - b_{n-1}) C_{n-1}(z) - a_{n-1} C_{n-2}(z) \ ,$$

Christoffel-Darboux:

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Resolvant of $\boldsymbol{J}$.

$$\boldsymbol{J} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & a_{n-1} & b_{n-1} & a_n & & \\ & & a_n & b_n & a_{n+1} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \qquad \text{bi-infinite } : n \in \mathbb{Z}.$$

Trace of $(z\boldsymbol{I}-\boldsymbol{J}_n)^{-1} = \sum (z- \text{ zeros})^{-1} = n\int_S \frac{d\mu_n(t)}{z-t}$, where $d\mu_n \to$ average density of states in solid-state physics (Haydock, Nex)

$B_n(z) = \prod_1^n (z- \text{zeros})$ and $B_n(z) \approx \exp(n\mathcal{G}(z))$ when $z \notin S \Rightarrow$

$$\text{average trace of } (z\boldsymbol{I}-\boldsymbol{J})^{-1} = \mathcal{G}'(z) = \frac{Q(z)}{\sqrt{Y(z)}}.$$

The j^{th} column of $(z\boldsymbol{I}-\boldsymbol{J})^{-1}$ is made of a solution of

$$a_n C_n(z) = (z-b_{n-1})C_{n-1}(z) - a_{n-1}C_{n-2}(z)\ ,$$

for $n \geqslant j$, such that the order in z tends to $-\infty$ when $n \to +\infty$: $C_n(z) = K_j(z)(f(z)B_n(z) - A_n(z))/(a_0\dots a_n)$, $n \geqslant j$;
and a solution of

$$a_n D_n(z) = (z-b_{n-1})D_{n-1}(z) - a_{n-1}D_{n-2}(z)\ ,$$

for $n \leqslant j$, such that the order in z tends to $-\infty$ when $n \to -\infty$: $D_n(z) = L_j(z)(f_{\text{conj.}}(z)B_n(z) - A_n(z))/(a_0\dots a_n)$, $n \leqslant j$,

thanks to the “Pell’s equation” (p. 21)

$$\begin{aligned}(B_{n-1}f - A_{n-1})(B_{n-1}f_{\text{conj.}} - A_{n-1}) &= f_0\cdots f_{n-1}f_{0,\text{conj.}}\cdots f_{n-1,\text{conj.}} \\ &= \frac{X_0-\sqrt{Y}}{Z_0}\cdots\frac{X_{n-1}-\sqrt{Y}}{Z_{n-1}}\frac{X_0+\sqrt{Y}}{Z_0}\cdots\frac{X_{n-1}+\sqrt{Y}}{Z_{n-1}} \\ &= a_0^2\cdots a_{n-1}^2 Z_n/Z_0.\end{aligned}$$

So,

$$\begin{aligned} a_0\dots a_n a_0\dots a_j (z\boldsymbol{I}-\boldsymbol{J})^{-1}_{n,j} &= M_j(B_n f - A_n)(B_j f_{\text{conj.}} - A_j), \qquad & n \geqslant j, \\ &= M_j(B_j f - A_j)(B_n f_{\text{conj.}} - A_n), \qquad & n \leqslant j, \end{aligned}$$

and $-a_j(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{j-1,j} + (z-b_j)(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{j,j} - a_{j+1}(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{j+1,j} = 1 \Rightarrow$

$M_j(a_0^2\dots a_{j-1}^2)^{-1}(A_jB_{j-1} - A_{j-1}B_j)(f_{\text{conj.}} - f) = 1 : M_j = \dfrac{Z_0}{2\sqrt{Y}}.$

Diagonal element:

$$(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{n,n} = \frac{Z_{n+1}(z)}{2\sqrt{Y(z)}}.$$

Trace average $= \dfrac{\text{average of } Z}{2\sqrt{Y(z)}}$. Finally,

$$\text{Average of } Z = 2Q. \tag{11}$$

On these matters, many thanks to J. Meinguet who spotted “Sur l’intégration de la formule différentielle $\dfrac{\rho\, dx}{\sqrt{R}}$, R et ρ étant des fonctions entières”, by N.H. Abel, *J. reine angew. Math.* **1** (1826) = *Œuvres* **1**, 104-144.

Abel says (p. 140) that $Q/\sqrt{Y}$ has an elementary integral only if the continued fraction expansion of $\sqrt{Y}$ is periodic. Then, if the period is p, the simplest instance is

$$pQ = \sum_0^{p-1} (X_k' - \mu_k s_k'),$$

where μ_k and s_k are our $\beta_k = z - b_k$ and Z_k, up to multiplicative constants. But one of the Abel's equations is $X_n = 2\mu_{n-1}s_{n-1} - X_{n-1}$, so that, on one period,

$$\sum_0^{p-1} X_k' = \sum_0^{p-1} (\mu_k s_k)' \Rightarrow pQ = \sum_0^{p-1} \mu_k' s_k,$$

which is obviously our (11). How did Abel overlook this simpler equation?

Some other elements:

$$1 = -a_n(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{n-1,n} + (z-b_n)\frac{Z_{n+1}(z)}{2\sqrt{Y(z)}} - a_{n+1}(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{n+1,n} \Rightarrow a_n(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{n-1,n} = \frac{\sqrt{Y(z)}-X_n(z)}{2\sqrt{Y(z)}} = -\frac{f_n(z)Z_n(z)}{2\sqrt{Y(z)}},$$

(from $X_{n+1}(z) = -X_n(z) - (z-b_n)Z_{n+1}(z)$).

$$-a_{n+1}\frac{Z_{n+1}(z)}{2\sqrt{Y(z)}} + (z-b_{n+1})\frac{\sqrt{Y(z)}-X_{n+1}(z)}{2a_{n+1}\sqrt{Y(z)}} - a_{n+2}(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{n+2,n} = 0 \Rightarrow$$

$$a_{n+1}a_{n+2}(z\boldsymbol{I}-\boldsymbol{J})^{-1}_{n+2,n} = \frac{(X_{n+1}(z)-\sqrt{Y(z)})(X_{n+2}(z)-\sqrt{Y(z)}}{2Z_{n+2}(z)\sqrt{Y(z)}} = \frac{f_{n+1}(z)f_{n+2}(z)Z_{n+1}(z)}{2\sqrt{Y(z)}}.$$

etc.

1.3. Terminators and (generalized) Turán determinants

Terminators (Haydock & Nex): if f is suspected to have branchpoints at the zeros of Y, its continued fraction expansion $\dfrac{\alpha_0}{\beta_0+\dfrac{\alpha_1}{\beta_1+\cdots}}$ is confortably accelerated by $f_{\text{approx}} = \dfrac{\alpha_0}{\beta_0+\dfrac{\alpha_1}{\ddots\,\dfrac{}{\beta_{n-1}+\tilde{f}_n}}}$, where $\tilde{f}_n = \dfrac{\tilde{\alpha}_n}{\tilde{\beta}_n+\dfrac{\tilde{\alpha}_{n+1}}{\tilde{\beta}_{n+1}+\cdots}}$ matches as well as possible the true $f_n = \dfrac{\alpha_n}{\beta_n+\dfrac{\alpha_{n+1}}{\beta_{n+1}+\cdots}}$.

f_{approx} is not a rational approximation, but an algebraic quadratic approximation to f. Of special interest is the jump of f_{approx} across S (weight function): $f_{\text{approx}} = \dfrac{A_n+A_{n-1}\tilde{f}_n}{B_n+B_{n-1}\tilde{f}_n} = \dfrac{A_{n-1}}{B_{n-1}} + \dfrac{(-1)^{n-1}\alpha_0\ldots\alpha_{n-1}}{B_{n-1}(B_n+B_{n-1}\tilde{f}_n)}$,

$$f_{\text{approx},+} - f_{\text{approx},-} = \frac{(-1)^{n-1}\alpha_0\ldots\alpha_{n-1}(\tilde{f}_{n,-}-\tilde{f}_{n,+})}{(B_n+B_{n-1}\tilde{f}_{n,+})(B_n+B_{n-1}\tilde{f}_{n,-})}.$$

Relation with Turàn's determinant $\begin{vmatrix} B_n & B_{n-1} \\ B_{n+p} & B_{n+p-1} \end{vmatrix}$ (Van Assche), where p is a quasi-period.

f_{approx} is the same if we put the cutoff at $n+p$ (or any other index $\geqslant n$) instead of n; also, $\tilde{f}_{n+p} = \tilde{f}_n$, so

$$(B_{n+p}+B_{n+p-1}\tilde{f}_{n,+})(B_{n+p}+B_{n+p-1}\tilde{f}_{n,-}) = \alpha_n\cdots\alpha_{n+p-1}(B_n+B_{n-1}\tilde{f}_{n,+})(B_n+B_{n-1}\tilde{f}_{n,-})$$

?

$$(B_n+B_{n-1}\tilde{f}_{n,+})(B_n+B_{n-1}\tilde{f}_{n,-}) = \left(B_n+\frac{X_nB_{n-1}}{Z_n}\right)^2 - \frac{B_{n-1}^2Y}{Z_n^2} = B_n^2 + 2\frac{X_nB_nB_{n-1}}{Z_n} - \alpha_n\frac{B_{n-1}^2Z_{n+1}}{Z_n}$$

?