MAPA 3072A Special topics in approximation theory.

1999-2000, 2002-2003, 2012-2013: Semi-classical orthogonal polynomials on the unit circle.

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New in 2002-2003: essentially the linear differential equation, more and better ordered experimental evidence about the generalized Jacobi polynomials, and various corrections. Thanks for remarks and kind words: L. Golinskii, M. Ismail, F. Marcellán, N. Witte. Earlier thanks to P. Nevai for sending me a copy of the Badkov paper [2] in a former century.

Er, most of these "new" items are still in construction, but there are new facts about the Grünbaum-Delsarte-Janssen-Vries problem in § 5.

New, new in 2012-2013: final asymptotic of unit circle gen. Jacobi

$$w(\theta) = \begin{cases} r_1 e^{-\gamma\theta} |\sin(\theta-\theta_1)/2|^{2\alpha} |\sin(\theta-\theta_2)/2|^{2\beta}, & \theta_1 < \theta < \theta_2 \\ r_2 e^{-\gamma\theta} |\sin(\theta-\theta_1)/2|^{2\alpha} |\sin(\theta-\theta_2)/2|^{2\beta}, & \theta_2 - 2\pi < \theta < \theta_1] \end{cases}$$

where r_1 and r_2 are positive, see § 4.3, p. 22.

This version: March 19, 2013 (incomplete and unfinished)

Rien n'est plus agréable que de savoir quelqu'un aux prises avec des difficultés insurmontables, surtout si l'on y voit du travail gâché, de l'argent perdu et des crises de nerfs[1].

J. Giono

Abstract: Semi-classical orthogonal polynomials on the unit circle are examined. Special care is given to generalized Jacobi polynomials.

CONTENTS

[1] There is nothing like knowing somebody having to struggle with a hopeless task, especially if it represents bad work, loss of money, and nervous tantrums.

1. Complex orthogonal polynomials.

1.1. **General scalar product.** For any scalar product (positive definite sesquilinear[2] hermitian symmetric form) defined at least on the polynomials of degree $\leqslant N \leqslant \infty$, there is exactly one sequence $\{\Phi_0, \Phi_1, \ldots, \Phi_N\}$ of *monic* polynomials, degree $\Phi_n = n, n = 0, 1, \ldots, n$, such that $(\Phi_n, \Phi_m) = 0, 0 \leqslant m \neq n \leqslant N$.

The n coefficients of Φ_n are determined by $(\Phi_n, p) = 0$ for any p of degree $< n$: $\Phi_n(z) = z^n + \sum_0^{n-1} \Phi_{n,z^k} z^k$,

$$\boldsymbol{G}_{n-1} \begin{bmatrix} \Phi_n(0) \\ \Phi_n'(0) \\ \vdots \\ \Phi_{n,z^{n-1}} \end{bmatrix} = \begin{bmatrix} (z^0,z^0) & (z^1,z^0) & \cdots & (z^{n-1},z^0) \\ (z^0,z^1) & (z^1,z^1) & \cdots & (z^{n-1},z^1) \\ \vdots & \vdots & & \vdots \\ (z^0,z^{n-1}) & (z^1,z^{n-1}) & \cdots & (z^{n-1},z^{n-1}) \end{bmatrix} \begin{bmatrix} \Phi_n(0) \\ \Phi_n'(0) \\ \vdots \\ \Phi_{n,z^{n-1}} \end{bmatrix} = - \begin{bmatrix} (z^n,z^0) \\ (z^n,z^1) \\ \vdots \\ (z^n,z^{n-1}) \end{bmatrix}.$$

$\boldsymbol{G}_{n-1}$ is the transposed of the *Gram matrix* of $\{z^0, z^1, \ldots, z^{n-1}\}$. The matrix is hermitian and positive definite (cf. [6]).

$\Phi_0 = 1 \qquad \Phi_1(z) = z - \dfrac{(z^1,z^0)}{(z^0,z^0)}$.

Also,

$$\boldsymbol{G}_n \begin{bmatrix} \Phi_n(0) \\ \Phi_n'(0) \\ \vdots \\ \Phi_{n,z^{n-1}} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (\Phi_n, z^n) = \|\Phi_n\|^2 \end{bmatrix}$$

i.e., the last column of $\boldsymbol{G}_n^{-1}$ is made of the coefficients of $\Phi_n / \|\Phi_n\|^2$.

[2] i.e., (f,g) involves the ***complex conjugate*** of g.

Φ_n is the only monic n^{th} degree polynomial of minimal norm: any n^{th} degree monic polynomial is $\Phi_n + p$ with degree $p < n$, and

$$\|\Phi_n + p\|^2 = (\Phi_n + p, \Phi_n + p) = \|\Phi_n\|^2 + 2\underbrace{\text{Re}\,(\Phi_n, p)}_{0} + \|p\|^2.$$

One has

$$\|\Phi_n\|^2 = \frac{\det \boldsymbol{G}_n}{\det \boldsymbol{G}_{n-1}}\,.$$

1.2. **Playing with the idea.**

1.2.1. The simplest case occurs when $\boldsymbol{G}_n$ happens to be a ***diagonal matrix***, such as for the simplest L^2 scalar products on a circle or on a disk:

$$(f,g) = \int_{|z=R\exp(i\theta)|=R} f(z)\overline{g(z)}\frac{d\theta}{2\pi R},\ (f,g) = \int_{|z|<R} f(z)\overline{g(z)}\,\frac{dx\,dy}{\pi R^2}.$$

This also happens with ***Sobolev*** scalar products $\int f(z)\overline{g(z)} + f'(z)\overline{g'(z)}$

1.2.2. The next simplest situation seems to be ***binomial*** polynomials: when do we have $\Phi_m(z) = z^m - \alpha_m$, $m = 1,2,\ldots$? Then one must have $(\Phi_m, z^k) = 0$, i.e., $(z^m, z^k) = \alpha_m(z^0, z^k)$ for $k = 0,1,\ldots,m-1 \geqslant 0$. In particular, for $k = 0$: $(z^m, z^0) = \alpha_n(z^0, z^0)$ for all $m > 0$. Whence

$$(z^m, z^0) = \alpha_m(z^0,z^0), (z^0,z^m) = \overline{\alpha_m}(z^0,z^0),\ (z^m,z^k) = \alpha_m\overline{\alpha_k}(z^0,z^0),\ m,k = 1,2,\ldots,m \neq k. \tag{1}$$

Oh, I see, this is just a familiar perturbation of rank one to a diagonal matrix, easily explainable by the addition of a masspoint to an elementary L^2 or Sobolev measure, and we invert $\boldsymbol{G}_n = \boldsymbol{D} + \boldsymbol{u}\boldsymbol{v}^T$ by Sherman-Morrison formula [15, p. 3] $\boldsymbol{G}_n^{-1} = \boldsymbol{D}^{-1} - \dfrac{\boldsymbol{D}^{-1}\boldsymbol{u}\boldsymbol{v}^T\boldsymbol{D}^{-1}}{1+\boldsymbol{v}^T\boldsymbol{D}^{-1}\boldsymbol{u}}$ and… hey! this is not sparse! Indeed, (1) is ***not*** a rank one perturbation to a diagonal matrix $\boldsymbol{D}$! From the nondiagonal elements of $\boldsymbol{G}_n$, $\alpha_m\overline{\alpha_k}(z^0,z^0) = u_m v_k$, (with $\alpha_0 = 1$),so, $u_m = \gamma\alpha_m, v_m = \gamma^{-1}(z^0,z^0)\overline{\alpha_m}$. But then the first diagonal element should be $(z^0,z^0) = d_0 + u_0 v_0 = d_0 + (z^0,z^0)$, or $d_0 = 0$, aaargh.
Well, there is nothing wrong with $d_0 = 0$, such as in

$$(f,g) = \int_0^{2\pi} f'(e^{i\theta})\,\overline{g'(e^{i\theta})}\,\frac{d\theta}{2\pi} + f(re^{ia})\overline{g(re^{ia})},$$

so that (1) holds with $\alpha_m = r^m e^{ima}$, $m = 0,1,\ldots$

Remark that, if $r > 1$, all the zeros but one of Φ_m are outside the convex hull of the support of the measures involved in the scalar product, a known feature of Sobolev orthogonal polynomials. Actually, I do not know if these zeros are useful in any respect.

$z^m - \alpha_m z^{m'}$

1.2.3. Here is an example where the (z^ℓ, z^k) satisfy special relations which make orthogonal polynomials immediately appear:

let α be an *irrational* positive number, and the scalar product be $(f,g) = \dfrac{4}{\pi^2}\displaystyle\sum_{\substack{m=-\infty\\ m\text{ odd}}}^{\infty}\frac{1}{m^2} f(e^{im\alpha\pi})\overline{g(e^{im\alpha\pi})}$,

so that $(z^\ell, z^k) = \dfrac{8}{\pi^2}\displaystyle\sum_{\substack{m=1\\ m\text{ odd}}}^{\infty}\frac{1}{m^2}\cos[(k-\ell)m\alpha\pi] = F((k-\ell)\alpha)$, where F is the even periodic function of period 2, with value $F(u) = 1 - 2|u|$ on $[-1,1]$. Remark that $F(u+k) = (-1)^k F(u)$ if $k \in \mathbb{Z}$.

Now, let m'/n' be the best rational approximant to α in the sense that $1 \leqslant n' \leqslant n$ and $|n'\alpha - m'| <$ any other $p\alpha - q$ with $1 \leqslant p \leqslant n$; m''/n'' the second best rational approximant to α. Then, $\Phi_n(z) = z^n +$ a combination of $z^{n-n'}$ and $z^{n-n''}$. Indeed, $(\Phi_n, z^{n-k}) = F(k\alpha) +$ the same combination of $F((k-n')\alpha)$ and $F((k-n'')\alpha)$.

more on "second best approx."

About this second best rational approximation [30] which most of us would ignore completely, if it were not available in B. Casselman's home page http://www.math.ubc.ca/people/faculty/cass/smith/smith.html

$F((k-n')\alpha) = (-1)^{m'} F(k\alpha - (n'\alpha - m')), F((k-n'')\alpha) = (-1)^{m''} F(k\alpha - (n''\alpha - m''))$.

As long as $k\alpha$, $k\alpha - (n'\alpha - m')$, and $k\alpha - (n''\alpha - m'')$ have the *same* integer (floor) part:

$$\frac{F(k\alpha) - F(k\alpha - (n'\alpha - m'))}{n'\alpha - m'} - \frac{F(k\alpha) - F(k\alpha - (n''\alpha - m''))}{n''\alpha - m''} = 0,$$

which is interpreted as $(\Phi_n, z^{n-k}) = 0$, where

$$\Phi_n(z) = \frac{\dfrac{z^n - z^{n-n'}}{(-1)^{m'}(n'\alpha - m')} - \dfrac{z^n - z^{n-n''}}{(-1)^{m''}(n''\alpha - m'')}}{\dfrac{(-1)^{m'}}{n'\alpha - m'} - \dfrac{(-1)^{m''}}{n''\alpha - m''}}$$

Check with the program used in p. 12:

```
{
 Foucoeff(k)=

   -(-1)^(floor(k*sqrt(2)))*( 2*frac(k*sqrt(2))-1 )
}
...
Script V1.1 session started Mon Sep 20 17:54:35 1999

C:\calc\pari>gp
                   GP/PARI CALCULATOR Version 2.0.12 (alpha)
                 ix86 running emx (ix86 kernel) 32-bit version
                (readline enabled, extended help not available)

                           Copyright (C) 1989-1998 by
          C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

   realprecision = 28 significant digits
   seriesprecision = 16 significant terms
   format = g0.28

parisize = 4000000, primelimit = 500000

? \r reflec
```

$\Phi_n(z)$			$\|\Phi_n\|^2$
1			1.0000000000
z	+ 0.1715728752		0.9705627484
z^2	+ 0.2928932188*x	+ 0.7071067811	0.4852813742
z^3	+ 0.5857864376*z	- 0.4142135623	0.4020202535
z^4	+ 0.5857864376*z^2	- 0.4142135623*z	0.4020202535
z^5	+ 0.2928932188*z^3	+ 0.7071067811	0.2010101267

```
 z^6  + 0.2928932188*z^4 + 0.7071067811*z    0.2010101267
 z^7  + 0.5857864376*z^2 - 0.4142136237      0.1665222413
...
 z^12 + 0.2928932188*z^7 + 0.7071067811      0.0832611206
...
 z^17 + 0.5857864376*z^5 - 0.4142135623      0.0689757708
? quit
Good bye!

C:\calc\pari>exit

Script completed Mon Sep 20 17:54:57 1999
```

1.3. **Cholesky factors.**

Considering all the monic orthogonal polynomials $\Phi_k(z) = z^k + \xi_{k-1,k} z^{k-1} + \cdots + \xi_{0,k}$, $k = 0, \ldots, n$:

$$\boldsymbol{G}_n \begin{bmatrix} \xi_{0,0}=1 & \xi_{0,1} & \cdots & \xi_{0,n} \\ & \xi_{1,1}=1 & \cdots & \xi_{1,n} \\ & & \ddots & \vdots \\ & & & \xi_{n,n}=1 \end{bmatrix} = \begin{bmatrix} (\Phi_0, z^0) = \|\Phi_0\|^2 & & & \\ (\Phi_0, z^1) & (\Phi_1, z^1) = \|\Phi_1\|^2 & & \\ \vdots & \vdots & \ddots & \\ (\Phi_0, z^n) & (\Phi_1, z^n) & \cdots & (\Phi_n, z^n) = \|\Phi_n\|^2 \end{bmatrix}$$

i.e., the *Cholesky* factorization of $\boldsymbol{G}_n$ is $\boldsymbol{L}\overline{\boldsymbol{L}}^T$, with $\boldsymbol{L}_{k,\ell} = (\varphi_\ell, z^k)$, where

$$\varphi_\ell = \frac{\Phi_\ell}{\|\Phi_\ell\|}$$

is the ℓ^{th} degree *orthonormal* polynomial.

Check (Riesz-Fisher- generalized Parseval):

$$(z^\ell, z^k) = \sum_{m=0}^{\min(k,\ell)} (z^\ell, \varphi_m)(\varphi_m, z^k).$$

And, as $\overline{\boldsymbol{L}}^T_{k,\ell} = (z^\ell, \varphi_k)$ is the coefficient of z^ℓ in the $\varphi-$basis: $[z^0, \ldots, z^n] = [\varphi_0, \ldots, \varphi_n]\overline{\boldsymbol{L}}^T$, we also confirm that $(\overline{\boldsymbol{L}}^{-1})^T$ contains the coefficients of the φ's, and that $\xi_{k,l} = (\overline{\boldsymbol{L}}^{-1})^T_{k,\ell} / \|\Phi_\ell\|$:

$$\boldsymbol{G}_n = \boldsymbol{L}\overline{\boldsymbol{L}}^T = \begin{bmatrix} (\varphi_0, z^0) & & & \\ (\varphi_0, z^1) & (\varphi_1, z^1) & & \\ \vdots & \vdots & \ddots & \\ (\varphi_0, z^n) & (\varphi_1, z^n) & \cdots & (\varphi_n, z^n) \end{bmatrix} \begin{bmatrix} (z^0, \varphi_0) & (z^1, \varphi_0) & \cdots & (z^n, \varphi_0) \\ & (z^1, \varphi_1) & \cdots & (z^n, \varphi_1) \\ & & \ddots & \vdots \\ & & & (z^n, \varphi_n) \end{bmatrix},$$

$$(\boldsymbol{G}_n)^{-1} = (\overline{\boldsymbol{L}}^{-1})^T \boldsymbol{L}^{-1} = \begin{bmatrix} \xi_{0,0} & \xi_{0,1} & \cdots & \xi_{0,n} \\ & \xi_{1,1} & \cdots & \xi_{1,n} \\ & & \ddots & \vdots \\ & & & \xi_{n,n} \end{bmatrix} \begin{bmatrix} \|\Phi_0\|^{-2} & & & \\ & \|\Phi_1\|^{-2} & & \\ & & \ddots & \\ & & & \|\Phi_n\|^{-2} \end{bmatrix} \begin{bmatrix} \overline{\xi_{0,0}} & & & \\ \overline{\xi_{0,1}} & \overline{\xi_{1,1}} & & \\ \vdots & \vdots & \ddots & \\ \overline{\xi_{0,n}} & \overline{\xi_{1,n}} & \cdots & \overline{\xi_{n,n}} \end{bmatrix}.$$

In particular, the *last* column of $(\boldsymbol{G}_n)^{-1}$ is

$$\frac{1}{\|\Phi_n\|^2} \begin{bmatrix} \xi_{0,n} & \xi_{1,n} & \cdots & \xi_{n,n} \end{bmatrix}^T. \qquad (2)$$

Example: Jacobi $(\alpha,0)$ and Gegenbauer polynomials on the unit circle. There are not so many explicit scalar products allowing a complete description of the Cholesky factors. Most known cases are related to combinatorial identities (Knuth? Wilf?). Here is such a case (Delsarte & Genin?):

The scalar product is $(f,g) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} f(e^{i\theta})\overline{g(e^{i\theta})}|\sin\theta/2|^{2\alpha}\,d\theta$ on the unit circle $\mathbb{T}$, $\alpha > -1/2$. The elements of $\boldsymbol{G}_n$ are $c_{k-\ell}$, with $c_k = (-1)^k \dfrac{\Gamma(\alpha+1)\Gamma(\alpha+1/2)}{\sqrt{\pi}\Gamma(\alpha+k+1)\Gamma(\alpha-k+1)} = \dfrac{(-1)^k\Gamma(2\alpha+1)}{4^\alpha\Gamma(\alpha+k+1)\Gamma(\alpha-k+1)}$ (see later on, § 2.1 for unit circle orthogonality)

One finds $\|\Phi_n\|^2 = \dfrac{n!\Gamma(2\alpha+n+1)}{4^\alpha[\Gamma(\alpha+n+1)]^2}$,

$$\boldsymbol{L}_{k,\ell} = \|\Phi_\ell\|(-1)^{k-\ell}\binom{k}{\ell}\frac{\Gamma(\alpha+1)\Gamma(\alpha+\ell+1)}{\Gamma(\alpha-k+\ell+1)\Gamma(\alpha+k+1)},\ \boldsymbol{L}^{-1}_{k,\ell} = \|\Phi_k\|^{-1}\binom{k}{\ell}\frac{\Gamma(\alpha+k-\ell)\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+k+1)\Gamma(\alpha)},$$

$$\Phi_n(z) = \sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\Gamma(\alpha+n-\ell)\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+n+1)\Gamma(\alpha)} z^\ell. \tag{3}$$

Gegenbauer: sieved polynomials $\dots,\Phi_n(z^2), z\Phi_n(z^2),\dots$

Indeed, first, the product of the two latter matrices is, for $k \geqslant \ell$,

$$\sum_{m=\ell}^{k}\|\Phi_m\|(-1)^{k-m}\binom{k}{m}\frac{\Gamma(\alpha+1)\Gamma(\alpha+m+1)}{\Gamma(\alpha-k+m+1)\Gamma(\alpha+k+1)}\|\Phi_m\|^{-1}\binom{m}{\ell}\frac{\Gamma(\alpha+m-\ell)\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+m+1)\Gamma(\alpha)}$$
$$= \alpha\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+k+1)}\sum_{m=\ell}^{k}(-1)^{k-m}\binom{k}{m}\binom{m}{\ell}\frac{\Gamma(\alpha+m-\ell)}{\Gamma(\alpha-k+m+1)}$$
$$= \alpha\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+k+1)}\binom{k}{\ell}\sum_{m=\ell}^{k}(-1)^{k-m}\binom{k-\ell}{m-\ell}\frac{\Gamma(\alpha+m-\ell)}{\Gamma(\alpha-k+m+1)}$$
$$= \alpha\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+k+1)}\binom{k}{\ell}\left[\Delta^{k-\ell}(\alpha+m-\ell-1)(\alpha+m-\ell-2)\cdots(\alpha-k+m+1)\right]_{m=\ell}$$

$= \delta_{k,\ell}$, as $(\alpha+m-\ell-1)(\alpha+m-\ell-2)\cdots(\alpha-k+m+1)$ is a polynomial of degree $k-\ell-1$ if $k>\ell$.

Next, the $(k,\ell)^{\text{th}}$ element of $\boldsymbol{L}\overline{\boldsymbol{L}}^T$ is

$$\sum_{m=0}^{\min(k,\ell)}\|\Phi_m\|(-1)^{k-m}\binom{k}{m}\frac{\Gamma(\alpha+1)\Gamma(\alpha+m+1)}{\Gamma(\alpha-k+m+1)\Gamma(\alpha+k+1)}\|\Phi_m\|(-1)^{\ell-m}\binom{\ell}{m}\frac{\Gamma(\alpha+1)\Gamma(\alpha+m+1)}{\Gamma(\alpha-\ell+m+1)\Gamma(\alpha+\ell+1)}$$
$$= \frac{(-1)^{k-\ell}\Gamma(2\alpha+1)}{4^\alpha\Gamma(\alpha+k-\ell+1)\Gamma(\alpha-k+\ell+1)}\,\frac{1}{(\alpha+k)\cdots(\alpha+1)(\alpha+\ell)\cdots(\alpha+\ell-k+1)}$$
$$\sum_{m=0}^{k}\frac{(2\alpha+m)\cdots(2\alpha+1)}{m!}k\cdots(k-m+1)\ell\cdots(\ell-m+1)\alpha\cdots(\alpha-k+m+1)(\alpha+k-\ell)\cdots(\alpha-\ell+m+1),\ \text{if } k\leqslant\ell.$$

The sum must be $(\alpha+k)\cdots(\alpha+1)(\alpha+\ell)\cdots(\alpha+\ell-k+1)$. It is indeed a polynomial $\alpha^{2k}+\cdots$ of degree $2k$ in α. The polynomial vanishes at $\alpha=-1,\dots,-k$, as it contains then the $(-2\alpha-1)^{\text{th}}$ difference of $(k-m-\alpha-1)\dots(k-m+1)(\ell-m-\alpha-1)\dots(\ell-m+1)$ which has degree $-2\alpha-2$ in m, etc.

It is current practice to accept that the scalar product may fail to be positive definite on polynomials of exact degree N, i.e., that some nonzero polynomials of degree N, but not $N-1$, may have zero norm: $\|\Phi_{N-1}\| > 0$, $\|\Phi_N\| = 0$, as Φ_N achieves least possible norm. If this happens, Φ_N, which is still orthogonal to any polynomial of degree $< N$, is the only monic polynomial of degree N with zero norm: any other monic polynomial of degree N is $\Phi_N + \Psi_{N-1}$, and $\|\Phi_N+\Psi_{N-1}\|^2 = \|\Phi_N\|^2 + 2\text{Re}\,(\Phi_N,\Psi_{N-1}) + \|\Psi_{N-1}\|^2 = \|\Psi_{N-1}\|^2 > 0$. See for instance [37].

What is stated above is still true, the positive semidefinite $\boldsymbol{G}_N$ still has a Cholesky factorization.

1.4. **Decreasing sequences.** In the semidefinite case just seen the sequence of norms $\{\|\Phi_n\|\}$ stops at $\|\Phi_N\| = 0$. Can we measure such a situation by producing a positive decreasing sequence up to the index $N-1$? The sequence of norms $\{\|\Phi_n\|\}$ is generally *not* decreasing (it is in the unit circle case…), but an interesting decreasing sequence is certainly the sequence of smallest eigenvalues of $\boldsymbol{G}_n$, from the minimizing property:

$$\lambda_{\min}(\boldsymbol{G}_n) = \min_{\|\boldsymbol{\eta}\|=1} \overline{\boldsymbol{\eta}}^T \boldsymbol{G}_n \boldsymbol{\eta},$$

where $\|\boldsymbol{\eta}\|$ is the Euclidian norm on $\mathbb{C}^n$.

Any item constructed as a minimum on larger and larger subspaces will do, such as norms of *kernel polynomials*:

$$\min \|p\|, \text{ on } p \in \mathscr{P}_n, \text{ with } p(z_0) = 1.$$

Solution [6]:

$$\frac{K_n(z,z_0)}{K_n(z_0,z_0)} = \frac{\sum_{k=0}^{n} \overline{\varphi_k(z_0)}\,\varphi_k(z)}{\sum_{k=0}^{n} |\varphi_k(z_0)|^2},$$

and the norm is $\dfrac{1}{\sqrt{\sum_{k=0}^{n} |\varphi_k(z_0)|^2}}$, $(n < N)$.

A very interesting feature of the coefficients of the kernel polynomial $K_n(z) := K_n(z,0)$ is that they build the *first* column of $\boldsymbol{G}_n^{-1}$:

$$\boldsymbol{G}_n \begin{bmatrix} K_n(0) \\ K_n'(0) \\ \vdots \\ K_{n,n} \end{bmatrix} = \begin{bmatrix} (K_n, z^0) \\ (K_n, z) \\ \vdots \\ (Kn, z^n) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4}$$

from the reproducing property $(K_n, f) = \overline{f(0)}$ for all f of degree $\leqslant n$.

A hint of Szegő's theory: the sequence $\{K_n(0)\}$ is of course increasing. Should it be bounded, the *functions* K_n make a Cauchy sequence: if $m \leqslant n$, $\|K_n - K_m\|^2 = \|K_n\|^2 - 2\,\mathrm{Re}\,(K_n, K_m) + \|K_m\|^2 = K_n(0) - K_m(0)$. What can $\lim_{n\to\infty} K_n$ be?

1.5. **Most general scalar product.** on polynomials is

$$\left(\sum a_k z^k, \sum b_\ell z^\ell\right) = \begin{bmatrix} \overline{b_0} & \overline{b_1} & \cdots & \overline{b_L} \end{bmatrix} \boldsymbol{G}^T_{(K,L)} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_K \end{bmatrix}$$

with any positive definite Hermitian tableau $\boldsymbol{G}$, whether it can be interpreted as a functional integral or not.

If one does not go further than the degree N, one may always write (f,g) through functional values: $(f,g) = \sum_{k,\ell=0}^{N} h_{k,\ell} f(a_\ell)\overline{g(a_k)}$, where $h_{k,\ell} = (\ell_\ell, \ell_k)$ (Lagrange interpolation).

A kind of "true support" could be deduced as the set of z_0 such that the sequence $\{K_n(z_0,z_0)\}$ is bounded or mildly increasing, for what the meaning of this may be.

1.6. $\int_S f(t)\overline{g(t)}\,d\mu(t)$ **scalar product.**

Here, $d\mu$ is a positive measure supported on a closed complex set S containing at least N points.

The two arguments scalar product form is now a simple linear form with the single argument $f\overline{g}$, written (Maroni, Marcellan):

$$(f,g) = u(f\overline{g}). \tag{5}$$

Theorem. (Fejér) [6] *All the zeros of Φ_n are in the convex closure of the support S.*

Indeed, let z_k be a zero of Φ_n, and $\Psi(z) = \dfrac{\Phi_n(z)}{z - z_k}$. As Ψ has degree $n-1$,

$$0 = \int_S \Phi_n(t)\overline{\Psi(t)}\,d\mu(t) = \int_S (t - z_k)|\Psi(t)|^2\,d\mu(t) \Rightarrow z_k = \frac{\int_S t|\Psi(t)|^2\,d\mu(t)}{\int_S |\Psi(t)|^2\,d\mu(t)}$$

is therefore in the convex closure of S, as center of mass of the non-negative distribution $|\Psi|^2\,d\mu$ on S. □

See other proof in [1]

A very interesting example of orthogonal polynomials on the unit circle are the ***Rogers-Szegő*** polynomials [21]

$$\Phi_n(z) := \sum_{j=0}^{n} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-j+1})}{(1-q)(1-q^2)\cdots(1-q^j)}\,z^j.$$

Zeros have strange properties if $|q| = 1$ and the argument of q is a *rational* multiple of π [21]. . .

2. $\displaystyle\int_{\mathbb{T}} f(t)\overline{g(t)}\,\frac{d\mu(t)}{2\pi}$ scalar product on the unit circle $\mathbb{T}$.

2.1. Toeplitz matrix and Fourier series.

Now, the Gram matrix is the ***Toeplitz matrix***

$$\boldsymbol{G}_n = \begin{bmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix},$$

with

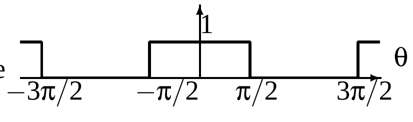
$$c_k = \frac{1}{2\pi}\int_{\mathbb{T}} t^{-k}\,d\mu(t), \tag{6}$$

$k \in \mathbb{Z}$ (or $k = -N, -N+1, \cdots, N-1, N$).

We shall only consider measures $d\mu$ without singularly continuous part, then $d\mu(t) = w(\theta)d\theta$, where w is $2\pi-$ periodic positive integrable, together with a countable set of positive Dirac masses. Then, c_k is the *Fourier* coefficient of w:

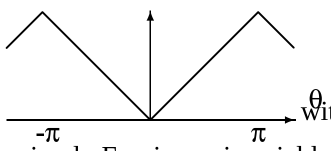
$$w(\theta) = \sum_{-\infty}^{\infty} c_k e^{ik\theta}.$$

Simplest Fourier series is the square wave



The Fourier coefficients are $c_0 = 1/2$, $c_k = (-1)^{(k-1)/2}/(\pi k)$ for odd k, $c_k = 0$ for even $k \neq 0$.

Another well known expansion is $w(\theta) = |\theta|$ on $[-\pi, \pi]$:



with $c_0 = \pi/2$, $c_k = -2/(\pi k^2)$ for odd k, $c_k = 0$ for even $k \neq 0$. Will these simple Fourier series yield easy orthogonal polynomials?

2.2. **Recurrence relation.**

An Hermitian Toeplitz matrix turns into its conjugate when elements are moved left-to-right and top-to-bottom

$\overline{\boldsymbol{G}_n} = \boldsymbol{S}\boldsymbol{G}_n\boldsymbol{S}$ $, \boldsymbol{S} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}.$

So, $\boldsymbol{G}_n^{-1} = \boldsymbol{S}\,\overline{\boldsymbol{G}_n^{-1}}\,\boldsymbol{S}$.

Remember from (4) that the first column is made of the coefficients of K_n. It is therefore the same than the conjugate upside-down *last* column: $\begin{bmatrix} K_n(0) \\ K_n'(0) \\ \vdots \\ K_{n,n} \end{bmatrix} = \frac{1}{\|\Phi_n\|^2} \begin{bmatrix} 1 \\ \overline{\xi_{n-1,n}} \\ \vdots \\ \overline{\xi_{0,n}} = \overline{\Phi_n(0)} \end{bmatrix}$ (from (2)), i.e.,

$$K_n(z) := \sum_{k=0}^{n} \frac{\overline{\Phi_k(0)}\,\Phi_k(z)}{\|\Phi_k\|^2} = \frac{z^n\overline{\Phi_n(1/\bar{z})}}{\|\Phi_n\|^2} := \frac{\Phi_n^*(z)}{\|\Phi_n\|^2}. \tag{7}$$

As Φ_n leads to the knowledge of $K_n =$ constant Φ_n^*, we use the obvious recurrence for the K's:

$$\frac{\Phi_n^*}{\|\Phi_n\|^2} - \frac{\Phi_{n-1}^*}{\|\Phi_{n-1}\|^2} = \frac{\overline{\Phi_n(0)}\,\Phi_n}{\|\Phi_n\|^2},$$

which is the * operator applied to

$$\Phi_n(z) - z\frac{\|\Phi_n\|^2}{\|\Phi_{n-1}\|^2}\Phi_{n-1}(z) = \Phi_n(0)\Phi_n^*(z), \tag{8}$$

and combine the two latter equations:

$$\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z), \tag{9}$$

together with an interesting identity

$$\frac{\|\Phi_n\|^2}{\|\Phi_{n-1}\|^2} = 1 - |\Phi_n(0)|^2. \tag{10}$$

Of course, $\Phi_n(0)$ is not yet known when we try to deduce Φ_n from Φ_{n-1}!

In matrix-vector form, (9) tells that $\begin{bmatrix} \Phi_n(0) \\ \Phi_n'(0) \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \Phi_{n-1}(0) \\ \vdots \\ 1 \end{bmatrix} + \Phi_n(0) \begin{bmatrix} 1 \\ \overline{\xi_{n-2,n-1}} \\ \vdots \\ 0 \end{bmatrix}$, which is not surprising, as a left-multiplication by $\boldsymbol{G}_n$ shows $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \|\Phi_n\|^2 \end{bmatrix} = \begin{bmatrix} \eta_{n-1} \\ 0 \\ \vdots \\ \|\Phi_{n-1}\|^2 \end{bmatrix} + \Phi_n(0) \begin{bmatrix} \|\Phi_{n-1}\|^2 \\ 0 \\ \vdots \\ \overline{\eta_{n-1}} \end{bmatrix}$, where η_{n-1} can be computed from Φ_{n-1}:

$\eta_{n-1} = c_{-1}\Phi_{n-1}(0) + c_{-2}\xi_{1,n-1} + \cdots + c_{-n+1}\xi_{n-2,n-1} + c_{-n} = (z\Phi_{n-1}(z), z^0)$.

Let us look at more rows of the matrix-vector relation:

$$\begin{bmatrix} c_{-2} & c_{-3} & \cdots & c_{-n-2} & c_{-n-3} \\ c_{-1} & c_{-2} & \cdots & c_{-n-1} & c_{-n-2} \\ c_0 & c_{-1} & \cdots & c_{-n} & c_{-n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 & c_{-1} \\ c_{n+1} & c_n & \cdots & c_1 & c_0 \\ c_{n+2} & c_{n+1} & \cdots & c_2 & c_1 \end{bmatrix} \begin{bmatrix} \xi_{0,n+1} & 0 & \overline{\xi_{n,n}} \\ \xi_{1,n+1} & \xi_{0,n} & \overline{\xi_{n-1,n}} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \xi_{n,n+1} & \xi_{n-1,n} & \overline{\xi_{0,n}} \\ \xi_{n+1,n+1} & \xi_{n,n} & 0 \end{bmatrix} = \begin{bmatrix} u(z^2\Phi_{n+1}(z)) & u(z^3\Phi_n(z)) & \overline{u(z^{-n-2}\Phi_n(z))} \\ u(z\Phi_{n+1}(z)) & u(z^2\Phi_n(z)) & \overline{u(z^{-n-1}\Phi_n(z))} \\ 0 & u(z\Phi_n(z)) & \|\Phi_n\|^2 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \|\Phi_{n+1}\|^2 & \|\Phi_n\|^2 & \overline{u(z\Phi_n(z))} \\ u(z^{-n-2}\Phi_{n+1}(z)) & u(z^{-n-1}\Phi_n(z)) & \overline{u(z^2\Phi_n(z))} \end{bmatrix}$$

and, as 1st column $=$ 2nd column $+\Phi_{n+1}(0)$ times 3rd column,
$u(z^2\Phi_n(z)) = u(z\Phi_{n+1}(z)) - \Phi_{n+1}(0)\overline{u(z^{-n-1}\Phi_n(z))}$
$u(z^{-n-2}\Phi_{n+1}(z)) = u(z^{-n-1}\Phi_n(z)) + \Phi_{n+1}(0)[\overline{u(z\Phi_{n+1}(z))} - \overline{\Phi_{n+1}(0)}u(z^{-n-1}\Phi_n(z))]$,
finally:

$$u(z\Phi_n(z)) = \eta_n = -\Phi_{n+1}(0)\|\Phi_n\|^2, \tag{11}$$

$$u(z^{-n-1}\Phi_n(z)) = -(\Phi_0(0)\overline{\Phi_1(0)} + \Phi_1(0)\overline{\Phi_2(0)} + \cdots + \Phi_n(0)\overline{\Phi_{n+1}(0)})\|\Phi_n\|^2, \tag{12}$$

Other forms of (9):

$$\begin{bmatrix} \Phi_{n+1}(z) \\ \Phi^*_{n+1}(z) \end{bmatrix} = \begin{bmatrix} z & \Phi_{n+1}(0) \\ z\overline{\Phi_{n+1}(0)} & 1 \end{bmatrix} \begin{bmatrix} \Phi_n(z) \\ \Phi^*_n(z) \end{bmatrix} ; \begin{bmatrix} \Phi_n(z) \\ \Phi^*_n(z) \end{bmatrix} = \frac{1}{1-|\Phi_{n+1}(0)|^2} \begin{bmatrix} z^{-1} & -\Phi_{n+1}(0)z^{-1} \\ -\overline{\Phi_{n+1}(0)} & 1 \end{bmatrix} \begin{bmatrix} \Phi_{n+1}(z) \\ \Phi^*_{n+1}(z) \end{bmatrix} \tag{13}$$

Also,

$$\begin{bmatrix} \Phi_{n+r}(z) \\ \Phi^*_{n+r}(z) \end{bmatrix} = \begin{bmatrix} zU_{r-1}(z;n) & V_{r-1}(z;n) \\ zV^*_{r-1}(z;n) & U^*_{r-1}(z;n) \end{bmatrix} \begin{bmatrix} \Phi_n(z) \\ \Phi^*_n(z) \end{bmatrix} \tag{14}$$

where U_{r-1} and V_{r-1} are polynomials of degrees $\leqslant r-1$:

r	$U_r(z;n)$	$V_r(z;n)$
0	1	x_{n+1}
1	$z + x_{n+2}\overline{x_{n+1}}$	$x_{n+1}z + x_{n+2}$
2	$z^2 + (x_{n+2}\overline{x_{n+1}} + x_{n+3}\overline{x_{n+2}})z + x_{n+3}\overline{x_{n+1}}$	$x_{n+1}z^2 + (x_{n+3}\overline{x_{n+2}}x_{n+1} + x_{n+2})z + x_{n+3}$

where $x_n = \Phi_n(0)$, etc.
One has $U_r(z;n) = zU_{r-1}(z;n) + x_{n+r-1}V^*_{r-1}(z;n), V_r(z;n) = zV_{r-1}(z;n) + x_{n+r-1}U^*_{r-1}(z;n)$.

The Cholesky factorization becomes

$$\boldsymbol{G}_n = \boldsymbol{L}\overline{\boldsymbol{L}}^T \;\Rightarrow\; \boldsymbol{G}_n^{-1} = (\boldsymbol{S}\overline{\boldsymbol{L}}^T\boldsymbol{S})^{-1}(\boldsymbol{SLS})^{-1}, \tag{15}$$

which is the Cholesky factorization of $\boldsymbol{G}_n^{-1}$!

2.3. **Multiplication operator: Toeplitz matrices in Grenander and Szegő's sense.**

A deep remark (Bultheel, Marcellan): the operator $\mathscr{M}$ of multiplication by z is represented in any polynomial basis by a Hessenberg matrix $\begin{bmatrix} x & x & x & x & x & x & x & x & \cdots \\ x & x & x & x & x & x & x & x & \cdots \\ & x & x & x & x & x & x & x & \cdots \\ & & x & x & x & x & x & x & \cdots \\ & & & x & x & x & x & x & \cdots \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$. On the real interval, the operator is *symmetric*, hence the matrix representation is tridiagonal $\Rightarrow$ the three terms recurrence relation; on the circle, the operator is *unitary*.

On curves $|P(z)| =$ constant (*lemniscates*), or $\mathrm{Re}\,P(z) =$ constant, one has to consider the polynomial $P(\mathcal{M})$ (Vigil, Marcellan).

Let $\{\varphi_0, \varphi_1, \dots\}$ be a complete orthonormal sequence in a Hilbert function space X, then

$$\mathcal{M}(f) = \begin{bmatrix} \langle \varphi_0 f, \varphi_0\rangle & \langle \varphi_1 f, \varphi_0\rangle & \langle \varphi_2 f, \varphi_0\rangle & \cdots \\ \langle \varphi_0 f, \varphi_1\rangle & \langle \varphi_1 f, \varphi_1\rangle & \langle \varphi_2 f, \varphi_1\rangle & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{16}$$

represents the operator of multiplication by f if an element F of X is reprensented by its sequence of coefficients $\{c_k(F)\}$ in the expansion $F = \sum_0^\infty c_K(F)\varphi_k$. Indeed [16, 22],

$$\begin{bmatrix} c_0(F) \\ c_1(F) \\ \vdots \end{bmatrix} = \begin{bmatrix} c_0(fF) \\ c_1(fF) \\ \vdots \end{bmatrix}.$$

One also have $\mathcal{M}(f)\mathcal{M}(g) = \mathcal{M}(fg)$.

The $\mathcal{M}$ of above is simply $\mathcal{M}(z)$. On the circle, we have $\mathcal{M}_{m,n} = \int_{\mathbb{T}} z\varphi_m(z)\,\overline{\varphi_n(z)}\,d\mu$, so that $\mathcal{M}(z^{-1})$ is the transpose of the complex cojugate of $\mathcal{M}$, and is also $\mathcal{M}^{-1}$: $\mathcal{M}$ is a unitary matrix.

The n^{th} column (starting at 0) of $\mathcal{M}$ is made with the coefficients of $z\varphi_n(z)$. From the recurrence relation (9),

$$\begin{aligned} z\varphi_n(z) = \frac{z\Phi_n(z)}{\|\Phi_n\|} &== \frac{\Phi_{n+1}(z) - \Phi_{n+1}(0)\Phi_n^*(z)}{\|\Phi_n\|} \\ &= \frac{\|\Phi_{n+1}\|}{\|\Phi_n\|}\varphi_{n+1}(z) - \Phi_{n+1}(0)\|\Phi_n\| \sum_{k=0}^{n} \overline{\varphi_k(0)}\varphi_k(z) \end{aligned}$$

from (7). Whence

$$\mathcal{M}_{n+1,n} = \frac{\|\Phi_{n+1}\|}{\|\Phi_n\|}\;, \mathcal{M}_{k,n} = -\frac{\|\Phi_{n+1}\|\Phi_{n+1}(0)\overline{\Phi_k(0)}}{\|\Phi_k\|}\;, k \leqslant n.$$

found in B.Simon's preprint [29], with reference to Nevai [] for some identities involving the Φ_n^*s.

2.4. First and last columns of $\boldsymbol{G}_n^{-1}$.

2.4.1. *First column.*

We already saw that the first column of $\boldsymbol{G}_n^{-1}$ is made of the coefficients of the kernel polynomial K_n, which is here $\Phi_n^*/\|\Phi_n\|^2$, so is

$$[\overline{\xi_{n,n}}, \overline{\xi_{n-1,n}}, \dots, \overline{\xi_{1,n}}, \overline{\xi_{0,n}}]^T / \|\Phi_n\|^2,$$

2.4.2. *Second column.*

Let us try $z\Phi_{n-1}^*(z)$, i.e.,

$$\begin{bmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix} \begin{bmatrix} 0 \\ \overline{\xi_{n-1,n-1}} \\ \overline{\xi_{n-2,n-1}} \\ \vdots \\ \overline{\xi_{1,n}} \\ \overline{\xi_{0,n}} \end{bmatrix}.$$

Thanks to the Toeplitz structure, the n last elements of the product are $\|\Phi_{n-1}\|^2, 0, \dots, 0$, and the first element of the product is $u(z\Phi_{n-1}^*(z)) = \overline{u(z^{-n}\Phi_{n-1}(z))}$, known from (11). So, second column is

$$\{[0, \overline{\xi_{n-1,n-1}}, \overline{\xi_{n-2,n-1}}, \dots, \overline{\xi_{1,n-1}}, \overline{\xi_{0,n-1}}]^T - \overline{u(z^{-n}\Phi_{n-1}(z))}[\overline{\xi_{n,n}}, \overline{\xi_{n-1,n}}, \dots, \overline{\xi_{1,n}}, \overline{\xi_{0,n}}]^T / \|\Phi_n\|^2\} / \|\Phi_{n-1}\|^2.$$

2.4.3. *Last column.* Known to be

$$[\xi_{0,n},\xi_{1,n},\ldots,\xi n,n]/\|\Phi_n\|^2.$$

2.4.4. *First before last column.*

$$\{[\xi_{0,n-1},\xi_{1,n-1},\ldots,\xi_{n-1,n-1},0]^T-u(z^{-n}\Phi_{n-1}(z))[\xi_{0,n},\xi_{1,n},\ldots,\xi_{n-1,n},\xi_{n,n}]^T/\|\Phi_n\|^2\}/\|\Phi_{n-1}\|^2.$$

2.5. **Behaviour of reflection coefficients.**

With the program PARI [3],

```
{
\\  reflec.gp    :  launch gp and make  \r reflec
\\
\\  Reflection coefficients given Fourier coefficients
\\
N=22;
}
\\ vector of Fourier coeff.  c_k = c[k+N+1]
{
 Foucoeff(k)=

   if(k==0,  1.0/2 ,
             (I^k-(-I)^k)/(2*Pi*I*k)
     )
}
{
c=vector(2*N+1,k,Foucoeff(k-N-1));
Phin=x^0;normPhin2=c[N+1];
for(n=1,N-1,
    print(n-1," ",polcoeff(Phin,0),"   ",normPhin2);
    scalPhizm1 = sum(k=1,n, c[N+1-k]*polcoeff(Phin,k-1) );
  \\print(scalPhizm1);
    Phin = x*Phin -(scalPhizm1/normPhin2)* polrecip(conj(Phin));
    normPhin2  -=  scalPhizm1*conj(scalPhizm1)/normPhin2;
   )
}
```

```
Script V1.1 session started Wed Aug 18 11:19:38 1999

C:\calc\pari>gp
                   GP/PARI CALCULATOR Version 2.0.12 (alpha)
                 ix86 running emx (ix86 kernel) 32-bit version
                (readline enabled, extended help not available)

                          Copyright (C) 1989-1998 by
          C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

Type ? for help, \q to quit.
Type ?12 for how to get moral (and possibly technical) support.
```

```
   realprecision = 28 significant digits
   seriesprecision = 16 significant terms
   format = g0.28

parisize = 4000000, primelimit = 500000
? \r reflec
```

One finds more and more complicated expressions, $\Phi_1(0) = -\frac{2}{\pi}$, $\Phi_2(0) = \frac{4}{\pi^2-4}$, $\Phi_3(0) = -\frac{2}{3\pi}\,\frac{16-\pi^2}{\pi^2-8}$, etc. which we can perhaps explore further through numerical values:

n	$\Phi_n(0)$	$\|\Phi_n\|^2$
0	1	0.5000000000000000000000000000
1	-0.6366197723675813430755350535	0.2973576327153244571122410735
2	0.6814769321178829752922525447	0.1592615339410585123538737276
3	-0.6958211852068310273669299707	0.08215233544212813372178866790
4	0.7010703573176610826474364438	0.04177449166168262201071826460
5	-0.7033737451875978766777148650	0.02110720417736141209933540938
6	0.7045635263840126108977529709	0.01062938196019166045331220303
7	-0.7052581523381021267708256617	0.005342443643331533145252544298
8	0.7057004468702704665764798408	0.002681836612283204103389461465
9	-0.7060001718947297527963793848	0.001345112047680199190623848503
10	0.7062129569336616331861078971	0.0006742552493577655387959406246
...		
19	-0.7068612070337355176642206532	0.000001331509045263803822704034814
20	0.7068852157013753769732944692	0.0000006661716735277939346122326538

```
? quit
Good bye!

C:\calc\pari>exit

Script completed Wed Aug 18 11:20:08 1999
```

And the $|\theta|$ example:

```
...
{
 Foucoeff(k)=

   if(k==0,  Pi/2 ,
             ((-1)^k-(1)^k)/(Pi*k^2)
     )
}
Script V1.1 session started Mon Sep 20 17:03:52 1999

C:\calc\pari>gp
? \r reflec
```

n	$\Phi_n(0)$	$\|\Phi_n\|^2$
0	1	1.570796326794896619231321691
1	0.4052847345693510857755178528	1.312784051329300705755945270
2	0.1965382464879410164306311434	1.262074771136928879634464159
3	0.1551854679951381617834429861	1.231680818259009766792617169

```
 4 0.10981250624809368230712298831   1.216828242200391418572744386
 5 0.095913512665188517541131261901  1.205634150142790986549470868
 6 0.076238285440464057240917826361  1.198626671505913288215311961
 7 0.069374661837324527824884379271  1.192857868675259251133628348
 8 0.058398932032048957274246564761  1.188789704136797789141563113
 9 0.054328902255899020226443797861  1.185280837233723055616638076
10 0.047329391322324741621375011201  1.182625723668014639762979918
...
19 0.026050765650854981267428710161  1.170344822783165775227484266
20 0.024306448690830646129894771311  1.169653379026563365817484145
? quit
Good bye!

C:\calc\pari>exit

Script completed Mon Sep 20 17:04:26 1999
```

The description and understanding of the behaviour of these coefficients $\Phi_n(0)$ is the *main subject* of the present lecture. The following completely explicit cases are known:

- Special relations, as in p. 3, also in [36, 37],
- Rogers-Szegő polynomials, p. 8,
- Bernstein-Szegő, § 2.8.1, p. 17,
- Bernstein-Szegő pol. on a circular arc [11],
- Gegenbauer, p. 6, and more generally:
- Jacobi, § 2.7,
- and others from known interval cases, from § 2.6, as in [36].

2.6. **Circle versus interval .**

Among other feats, Szegő achieved the description of the connection between orthogonal polynomials on $[-1,1]$ and orthogonal polynomials on the unit circle [31, § 11.5]. The connection is usually interpreted as a reduction of interval polynomials to the "more basic" circle polynomials, but wait.

Ah, let P_n be the n^{th} degree monic polynomial orthogonal with respect to a measure $d\nu$ on $[-1,1]$, i.e., a (real) scalar product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,d\nu(x)$. We express orthogonality to lower degree polynomials through *Chebyshev* polynomials of *first kind*: let

$$P_n(x) = \frac{d_0}{2} + d_1T_1(x) + \cdots + d_nT_n(x),$$

which must be $\langle,\rangle-$orthogonal to $T_0,\ldots,T_{n-1}$:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \langle P_n,T_n\rangle \end{bmatrix} = \begin{bmatrix} \frac12\langle T_0,T_0\rangle & \cdots & \langle T_n,T_0\rangle \\ \vdots & & \vdots \\ \frac12\langle T_0,T_{n-1}\rangle & \cdots & \langle T_n,T_{n-1}\rangle \\ \frac12\langle T_0,T_n\rangle & \cdots & \langle T_n,T_n\rangle \end{bmatrix} \begin{bmatrix} d_0 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}.$$

As $\langle T_\ell,T_k\rangle = \langle T_\ell T_k,1\rangle$, and $T_\ell T_k = \dfrac{T_{\ell+k}+T_{\ell-k}}{2}$, let $c_k = \langle T_k,1\rangle$. Remark that $c_{-k}=c_k$, also that $c_k = \int_{-1}^{1} T_k(x)\,d\nu(x) = \int_{\pi}^{0} \cos k\theta\,d\nu(\cos\theta) = \dfrac12\int_{\pi}^{-\pi} e^{-ik\theta}\,d\nu(\cos\theta) = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\,d\mu(e^{i\theta})$, with

$d\mu(e^{i\theta}) = -\pi d\nu(\cos\theta)$. Then, the equations above may be rearranged as

$$\begin{bmatrix} 2\langle P_n, T_n\rangle \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 2\langle P_n, T_n\rangle \end{bmatrix} = \begin{bmatrix} c_0 & \cdots & c_{1-n} & c_{-n} & c_{-n-1} & \cdots & c_{-2n} \\ c_1 & \cdots & c_{2-n} & c_{1-n} & c_{-n} & \cdots & c_{-2n+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_n & \cdots & c_1 & c_0 & c_{-1} & \cdots & c_{-n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{2n-1} & \cdots & c_n & c_{n-1} & c_{n-2} & \cdots & c_{-1} \\ c_{2n} & \cdots & c_{n+1} & c_n & c_{n-1} & \cdots & c_0 \end{bmatrix} \begin{bmatrix} d_n \\ \vdots \\ d_1 \\ d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix},$$

which is exactly a Toeplitz system! The solution is $\langle P_n, T_n\rangle$ times the sum of the first and the last columns of $\boldsymbol{G}_{2n}^{-1}$, so, $2z^nP_n(x) =$

$$d_nz^{2n} + \cdots + d_1z^{n+1} + d_0z^n + d_1z^{n-1} + \cdots d_n = 2\langle P_n, T_n\rangle \left(K_{2n}(z) + \frac{\Phi_{2n}(z)}{\|\Phi_{2n}\|^2}\right) = 2\frac{\langle P_n, T_n\rangle}{\|\Phi_{2n}\|^2}(\Phi_{2n}(z) + \Phi_{2n}^*(z)),$$

with $d_n = 2^{1-n} \Rightarrow \|\Phi_{2n}\|^2 = 2^n\langle P_n, T_n\rangle(1 + \Phi_{2n}(0))$. (N.B., $\Phi_{2n}(0)$ is real).

Conversely, can we recover the Φ's from the P's?? We need the *difference* of the first and last columns of $\boldsymbol{G}_{2n}^{-1}\ldots$ Let Q_{n-1} be the monic orthogonal polynomial of degree $n-1$ with respect to some scalar product $\{\ ,\ \}$ on $[-1,1]$. We use the basis of Chebyshev polynomials of *second* kind: $Q_{n-1} = \sum_{k=0}^{n-1} e_kU_k$,

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \{Q_{n-1}, U_{n-1}\} \end{bmatrix} = \begin{bmatrix} \{U_0, U_0\} & \cdots & \{U_{n-1}, U_0\} \\ \vdots & & \vdots \\ \{U_0, U_{n-2}\rangle & \cdots & \{U_{n-1}, U_{n-2}\} \\ \{U_0, U_{n-1}\} & \cdots & \{U_{n-1}, U_{n-1}\} \end{bmatrix} \begin{bmatrix} e_0 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{bmatrix}.$$

Now, $U_\ell(\cos\theta)U_k(\cos\theta) = \dfrac{\sin(\ell+1)\theta\ \sin(k+1)\theta}{\sin^2\theta} = \dfrac{\cos(\ell-k)\theta - \cos(k+\ell+2)\theta}{\sin^2\theta} = \dfrac{T_{\ell-k}(x) - T_{k+\ell+2}(x)}{1-x^2}$, and we recover the c_k's if $\{f(x), 1\} = \langle (1-x^2)f(x), 1\rangle$, or $\{f, g\} = \int_{-1}^{1} f(x)g(x)\,(1-x^2)\,d\nu(x)$. The convenient rearrangement of the equations is

$$\begin{bmatrix} 2\{Q_{n-1}, U_{n-1}\} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -2\{Q_{n-1}, U_{n-1}\} \end{bmatrix} = \begin{bmatrix} c_0 & \cdots & c_{1-n} & c_{-n} & c_{-n-1} & \cdots & c_{-2n} \\ c_1 & \cdots & c_{2-n} & c_{1-n} & c_{-n} & \cdots & c_{-2n+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & \cdots & c_0 & c_{-1} & c_{-2} & \cdots & c_{-n-1} \\ c_n & \cdots & c_1 & c_0 & c_{-1} & \cdots & c_{-n} \\ c_{n+1} & \cdots & c_2 & c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{2n-1} & \cdots & c_n & c_{n-1} & c_{n-2} & \cdots & c_{-1} \\ c_{2n} & \cdots & c_{n+1} & c_n & c_{n-1} & \cdots & c_0 \end{bmatrix} \begin{bmatrix} e_{n-1} \\ \vdots \\ \vdots \\ e_0 \\ 0 \\ -e_0 \\ \vdots \\ \vdots \\ -e_{n-1} \end{bmatrix},$$

so, $-z^n(z - z^{-1})Q_{n-1}(x) =$

$$e_{n-1} + \cdots + e_0z^{n-1} + 0z^n - e_0z^{n+1} - \cdots - e_{n-1}z^{2n} = 2\{Q_{n-1}, U_{n-1}\}\left(K_{2n}(z) - \frac{\Phi_{2n}(z)}{\|\Phi_{2n}\|^2}\right)$$

$$= 2\frac{\{Q_{n-1}, U_{n-1}\}}{\|\Phi_{2n}\|^2}(-\Phi_{2n}(z) + \Phi_{2n}^*(z)),$$

with $e_{n-1} = 2^{1-n} \Rightarrow \|\Phi_{2n}\|^2 = 2^n\{Q_{n-1}, U_{n-1}\}(1 - \Phi_{2n}(0))$.

Finally,

$$\Phi_{2n}(z) = \frac{\|\Phi_{2n}\|^2}{2\langle P_n, T_n\rangle} z^n P_n(x) + \frac{\|\Phi_{2n}\|^2}{4\{Q_{n-1}, U_{n-1}\}} z^n (z - z^{-1}) Q_{n-1}(x),$$

$$\Phi^*_{2n}(z) = \frac{\|\Phi_{2n}\|^2}{2\langle P_n, T_n\rangle} z^n P_n(x) - \frac{\|\Phi_{2n}\|^2}{4\{Q_{n-1}, U_{n-1}\}} z^n (z - z^{-1}) Q_{n-1}(x),$$

$$\Phi_{2n} \text{ monic } \Rightarrow \|\Phi_{2n}\|^2 = \frac{2^{n+1}}{\dfrac{1}{\langle P_n, T_n\rangle} + \dfrac{1}{\{Q_{n-1}, U_{n-1}\}}},$$

$$\Phi_{2n}(0) = \frac{\{Q_{n-1}, U_{n-1}\} - \langle P_n, T_n\rangle}{\{Q_{n-1}, U_{n-1}\} + \langle P_n, T_n\rangle},$$

$$\|\Phi_{2n-1}\|^2 \; \|\Phi_{2n}\|^2 = 2^{n-1}\{Q_{n-1}, U_{n-1}\}\langle P_n, T_n\rangle\,,$$

$$\Phi_{2n-1}(z) = \frac{\|\Phi_{2n-1}\|^2}{\|\Phi_{2n}\|^2}\, \frac{\Phi_{2n}(z) - \Phi_{2n}(0)\Phi^*_{2n}(z)}{z} = 2^{n-1} z^{n-1}\left[P_n(x) + \frac{z - z^{-1}}{2} Q_{n-1}(x)\right].$$

See also [36]

2.7. **Jacobi polynomials on the unit circle.**

Here, $d\mu(e^{i\theta}) = \left(\cos\dfrac{\theta}{2}\right)^{2\beta}\left|\sin\dfrac{\theta}{2}\right|^{2\alpha} d\theta \quad (-\pi < \theta < \pi,\ \alpha \text{ and } \beta > -1/2).$

$d\nu(\cos\theta) = (-1/\pi)d\mu(e^{i\theta})$, so $d\nu(x) = (1/\pi)\left(\dfrac{1+x}{2}\right)^{\beta}\left(\dfrac{1-x}{2}\right)^{\alpha}(1-x^2)^{-1/2}\,dx,$

$P_n = P^{\alpha-1/2,\beta-1/2}_{n,\text{monic}} = x^n + \dfrac{n(\alpha-\beta)}{2n+\alpha+\beta-1}x^{n-1} + \cdots, Q_{n-1} = P^{\alpha+1/2,\beta+1/2}_{n-1,\text{monic}} = x^{n-1} + \dfrac{(n-1)(\alpha-\beta)}{2n+\alpha+\beta-1}x^{n-2} + \cdots,$

$\langle P_n, T_n\rangle = 2^{n-1}\langle P_n, P_n\rangle = \frac{2^{3n+\alpha+\beta-1}\Gamma(n+\alpha+1/2)\Gamma(n+\beta+1/2)\Gamma(n+\alpha+\beta)n!}{\Gamma(2n+\alpha+\beta)\Gamma(2n+\alpha+\beta+1)},$

$\{Q_{n-1}, U_{n-1}\} = 2^{n-1}\{Q_{n-1}, Q_{n-1}\} = \frac{2^{3n+\alpha+\beta-1}\Gamma(n+\alpha+1/2)\Gamma(n+\beta+1/2)\Gamma(n+\alpha+\beta+1)(n-1)!}{\Gamma(2n+\alpha+\beta)\Gamma(2n+\alpha+\beta+1)},$

$$\Phi_{2n}(0) = \frac{\alpha+\beta}{2n+\alpha+\beta},$$

$$\Phi_{2n-1}(z) = 2^{n-1}z^{n-1}\left[\left(\frac{z+z^{-1}}{2}\right)^n + \frac{n(\alpha-\beta)}{2n+\alpha+\beta-1}\left(\frac{z+z^{-1}}{2}\right)^{n-1} + \frac{z-z^{-1}}{2}\left(\frac{z+z^{-1}}{2}\right)^{n-1} + \frac{z-z^{-1}}{2}\,\frac{(n-1)(\alpha-\beta)}{2n+\alpha+\beta-1}\left(\frac{z+z^{-1}}{2}\right.\right.$$

$$= z^{2n-1} + \frac{(2n-1)(\alpha-\beta)}{2n+\alpha+\beta-1}z^{2n-2} + \cdots + \frac{\alpha-\beta}{2n+\alpha+\beta-1}$$

so,

$$\Phi_n(0) = \frac{\alpha + (-1)^n\beta}{n+\alpha+\beta} \tag{17}$$

[2]

2.8. **The Szegő-Geronimus theory.**

Szegő [31] investigated the cases where the increasing sequence $\{K_n(0)\}$ remains bounded, the sequence of functions

$$K_n(z;0) = \sum_{k=0}^{n}\overline{\varphi_k(0)}\varphi_k(z) = \sum_{k=0}^{n}\overline{\Phi_k(0)}\Phi_k(z)/\|\Phi_k\|^2$$

is a Cauchy sequence. What about the limit?

2.8.1. *The Bernstein-Szegő polynomials* . If $p(z) = p_0 z^{-d} + \cdots + p_{2d} z^d$ is a Laurent polynomial, with real values on the unit circle, $p_k = \overline{p_{2d-k}}$: $p_{2d-k} = (2\pi)^{-1} \int_{\mathbb{T}} p(e^{i\theta}) e^{i(-d+k)\theta} d\theta$ is the complex conjugate of $p_k = (2\pi)^{-1} \int_{\mathbb{T}} p(e^{i\theta}) e^{i(d-k)\theta} d\theta$. The $2d$ zeros of p come in inverse pairs: if $p(\zeta) = 0$, $p(1/\overline{\zeta}) = \sum_k p_k (1/\overline{\zeta})^{-d+k} = \overline{\sum_k p_{2d-k} \zeta^{d-k}} = 0$.

So, $p(z) = q(z)r(z)$ (Fejér), where $q(z) = q_0 + q_1 z^{-1} + \cdots + q_d z^{-d}$ has its zeros in the unit disk (in the open unit disk if p does not vanish on the unit circle), and $r(z) = r_0 + \cdots + r_d z^d$ has its zeros outside the unit disk.

When $w(\theta) = 1/p(e^{i\theta})$, where p is a positive Laurent polynomial of degree d,

$$\Phi_n(z) = \frac{1}{q_0} q(z) z^n$$

from $n = d$ onwards.
Indeed, $(\Phi_n(z), z^k) = (1/2\pi i q_0) \int_{\mathbb{T}} q(z) z^{n-k} \frac{1}{q(z)r(z)} \frac{dz}{z} = 0$ for $k = 0, \ldots, n-1$, as there is no residue. When $k = n$, we find the residue $\|\Phi_n\|^2 = 1/(q_0 r_0)$.

And, when $n \geqslant d$, K_n is the polynomial

$$K_n(z) = \frac{\Phi_n^*(z)}{\|\Phi_n\|^2} = q_0 r(z).$$

2.8.2. *The Szegő theory.*

A necessary condition to get a generalization of the factorization of w is $\log w \in L^1$, so that $\log w$ has a Fourier series, and we take

$$\log w = \underbrace{\cdots + \lambda_{-2} z^{-2} + \lambda_{-1} z^{-1} + \lambda_0}_{\log q(z)} + \underbrace{\lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots}_{\log r(z)},$$

and we expect $\Phi_n^*(z) \to r(z)/exp(\lambda_0)$ when $n \to \infty$, for $z \in \mathbb{D}$. For the coefficients:

$$\Phi_n^*(z) = 1 + \overline{\xi_{n-1,n}}\, z + \overline{\xi_{n-2,n}}\, z^2 + \cdots \to_{n\to\infty} \exp(\lambda_1 z + \lambda_2 z^2 + \cdots).$$

Proofs and extensions: [31, 8, etc.].

2.8.3. *Some theorems by Geronimus.* cf. [13]

The following statements are equivalent

- $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$,
- $\sum_{n=0}^{\infty} |\varphi_n(z)|^2 < \infty$ for at least one point $z \in \mathbb{D}$,
- there is a subsequence n_k such that $\varphi_{n_k}^*(z)$ converges at least for one point $z \in \mathbb{D}$,
- $\lim_{n\to\infty} \varphi_n^*(z) = S(z)$ uniformly inside $\mathbb{D}$.

2.9. **Formal unit circle orthogonal polynomials: non hermitian Toeplitz matrices.**

3. Semi-classical orthogonal polynomials on unit circle.

3.1. **Definition, forms, and differential equation for weight function.**

Cf. [1] and references therein

We can not guess easily the behaviour, especially the asymptotic behaviour of the reflection coefficients $-\Phi_n(0)$ from the entries c_k with the recurrence relation (9): this relation is a universal relation for all unit circle orthogonal polynomials. We only need $\Phi_n(0) = \xi_{0,n}$, but (9) compels us to get all the two-dimensional tableau of the $\xi_{k,n}$'s.

It is very useful to select classes of orthogonal polynomials whose coefficients $\Phi_n(0)$ satisfy a kind of one-dimensional recurrence relation. This will happen with the class defined hereafter, where such recurrence relations will be associated to *differential relations*.

Speaking of differential relations, the famous Sonine-Hahn characterization of *classical* orthogonal polynomials as having orthogonal derivatives falls short on the unit circle, as it only works for z^n [1,24].

In the language of forms defined on Laurent polynomials [1], one defines the product of a function and a form fu as the form such that $\forall p, (fu)(p) = u(fp)$; and the derivative Du of u as the form such that $\forall p, (Du)(p) = -iu(tp'(t))$. This latter strange definition corresponds actually to a simple relation for the *weight functions*: let $d\mu(t = e^{i\theta}) = w(\theta)d\theta$, where w has at most a finite number of Dirac masses, and let P be a Laurent polynomial vanishing at these masses, then if $u(p) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} p(e^{i\theta})d\mu(e^{i\theta})$,
$\forall p : (Pu)(p) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P(e^{i\theta})p(e^{i\theta})w(\theta)\,d\theta$;

$$(D(Pu))(p) = -i(Pu)(tp'(t)) = \frac{1}{2\pi i}\int_0^{2\pi} P(e^{i\theta})e^{i\theta}\frac{dp(e^{i\theta})}{de^{i\theta}}w(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} p(e^{i\theta})\frac{dP(e^{i\theta})w(\theta)}{d\theta}\,d\theta,$$

i.e., $D(Pu)$ has the integral representation involving $d[P(e^{i\theta})w(\theta)]/d\theta$. Remark that, in order to get rid of annoying boundary terms in the integration by parts, P must vanish at all Dirac points and other discontinuities of w.

Definition. A semi-classical form on the circle satisfies

$$D(A(z)u) = B(z)u, \tag{18}$$

with Laurent polynomials A and B, which means, if u has an integral representation involving $w(\theta)d\theta$, that

$$\frac{d[A(e^{i\theta})w(\theta)]}{d\theta} = B(e^{i\theta})w(\theta)\,,$$

or

$$\frac{dw(\theta)/d\theta}{w(\theta)} = \frac{B(e^{i\theta}) - dA(e^{i\theta})/d\theta}{A(e^{i\theta})} = \frac{B(e^{i\theta}) - ie^{i\theta}\,dA(e^{i\theta})/de^{i\theta}}{A(e^{i\theta})}$$

with $A(e^{i\theta}) = 0$ at the singular points of w.

Jacobi:

$$u(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(e^{i\theta})\left(\cos\frac{\theta}{2}\right)^{2\beta}\left|\sin\frac{\theta}{2}\right|^{2\alpha}\,d\theta\,,$$

$$\frac{dw(\theta)/w(\theta)}{w(\theta)} = \alpha\cot\frac{\theta}{2} - \beta\tan\frac{\theta}{2} = i\alpha\frac{z+2+z^{-1}}{z-z^{-1}} + i\beta\frac{z-2+z^{-1}}{z-z^{-1}}\,,\ (z = e^{i\theta})\,,$$

so, $A(z) = z - z^{-1}$, $B(z) = A(z)(dw(\theta)/d\theta)/w(\theta) + izdA(z)/dz = i(\alpha+\beta+1)z + 2i(\alpha-\beta) + i(\alpha+\beta-1)z^{-1}$, or, if we prefer polynomials without negative powers,

$$A(z) = z^2 - 1; B(z) = i\{(\alpha+\beta+2)z^2 + 2(\alpha-\beta)z + \alpha+\beta\}$$

Exercise. Show that the square wave exemple is semi-classical, but that $w(\theta) = |\theta|$ is not semi-classical.

3.2. **Recurrence relation for the moments.**

From (6) (p. 8), the moment c_k is just $u(t^{-k})$ when u has the integral representation $u(p) = (2\pi)^{-1}\int_{\mathbb{T}} p(e^{i\theta})d\mu$. So, we define $c_k = u(t^{-k})$, $k \in \mathbb{Z}$ for any form defined on Laurent polynomials.

Let us apply (18) to momomials t^{-k}, $k \in \mathbb{Z}$:

$$iu(A(t)kt^{-k}) = u(B(t)t^{-k}), \qquad k \in \mathbb{Z}.$$

If $A(t) = \sum_{p=0}^{d} a_p t^{p-p_0}$ and $B(t) = \sum_{p=0}^{d} b_p t^{p-p_0}$, then

$$\sum_{p=0}^{d}(ika_p - b_p)c_{k-p+p_0} = 0, \qquad k \in \mathbb{Z}. \tag{19}$$

A linear recurrence relation of the form (19) is another way to recognize a semi-classical functional.

Without more information than (19), we may find differential equations for the generating functions $G_{\pm(z)} = \sum_{m=0}^{\infty} c_{\pm m} z^m$, and contour integral representations for c_k

to be continued

3.3. **Differential relations for the orthogonal polynomials and recurrence relations for the reflection coefficients.**

Let A and B be (plain, i.e., without negative powers) polynomials of degree $\leqslant d$. Then, the product $A(z)d\Phi_n(z)/dz$ is a remarkably short combination of the Φ and the Φ^*s.

The matrix $\boldsymbol{G}_{n+d-1}$ times the vector $[A(0)\Phi_n'(0), A(0)\Phi_n''(0)+A'(0)\Phi_n'(0), \ldots, (n-1)A_d\xi_{n-1,n}+nA_{d-1}, nA_d]^T$ of the coefficients of $A\Phi_n'$ is the vector of

$$(A(z)\Phi_n'(z), z^k) = u(A(z)\Phi_n'(z)z^{-k}) = u(zA(z)[\Phi_n(z)z^{-k-1}]') - u(zA(z)\Phi_n(z)[z^{-k-1}]')$$
$$= i(D(Au))(\Phi_n(z)z^{-k-1}) + (k+1)u(\Phi_n(z)A(z)z^{-k-1}) = iu(B(z)\Phi_n(z)z^{-k-1}) + (k+1)u(A(z)\Phi_n(z)z^{-k-1})$$
$$= u\left(\Phi_n(z)\frac{iB(z)+(k+1)A(z)}{z^{k+1}}\right)$$

for $k = 0, \ldots, n+d-1$. The latter form vanishes as soon as $(iB(z)+(k+1)A(z))z^{-k-1}$ contains only nonpositive powers, downto z^{-n+1}, which makes $k = d-1, \ldots, n-2$. Therefore, $A\Phi_n'$ is a combination of $\Phi_{n+d-1}, \ldots, \Phi_{n-1}$, and $\Phi^*_{n+d-1}, z\Phi^*_{n+d-2}, \ldots, z^{d-2}\Phi^*_{n+1}$.

A more detailed look:

$$\boldsymbol{G}_{n+d-1}\begin{bmatrix} A(0)\Phi_n'(0) \\ A(0)\Phi_n''(0)+A'(0)\Phi_n'(0) \\ \vdots \\ (n-1)A_d\xi_{n-1,n}+nA_{d-1} \\ nA_d \end{bmatrix} = \begin{bmatrix} \vdots \\ u((iB_d+(d-1)A_d)z\Phi_n(z)) \\ 0 \\ \vdots \\ 0 \\ u((iB(0)+nA(0))\Phi_n(z)/z^n) \\ \vdots \end{bmatrix} \begin{matrix} \\ k=d-2 \\ \\ \\ \\ k=n-1 \\ \\ \end{matrix}$$

$$\begin{aligned} A(z)\Phi_n'(z) = \cdots\Phi^*_{n+d-1}(z) + \cdots + (iB_d+(d-1)A_d)\eta_n z^{d-2}\frac{\Phi^*_{n+1}(z)}{\|\Phi_{n+1}\|^2} \\ + (iB(0)+nA(0))\|\Phi_n\|^2\frac{\Phi_{n-1}(z)}{\|\Phi_{n-1}\|^2} + \cdots + \cdots\Phi_{n+d-1}(z), \end{aligned} \tag{20}$$

where we already encountered $\eta_n = -\Phi_{n+1}(0)\|\Phi_n\|^2$ p. 9

Manifesto.

As all the polynomials Φ_n depend, from the recurrence relations (9)*, on the reflection coefficients* $-\Phi_m(0)$*, the semi-classical identities will yield equations for these reflection coefficients.*

How Φ_n depends on the $\Phi_m(0)$: with $x_n := \Phi_n(0)$,

$$\Phi_n(z) = z^n + \xi_{n-1,n} z^{n-1} + \cdots + \Phi_n'(0)z + \Phi_n(0) \text{ with } \xi_{n-1,n} = x_1\overline{x_0} + x_2\overline{x_1} + \cdots + x_n\overline{x_{n-1}}\,,$$
$$\Phi_n'(0) = \Phi_{n-1}(0) + \Phi_n(0)\overline{\xi_{n-2,n-1}}.$$

Simplest case: Jacobi $(\alpha, 0)$, $d = 1$, $A(z) = z-1$, $B(z) = i(\alpha+1)z + i\alpha$, $iB(0)+nA(0) = -(n+\alpha)$, one finds, from (3)

$$(z-1)\Phi_n'(z) = n\Phi_n(z) - \frac{n(n+2\alpha)}{n+\alpha}\Phi_{n-1}(z).$$

remember (10): $\|\Phi_n\|^2/\|\Phi_{n-1}\|^2 = 1 - |\Phi_n(0)|^2$.

3.4. **More Jacobi polynomials on the unit circle.**

Let us see how the features of the polynomials Φ_n, in particular the reflection coefficients $-\Phi_n(0)$, can be recovered (and, why not, *discovered*) from the semi-classical identities:

$d = 2$, we know $A(z) = z^2 - 1$, $B(z) = i(\alpha+\beta+2)z^2 + 2i(\alpha-\beta)z + i(\alpha+\beta)$.

$(z^2-1)\Phi_n'(z) = nz^{n+1} + (n-1)\xi_{n-1,n}z^n + [(n-2)\xi_{n-2,n} - n]z^{n-1} + \cdots =$
$X_n\Phi_{n+1}^*(z) + Y_n\Phi_{n+1}(z) + Z_n\Phi_n(z) + W_n\Phi_{n-1}(z)$,

We know that $X_n = -(\alpha+\beta+1)\Phi_{n+1}(0)\|\Phi_n\|^2/\|\Phi_{n+1}\|^2 = -(\alpha+\beta+1)\dfrac{\Phi_{n+1}(0)}{1-|\Phi_{n+1}(0)|^2}$,
$W_n = -(n+\alpha+\beta)\|\Phi_n\|^2/\|\Phi_{n-1}\|^2 = -(n+\alpha+\beta)(1-|\Phi_n(0)|^2)$.

Comparing the terms in

- z^{n+1}: $X_n\overline{\Phi_{n+1}(0)} + Y_n = n$,
- z^n: $X_n\overline{\Phi_{n+1}'(0)} + Y_n\xi_{n,n+1} + Z_n = (n-1)\xi_{n-1,n}$,
- z^{n-1}: $X_n\overline{\xi_{2,n+1}} + Y_n\xi_{n-1,n+1} + Z_n\xi_{n-1,n} = (n-2)\xi_{n-2,n} - n$.

The two first equations merely yield expressions for Y_n and Z_n, and the third equation is actually an equation for the $\Phi_m(0)$… after all the ξ's have been expanded!

We avoid to have to bother with the $\xi_{2,n}$'s by
$Z_n\|\Phi_n\|^2 + W_n u(z^{-n}\Phi_{n-1}(z)) = (iB(0) + (n+1)A(0))u(z^{-n-1}\Phi_n(z)) + (iB'(0) + nA'(0))\|\Phi_n\|^2$

$x_n := \Phi_n(0)$:

$$(n+\alpha+\beta+1)x_{n+1} = (n+\alpha+\beta-1)x_{n-1} + \frac{2x_n \operatorname{Im}(\overline{x_0}x_1 + \overline{x_1}x_2 + \cdots + \overline{x_{n-1}}x_n)}{1-|x_n|^2}$$

if x_1 is real, so are the next x_n's, and one recovers (17)
what if x_1 is *not* real?

3.5. **The second order differential equation.**

The most attractive feature for most people is obviously the linear second order differential equation

$$zA(z)A^*(z)\Theta_n(z)\Phi_n''(z) = R_n(z)\Phi_n'(z) + S_n(z)\Phi_n(z) \tag{21}$$

with Θ_n of degree $\leqslant d$, R_n and S_n

Indeed, we put (14) in (20), to have $A\Phi_n' =$ polynomials times Φ_n and Φ_n^*, and also a constant times Φ_{n-1}, which is eliminated from the second equation of (13), the net result is

$$zA(z)\Phi_n'(z) = \mathfrak{A}_n(z)\Phi_n(z) + \mathfrak{B}_n(z)\Phi_n^*(z) \tag{22}$$

with $\mathfrak{A}_n$ and $\mathfrak{B}_n$ of degree $\leqslant d$. Considering the contribution of Φ_{n-1} in (20), one has $\mathfrak{A}_n(0) = iB(0) + nA(0)$ and $\mathfrak{B}_n(0) = -\Phi_n(0)[iB(0)+nA(0)]$.

With

$$zA^*(z)(\Phi_n^*(z))' = -\mathfrak{B}_n^*(z)\Phi_n(z) + (nA^*(z) - \mathfrak{A}_n^*(z))\Phi_n^*(z), \tag{23}$$

using[3] $(z\Phi_n')^*(z) = n\Phi_n^*(z) - z(\Phi^*)'(z)$, we have a linear differential system of the first order for $[\Phi_n, \Phi_n^*]$. And we eliminate Φ_n^* between (22)-(23)

3.6. Linear differential equation for Jacobi polynomials on the unit circle: a bigger flop.

Of course we want to see the differential equation for Jacobi polynomials on the unit circle, to compare with the differential equation for plain Jacobi polynomials. What a thrill.

...

to be completed

...

So, a unit circle Jacobi polynomial is a more composite object than a plain Jacobi polynomial. Thats's why I do not believe that reduction to unit circle is always the simplest thing to do.

4. Generalized Jacobi polynomials on the unit circle, with two singular points.

4.1. The recurrence relation for the $\Phi_n(0)$'s.

We must have $\dfrac{dw/d\theta}{w} = \dfrac{az^2+bz+c}{(z-e^{i\theta_1})(z-e^{i\theta_2})}$, real on the unit circle. As denominator $/(z\exp(i(\theta_1+\theta_2)/2))$ is real, $(az+b+cz^{-1})/\exp(i(\theta_1+\theta_2)/2)$ must remain real for z on the unit circle $\Rightarrow |a| = |c|, ac\exp(-i(\theta_1+\theta_2)) > 0$ and $b\exp(-i(\theta_1+\theta_2)/2)$ real.

We solve now for w:

$$\frac{dw/dz}{w} = \frac{dw/d\theta}{izw} = \frac{az+b+cz^{-1}}{i(z-e^{i\theta_1})(z-e^{i\theta_2})}$$
$$= \frac{2\alpha}{z-e^{i\theta_1}} + \frac{2\beta}{z-e^{i\theta_2}} + \frac{\tilde{\gamma}}{z},$$

with the residues

$$2\alpha = \frac{ae^{i\theta_1}+b+ce^{-i\theta_1}}{i(e^{i\theta_1}-e^{i\theta_2})} = \frac{ae^{i\theta_1}+b+ce^{-i\theta_1}}{2e^{i(\theta_1+\theta_2)/2}\sin((\theta_2-\theta_1)/2)}, 2\beta = \frac{ae^{i\theta_2}+b+ce^{-i\theta_2}}{2e^{i(\theta_1+\theta_2)/2}\sin((\theta_1-\theta_2)/2)}, \tilde{\gamma} = \frac{c}{ie^{i(\theta_1+\theta_2)}},$$

so, $w =$ constant $z^{\tilde{\gamma}}(z-e^{i\theta_1})^{2\alpha}(z-e^{i\theta_2})^{2\beta} =$ constant $\exp(i(\tilde{\gamma}+\alpha+\beta)\theta)|\sin((\theta-\theta_1)/2)|^{2\alpha}|\sin((\theta-\theta_2)/2)|^{2\beta}$, with two constants on the two arcs of endpoints $\exp(i\theta_1)$ and $\exp(i\theta_2)$.

α and β are real, $\tilde{\gamma}+\alpha+\beta = \dfrac{a}{2i} + \dfrac{c}{2ie^{i(\theta_1+\theta_2)}} = i\gamma$ is a pure imaginary number. So,

$w =$ constants $\exp(-\gamma\theta)|\sin((\theta-\theta_1)/2)|^{2\alpha}|\sin((\theta-\theta_2)/2)|^{2\beta}$,

$d = 2, A(z) = (z-e^{i\theta})(z-e^{i\theta_2}) = z^2 - (e^{i\theta_1}+e^{i\theta_2})z + e^{i(\theta_1+\theta_2)}$,

$B(z) = izdA/dz + A[(dw/d\theta)/w] = izdA/dz + Aiz[(dw/dz)/w] = iz(2z-e^{i\theta_1}-e^{i\theta_2}) + i[2\alpha z(z-e^{i\theta_2}) + 2\beta z(z-e^{i\theta_1})(i\gamma-\alpha-\beta)(z-e^{i\theta_1})(z-e^{i\theta_2})]$,

$B(z) = i(\alpha+\beta+2+i\gamma)z^2 + [i(\alpha-\beta)(e^{i\theta_1}-e^{i\theta_2}) + (\gamma-i)(e^{i\theta_1}+e^{i\theta_2})]z - (i(\alpha+\beta)+\gamma)e^{i(\theta_1+\theta_2)}$.

The building of the equations proceeds as before,

$$A(z)\Phi_n'(z) = X_n\Phi_{n+1}^*(z) + Y_n\Phi_{n+1}(z) + Z_n\Phi_n(z) + W_n\Phi_{n-1}(z) = X_n'x_{n+1}\Phi_n^*(z) + n\Phi_{n+1}(z) + Z_n\Phi_n(z) + W_n\Phi_{n-1}(z),$$

$$X_n'x_{n+1} = X_n(1-|x_{n+1}|^2) = iB_2 + A_2 = -(\alpha+\beta+1+i\gamma),$$

---

[3] If $\Phi(z) = az^n + bz^{n-1} + \cdots + uz + v$, $\Phi^*(z) = \overline{v}z^n + \overline{u}z^{n-1} + \cdots + \overline{b}z + \overline{a}$, $(\Phi^*)'(z) = n\overline{v}z^{n-1} + (n-1)\overline{u}z^{n-2} + \cdots + \overline{b}$, $(z\Phi')^*(z) = \overline{u}z^{n-1} + \cdots + (n-1)\overline{b}z + n\overline{a}$.

$W_n = (iB(0)+nA(0))(1-|x_n|^2) = (n+\alpha+\beta-i\gamma)e^{i(\theta_1+\theta_2)}(1-|x_n|^2)$,
and we look at the coefficients of $A\Phi_n'$:
z^n: $(n-1)\xi_{n-1,n} - n(e^{i\theta_1}+e^{i\theta_2}) = X_n' x_{n+1}\overline{x_n} + n\xi_{n,n+1} + Z_n$,
z^0: $A(0)\Phi_n'(0) = X_n' x_{n+1} + n x_{n+1} + Z_n x_n + W_n x_{n-1}$,
elimination of Z_n:
$A(0)[x_{n-1}+x_n\overline{\xi_{n-2,n-1}}] = (n+X_n')x_{n+1} + (n-1)\xi_{n-1,n}x_n - n(e^{i\theta_1}+e^{i\theta_2})x_n - X_n' x_n x_{n+1}\overline{x_n} - n x_n \xi_{n,n+1} + W_n x_{n-1}$,
which is a linear equation for x_{n+1} if we know all the x_m with $m \leqslant n$,
$(n+X_n')(1-|x_n|^2)x_{n+1} = [A(0)\overline{\xi_{n-1,n}} + \xi_{n-1,n} + n(e^{i\theta_1}+e^{i\theta_2})]x_n + A(0)(1-|x_n|^2)x_{n-1} - W_n x_{n-1}$,

$$(n+1+\alpha+\beta+i\gamma)x_{n+1} = \frac{e^{i(\theta_1+\theta_2)}\overline{\xi_{n-1,n}} + \xi_{n-1,n} + n(e^{i\theta_1}+e^{i\theta_2})}{1-|x_n|^2}x_n - (n-1+\alpha+\beta-i\gamma)e^{i(\theta_1+\theta_2)}x_{n-1}, \quad (24)$$

with $\xi_{n-1,n} = x_1\overline{x_0} + x_2\overline{x_1} + \cdots + x_n\overline{x_{n-1}}$.

4.2. **Numerical experiments with the asymptotics of** (24).

4.2.1. . We now proceed to extract the last asymptotic secrets of the $\Phi_n(0) = x_n$'s of (24).

One obviously expects $\Phi_n(0)$ to behave like a combination of $e^{in\theta_1}$ and $e^{in\theta_2}$ with slowly varying coefficients. What works is

$$\Phi_n(0) \sim A_1 n^{\kappa_1} e^{in\theta_1} + A_2 n^{\kappa_2} e^{in\theta_2}$$

as guessed from [2, 9, 35, etc.], also [23]. The powers of n are deduced from (24) looking as a linear recurrence relation when $|x_n|^2$ is neglected, and where $\xi_{n-1,n}$ is replaced by its limit, say ξ. Then each particular solution $A_j n^{\kappa_j} e^{in\theta_j}$ satisfies $\dfrac{x_{n+1}}{x_n} \sim (1+\kappa_j/n)e^{i\theta_j}$, so (24) becomes $[1+(1+\alpha+\beta+i\gamma)/n](1+\kappa_j/n)e^{i\theta_j} \sim [e^{i(\theta_1+\theta_2)}\overline{\xi}+\xi]/n + e^{i\theta_1}+e^{i\theta_2}) - [1+(-1+\alpha+\beta-i\gamma)/n]e^{i(\theta_1+\theta_2)}(1-\kappa_j/n)e^{-i\theta_j}$, match the n^{-1} contributions: $(1+\alpha+\beta+i\gamma+\kappa_j)e^{i\theta_j} = e^{i(\theta_1+\theta_2)}\overline{\xi}+\xi+(1-\alpha-\beta+i\gamma+\kappa_j)e^{-i\theta_j}$, or

κ_1 and $\kappa_2 = -1-i\gamma \pm i\delta$, with $i\delta = -(\alpha+\beta)\dfrac{e^{i\theta_1}+e^{i\theta_2}}{e^{i\theta_1}-e^{i\theta_2}} + \dfrac{\overline{\xi}}{e^{-i\theta_2}-e^{-i\theta_1}} + \dfrac{\xi}{e^{i\theta_1}-e^{i\theta_2}}$.

What can A_1 and A_2 be? We perform numerical explorations. First with known cases: when $\theta_2-\theta_1=\pi$, and when $w(\theta)/[|\sin(\theta-\theta_1)/2|^{2\alpha}|\sin(\theta-\theta_2)/2|^{2\beta}]$ is the same constant on the whole circle, one must have $A_1=\alpha$, $A_2=\beta$, and $\delta=0$.

When $\gamma=0$, and where is no jump, more experiments lead to

$$A_1 = \alpha\exp(i\beta(\pi-\theta_2+\theta_1)), A_2 = \beta\exp(-i\alpha(\pi-\theta_2+\theta_1)), \quad (25)$$

where $\pi-\theta_2+\theta_1$ must be understood modulo 2π, with a value between $-\pi$ and π.

4.3. **Final (so far) experiments.**

4.3.1. *Discontinuous case.* Now, $w(\theta)/[|\sin(\theta-\theta_1)/2|^{2\alpha}|\sin(\theta-\theta_2)/2|^{2\beta}] =$ discontinuous, i.e.:

$$w(\theta) = \begin{cases} r_1 e^{-\gamma\theta}|\sin(\theta-\theta_1)/2|^{2\alpha}|\sin(\theta-\theta_2)/2|^{2\beta}, & \theta_1<\theta<\theta_2 \\ r_2 e^{-\gamma\theta}|\sin(\theta-\theta_1)/2|^{2\alpha}|\sin(\theta-\theta_2)/2|^{2\beta}, & \theta_2-2\pi<\theta<\theta_1] \end{cases} \quad (26)$$

where r_1 and r_2 are positive, possibly different. Of course, it's the ratio r_2/r_1 which is important.

The multiplicative jump is $\dfrac{r_1}{r_2}$ at θ_1, and $\dfrac{r_2 e^{-\gamma(\theta_2-2\pi)}}{r_1 e^{-\gamma\theta_2}} = e^{2\pi\gamma}\dfrac{r_2}{r_1}$ at θ_2.

We proceed with examining the output of (24), but where some tricks must be explained: (24) needs a complex number $x_1=\Phi_1(0)$ which is not so easy to get, as it is the ratio $-c_{-1}/c_0$ of the two Fourier coefficients $-\int_0^{2\pi} w(\theta)\exp(i\theta)\,d\theta/\int_0^{2\pi} w(\theta)\,d\theta$.

However, from the already encountered differential equation for $w =$ constants $z^{\tilde\gamma}(z-e^{i\theta_1})^{2\alpha}(z-e^{i\theta_2})^{2\beta}$,

$$\frac{dw/dz}{w} = \frac{\sum_{-\infty}^{\infty} k c_k z^{k-1}}{\sum_{-\infty}^{\infty} c_k z^k} = \frac{\tilde\gamma}{z} + \frac{2\alpha}{z-e^{i\theta_1}} + \frac{2\beta}{z-e^{i\theta_2}},$$

(almost) everywhere on $(0,2\pi)$, with $\tilde\gamma = i\gamma-\alpha-\beta$: $(\sum_{-\infty}^{\infty} k c_k z^k)(z^2-(e^{i\theta_1}+e^{i\theta_2})z+e^{i(\theta_1+\theta_2)}) =$ $(\sum_{-\infty}^{\infty} c_k z^k)[(i\gamma+\alpha+\beta)z^2+[(\beta-\alpha)(e^{i\theta_2}-e^{i\theta_1})-i\gamma(e^{i\theta_2}+e^{i\theta_1})]z+(i\gamma-\alpha-\beta)e^{i(\theta_1+\theta_2)}]$ whence the ***linear recurrence relation*** for the c_k's (see also (19), p. 19)

$(k-1-i\gamma-\alpha-\beta)c_{k-1}-[k(e^{i\theta_1}+e^{i\theta_2})+(\beta-\alpha)(e^{i\theta_2}-e^{i\theta_1})-i\gamma(e^{i\theta_2}+e^{i\theta_1})]c_k+(k+1+\alpha+\beta-i\gamma)e^{i(\theta_1+\theta_2)}c_{k+1}=0.$

At $k=0$, knowing that $c_1=\overline{c_{-1}}$: $(\alpha+\beta+1+i\gamma)x_1-(\beta-\alpha)(e^{i\theta_2}-e^{i\theta_1})+i\gamma(e^{i\theta_2}+e^{i\theta_1})-(\alpha+\beta+1-i\gamma)e^{i(\theta_1+\theta_2)}\overline{x_1}=0$, or $(\alpha+\beta+1+i\gamma)\frac{x_1}{e^{i\theta_2}-e^{i\theta_1}}+\alpha-\beta+\gamma\cot((\theta_2-\theta_1)/2)-(\alpha+\beta+1-i\gamma)\frac{\overline{x_1}}{e^{-i\theta_1}-e^{-i\theta_2}}$ amounting to

$$x_1 = (e^{i\theta_2}-e^{i\theta_1})\left[-\frac{\alpha-\beta+\gamma\cot((\theta_2-\theta_1)/2)}{2(\alpha+\beta+1+i\gamma)}+(\gamma+(\alpha+\beta+1)i)t\right], \quad t\in\mathbb{R}$$

so that we only have to experiment with various (real) values of t, which correspond to various real ratios r_2/r_1, which we recover in the Christoffel function-aided weight function reconstruction.

Check that $|x_1|<1$ when $t=0$: $|x_1| = \frac{|(\alpha-\beta)\sin((\theta_2-\theta_1)/2)+\gamma\cos((\theta_2-\theta_1)/2)|}{|\alpha+\beta+1+i\gamma|}$, OK if $|\alpha-\beta| < |\alpha+\beta+1|$.

Simple check with $\alpha=\beta=\gamma=0$: we immediately have $x_1=\Phi_1(0)=-c_{-1}/c_0=i(e^{i\theta_2}-e^{i\theta_1})(r_1-r_2)/[r_1(\theta_2-\theta_1)+r_2(2\pi-\theta_2+\theta_1)]$. When $\gamma\neq 0$, $x_1 = \frac{(i-\gamma)^{-1}[r_1(e^{(i-\gamma)\theta_2}-e^{(i-\gamma)\theta_1})+r_2(e^{(i-\gamma)\theta_1}-e^{(i-\gamma)(\theta_2-2\pi)})]}{\gamma^{-1}[r_1(e^{-\gamma\theta_2}-e^{-\gamma\theta_1})+r_2(e^{-\gamma\theta_1}-e^{-\gamma(\theta_2-2\pi)})]} =$

$$e^{i(\theta_1+\theta_2)/2}\frac{\gamma}{i-\gamma}\left[\cos((\theta_2-\theta_1)/2)+i\sin((\theta_2-\theta_1)/2)\underbrace{\frac{r_1(e^{-\gamma\theta_2}+e^{-\gamma\theta_1})-r_2(e^{-\gamma\theta_1}+e^{-\gamma(\theta_2-2\pi)})}{r_1(e^{-\gamma\theta_2}-e^{-\gamma\theta_1})+r_2(e^{-\gamma\theta_1}-e^{-\gamma(\theta_2-2\pi)})}}_{-2(\gamma^{-1}+\gamma)t}\right]$$

Check with $r_1/r_2=\exp(2\pi\rho)$ with $\rho=0.25$:

```
reflecj2 unit circle gen Jacobi  Sat 9 Mar 2013 19:26

 alpha=0 beta=0 , gamma=0.50000; theta1,2/pi=-0.35000 0.25000
 Phi1(0)= -0.063355 - 0.27785*I
A1=-0.0089576 - 0.24984*I A2=-0.0089729 - 0.24984*I
 xi=-0.045993 - 0.29027*I i delta= - 0.000022164*I
r1=0.79468 r2=0.16520 rho=0.25000
A1/(alpha-rho i)=0.99936 - 0.035830*I A2/(beta+(rho-gamma) i)=0.99936 - 0.035892*I
 abs: 1.0000 1.0000
```

```
2.4313888 |"'''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''|
          |:"                                                            |
          |: "                                                           |
          |:  "_                                                         |
          |:    _                                                        |
          :      x                                                       |
          :       x                                                      |
          :        "_                                                    |
          :          x                                                   |
          :           "_                                                 |
          "             x_                             "_                "
          |               x_                           : "_              :
          |                 x_                         :   "_            :
          |                   x_                       :     "x_        :|
          |                     "x_                    :        x_      :|
          |                        "x_                :           "x__  :|
          |                           "x__            :               "x_|
          |                               "xx_        :                  |
          |                                   ""xx___ :                  |
          |                                          ""                  |
          |                                                              |
        0 ''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''
          -5.497787                                      0.7853982
(19:26) gp > t
%7 = -0.060843
(19:27) gp > quit
Goodbye!
```

For general α and β, we could use that, as c_0 and c_{-1} are linear in r_1 and r_2, $x_1 = \dfrac{X + Y(r_2/r_1)}{1 + Z(r_2/r_1)}$, so that we could extract X, Y, and Z from three different trials, without having to compute X, Y, and Z, which are complicated integrals, and arrive at a full algorithm for computing long sequences of reflection coefficients for (26), given α, β, θ_1, θ_2, r_1, and r_2, but we just finish now asymptotic experiments.

In the recurrence (24), we know that $x_n \to 0$, and even that $\overline{\xi_{n-1,n}} \to_{n\to\infty} \lambda_1$, from Szegő-Geronimus theory, where

$$\log w = \underbrace{\cdots + \lambda_{-2} z^{-2} + \lambda_{-1} z^{-1} + \lambda_0}_{\log q(z)} + \underbrace{\lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots}_{\log r(z)},$$

We recover A_1 and A_2 from numerical runs and guess:

$$A_1 = (\alpha - \rho_1 i) e^{i\beta(\pi - \theta_2 + \theta_1) + i\psi_1}, \qquad A_2 = (\beta - \rho_2 i) e^{-i\alpha(\pi - \theta_2 + \theta_1) + i\psi_2},$$

with $\rho_1 = \dfrac{\log(r_1/r_2)}{2\pi}$ and $\rho_2 = \gamma - \rho_1$, the jumps of $(2\pi)^{-1} \log w$ at θ_1 and θ_2, as seen above.

The absolute values of A_1 and A_2 are probably correct, as they agree with what is needed in the Hartwig-Fisher formula of next section; the phases ψ_1 and ψ_2 are still unknown, they will be examined later on.

4.3.2. *Relation with Fisher-Hartwig determinants.*

We may ***conjecture*** that the main behaviour of $\Phi_n(0)$ for a weight which is smooth except at $\theta_1,\dots,\theta_p$ where it behaves like $r_{k,1}|\theta-\theta_k|^{2\alpha_k}$ for $\theta\to\theta_k, \theta<\theta_k$, and like $r_{k,2}|\theta-\theta_k|^{2\alpha_k}$ for $\theta\to\theta_k, \theta>\theta_k$, will be

$$x_n=\Phi_n(0)=\sum_{k=1}^{p}\frac{A_k n^{i\delta_k}}{n}e^{in\theta_k}+o(1/n),$$

with $|A_k|^2=\alpha_k^2+\rho_k^2$, where $\rho_k=\dfrac{\log(r_{k,1}/r_{k,2})}{2\pi}$. From (10),

$$\|\Phi_n\|^2=\prod_{m=1}^{n}(1-|\Phi_m(0)|^2)=\text{const.}\left(1+\frac{\sum_{k=1}^{p}|A_k|^2}{n}+\cdots\right),$$

as the only non-oscillating terms in the expansion of $|\Phi_m(0)|^2=\Phi_m(0)\overline{\Phi_m(0)}=\sum_k\sum_j A_k\overline{A_j}m^{i(\delta_k-\delta_j)}\exp(im(\theta_k-\theta_j))/m^2$ are the $|A_k|^2/m^2$ terms. The constant is actually very well known from the Szegő theory, as it must be $\exp(2\lambda_0)$.

Finally, the product of these square norms yields the determinant of the Gram matrix (here, a Toeplitz matrix) $\det \boldsymbol{G}_n=\prod_{m=1}^{n}\|\Phi_m\|^2=\text{const. } n^{\sum_k|A_k|^2}\exp(2n\lambda_0)$, (with another constant). This formula is exactly the Hartwig-Fisher asymptotic formula for such Toeplitz determinants, see [4] for details and full history!

4.3.3. *Last last calculations, Spring 2012 and March 2013.*

We look at the influence of the jump $\rho=\rho_1$ in $A_1=(\alpha-\rho_1 i)e^{i\beta(\pi-\theta_2+\theta_1)+i\psi_1}$, $A_2=(\beta-\rho_2 i)e^{-i\alpha(\pi-\theta_2+\theta_1)+i\psi_2}$, when $\alpha=\beta=\gamma=0$ and $\theta_1,\theta_2=\mp\pi/2$.

ρ	ψ_1	$d\psi_1/d\rho$	$\rho\dfrac{d^2\psi_1}{d\rho^2}$	$2\arg\Gamma(1+\rho i)-\rho\log 4$
-0.10347	0.26208			0.26201
0	0			0
0.10347	- 0.26208	-2.533		-0.26201
0.23504	- 0.58723			-0.58705
0.55998	- 1.3012			-1.30086
0.68822	- 1.5363			1.53590
0.98394	- 1.9705			-1.97015
1.8739	- 2.5110			-2.51096
1.9103	- 2.5133			-2.51325
1.9576	- 2.5142			-2.51425
2.0253	- 2.5119			-2.51191
4.4543	- 0.23977			-0.24197
4.8284	0.39333			0.39062
5.3055	1.2836			1.28040
5.9850	2.6974			2.69334
7.0797	5.2921			5.28652
8.1195	8.071			8.06407
8.9465	10.4736			10.4649
9.9983	13.7476			13.7374
11.009	17.1021	3.3190	2.006	17.0908
11.999	20.5684	3.5013	2.021	20.5555
13.015	24.2974	3.6703	1.954	24.2810
14.005	28.0801	3.8209	2.043	28.0638
15.000	32.0260	3.9657		32.0068

Various values of t, so of x_1 are entered in the recurrence (24), A_1 and A_2 are extracted from x_n with large n, and ρ from weight reconstruction. One finds $\psi_2=-\psi_1$, and we look at ψ_1 as a continuous function of ρ, which may ask for adding integer multiples of 2π.

The second order divided difference on almost uniformly spaced sets of three values of ρ yield a good estimate of (half the) second derivative at the middle point. The formula $d^2\psi_1/d\rho^2\sim 2/\rho$ seems to hold.

This means a main behaviour $2\rho\log\rho$ for the function itself. The simplest special function with this behaviour is the Gamma function. More precisely, an exploration of the pages of Abramowitz & Stegun's masterpiece, where even the pages with tables are wonderfully inspiring, shows that the imaginary part of $\log\Gamma(1+i\rho)$ is the needed odd continuous function of ρ. From Stirling formula, Abr 6.1.44, the behaviour for large positive ρ is $\rho\log\rho-\rho+\pi/4+o(1)$.So the first and second derivatives are close to $\log\rho$ and $1/\rho$.

The derivative at $\rho=0$ of two times the argument of the Gamma function of $1+i\rho$

is $-1.15443 = -2$ times the Euler constant. The derivative of our function there is estimated to be -2.533. The difference is -1.379, close enough to $-\log 4$. GOT IT! (Spring 2012). Music, at last!

For general θ_1 and θ_2, we find readily the term $-2\rho \log \sin((\theta_2-\theta_1)/2)$ to be present n ψ_1.

GAMMA FUNCTION FOR COMPLEX ARGUMENTS **Table 6.7**

$x=1.0$

y	$\mathscr{R} \ln \Gamma(z)$	$\mathscr{I} \ln \Gamma(z)$	y	$\mathscr{R} \ln \Gamma(z)$	$\mathscr{I} \ln \Gamma(z)$
0.0	0.00000 00000 00	0.00000 00000 00	5.0	- 6.13032 41445 53	3.81589 85746 15
0.1	- 0.00819 77805 65	- 0.05732 29404 17	5.1	- 6.27750 24635 84	3.97816 38691 88
0.2	- 0.03247 62923 18	- 0.11230 22226 44	5.2	- 6.42487 30533 35	4.14237 74050 86
0.3	- 0.07194 62509 00	- 0.16282 06721 68	5.3	- 6.57242 85885 29	4.30850 21885 83
0.4	- 0.12528 93748 21	- 0.20715 58263 16	5.4	- 6.72016 21547 03	4.47650 25956 68
0.5	- 0.19094 54991 87	- 0.24405 82989 05	5.5	- 6.86806 72180 48	4.64634 42978 70
0.6	- 0.26729 00682 14	- 0.27274 38104 91	5.6	- 7.01613 75979 76	4.81799 41933 05
0.7	- 0.35276 86908 60	- 0.29282 63511 87	5.7	- 7.16436 74421 06	4.99142 03424 89
0.8	- 0.44597 87835 49	- 0.30422 56029 76	5.8	- 7.31275 12034 30	5.16659 19085 37
0.9	- 0.54570 51286 05	- 0.30707 43756 42	5.9	- 7.46128 36194 29	5.34347 91013 53
1.0	- 0.65092 31993 02	- 0.30164 03204 68	6.0	- 7.60995 96929 51	5.52205 31255 15
1.1	- 0.76078 39588 41	- 0.28826 66142 39	6.1	- 7.75877 46746 55	5.70228 61315 35
1.2	- 0.87459 04638 95	- 0.26733 05805 81	6.2	- 7.90772 40468 98	5.88415 11702 39
1.3	- 0.99177 27669 59	- 0.23921 67844 65	6.3	- 8.05680 35089 04	6.06762 21500 13
1.4	- 1.11186 45664 26	- 0.20430 07241 49	6.4	- 8.20600 89631 00	6.25267 37967 05
1.5	- 1.23448 30515 47	- 0.16293 97694 80	6.5	- 8.35533 65025 11	6.43928 16159 76
1.6	- 1.35931 22484 65	- 0.11546 87935 89	6.6	- 8.50478 23991 25	6.62742 18579 12
1.7	- 1.48608 96127 57	- 0.06219 86983 29	6.7	- 8.65434 30931 23	6.81707 14837 44
1.8	- 1.61459 53960 00	- 0.00341 66314 77	6.8	- 8.80401 51829 10	7.00820 81345 02
1.9	- 1.74464 42761 74	+ 0.06061 28742 95	6.9	- 8.95379 54158 79	7.20081 01014 93
2.0	- 1.87607 87864 31	0.12964 63163 10	7.0	- 9.10368 06798 32	7.39485 62984 36
2.1	- 2.00876 41504 71	0.20345 94738 33	7.1	- 9.25366 79950 15	7.59032 62351 84
2.2	- 2.14258 42092 96	0.28184 56584 26	7.2	- 9.40375 45067 08	7.78719 99928 77
2.3	- 2.27743 81922 04	0.36461 40489 50	7.3	- 9.55393 74783 21	7.98545 82004 68
2.4	- 2.41323 81411 84	0.45158 81524 41	7.4	- 9.70421 42849 72	8.18508 20125 03
2.5	- 2.54990 68424 95	0.54260 44058 52	7.5	- 9.85458 24074 86	8.38605 30880 89
2.6	- 2.68737 61537 50	0.63751 09190 46	7.6	-10.00503 94267 90	8.58835 35709 62
2.7	- 2.82558 56411 91	0.73616 63516 79	7.7	-10.15558 30186 86	8.79196 60705 87
2.8	- 2.96448 14617 89	0.83843 89130 96	7.8	-10.30621 09489 48	8.99687 36442 29
2.9	- 3.10401 54399 01	0.94420 54730 39	7.9	-10.45692 10687 39	9.20305 97799 25
3.0	- 3.24414 42995 90	1.05335 07710 69	8.0	-10.60771 13103 15	9.41050 83803 12
3.1	- 3.38482 90223 77	1.16576 67132 86	8.1	-10.75857 96829 95	9.61920 37472 42
3.2	- 3.52603 43067 09	1.28135 17459 32	8.2	-10.90952 42693 78	9.82913 05671 62
3.3	- 3.66772 81104 88	1.40001 02965 76	8.3	-11.06054 32217 92	10.04027 38971 80
3.4	- 3.80988 12618 23	1.52165 22746 73	8.4	-11.21163 47589 48	10.25261 91518 09
3.5	- 3.95246 71261 89	1.64619 26242 69	8.5	-11.36279 71628 04	10.46615 20903 24
3.6	- 4.09546 13204 51	1.77355 09225 91	8.6	-11.51402 87756 02	10.68085 88047 12
3.7	- 4.23884 14660 71	1.90365 10190 19	8.7	-11.66532 79970 81	10.89672 57081 77
3.8	- 4.38258 69752 28	2.03642 07096 93	8.8	-11.81669 32818 48	11.11373 95241 57
3.9	- 4.52667 88647 16	2.17179 14436 05	8.9	-11.96812 31369 01	11.33188 72758 53
4.0	- 4.67109 95934 09	2.30969 80565 73	9.0	-12.11961 61192 81	11.55115 62762 02
4.1	- 4.81583 29197 96	2.45007 85299 47	9.1	-12.27117 08338 67	11.77153 41183 09
4.2	- 4.96086 37766 87	2.59287 37713 19	9.2	-12.42278 59312 81	11.99300 86662 85
4.3	- 5.10617 81606 63	2.73802 74148 20	9.3	-12.57446 01059 08	12.21556 80464 79
4.4	- 5.25176 30342 30	2.88548 56389 27	9.4	-12.72619 20940 29	12.43920 06390 90
4.5	- 5.39760 62389 84	3.03519 69999 22	9.5	-12.87798 06720 44	12.66389 50701 28
4.6	- 5.54369 64183 04	3.18711 22793 89	9.6	-13.02982 46547 89	12.88964 02037 08
4.7	- 5.69002 29483 73	3.34118 43443 27	9.7	-13.18172 28939 51	13.11642 51346 66
4.8	- 5.83657 58764 54	3.49736 80186 15	9.8	-13.33367 42765 47	13.34423 91814 77
4.9	- 5.98334 58655 32	3.65561 99647 12	9.9	-13.48567 77234 95	13.57307 18794 55
5.0	- 6.13032 41445 53	3.81589 85746 15	10.0	-13.63773 21882 47	13.80291 29742 30

Linear interpolation will yield about three figures; eight-point interpolation will yield about eight figures.

For z outside the range of the table, see **Examples 5–8.**

$\mathscr{R} \ln \Gamma(z) = \ln |\Gamma(z)|$ $\qquad$ $\mathscr{I} \ln \Gamma(z) = \arg \Gamma(z)$

We now must consider the influence of α, β, and γ.

4.3.4. *The final conjecture. Consider the special weight on the unit circle*

$$w(\theta) = \begin{cases} r_1 e^{-\gamma\theta} |\sin(\theta-\theta_1)/2|^{2\alpha} |\sin(\theta-\theta_2)/2|^{2\beta}, & \theta_1 < \theta < \theta_2 \\ r_2 e^{-\gamma\theta} |\sin(\theta-\theta_1)/2|^{2\alpha} |\sin(\theta-\theta_2)/2|^{2\beta}, & \theta_2 - 2\pi < \theta < \theta_1 \end{cases}$$

where $\theta_1 < \theta_2 < \theta_1 + 2\pi$, r_1 *and* r_2 *are positive,* α, β, *and* γ *real,* α *and* $\beta > -1/2$. *Then*

$$\Phi_n(0) = A_1 n^{\kappa_1} e^{in\theta_1} + A_2 n^{\kappa_2} e^{in\theta_2} + o(n^{-1}),$$

with κ_1 *and* $\kappa_2 = -1 - i\gamma \pm i\delta$, *where* $i\delta = -(\alpha+\beta)\dfrac{e^{i\theta_1}+e^{i\theta_2}}{e^{i\theta_1}-e^{i\theta_2}} + \dfrac{\overline{\xi}}{e^{-i\theta_2}-e^{-i\theta_1}} + \dfrac{\xi}{e^{i\theta_1}-e^{i\theta_2}}$, ξ *being the coefficient of* $e^{-i\theta}$ *in the Fourier expansion* $\log w(\theta) = \cdots + \xi e^{-i\theta} + \lambda_0 + \overline{\xi} e^{i\theta} + \cdots$, ξ *is also the limit when* $n \to \infty$ *of* ξ_n *in* $\Phi_n(z) = z^n + \xi_n z^{n-1} + \cdots$. *And the big conjecture is*

$A_1 = (\alpha - \rho_1\, i)\exp[i[2\arg\Gamma(1+\alpha+\rho_1\, i) + 2(\gamma-\rho_1)\log(2\sin((\theta_2-\theta_1)/2)) + \beta(\pi-\theta_2+\theta_1)\,]]$,
$A_2 = (\beta - \rho_2\, i)\exp[i[2\arg\Gamma(1+\beta+\rho_2\, i) + 2(\gamma-\rho_2)\log(2\sin((\theta_2-\theta_1)/2)) - \alpha(\pi-\theta_2+\theta_1)\,]]$,
where $\rho_1 = \dfrac{\log(r_1/r_2)}{2\pi}$ *and* $\rho_2 = \gamma - \rho_1$, *the jumps of* $(2\pi)^{-1}\log w$ *at* θ_1 *and* θ_2.

4.3.5. *The GP-PARI program.* Given $\alpha, \beta, \gamma, \theta_1$, and θ_2, the program computes $x_1 = \Phi_1(0) = (e^{i\theta_2} - e^{i\theta_1})\left[-\dfrac{\alpha-\beta+\gamma\cot((\theta_2-\theta_1)/2)}{2(\alpha+\beta+1+i\gamma)} + (\gamma + (\alpha+\beta+1)i)t\right]$, $\quad t \in \mathbb{R}$, with various values of t which will allow experiments with various values of ρ. Then, several thousands of the next x_ns are computed with (24). At each power of two, A_1 and A_2 are estimated from $x_n \sim A_1 n^{\kappa_1} e^{in\theta_1} + A_2 n^{\kappa_2} e^{in\theta_2}$ and $x_{n+1} \sim A_1 (n+1)^{\kappa_1} e^{i(n+1)\theta_1} + A_2 (n+1)^{\kappa_2} e^{i(n+1)\theta_2} \Rightarrow n^{-\kappa_2} e^{i\theta_2} x_n - (n+1)^{-\kappa_2} x_{n+1} \sim [n^{\kappa_1-\kappa_2} e^{i\theta_2} - (n+1)^{\kappa_1-\kappa_2} e^{i\theta_1}] e^{in\theta_1} A_1$,etc. A further simple acceleration is performed with A_1 replaced by $2A_1$ minus the former A_1 from the step $n/2$. Several thousands of $\Phi_n(0)$s allow an accurate reconstruction of the weight function through Christoffel function (P.Nevai, the big 1986 paper, p. 26) $w(\theta) = \lim_{n\to\infty} n\omega_n(e^{i\theta}) = \dfrac{n}{\sum_0^{n-1} |\varphi_k(e^{i\theta})|^2}$, from which $\rho = \rho_1 = (2\pi)^{-1}\log(r_1/r_2)$ is found, and the formulas for A_1 and A_2 are checked.

```
{
/*
  reflecj2.gp    :  launch gp and make  \r reflecj2

  Reflection coefficients and reconstruction of gen. Jacobi
  weight on unit circle with 2 singular points exp(i theta1)
     and exp(i theta2).

  w(theta) = A or B   exp(-gamma theta) (sin(theta-theta1)/2)^(2alpha)
                                        (sin(theta-theta2)/2)^(2 beta)


*/
sextern=extern("date +%a%e%b20%y__%H-%M");print("reflecj2 unit circle gen Jacobi ",sextern);
\\default(realprecision,55);
default(format,"g4.5");
N=2^15+1;
Nr=2^13+1;  \\ only for reconstruction
tp1=-0.15; tp2=0.275;  \\  theta1/pi and theta2/pi    albe=alpha+beta
al=1.6;be=0.9;albe=al+be;ga=2;    \\  alpha, beta, gamma
et1=exp(I*Pi*tp1);et2=exp(I*Pi*tp2);print(" alpha=",al," beta=",be," , gamma=",ga,"; theta1,2/pi=",tp1," ",tp2);
st1t2=sin(Pi*(tp2-tp1)/2);L12=2*log(2*st1t2);ct1t2=cos(Pi*(tp2-tp1)/2);

\\ P.Nevai's algorithm of weight reconstruction through Christoffel function
}
```

```
{
 Chris(z)=

       locp=1;locps=1;locsom=1;
       for(n=1,Nr,locp=z*locp+ref[n+1]*locps;
                 locps=(1-(abs(ref[n+1]))^2)*locps+conj(ref[n+1])*locp;
                 locsom=locsom+(abs(locp))^2/norm2[n+1]
          );
   1/locsom
}

{
ref=vector(Nr+1,k,0);
\\cas alpha=beta=0
\\rr=exp(2*Pi*0.25);t=-( (rr-1)*exp(-ga*Pi*tp1)+(rr-exp(2*ga*Pi))*exp(-ga*Pi*tp2)  )/(2*(ga+1/ga)*(  (1-rr)*exp(-ga*Pi
t=0.1;ref[2]=(et2-et1)*( (be-al-ga*ct1t2/st1t2)/(2*(albe+1+ga*I)) +(ga+(albe+1)*I)*t );
\\ vector of reflection coeff.
ref[1]=1;xi=ref[2];
print1(" Phi1(0)= ",xi);default(format,"g4.30");print(" , t=",t);default(format,"g4.5");
stop=0;
p2=1;
refn=ref[1]; refn1=ref[2];refn1o=refn1;cr1o=0;cr2o=0;
for(n=1,N-1,
 if( abs(refn1)>1,stop=1;return);
 refn2=(  refn1*(et1*et2*conj(xi)+xi+n*(et1+et2))/(1-(abs(refn1))^2)
             -(n-1+albe-I*ga)*et1*et2*refn
          )/(n+1+albe+I*ga);
 \\   xn = A1 n^(-1+i delta) e^(i n theta1) +A2 n^(-1-i delta) e^(i n theta2)
 xi=xi+conj(refn1)*refn2;la1=conj(xi);id1=la1/( 1/et2-1/et1 )+conj(la1)/( et1-et2 )
    -albe*( et1+et2 )/( et1-et2 );
 if(n+2<= Nr+1,  ref[n+2]=refn2);
 \\ asymptotic behaviour  cr1 et1^n/n^(1+i gamma -i delta)  + cr2 et2^n /n^(1+i gamma +i delta)
 if(n==p2,dn=n;dn1=n+1;
 cr1=(dn^(1+ga*I+id1)*refn1*et2-dn1^(1+ga*I+id1)*refn2)/(et1^n*(dn^(2*id1)*et2-dn1^(2*id1)*et1));
 cr2=(dn^(1+ga*I-id1)*refn1*et1-dn1^(1+ga*I-id1)*refn2)/(et2^n*(dn^(-2*id1)*et1-dn1^(-2*id1)*et2));
     cr12=2*cr1-cr1o; cr22=2*cr2-cr2o;errx=refn1-cr12*et1^n/(dn^(1+ga*I-id1))-cr22*et2^n/(dn^(1+ga*I+id1));
     print(n," A1=",cr12," A2=",cr22," relerr=10^",log(abs(errx/refn1))/2.3026  );
      p2=p2*2;cr1o=cr1;cr2o=cr2;
     ); \\ end if  p2
 refn=refn1; refn1=refn2
   );    \\  end for n
if(stop,return);
cr1=cr12;cr2=cr22;
\\  norms
norm2=vector(Nr+1,k,0);norm2[1]=1;
for(n=2,Nr+1,  norm2[n]= (1 -(abs(ref[n]))^2)*norm2[n-1] );
if(stop,return);
\\  B/A, al, be:
\\    dw/d theta =w[  alpha cot((theta-theta1)/2)+beta cot((theta-theta2)/2) -gamma  ]
  th1=Pi*(tp1+tp2)/2;w1=Nr*Chris(exp(I*th1));
  tans=tan( Pi*(tp2-tp1)/4 ); cots=1/tans;
 dw1=Nr*( Chris(exp(I*(th1+0.0001)))-Chris(exp(I*(th1-0.0001))))/(0.0002*w1)+ga;
al1=(dw1*tans+albe)/2;be1=(-dw1*tans+albe)/2;print1(" check alpha, beta= ",al1," ",be1);
th2=Pi*(-2+tp1+tp2)/2;w2=Nr*Chris(exp(I*th2));
dw2=Nr*( Chris(exp(I*(th2+0.0001)))-Chris(exp(I*(th2-0.0001))))/(0.0002*w2) +ga;
al2=(-dw2*cots+albe)/2;be2=(dw2*cots+albe)/2;print(" ;  ",al2," ",be2);
\\al1=(albe-(abs(cr2)^2-abs(cr1)^2)/albe)/2;be1=(albe+(abs(cr2)^2-abs(cr1)^2)/albe)/2;
print(" xi=",xi," i delta=",id1);
r1=w1/( abs(sin((th1-Pi*tp1)/2))^(2*al) *
```

```
        abs(sin((th1-Pi*tp2)/2))^(2*be) *
        exp(-ga*th1)   );
r2=w2/( abs(sin((th2-Pi*tp1)/2))^(2*al) *
        abs(sin((th2-Pi*tp2)/2))^(2*be) *
        exp(-ga*th2) );
rho1=log(r1/r2)/(2*Pi);rho2=ga-rho1;errho=r1/r2;
Astart=cr1/(al-rho1*I);Bstart=cr2/(be-rho2*I) ;
print("r1=",r1," r2=",r2, " rho=",rho1 , " A1/(alpha-rho i)=",Astart, " A2/(beta-(gamma-rho) i)=",Bstart," abs: ",abs
psi1=-rho1*L12+2* arg(gamma(1+al+rho1*I)) ;psi11=be*Pi*(1-tp2+tp1)+psi1+ga*L12;
psi11=psi11-2*Pi*round(0.5*psi11/Pi);
psi2=-rho2*L12+2* arg(gamma(1+be+rho2*I)) ;psi12= -al*Pi*(1-tp2+tp1)+psi2+ga*L12;
psi12=psi12-2*Pi*round(0.5*psi12/Pi);
print(" logarithmes: ",log(Astart)," ",log(Bstart)," ",psi11," ",psi12    );
plot(th=Pi*tp2-2*Pi,Pi*tp2,Nr*Chris(exp(I*th)));

}

          GP/PARI CALCULATOR Version 2.2.8 (development CHANGES-1.887)
                i686 running cygwin (ix86 kernel) 32-bit version
               compiled: Jan 13 2004, gcc-3.3.1 (cygming special)
                (readline v4.3 enabled, extended help available)

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comes WITHOUT ANY WARRANTY WHATSOEVER.

Type ? for help, \q to quit.
Type ?12 for how to get moral (and possibly technical) support.

   realprecision = 28 significant digits
   seriesprecision = 16 significant terms
   format = g0.28

parisize = 4000000, primelimit = 500000

reflecj2 unit circle gen Jacobi Tue 19 Mar 2013 15:15
 alpha=1.6000 beta=0.90000 , gamma=2; theta1,2/pi=-0.15000 0.27500
 Phi1(0)= -0.63105 - 0.31312*I , t=0.1000000000000000000000000000

    1 A1=-5.6514 + 2.8024*I A2=-2.7015  - 5.5964*I  relerr=10^4.6045 E-28
    2 A1=1.0605  + 1.4044*I A2= 0.24594 + 0.78057*I relerr=10^-1.3150
    4 A1=-1.6032 - 3.7685*I A2= 0.72380 - 1.1202*I  relerr=10^0.20871
    8 A1=2.7833  - 2.5577*I A2=-1.4613  + 0.20793*I relerr=10^-0.32309
   16 A1=2.7643  - 1.2968*I A2=-0.58543 - 0.23763*I relerr=10^-0.50384
   32 A1=2.0948  - 2.0138*I A2=-0.68737 - 0.78442*I relerr=10^-0.93469
   64 A1=1.5325  - 2.5385*I A2=-0.42672 - 0.90166*I relerr=10^-0.56313
  128 A1=1.0397  - 2.8077*I A2=-0.31776 - 0.98557*I relerr=10^-0.56055
  256 A1=0.78972 - 2.8517*I A2=-0.22935 - 0.98711*I relerr=10^-0.64465
  512 A1=0.66794 - 2.8593*I A2=-0.18537 - 0.99123*I relerr=10^-0.80112
 1024 A1=0.61129 - 2.8605*I A2=-0.16559 - 0.99087*I relerr=10^-1.5224
 2048 A1=0.58541 - 2.8612*I A2=-0.15641 - 0.99045*I relerr=10^-1.7537
 4096 A1=0.57276 - 2.8622*I A2=-0.15201 - 0.99067*I relerr=10^-2.0813
 8192 A1=0.56663 - 2.8629*I A2=-0.14989 - 0.99080*I relerr=10^-2.3460
16384 A1=0.56362 - 2.8633*I A2=-0.14885 - 0.99090*I relerr=10^-2.0060
32768 A1=0.56213 - 2.8635*I A2=-0.14833 - 0.99096*I relerr=10^-2.4234
 check alpha, beta= 1.6000 0.89998 ;  1.6000 0.90002
 xi=0.56702 - 1.9303*I i delta=4.1709 E-29 - 2.8812*I
r1=2934.6 r2=0.00064375 rho=2.4402 A1/(alpha-rho i)=0.92629 - 0.37698*I A2/(beta-(gamma-rho) i)=-0.56760 - 0.82341*I a
```

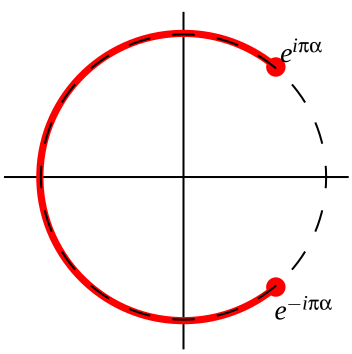
```
 logarithmes: 0.000059741 - 0.38651*I 0.000087415 - 2.1743*I -0.38733 -2.1739

7.4349977 |''''''''''''''''''''''''''''''''''''''''''''''''''''''''"x'''''|
          |                                                         :     |
          |                                                       "  :    |
          |                                                       :  "    |
          |                                                      :   :    |
          |                                                      :    :   |
          |                                                      x    :   |
          |                                                      :    x   |
          |                                                      :    :   |
          |                                                     :     :   |
          |                                                     :      :  |
          |                                                     _      :  |
          |                                                     :      x  |
          |                                                     :      :  |
          |                                                    :        : |
          |                                                    :        : |
          |                                                    :        _ |
          |                                                    "        : |
          |                                                    :         :|
          |      _x"""""xx__                                  :          :|
          |   _x"           "xx__                             _          x|
        0 __x",,,,,,,,,,,,,,,,,,,""xxxxx_____________________,,,,,,,,,,,_
          -5.419247                                               0.863938
(15:16) gp > quit
Goodbye!
```

5. The Grünbaum-Delsarte-Janssen-Vries problem.

5.1. The problem.

.



"3. The following Toeplitz matrix arises in several applications. Define for $i \neq j$, $A_{i,j}(\alpha) = \dfrac{\sin \pi\alpha(i-j)}{\pi(i-j)}$ and set $A_{i,i} = \alpha$. Conjecture: the matrix $M = (I-A)^{-1}$ has positive entries. A proof is known for $1/2 \leqslant \alpha < 1$. Can one extend this to $0 < \alpha < 1$? Submitted by Alberto Grünbaum, November 3, 1992. (grunbaum@math.berkeley.edu)" [17].

$I - A$ is the $\boldsymbol{G}_N$ of the weight $w = 1$ on the circular arc shown left. For all the entries of all the $\boldsymbol{G}_N^{-1}$ to be positive, it is necessary that all the reflection coefficients $\Phi_n(0) > 0$, $n = 1, \dots, N$, and the condition is sufficient: from (9), all the Φ_n's have positive coefficients, so does $\boldsymbol{L}^{-1}$, and $\boldsymbol{G}_N^{-1}$, from (15) [7, p.645].

In [7], Delsarte & *al.* study the robustness of a signal recovery procedure amounting to find the polynomial $p = p_0 + \cdots + p_N z^N$ minimizing the integral of $|f(\theta) - p(e^{i\theta})|^2$ on the circular arc shown above. This elementary least-squares problem involves the Gram matrix $\boldsymbol{G}_N$, and the stability of the recovery procedure is related to the size of the smallest eigenvalue of the matrix. The corresponding eigenvector is shown to have elements of the same sign. The theory of this eigenvalue-eigenvector pair sould be more complete if it could be shown that $\boldsymbol{G}_N^{-1}$ has only positive elements, for any $N = 1, 2, \ldots$, and any $\alpha \in (0, 1)$. It is also reported in [7, p. 644] that Grünbaum stated this conjecture as early as 1981.

If $\alpha \geqslant 1/2$, all the zeros of Φ_n have negative real part (Fejér), so $\Phi_n(0) = (-1)^n$ times the product of all the zeros must be > 0 (conjugate pairs have no influence on the sign, and the number of real zeros is $n-$ an even number).

From continuity of the zeros with respect to α, we are trying to show that the ***real zeros*** of Φ_n all remain negative for all $0 < \alpha < 1$. Most zeros are close to the support anyhow, and there are probably only a small number of real zeros which are not close to -1.

In order to remove the insufferable tension, here is some numerical evidence, where the relevant $\Phi_n(0)$'s are computed either with the all-purpose algorithm of p. 12, or from the formula (24) which will be further worked in (27):

α	$\Phi_1(0)$	$\Phi_2(0)$	$\Phi_3(0)$	$\Phi_4(0)$	$\Phi_5(0)$	$\Phi_6(0)$	$\Phi_7(0)$	$\Phi_8(0)$	$\Phi_9(0)$	$\Phi_{10}(0)$	Φ_1
0.1	0.109292	0.117289	0.124056	0.129708	0.134380	0.138214	0.141343	0.143889	0.145959	0.147640	0.1
0.2	0.233872	0.258015	0.274428	0.285306	0.292452	0.297160	0.300299	0.302429	0.303905	0.304953	0.3
0.3	0.367883	0.406603	0.427202	0.438032	0.443863	0.447146	0.449098	0.450327	0.451143	0.451711	0.4
0.4	0.504551	0.550672	0.569686	0.577835	0.581659	0.583657	0.584812	0.585537	0.586023	0.586365	0.5
0.5	0.636619	0.681477	0.695821	0.701070	0.703373	0.704563	0.705258	0.705700	0.706000	0.706213	0.7
0.6	0.756826	0.793314	0.802464	0.805497	0.806813	0.807503	0.807911	0.808173	0.808352	0.808479	0.8
0.7	0.858393	0.882688	0.887579	0.889138	0.889826	0.890192	0.890410	0.890551	0.890647	0.890716	0.8
0.8	0.935489	0.947570	0.949601	0.950254	0.950547	0.950704	0.950798	0.950859	0.950901	0.950930	0.9
0.9	0.983631	0.986853	0.987333	0.987491	0.987563	0.987601	0.987625	0.987639	0.987650	0.987657	0.9

```
? quit
Good bye!

C:\calc\pari>exit    Script completed Wed Oct 13 14:09:47 1999
```

All the $\Phi_n(0)$'s seem indeed to be positive. Moreover, it is known that $\Phi_n(0) \to \sin(\pi\alpha/2)$ when $n \to \infty$ [10, 11, 12, etc.]. N.B.: the capacity of the support is $\cos(\pi\alpha/2)$ [32]. A possible strategy, although not extremely elegant, is to look for a more accurate asymptotic behaviour, so to prove the positivity for $n >$ some finite n_0, and to show positivity of the finite number of remaining $\Phi_n(0)$, $n \leqslant n_0 \ldots$ But perhaps we may learn something from a (nonlinear) recurrence relation between the $x_n = \Phi_n(0)$:

5.1.1. *A recurrence relation for the* $\Phi_n(0)$*'s.* .

(24) becomes

$$(n+1)x_{n+1} = \frac{2(x_1x_0 + \cdots + x_nx_{n-1}) + 2n\cos\pi\alpha}{1 - x_n^2}\, x_n - (n-1)x_{n-1}, \tag{27}$$

with $x_0 = 1$ and $x_1 = \dfrac{\sin\pi\alpha}{(1-\alpha)\pi}$.

Question: are all the x_n's positive??

The most elegant proof should establish that the sequence of the $\Phi_n(0)$'s is ***increasing***, as suggested by the numerical tests, but how to achieve that?

OK, it will be achieved in section 5.1.10, p. 44!

5.1.2. *Trying to solve* (27). .

From numerical runs of (27), the following empiric asymptotic formula:

$$x_n = \sin(\pi\alpha/2) - \frac{\cos^2(\pi\alpha/2)}{8n^2\sin(\pi\alpha/2)} + O(n^{-4}).$$

quite in agreement with formula (56) of Golinskii, Nevai, and Van Assche [10] stating that

$$\Phi_n(0) = \sin(\pi\alpha/2) + (-1)^n \frac{\delta\cos^2(\pi\alpha/2)}{2n} - \frac{\cos^2(\pi\alpha/2)}{8n^2\sin(\pi\alpha/2)}\left[1+\delta^2\sin^2(\pi\alpha/2) - 4\gamma^2 + 2(-1)^n\delta\sin(\pi\alpha/2)(1+\delta+2\gamma-\sin\pi\alpha/2)\right] + O(n^{-3})$$

for the weight function $(\cos\pi\alpha-\cos\theta)^\gamma|\cos\theta/2|^\delta\sin\theta/2$ on $(\pi\alpha, 2\pi-\pi\alpha)$. The formula is based on values of Jacobi polynomials, as the problem is reduced to the weight $(\cos\pi\alpha - x)^\gamma(1+x)^{(\delta-1)/2}$ on the real interval $(-1,\cos\pi\alpha)$. Unfortunately, the same technique would lead here to $(1-x^2)^{-1/2}$, which is not related to known real line orthogonal polynomials. Actually, we are struggling here like mad to make these polynomials known!!!

What can be shown from (27) are -probably useless- expansions in powers of α:

$$x_n = \alpha + n\alpha^2 - n^2(\pi^2-6)\alpha^3/6 - [2(\pi^2-9)n^3+\pi^2 n]\alpha^4/18 + \cdots$$

5.1.3. *First exploration of the solutions.* Making some numerical runs of (27) with a definite precision leads to troubles in the long run:

```
ubasic
 20   ' grunbaum
 25   print=print+"grunbaum.l"
 30   point 2:print point
 40   A=0.25
201   X0=1:X1=sin(A*#pi)/((1-A)*#pi):C=cos(A*#pi):N=100
202   ' recurrence xi
205   Xi=X1:for M=1 to N-1:Xp1=(2*(Xi+M*C)*X1/(1-X1^2)-(M-1)*X0)/(M+1)
207   Xi=Xi+X1*Xp1
208   print M;" ";X1
210   X0=X1:X1=Xp1:next M
```

```
                                          POINT 20:N=1000
Words for fractionals 2          Words for fractionals 20
 (Decimals for display 9)        (Decimals for display 96)

   1   0.300105438    0.300105438719035356518399730335580194221524716651740208523812490410486243573015448332694259 25
   2   0.332187784    0.332187784729316652882457311808060176581793173463829176801791548754833121871761352331081804 10
   3   0.351521621    0.351521622944229130230779177777568104547706234023029249918628500160804783230458059315851552 86
   4   0.362911325    0.362911328411337035591480997286720893276610370864906586956010249324182423780756266203497441 61
   5   0.369642298    0.369642304146171000824558797145386999571114241163682822347130495242252026610759141257008687 93
...
  10   0.379705895    0.379706107321706676276909192156965441252351691683364935128330902675216176879955158028808611 53
...
  15   0.38140294     0.381412605388367735279737946411155252997049268684891411731472630807613727505757096261248973 50
...
  20   0.381507291    0.381977095319547457088724621880884690851310343429939475651883736964631959282185813383641450 94
...
  25   0.359328655    0.382233631131524225665394463139252947181740359520899165926617696071857793677077229535761392 06
  26   0.33380412     0.382267835879970697545013441331993004148898058825029624394251751031440005541733388544649827 93
  27   0.283680342    0.382298272241786482488977563420885395831207045532374260227696140130798333139018498902672564 46
  28   0.19583214     0.382325475582535900905835127899151269669792416349989848038142801586201681186591157286180948 01
  29   0.06894989     0.382349889199973187070667721218694583629355500346001470369578737062748878379638504424051778 31
  30  -0.069915877    0.382371882746964408647578858126608685637007444569753291725168347469398733103580471388794753 98
  31  -0.178450092    0.382391766478771822608282214017601119299520985258580905855093679950157733260203245941596930 5
  32  -0.232336628    0.382409802376152311009284842542391099911901759090421967331623488214726041253072508185263231 31
  33  -0.228388816    0.382426212905830032178925651503557285812376149639979010755870489501431543526965361330523328 30
  34  -0.169224556    0.382441187976535001913975623756566448820215384921110339829412076390188377351663892850831446 06
  35  -0.06452273     0.382454890504252853498411400564693922512454933309252993728738535616386636307400837122025343 68
  36   0.056045639    0.382467460896345942368811864350089463890419602466812325132690300256030050759753272012875467 68
```

```
  37   0.150789831    0.3824790206885602180954252662456751716460810716006723341976826846310247675769561404160247194 9
  38   0.192559004    0.38248967551332896042000271238809590502281674099493684721299554818511437818292493373748237581
  39   0.174702805    0.38249951753651672968156468259575246726644840148792253730571079200644531605083678549433684531
  40   0.102928098    0.38250862746884118758802777537257439941615157615474177408716915704407258161586912556317338089
  41  -0.00150102     0.38251707623486851027186672908137167366417593635412494409608333257617189561148172785924684577
  42  -0.100396835    0.38252492636470828130111112092949665986452357740717420125401438705769567876753379880205431437
  43  -0.158318881    0.38253223315990433815802515153331897925942421237549662074542351498305157692084071705241339428
  44  -0.159476973    0.38253904567449028763179388703498779736063673103023585505165204569309473098715206680150057840
  45  -0.105601602    0.38254540754399199970670582104074210039074420829458282464158064325046969281025221079348111888
...
 270                  0.3827446974295452142055697102529721450111058862344321410001231314485705160446864159923954197338
 271                  0.3828251715939946435739648724529429890725488892307700448585187621522616032902178143198059293784
 272                  0.3830051151786039082977296592527741324378061485450737502059375611257620252836479433456808199598
 273                  0.3834076931234117627810813088649269818828540786524049080162367018499051428680794492066442668627
 274                  0.3843092602773831997409668867732470870765017328631604069744220294563057683232797687543853468900
 275                  0.3863326454193672293985558949462663582860557174248230173738202249357001314372809299431594267942
 276                  0.3908954722363944262226318869995802256570317686718730469630040862009891598218750444520423067016 5
 277                  0.4012963828048901550011675688335964129279587580906529614054621666195462854236230515273688528505
 278                  0.4255987083983030107856646533466844287814763094073921947381537096257140004048979326499345876334
 279                  0.4858178966547720811801825929439491197910279738661309029394268566299920336644133770270083101281
 280                  0.6595916320130780764952796921601177280291097320453367358159176200924319319557076786803742026928
 281                  1.5059819322185246932486792850805501859851682749523910794364737945220351202448964737014155243631
 282                 -2.6847309052534040768570083599130516900308028594699637004447269165446308205080251169207346628532
 283                 -0.7690276005780758139985429965527641625529274994202697104073868753850489692104870271916001194232
 284                 -0.5206461079786216537294418900474114022291827147859340466770828570679257704049802853285134679637
 285                 -0.4469405623521448235939564863317985075815055041299639138564435194457419318592407262004963618356
 286                 -0.4299403004942594757821189202164925705182887179838919698286017530992681850057375122516838295605
 287                 -0.4506312341977500357393482911177227455600110991250230265200769293060201869701295278784173199983
 288                 -0.5321899756654534286702397485846467109624861534270923886460930428307061095786422284396499848290
...
```

That happens because the general solution of (27), keeping only $x_0 = 1$, is related to the piecewise constant weight function $w = A$ on $\pi\alpha < \theta < \pi(2-\alpha)$, and $w = B$ on $-\pi\alpha < \theta < \pi\alpha$. The overwhelming majority of solutions have A and $B \neq 0$, are related to a Szegő weight if A and $B > 0$, and have therefore $x_n \to 0$ when $n \to \infty$. Taking x_1 just a trifle above or below the ideal value $\sin(\pi\alpha)/((1-\alpha)\pi)$ will therefore end up with unsatisfactory values of x_n for large n. More precisely, x_1 is the $\Phi_1(0)$ related to the (A,B) weight function:

$$x_1 = -\frac{\mu_1}{\mu_0} = -\frac{A\int_{\text{left}} \cos\theta\, d\theta + B\int_{\text{right}} \cos\theta\, d\theta}{A\int_{\text{left}} d\theta + B\int_{\text{right}} d\theta} = \frac{(1-B/A)\sin\pi\alpha}{\pi[1-\alpha+(B/A)\alpha]}.$$

Values of x_1 smaller than $\sin\pi\alpha/((1-\alpha)\pi)$ correspond indeed to a Szegő weight (as long as $x_1 > -\sin\pi\alpha/(\pi\alpha)$); larger values of x_1 does not even correspond to positive weights and will have some $x_n \notin (-1,1)$!

These numerical difficulties show that a direct study of the recurrence relation (27), trying to get an induction of the form "if some inequality is valid for x_n, it is also valid for x_{n+1}", will lead nowhere.

```
Script V1.1 session started Tue Mar 18 14:45:45 2003

type grunbg.gp

{
/*
  grunbg.gp    :  launch gp and make  \r grunbg

  Reflection coefficients for Grunbaum- Delsarte et al.  problem
```

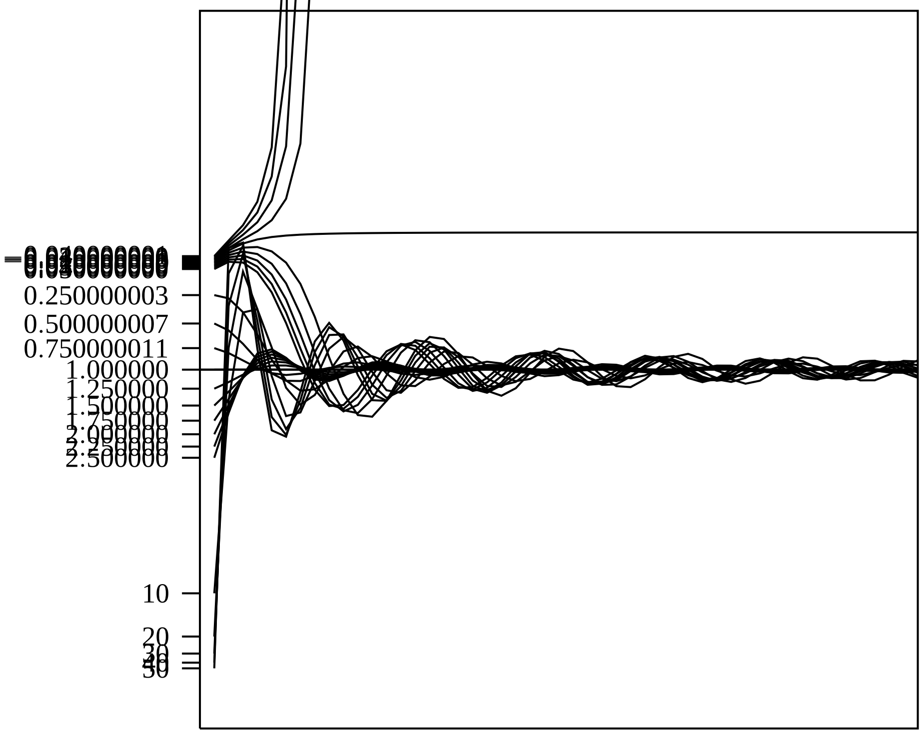
```
 by modified iteration,   January 2003.

  weight= (cos pi a - cos theta)^(beta)  on pi a < theta < 2pi -pi a
*/
default(format,"g2.7");

for(bigN=50,50,print("N= ",bigN);
for(ia=1,1,a=0.25*ia;
\\  a=2*asin(a)/Pi;
print1("alpha=",a," ");    sa=sin(Pi*a/2);
print1(" sigma=sin pi a/2= ",sa);
ca=cos(a*Pi);
print(" cos pi a= ",ca);
\\ vector of reflection coeff.
v=vector(bigN+2,k,0);
print("\\psline(0,-5)(0,0)(10,0)(0,0)(0,5)(10,5)(10,-5)(0,-5)");
v[1]=1;
for(iba=1,25,
   ba=10*(iba-20);
   if(iba<=20,ba=0.25*(iba-10));
   if(iba<=10,ba=0.01*(iba-5));
v[2]=(1-ba)*sin(a*Pi)/(Pi*(1-a+ba*a)); \\print(v[2]);
print("\\psline(0,",5*v[2],")(-0.25,",5*v[2],")");
print("\\uput[180](-0.25,",5*v[2],"){$",ba,"$}");
print("\\psline%");
som=0; kpr=1;
for(n=1,bigN,
   som=som+v[n+1]*v[n];
  v[n+2]=(  2*(som+n*ca)*v[n+1]/(1-v[n+1]^2) -(n-1)*v[n] )/(n+1);
  if(kpr==1,print1("(",0.2*n,",",5*v[n+1],")"));
  if(divrem(n,5)[2]==0,print("%"));
  if(abs(v[n+1])>1.2,kpr=0);
   );
)))
}

C:\calc\pari>gp-sta
                  GP/PARI CALCULATOR Version 2.1.3 (released)
                      Copyright (C) 2000 The PARI Group PARI/GP
 \r grunbg
N= 50
alpha=0.250000003  sigma=sin pi a/2= 0.382683438 cos pi a= 0.707106791

-0.200000005 0.385849855
-0.100000002 0.341499297
 0           0.300105443
 0.100000001 0.261382160
 0.200000002 0.225079082
```




```
? quit
Good bye!

C:\calc\pari>exit

Script completed Tue Mar 18 14:54:01 2003
```

Check that $x_n \to \sigma \neq 0$ in (27) yields $\sigma = \pm\sin(\pi\alpha/2)$. There is also an alternating solution $(-1)^n x_n \to \sigma$ which yields $\sigma = \pm\cos(\pi\alpha/2)$, corresponding to the arc of circle joining $e^{i\pi\alpha}$ to $e^{-i\pi\alpha}$ and containing $+1$. These two exceptional solutions correspond to $B = 0$ and $A = 0$. So, we are looking for the only solution of (27) satisfying $x_n \to \sin\pi\alpha/2$ when $n \to \infty$, and we are wondering if all the x_n's are positive.

Of course, each x_n is a rational function of x_1:

$$x_2 = \frac{x_1(\cos\pi\alpha + x_1)}{1-x_1^2}\ , x_3 = \frac{x_1(4\cos^2\pi\alpha - 1 + 6\cos\pi\alpha x_1 + (4-\cos^2\pi\alpha)x_1^2)}{3(1-(2+\cos^2\pi\alpha)x_1^2 - 2\cos\pi\alpha x_1^3)}\ \ldots$$

Here is an information on the variation of each x_n with respect to x_1, while x_n remains positive:
We look at the influence of x_1 on x_n, i.e., at $\partial x_n/\partial x_1$, which we write $\dot{x}_n$.

5.1.4. *Influence of x_1 on x_n.* **Proposition.** *If $x_1, x_2, \ldots, x_n \in (0,1)$, then $\dot{x}_n > 0$.*

Derivating (27) for $i = 1, 2, \ldots, n-1$:

$$(i+1)\dot{x}_{i+1} = \frac{2x_i}{1-x_i^2}\left[\dot{x}_1 + \sum_{j=1}^{i-1}(x_j\dot{x}_{j+1} + x_{j+1}\dot{x}_j)\right] + 2\frac{\text{num.}}{(1-x_i^2)^2}(1+x_i^2)\dot{x}_i - (i-1)\dot{x}_{i-1},$$

where “num.” is the numerator in the right-hand side of (27). We now use precisely this equation (27) to replace “2num.”$/(1-x_i^2)$ by $((i+1)x_{i+1} + (i-1)x_{i-1})/x_i$:

$$(i+1)\dot{x}_{i+1} = \frac{2x_i}{1-x_i^2}\left[\dot{x}_1 + \sum_{j=1}^{i-1}(x_j\dot{x}_{j+1} + x_{j+1}\dot{x}_j)\right] + \frac{(i+1)x_{i+1} + (i-1)x_{i-1}}{x_i(1-x_i^2)}(1+x_i^2)\dot{x}_i - (i-1)\dot{x}_{i-1},$$

All the $\dot{x}$'s are positive: if true up to i,

$$(i+1)\dot{x}_{i+1} > \frac{(i+1)x_{i+1}+(i-1)x_{i-1}}{x_i}\dot{x}_i - (i-1)\dot{x}_{i-1},$$

$$(i+1)x_{i+1}\left[\frac{\dot{x}_{i+1}}{x_{i+1}} - \frac{\dot{x}_i}{x_i}\right] > (i-1)x_{i-1}\left[\frac{\dot{x}_i}{x_i} - \frac{\dot{x}_{i-1}}{x_{i-1}}\right].$$

whence indeed $\dot{x}_{i+1} > 0$ too. □

Incidentally, we also have $\dot{x}_i/x_i$ increasing with i, as also $i(i-1)[x_{i-1}\dot{x}_i - x_i\dot{x}_{i-1}]$.

Remark that, if x_1 is very small,

$$x_n \sim \frac{\sin n\pi\alpha}{n\sin\pi\alpha}x_1,$$

seeing that (27) is $(n+1)x_{n+1} \sim 2nx_n\cos\pi\alpha - (n-1)x_{n-1}$ at first order. So, the first x_n's are positive as long as $n\alpha < 1$ if x_1 is a small positive number. And when x_1 increases, the length of the initial sequence of positive x_n's must increase too, and we hope to have the whole infinite sequence at $x_1 = \sin\alpha\pi/((1-\alpha)\pi)$. If this is true, what happens when x_1 still increases a little bit, may we ask? According to the Proposition above, all the $\dot{x}_n$,s are positive, so that all the x_n's will increase by an infinitesimal amount and still keep a valid value < 1. What about the unicity of the positive solution? Answer: for any small but nonzero increase of x_1, there is an n large enough where x_n will increase so much as to be > 1. A strong hint is the fast exponential increase of $\dot{x}_n$ with n:

```
ubasic

 20 ' grunbaum
 25 print=print+"grunbaum.l"
 30 point 21:word 74
 40 A=0.25
201 X0=1:X1=sin(A*#pi)/((1-A)*#pi):Dx0=0:Dx1=1:C=cos(A*#pi):N=100
202 ' recurrence xi  and dxi/x1
205 Xi=X1:Dxi=1:for M=1 to N-1:Xp1=(2*(Xi+M*C)*X1/(1-X1^2)-(M-1)*X0)/(M+1)
206 Dxp1=((2*DXi*X1+((M+1)*Xp1+(M-1)*X0)*(1+X1^2)*Dx1/X1)/(1-X1^2)-(M-1)*Dx0)/(M+1)
207 Xi=Xi+X1*Xp1:Dxi=Dxi+Dx1*Xp1+X1*Dxp1
208 print M;" ";X1;" ";Dx1
210 X0=X1:X1=Xp1:Dx0=Dx1:Dx1=Dxp1:next M
```

n	1	2	3	4	5	6	7	8	9	10
x_n	0.3001	0.3322	0.3515	0.3629	0.3696	0.3737	0.3762	0.3779	0.3789	0.3797
$\dot{x}_n$	1.0000	1.6558	2.9312	5.5059	10.7885	21.7452	44.6731	93.0072	195.5287	414.1064

The lesson here is that the above Proposition is useful only with respect to a finite sequence.

More on the nonpositivity of the Szegő's solutions:

When A and $B > 0$, we know that $x_n \to 0$ and ξ_n has a bounded limit, hence (27) "looks like" $(n+1)x_{n+1} = 2n\cos(\pi\alpha)x_n - (n-1)x_{n-1}$, and we expect a $\sin(n\pi\alpha)$ behaviour as seen before. What is true is that in $\begin{bmatrix} x_n \\ x_{n+1}\end{bmatrix} = A_n\begin{bmatrix} x_{n-1} \\ x_n\end{bmatrix}$, the matrix A_n is close to $A_\infty := \begin{bmatrix} 0 & 1 \\ -1 & 2\cos\pi\alpha\end{bmatrix}$ for large n, so that, for a given p, $\begin{bmatrix} x_{n+p} \\ x_{n+p+1}\end{bmatrix} = A_{n+p}A_{n+p-1}\cdots A_{n+1}\begin{bmatrix} x_n \\ x_{n+1}\end{bmatrix}$, with $A_{n+p}A_{n+p-1}\cdots A_{n+1}$ close to $A_\infty^p = \dfrac{1}{\sin\pi\alpha}\begin{bmatrix} -\sin(p-1)\pi\alpha & \sin p\pi\alpha \\ -\sin p\pi\alpha & \sin(p+1)\pi\alpha\end{bmatrix}$. So, x_{n+p} is close to $C_n\sin(p\pi\alpha+\zeta_n)$,

We shall need a finite, non infinitesimal, version of the Proposition:

5.1.5. *Difference of two solutions.* **Proposition.** *If $x_1', x_2', \dots$ and $x_1'', x_2'', \dots$ are two solutions of* (27)*, with $0 < x_i' < 1$ and $0 < x_i'' < 1$ for $i = 1, 2, \dots, n$, then $x_i' = x_i''$ or all the $x_i'' - x_i'$ have the same sign for $i = 1, 2, \dots, n$.*

Indeed, let us write (27) as

$$(i+1)x_{i+1} = C_i x_i - (i-1)x_{i-1}, i = 1,2,\dots \qquad (28)$$

where $C_i = 2\dfrac{x_1 + x_2 x_1 + \cdots + x_i x_{i-1} + i\cos\pi\alpha}{1-x_i^2}$.

Suppose $x_1'' > x_1'$, and that $x_j'' > x_j'$ for $j = 1,2,\dots,i$.

$$\begin{aligned}(i+1)[x''_{i+1} - x'_{i+1}] &= C''_i[x''_i - x'_i] + [C''_i - C'_i]x'_i - (i-1)[x''_{i-1} - x'_{i-1}] \\ &> C''_i[x''_i - x'_i] - (i-1)[x''_{i-1} - x'_{i-1}] \text{ as } C''_i > C'_i \\ &> \frac{(i+1)x''_{i+1} + (i-1)x'_{i-1}}{x''_i}[x''_i - x'_i] - (i-1)[x''_{i-1} - x'_{i-1}] \text{ from (28)}\end{aligned}$$

and $x''_{i+1} > x'_{i+1}$ follows. □

As we still do not know if the $\Phi_n(0)$'s are all positive, we shall try to build a positive solution of (27).

> CHICO: Go to the house next door.
> GROUCHO: That's great. Suppose there isn't any house next door.
> CHICO: Well, then of course we gotta build one!
> from *Animal Crackers*[4].

5.1.6. *It seems that a smart idea.* is to look at (27) as a relation between positive sequences. We use (27) to extract x_n through the only positive root of

$$\frac{x_n^{-1} - x_n}{2} = \frac{x_1 x_0 + \cdots + x_n x_{n-1} + n\cos\pi\alpha}{(n+1)x_{n+1} + (n-1)x_{n-1}}, n = 1,2,\dots \quad x_0 = 1, \qquad (29)$$

recomputing new estimates of $x_1, x_2, \dots, x_N$ from old ones in the right-hand side of (29), with $x_0 = 1$ and x_{N+1} = the limit $\sin(\pi\alpha/2)$. The first estimate of the sequence $\{x_n\}$ may simply be $x_n = \sin(\pi\alpha/2)$ for all $n > 0$. Remark that, as a bonus, $x_n < 1$ if all the x_i's on the right-hand side are positive.

It works!! Here is a test with $\alpha = 0.3$, applying first (27) directly with $x_1 = (\sin\alpha\pi)/((1-\alpha)\pi) =$ 0.367883098..., and using several iterations of (29) with $x_n = \sin(\alpha\pi/2) = 0.453990607...$, $n = 1,2,\dots$ as starting sequence:

```
{
/*
  reflecjg.gp    :  launch gp and make  \r reflecjg

  Reflection coefficients for Delsarte et al. and Grunbaum problem

*/
default(realprecision,50);
flone=1.0;pr=precision(flone);ep=10^(-pr);
default(format,"g1.6");
N=20;

for(ia=3,3,alpha=0.1*ia;print1(alpha," ");ca=cos(alpha*Pi);
\\ vector of reflection coeff.
ref=vector(N+1,k,0);
ref[1]=1;ref[2]=sin(alpha*Pi)/(Pi*(1-alpha));
xi=ref[2];
 for(n=1,N-1,
```

[4]considered by J. Adamson, in *Groucho, Harpo, Chico and sometimes Zeppo*, Simon & Schuster 1973 = Pocket Book 1976, as an overestimated sample of the Marx Brothers humor. Perhaps a better instance is the much repeated scene where a handshake produces a flood of knives falling from Harpo sleeves... when one should have expected mere spoons.

```
  print1(ref[n+1]," ");
  ref[n+2]=(  ref[n+1]*(2*xi+2*n*ca)/(1-(ref[n+1])^2)
             -(n-1)*ref[n]     )/(n+1);
  xi=xi+ref[n+1]*ref[n+2]
    ) ;
print(" ");

   sa=sin(Pi*alpha/2);    print(" sin pi alpha/2= ",sa);
  \\  par iteration de suites
   newref=vector(N+1,k,0);
   for(n=2,N,ref[n]=sa);ref[1]=1;
    for(ns=1,99,   newref[1]=1; residm=0; xi=0;  print(ns," ");
     for(n=2,N-1,
            print1(ref[n]," ");
             xi=xi+ref[n-1]*ref[n];
      resid=
      ref[n+1]-(  ref[n]*(2*xi+2*(n-1)*ca)/(1-(ref[n])^2)
                       -(n-2)*ref[n-1]     )/n  ;
         residm=max(residm,abs(resid));
         newref[n]=
         (xi+(n-1)*ca) / (  n*ref[n+1]+(n-2)*ref[n-1] )  ;
         newref[n]= 1/( sqrt(  1+newref[n]^2 )+newref[n]  );
        ) ;  print(" ",residm);
       for(n=2,N-1,ref[n]=newref[n]);
       );
 );
}

Script V1.1 session started Thu Dec 19 11:38:53 2002

C:\calc\pari>gp-2-1
                  GP/PARI CALCULATOR Version 2.1.0 (released)
             ix86 running Windows 3.2 (ix86 kernel) 32-bit version
              (readline v4.0 enabled, extended help not available)

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comes WITHOUT ANY WARRANTY WHATSOEVER.

? \r reflecjc
alpha= 0.3000000  sin pi alpha/2= 0.453990607
```

iter.	$\Phi_1(0)$	$\Phi_2(0)$	$\Phi_3(0)$	$\Phi_4(0)$	$\Phi_5(0)$	$\Phi_6(0)$	$\Phi_7(0)$	$\Phi_8(0)$	$\Phi_9(0)$
	0.367883098	0.406603001	0.427202438	0.438032120	0.443863125	0.447146049	0.449098425	0.450327477	0.451143664
iter.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
1	0.453990607	0.453990607	0.453990607	0.453990607	0.453990607	0.453990607	0.453990607	0.453990607	0.453990607
2	0.374625426	0.411055959	0.424578085	0.431625072	0.435948566	0.438871514	0.440979521	0.442571680	0.443816684
3	0.368965781	0.404493014	0.423836837	0.433240534	0.438634789	0.442072607	0.444425731	0.446121465	0.447391852
4	0.366110087	0.404880268	0.424682308	0.435405794	0.441214990	0.444727398	0.447021691	0.448606494	0.449748319
5	0.367214055	0.405488905	0.426180133	0.436916193	0.442878899	0.446324018	0.448484612	0.449923774	0.450926738
...									
10	0.367870659	0.406611437	0.427205887	0.438050902	0.443884096	0.447176753	0.449133606	0.450367787	0.451186558
...									
20	0.367883024	0.406603045	0.427202390	0.438032144	0.443863094	0.447146049	0.449098403	0.450327458	0.451143670
...									
30	0.367883097	0.406603002	0.427202438	0.438032121	0.443863126	0.447146052	0.449098433	0.450327496	0.451143714
...									
50	0.367883098	0.406603001	0.427202438	0.438032121	0.443863127	0.447146052	0.449098433	0.450327497	0.451143714

```
? quit   Good bye!   C:\calc\pari>exit    Script completed Thu Dec 19 11:39:56 2002
```

Hmm, we should now show that (29) leads to a ***contraction*** of positive sequences, so to allow a proof of a unique fixed point. We can see (29) as

$$\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x}),$$

acting on positive sequences $\boldsymbol{x} = \{x_1, x_2 \dots\}$, with $F_n(\boldsymbol{x}) = \sqrt{A_n(\boldsymbol{x})^2+1} - A_n(\boldsymbol{x})$, and where $A_n(\boldsymbol{x})$ is the right-hand side of (29).

We estimate the contraction in the $\| \, \|_\infty$ norm by looking at the sum of all the $|\partial F_n/\partial x_i|$. For $i = 1, 2, \dots, n-2$ and $i = n$, one finds $\dfrac{\partial A_n}{\partial x_i} = \dfrac{x_{i-1}+x_{i+1}}{(n+1)x_{n+1}+(n-1)x_{n-1}}$. For $i = n \pm 1$, one must subtract $(n\pm 1)\dfrac{\text{num.}}{[(n+1)x_{n+1}+(n-1)x_{n-1}]^2}$, where "num" is the numerator of the right-hand side of (29). The sum of absolute values is bounded by

$$\frac{1+2x_1+2x_2+\cdots+2x_{n-2}+x_{n-1}+x_n+n(F_n^{-1}-F_n)}{(n+1)x_{n+1}+(n-1)x_{n-1}},$$

having introduced $F_n^{-1} - F_n$ from the left-hand side of (29). For the derivatives of F_n, we multiply by $dF_n/dA_n = -F_n/\sqrt{A_n^2+1} = -2F_n/(F_n^{-1}+F_n)$. The sum of the $|\partial F_n/\partial x_i|$ is therefore bounded by

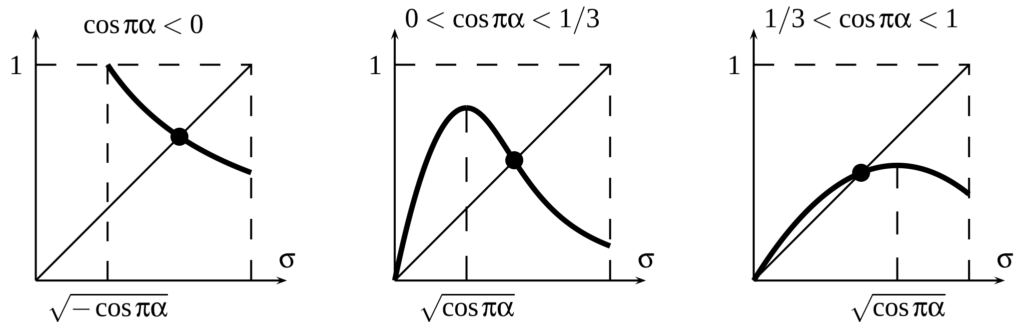
$$2F_n \frac{1+2x_1+2x_2+\cdots+2x_{n-2}+x_{n-1}+x_n+n(F_n^{-1}-F_n)}{(F_n^{-1}+F_n)[(n+1)x_{n+1}+(n-1)x_{n-1}]}.$$

For large n, when most of the x_i's and F_n itself are close to the limit σ, we find asymptotically $2\sigma\dfrac{2n\sigma+n(\sigma^{-1}-\sigma)}{(\sigma^{-1}+\sigma)2n\sigma}$ which is… 1!

If the positive sequence $\boldsymbol{x} = \{x_n\}$ has a limit σ with $\sigma^2+\cos\pi\alpha > 0$, then $A_n(\boldsymbol{x}) \to \dfrac{\sigma^2+\cos\pi\alpha}{2\sigma}$, and

$$F_n(\boldsymbol{x}) \to f(\sigma) := \sqrt{\left(\frac{\sigma}{2}+\frac{\cos\pi\alpha}{2\sigma}\right)^2+1} - \frac{\sigma}{2} - \frac{\cos\pi\alpha}{2\sigma}$$

which has $\sin(\pi\alpha/2)$ as attractive fixed point, as $f'(\sigma) = \dfrac{3\cos\pi\alpha - 1}{3-\cos\pi\alpha}$ at the fixed point $\sigma = \sin(\pi\alpha/2)$.



However, contraction cannot be established on the set of all the positive sequences. A convenient subset must be found. Let us try the already suspected set of the ***increasing*** positive sequences. Does $\boldsymbol{F}$ transforms an increasing sequence into an increasing sequence? Alas no, some iterated sequences may show decreasing episodes:

```
alpha= 0.4000   sin pi alpha/2= 0.587785261    cos pi alpha= 0.309016998

     x1                x23          x24         x25            x26         x27          x28          x29            x30
```

```
0.587785261 ...  0.587785261 0.587785261 0.587785261 0.587785261 0.587785261 0.587785261 0.587785261 0.587785261
0.494895897 ...  0.583209880 0.583399460 0.583573954 0.583735095 0.583884358 0.584023011 0.584152146 0.584272710
0.501288967 ...  0.589828341 0.589838555 0.589844230 0.589846167 0.589845023 0.589841333 0.589835538 0.589828002
0.504031732 ...  0.588181205 0.588167246 0.588152427 0.588137067 0.588121407 0.588105632 0.588089882 0.588074262
0.504781340 ...  0.587432119 0.587439806 0.587446203 0.587451582 0.587456152 0.587460079 0.587463494 0.587466499
...
0.504551159 ...  0.587521237 0.587542850 0.587561911 0.587578807 0.587593854 0.587607313 0.587619400 0.587630295
```

5.1.7. *Anticlimax.* . Moreover, some intermediate x_i's are larger that the limit $\sigma = \sin\pi\alpha/2$, whereas all the final values appear to be $< \sigma$. Finally, asymptotic behaviour of intermediate x_n's is not as expected: with $x_n = \sigma$, $n = 1, 2, \dots$ as first iterate, the next one is $F_n(x)$ such that $\dfrac{F_n(x)^{-1} - F_n(x)}{2} = \dfrac{\xi_n + n\cos\pi\alpha = x_1x_0 + \cdots + x_nx_{n-1} + n\cos\pi\alpha}{(n+1)x_{n+1} + (n-1)x_{n-1}} = \dfrac{\sigma + (n-1)\sigma^2 + n\cos\pi\alpha}{2n\sigma} \sim \dfrac{\sigma^{-1} - \sigma}{2} + \dfrac{1-\sigma}{2n}$, whence $F_n(x) = \sigma + O(n^{-1})$, whereas $O(n^{-2}$ is expected.

Obviously, we have a problem with $\xi_n = x_1x_0 + \cdots + x_nx_{n-1}$ which is numerically found to be $n\sigma^2 + O(n^{-1})$, a feature which is impossible to catch.

5.1.8. *More smart ideas.* ***Aha!*** In order to be sure of the sought feature for ξ_n, I intend to compute x_1, $x_2, \dots, x_N$ with boundary values $x_0 = 1$ and $x_{N+1} = \sigma$, and where ξ_n is approximated by $N\sigma^2 - x_{n+1}x_n - x_{n+2}x_{n+1} - \cdots - x_Nx_{N-1}$, which has the expected right behaviour.

So, the iteration is $\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x})$, with

$$\frac{F_n(\boldsymbol{x})^{-1} - F_n(\boldsymbol{x})}{2} = A_n((x) = \frac{\xi_n + n\cos\pi\alpha = N\sigma^2 - x_{n+1}x_n - x_{n+2}x_{n+1} - \cdots - x_Nx_{N-1} + n\cos\pi\alpha}{(n+1)x_{n+1} + (n-1)x_{n-1}} \quad (30)$$

for $n = 1, 2, \dots, N$, with $x_0 = 1$, $x_{N+1} = \sigma$.

Let us try a run, I can't wait:

```
{
/*
  grunb2.gp    :  launch gp and make  \r grunb2

  Reflection coefficients for Grunbaum- Delsarte et al.  problem

 by modified iteration,   January 2003.

*/
default(realprecision,75);
flone=1.0;pr=precision(flone);ep=10^(-pr);
default(format,"g1.7");
N=50;
a=0.2;
print1(a," ");    sa=sin(Pi*a/2);
print(" sigma=sin pi a/2= ",sa);
ca=cos(a*Pi);
print(" cos pi a= ",ca);
\\ vector of reflection coeff.
ref=vector(N+2,k,0);
newref=vector(N+2,k,0);
   print(" iteration");
   for(n=2,N+2,ref[n]=sa);ref[1]=1;
    for(ns=1,5,   newref[1]=1; xi=N*sa^2;  print(ns," ");
          for(ni=1,N,
            n=N+1-ni;
```

```
    An= (xi+n*ca) / (  (n+1)* ref[n+2]+(n-1)*ref[n] )  ;
    newref[n+1]= 1/( sqrt(  1+An^2 )+An  );
             xi=xi-ref[n+1]*ref[n];
            ) ;
   print(" final xi=", xi);
   resid=0;  for(n=2,N+1, print1(ref[n]," ");
              resid=max(resid,abs(newref[n]-ref[n]));ref[n]=newref[n]);
             print(" ",resid);
       );
}

Script V1.1 session started Mon Jan 13 09:55:38 2003

? \r grunb2
0.200000 sigma=sin pi a/2= 0.309016998     cos pi a= 0.809017006

iteration x1       x2          x3         x4          x5          x6
1 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998
2 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998
3 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998
...
 final xi=-0.213525497

Script completed Mon Jan 13 10:01:06 2003
```

This is ridiculous! Indeed, $x_n = \sigma$ for all $n \geqslant 1$ is a solution of (30). The mistake was to input the wrong value $\xi_N = N\sigma^2$ and, even as the error is small when N is large, it induced a big error on $\xi_0 = 0$.

I now try $\xi_N = N\sigma^2 + \varepsilon$, with a small value of ε:

$$\frac{F_n(\boldsymbol{x})^{-1} - F_n(\boldsymbol{x})}{2} = A_n(\boldsymbol{x}) = \frac{\xi_n + n\cos\pi\alpha = N\sigma^2 + \varepsilon - x_{n+1}x_n - x_{n+2}x_{n+1} - \cdots - x_N x_{N-1} + n\cos\pi\alpha}{(n+1)x_{n+1} + (n-1)x_{n-1}} \tag{31}$$

for $n = 1, 2, \ldots, N$, with $x_0 = 1$, and x_{N+1} given in $(0, \sigma]$.

```
   ...
   epsN=0.01;
   for(n=2,N+2,ref[n]=sa);ref[1]=1;
    for(ns=1,99,   newref[1]=1; xi=N*sa^2+epsN;  print(ns," ");
   ...
Script V1.1 session started Mon Jan 13 10:15:37 2003

C:\calc\pari>gp-2-1
? \r grunb2
0.20000  sigma=sin pi a/2= 0.309016998     cos pi a= 0.809017006
iteration  x1       x2          x3         x4          x5          x6          x7          x8
 1 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998
 2 0.306219520 0.307612449 0.308079335 0.308313263 0.308453776 0.308547516 0.308614505 0.308664765
 3 0.303157786 0.305671680 0.306848166 0.307423936 0.307763642 0.307987009 0.308144695 0.308261757
 4 0.299463531 0.303418696 0.305337763 0.306365638 0.306962124 0.307349253 0.307619605 0.307818446
 5 0.295361338 0.300807905 0.303625347 0.305154810 0.306064180 0.306647176 0.307050272 0.307344243
10 0.270967997 0.284787140 0.292876381 0.297689581 0.300651827 0.302561102 0.303851791 0.304765798
20 0.225981032 0.252575546 0.270358158 0.281962161 0.289514290 0.294493259 0.297852035 0.300183570
50 0.179686381 0.215487507 0.242549751 0.261914544 0.275338719 0.284519575 0.290798615 0.295135010

 final xi=0.174535940
...
? quit      Good bye!    C:\calc\pari>exit
Script completed Mon Jan 13 10:25:40 2003
```

Now, we get a slightly better estimate, and we try other values of ε, trying to have a vanishing ξ_0. As the iteration has become so slow, only the 50^{th} and the 75^{th} are shown:

```
Script V1.1 session started Mon Jan 13 10:51:51 2003

epsN  it.     xi0          x1          x2          x3          x4          x5          x6          x7          x8
0.001 50 -0.163917929  0.291013681 0.297533706 0.301534030 0.304008217 0.305561057 0.306554982 0.307206219 0.30764414
      75 -0.161816339  0.290153919 0.296967949 0.301165472 0.303769074 0.305405851 0.306453912 0.307140014 0.30760042

0.002 50 -0.117394143  0.274474296 0.286600667 0.294248505 0.299064334 0.302122190 0.304094074 0.305392139 0.30626748
      75 -0.113484834  0.272896542 0.285524758 0.293531093 0.298591648 0.301812283 0.303890901 0.305258463 0.30617896

0.003 50 -0.073636237  0.259241672 0.276185619 0.287156206 0.294186150 0.298701449 0.301634917 0.303575083 0.30488717
      75 -0.068164090  0.257063455 0.274649366 0.286108540 0.293485440 0.298237418 0.301328662 0.303372693 0.30475276

0.004 50 -0.032368428  0.245179644 0.266258421 0.280252791 0.289374295 0.295299816 0.299178134 0.301755372 0.30350335
      75 -0.025538800  0.242498971 0.264306568 0.278892477 0.288451021 0.294682317 0.298767874 0.301483053 0.30332199

0.005 50  0.006649610  0.232169733 0.256790922 0.273533841 0.284629255 0.291918212 0.296724326 0.299933321 0.30211617
      75  0.014663500  0.229068764 0.254463778 0.271877573 0.283488829 0.291147982 0.296209200 0.299589886 0.30188680

0.006 50  0.043628206  0.220108412 0.247756840 0.266994894 0.279951380 0.288557493 0.294274072 0.298109238 0.30072577
      75  0.052679462  0.216656357 0.245090801 0.265058525 0.278599174 0.287635359 0.293653287 0.297693533 0.30044734

...

Script completed Mon Jan 13 11:00:21 2003
```

Still further detail, keeping now 100^{th} and the 200^{th} iterations:

```
Script V1.1 session started Mon Jan 13 12:05:34 2003


  epsN  it.      res.          xi0          x1          x2          x3          x4          x5          x6          x
0.0044 100 0.00003326434 -0.008321663 0.236659067 0.260060575 0.275886322 0.286337328 0.293185851 0.297691239 0.30069
       200 0.00000000397 -0.008217685 0.236617948 0.260030122 0.275864899 0.286322753 0.293176147 0.297684867 0.30068

0.0045 100 0.00003413821 -0.004276811 0.235306088 0.259069462 0.275180283 0.285838125 0.292830474 0.297434110 0.30050
       200 0.00000000408 -0.004171089 0.235264338 0.259038445 0.275158416 0.285823227 0.292820544 0.297427584 0.30049

0.0046 100 0.00003501818 -0.000255118 0.233963922 0.258083236 0.274476246 0.285339652 0.292475309 0.297177003 0.30031
       200 0.00000000420 -0.000147672 0.233921551 0.258051661 0.274453938 0.285324430 0.292465153 0.297170323 0.30030

0.0047 100 0.00003590429  0.003743652 0.232632453 0.257101867 0.273774206 0.284841907 0.292120358 0.296919919 0.30012
       200 0.00000000431  0.003852803 0.232589470 0.257069738 0.273751457 0.284826361 0.292109975 0.296913085 0.30011

0.0048 100 0.00003679661  0.007719737 0.231311564 0.256125325 0.273074157 0.284344892 0.291765620 0.296662858 0.29993
       200 0.00000000442  0.007830572 0.231267979 0.256092646 0.273050970 0.284329022 0.291755010 0.296655870 0.29992

0.0049 100 0.00003769518  0.011673364 0.230001143 0.255153580 0.272376095 0.283848606 0.291411097 0.296405822 0.29974
       200 0.00000000454  0.011785865 0.229956965 0.255120356 0.272352470 0.283832413 0.291400260 0.296398679 0.29973

0.0050 100 0.00003860006  0.015604763 0.228701077 0.254186601 0.271680013 0.283353050 0.291056790 0.296148811 0.29955
       200 0.00000000466  0.015718911 0.228656316 0.254152836 0.271655952 0.283336534 0.291045725 0.296141513 0.29954

? quit  Good bye! C:\calc\pari>exit   Script completed Mon Jan 13 13:54:13 2003
```

where “res.” is the residue norm $\max_n |(x_n^{-1} - x_n)/2 - A_n(\boldsymbol{x})|$. It seems indeed that we get the correct sequence when $\xi_0 = 0$, if one reminds that one should find

```
      x1         x2          x3          x4          x5          x6          x7          x8          x9          x10
0.233872324 0.258015427 0.274428042 0.285306081 0.292452074 0.297160853 0.300299896 0.302429553 0.303905700 0.30495330
```

The iteration based on (31) is numerically rather poor, but has very interesting monotony properties:

5.1.9. ***Lemma.*** For any $\alpha \in (0,1)$, the map $\boldsymbol{F}$ acting on $\mathbb{R}^N$ through (31) has the properties:

(1) For any $\varepsilon \geqslant 0$, a positive sequence bounded by $\sigma = \sin\pi\alpha/2$ is transformed by $\boldsymbol{F}$ in a sequence with the same properties.

(2) If $\boldsymbol{x}$ is a positive sequence bounded by σ, and if $\boldsymbol{F}(\boldsymbol{x}) \leqslant \boldsymbol{x}$, then $\boldsymbol{F}(\boldsymbol{F}(\boldsymbol{x})) \leqslant \boldsymbol{F}(\boldsymbol{x})$, where $\boldsymbol{x} \leqslant \boldsymbol{y}$ means $x_n \leqslant y_n$, $n = 1, 2, \dots, N$.

(3) ξ_0 is an increasing function of ε when one starts (31) with the constant sequence $x_n = \sigma$, $n = 1, \dots, N$.

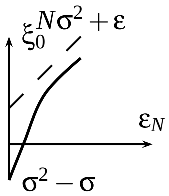
(4) $\xi_0 = \sigma^2 - \sigma < 0$ if $\varepsilon = 0$ and $x_{N+1} = \sigma$; $\xi_0 \geqslant \varepsilon + \sigma^2 - \sigma$. There is therefore exactly one $\varepsilon > 0$ such that $\xi_0 = 0$ if $x_{N+1} = \sigma$.

Indeed,

(1) If all the x_n's are positive and $\leqslant \sigma = \sin\pi\alpha/2$, the numerator of A_n in (31) is $\geqslant n\sigma^2 + n\cos\pi\alpha = n\cos^2(\pi\alpha/2)$, so that $A_n \geqslant \cos^2(\pi\alpha/2)/(2\sin\pi\alpha/2 = (\sigma^{-1} - \sigma)/2$, and $0 < F_n(\boldsymbol{x}) \leqslant \sigma$.

(2) We show $\boldsymbol{y} \leqslant \boldsymbol{x} \Rightarrow \boldsymbol{F}(\boldsymbol{y}) \leqslant \boldsymbol{F}(\boldsymbol{x})$ on positive sequences such that A_n is positive. Indeed: $A_n(\boldsymbol{y}) \geqslant A_n(\boldsymbol{x})$, as A_n is a decreasing function of its x_i's arguments.

(3) At each iteration, $x'_n \geqslant x''_n$ if x'_n and x''_n correspond to ε' and ε'', with $\varepsilon' \leqslant \varepsilon''$. This is true for the first step; if true at some iteration, $A'_n \leqslant A''_n$, and $x'_n \geqslant x''_n$ again at the next iteration. Finally, $\xi'_0 = N\sigma^2 + \varepsilon' - x'_1 - x'_1 x'_2 - \cdots - x'_{N-1} x'_N \leqslant \xi''_0$.



That there is always an $\varepsilon_N > 0$ ensuring $\xi_0 = 0$ follows from 1) that $\xi_0 = N\sigma^2 - x_1 - x_1 x_2 - \cdots - x_{N-1} x_N = \sigma^2 - \sigma < 0$ at $\varepsilon = 0$ (as all the x_i's $= \sigma$); 2) when ε is large, the x_n's are very small, so that ξ_0 is close to $N\sigma^2 + \varepsilon$.

See here a computer run with $N = 50$ and $\sigma = 0.309...$, so that ξ_0 is expected near $4.774 + \varepsilon$:

```
Script V1.1 session started Wed Jan 15 13:51:32 2003

C:\calc\pari>gp-2-1
? \r grunb2
0.200000002 0.200000002  sigma=sin pi a/2= 0.309016998   cos pi a= 0.809017006

it.    res.     xi0        x1             x2          x3             x4             x5
 epsN= 1
100 43632.74 4.311780 1.020088 E-07 5.177575 E-07 1.995604 E-06 6.411419 E-06 1.801285 E-05
200 611.2635 4.316579 8.342916 E-08 4.275806 E-07 1.663710 E-06 5.395415 E-06 1.530021 E-05
300 7.097863 4.316634 8.323656 E-08 4.266411 E-07 1.660235 E-06 5.384725 E-06 1.527153 E-05
400 0.082361 4.316635 8.323433 E-08 4.266302 E-07 1.660194 E-06 5.384602 E-06 1.527120 E-05
500 0.000955 4.316635 8.323430 E-08 4.266301 E-07 1.660194 E-06 5.384600 E-06 1.527119 E-05

 epsN= 2
400 3010.520 6.101422 5.711563 E-11 3.946940 E-10 2.012176 E-09 8.383047 E-09 3.010333 E-08
500 75.78560 6.101422 5.711510 E-11 3.946905 E-10 2.012159 E-09 8.382978 E-09 3.010309 E-08

 epsN= 3
400 1665142. 7.455424 2.227993 E-13 1.841310 E-12 1.106376 E-11 5.374799 E-11 2.232201 E-10
500 44603.46 7.455424 2.227948 E-13 1.841274 E-12 1.106355 E-11 5.374702 E-11 2.232162 E-10
```

```
? quit    Good bye!     C:\calc\pari>exit     Script completed Wed Jan 15 13:58:43 2003
```

The amazing feature is the small size of the first x_n's when ε is large.

We are far enough to get the

5.1.10. ***Theorem***. *For any* $\alpha \in (0,1)$*, the positive solution -known to be unique- of* (27) *with the sole boundary condition* $x_0 = 1$ *is the limit (in the* ℓ_∞ *norm) when* $N \to \infty$ *of the unique positive solution of the system of* N *equations from* (29) *with* $n = 1, 2, \dots, N$*, where* $x_{N+1}, x_{N+2}, \dots$ *are replaced by* $\sigma = \sin(\pi\alpha/2)$*.* *One also has* $\Phi_n(0) < \sigma = \sin(\pi\alpha/2),\ n = 1, 2, \dots$

$\Phi_1(0) + \Phi_2(0)\Phi_1(0) + \cdots + \Phi_n(0)\Phi_{n-1}(0) > n\sigma^2, \quad n = 1, 2, \dots$

Of course, all this means that the sequence $\{\Phi_n(0)\}$ is positive

I would be more at ease with a lower bound of x_n, to be sure that we do not fall towards the black hole which is the trivial solution $x_n = 0, n = 1, 2, \dots$ of (27)! The example above shows that the x_n's may become very small.

No problem if $\xi_0 = 0$: then, $x_1 + x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n > n\sigma^2$ and all the x_n's $< \sigma \Rightarrow x_1 > \sigma^2$.

Now, the part "limit when $N \to \infty$":

The step $N \to N+1$**.**

To show: $x_i^{(N+1)} < x_i^{(N)}, i = 1, 2, \dots, N+1$.

$\{x_1^{(N)}, \dots, x_N^{(N)}\}$ is the unique positive solution of the N first equations of (27), when $x_0 = 1$ and $x_{N+1} = \sigma$ are given.

Let $\{x_i^{(N+1)}\}$, $i = 1, \dots, N+1$, be as above with $x_{N+2} = \sigma$. We know that $x_{N+1}^{(N+1)} < \sigma$, as all the other $x_i^{(N+1)}$ for $i = 1, \dots, N$.

A first point is that $x_i^{(N+1)}$ for $i = 1, 2, \dots, N$ is another positive solution of the N first equations of (27), with the boundary conditions $x_0^{(N+1)} = 1$ and $x_{N+1}^{(N+1)} < \sigma$.

Then, $x_i^{(N+1)} < x_i^{(N)}$ for $i = 1, 2, \dots, N+1$ from Proposition 5.1.5, p. 36!

5.1.11. ***Extended conjecture.*** Another generalized Jacobi weight giving rise to real $\Phi_n(0)$'s is $2^{2\beta}\left|\sin\dfrac{\theta-\theta_1}{2}\right|^{2\beta}\left|\sin\dfrac{\theta+\theta_1}{2}\right|^{2\beta} = (\cos\theta_1 - \cos\theta)^{2\beta}$ on the same arc $\theta_1 < \theta < 2\pi - \theta_1$ as above.

Then we apply (24) with $\alpha = \beta$ (the α of (24), of course), and $\theta_1 = \pi\alpha$:

$$(n+1+2\beta)x_{n+1} = \frac{2\xi_{n-1,n} + 2n\cos\pi\alpha}{1-x_n^2}x_n - (n-1+2\beta)x_{n-1}, \tag{32}$$

with $\xi_{n-1,n} = x_1x_0 + x_2x_1 + \cdots + x_nx_{n-1}$.

Everything should follow as previously, at least when $\beta \geqslant 0$. If $-1/2 < \beta < 0$, x_1 may be negative. Of course: when $\alpha \to 0$, $x_n = \Phi_n(0) \to \dfrac{2\beta}{n+2\beta+1}$, from the Jacobi case (3). Check that it is a solution of (32), and that $\xi_n \to \dfrac{2n}{n+2\beta}$.

The algorithm based on (31) still works, but monotonicity is lost if $\beta < 0$:

```
Script V1.1 session started Wed Jan 15 19:26:42 2003

C:\calc\pari>gp-2-1
? \r grunb2
 beta= -0.4000
a=0.200  sigma=sin pi a/2= 0.309016998    cos pi a= 0.809017006

  epsN   it.   res.         xi0          x1          x2          x3          x4          x5          x6          x7
-0.7300   1 1.870135   -0.943525519  0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016
```

```
        2 0.7042453   0.338622633 -0.672895643 0.310526529 0.309838715 0.309581500 0.309446924 0.309364162 0.309308
        3 0.3774387  -0.049577267 -0.383801812 0.239853530 0.310968644 0.310318867 0.309987500 0.309787483 0.309654
        4 0.1884398   0.111600434 -0.506704566 0.259264601 0.295726171 0.311221992 0.310629146 0.310277695 0.310046
        5 0.0789971   0.050028948 -0.463622974 0.237411815 0.300642846 0.307529406 0.311362442 0.310825839 0.310477
        6 0.0719641   0.088081151 -0.493017577 0.244316002 0.293604070 0.309517456 0.310682941 0.311423509 0.310939
        7 0.0340438   0.068994451 -0.481645602 0.236455395 0.296619959 0.307394269 0.311786747 0.311564930 0.311426
        8 0.0248932   0.081868112 -0.492970739 0.240113465 0.293811476 0.309078412 0.311378238 0.312301968 0.311760
        9 0.0254564   0.073029980 -0.489236169 0.237428235 0.296032722 0.308307282 0.312454837 0.312461032 0.312298
       10 0.0120816   0.077756786 -0.494995718 0.240174885 0.295322860 0.309772801 0.312470898 0.313205727 0.312608
      ...
      250 2.157 E-11  0.015164409 -0.513154068 0.305572573 0.341790084 0.339335893 0.332603304 0.326763176 0.322394
      500 1.025 E-21  0.015164409 -0.513154068 0.305572573 0.341790084 0.339335893 0.332603304 0.326763176 0.322394

-0.7305 250 2.236 E-11  0.006866933 -0.516129642 0.315181023 0.347903715 0.343117240 0.335033848 0.328400153 0.323546
        500 1.027 E-21  0.006866933 -0.516129642 0.315181023 0.347903715 0.343117240 0.335033848 0.328400153 0.323546
-0.7310 250 2.306 E-11 -0.001575852 -0.519134554 0.325188111 0.354143600 0.346933581 0.337471851 0.330036888 0.324697
        500 1.025 E-21 -0.001575852 -0.519134554 0.325188111 0.354143600 0.346933581 0.337471851 0.330036888 0.324697

? quit  Good bye! C:\calc\pari>exit  Script completed Wed Jan 15 19:32:49 2003
```

The true values are

```
      x1          x2          x3          x4          x5          x6          x7          x8          x9         x10
-0.518575525 0.323302616 0.352977636 0.346223740 0.337019515 0.329733610 0.324484652 0.320776200 0.318145256 0.316253
```

6. Generalized Jacobi polynomials on the unit circle, with > 2 singular points.

[34]

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