New in 2002-2003: essentially the linear differential equation, more and better ordered experimental evidence about the generalized Jacobi polynomials, and various corrections. Thanks for remarks and kind words: L. Golinskii, M. Ismail, F. Marcellán, N. Witte. Earlier thanks to P. Nevai for sending me a copy of the Badkov paper in a former century.

Er, most of these “new” items are still in construction, but there are new facts about the Grünbaum-Delsarte-Janssen-Vries problem in § 5.

New, new in 2012-2013: final asymptotic of unit circle gen. Jacobi

\[ w(\theta) = \begin{cases} r_1 e^{-|\theta|} \sin(\theta - \theta_1)/2^{2\alpha} \sin(\theta - \theta_2)/2^{2\beta}, & \theta_1 < \theta < \theta_2 \\ r_2 e^{-|\theta|} \sin(\theta - \theta_1)/2^{2\alpha} \sin(\theta - \theta_2)/2^{2\beta}, & \theta_2 - 2\pi < \theta < \theta_1 \end{cases} \]

where \( r_1 \) and \( r_2 \) are positive, see § 4.3, p. 22.

This version: March 19, 2013 (incomplete and unfinished)

Rien n’est plus agréable que de savoir quelqu’un aux prises avec des difficultés insurmontables, surtout si l’on y voit du travail gâché, de l’argent perdu et des crises de nerfs.

J. Giono

Abstract: Semi-classical orthogonal polynomials on the unit circle are examined. Special care is given to generalized Jacobi polynomials.

Contents

1. Complex orthogonal polynomials
   1.1. General scalar product .......................................................... 2
   1.2. Playing with the idea ............................................................ 3
   1.3. Cholesky factors ................................................................. 5
   1.4. Decreasing sequences .......................................................... 6
   1.5. Most general scalar product ............................................... 7
   1.6. \( \int f(t)g(t) \, dt \) scalar product ..................................... 7
2. \( \int f(t)g(t) \, d\mu(t) \) scalar product on the unit circle .............. 8
   2.1. Toeplitz matrix and Fourier series ................................... 8
   2.2. Recurrence relation ......................................................... 9
   2.3. Multiplication operator: Toeplitz matrices in Grenander and Szegő’s sense ......................................................... 10
   2.4. First and last columns of \( G^{-1} \) ..................................... 11
   2.5. Behaviour of reflection coefficients ................................... 12
   2.6. Circle versus interval ....................................................... 14
   2.7. Jacobi polynomials on the unit circle .......................... 16

\(^1\)There is nothing like knowing somebody having to struggle with a hopeless task, especially if it represents bad work, loss of money, and nervous tantrums.
1. Complex orthogonal polynomials.

1.1. General scalar product. For any scalar product (positive definite sesquilinear hermitian symmetric form) defined at least on the polynomials of degree \( \leq N \leq \infty \), there is exactly one sequence \( \{ \Phi_0, \Phi_1, \ldots, \Phi_N \} \) of monic polynomials, degree \( \Phi_n = n, n = 0, 1, \ldots, n \), such that \( (\Phi_n, \Phi_m) = 0, 0 \leq m \neq n \leq N \).

The \( n \) coefficients of \( \Phi_n \) are determined by \( (\Phi_n, p) = 0 \) for any \( p \) of degree \( < n \): \( \Phi_n(z) = z^n + \sum_{k=0}^{n-1} \Phi_{n,k} z^k \).

\[
G_{n-1} = \begin{bmatrix}
\Phi_n(0) \\
\Phi_n'(0) \\
\vdots \\
\Phi_{n,n-1}
\end{bmatrix} = \begin{bmatrix}
(z^0, z^0) & (z^1, z^0) & \cdots & (z^{n-1}, z^0) \\
(z^0, z^1) & (z^1, z^1) & \cdots & (z^{n-1}, z^1) \\
\vdots & \vdots & \ddots & \vdots \\
(z^0, z^n) & (z^1, z^n) & \cdots & (z^{n-1}, z^n)
\end{bmatrix} = -\begin{bmatrix}
(z^n, z^0) \\
(z^n, z^1) \\
\vdots \\
(z^n, z^n)
\end{bmatrix}.
\]

\( G_{n-1} \) is the transposed of the Gram matrix of \( \{z^0, z^1, \ldots, z^{n-1}\} \). The matrix is hermitian and positive definite (cf. [10]).

\[
\Phi_0 = 1 \quad \Phi_1(z) = z \cdot \frac{(z^1, z^0)}{(z^0, z^0)}
\]

Also,

\[
G_n = \begin{bmatrix}
\Phi_n(0) \\
\Phi_n'(0) \\
\vdots \\
\Phi_{n,n-1}
\end{bmatrix}^T = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

i.e., the last column of \( G_{n}^{-1} \) is made of the coefficients of \( \Phi_n/\|\Phi_n\|^2 \).

\( \Phi_n \) is the transposed of the complex conjugate of \( \Phi_n \).

\[\text{i.e., } (f,g) \text{ involves the complex conjugate of } g.\]
\[ \Phi_n \] is the only monic \( n \)th degree polynomial of minimal norm: any \( n \)th degree monic polynomial is \( \Phi_n + p \) with degree \( p < n \), and
\[ \| \Phi_n + p \|^2 = (\Phi_n + p, \Phi_n + p) = \| \Phi_n \|^2 + 2 \Re \left( \Phi_n, p \right) + \| p \|^2. \]

One has
\[ \| \Phi_n \|^2 = \det G_n \over \det G_{n-1}. \]

1.2. Playing with the idea.

1.2.1. The simplest case occurs when \( G_n \) happens to be a diagonal matrix, such as for the simplest \( L^2 \) scalar products on a circle or on a disk:
\[ (f, g) = \int_{|z|=\exp(i\theta)=R} f(z)g(z) \frac{d\theta}{2\pi R}, \quad (f, g) = \int_{|z|<R} f(z)g(z) \frac{dx dy}{2\pi R^2}. \]
This also happens with Sobolev scalar products
\[ \int f(z)\overline{g(z)} + \frac{1}{2} \int f'(z)\overline{g'(z)}. \]

1.2.2. The next simplest situation seems to be binomial polynomials: when do we have \( \Phi_m(z) = z^m - \alpha_m \), \( m = 1, 2, \ldots \)? Then one must have \( \Phi_m(z, z^k) = 0 \), i.e., \( \alpha_m(z^k, z^k) = \alpha_m(z^k, z^k) \) for \( k = 0, 1, \ldots, m - 1 \). In particular, for \( k = 0 \): \( (z^m, z^d) = \alpha_m(z^d, z^0) \) for all \( m > 0 \). Whence
\[ (z^m, z^0) = \alpha_m(z^0, z^0), (z^0, z^m) = \overline{\alpha_m(z^0, z^0)}, (z^m, z^k) = \alpha_m(z^0, z^0), m, k, 1, 2, \ldots, m \neq k. \]

Oh, I see, this is just a familiar perturbation of rank one to a diagonal matrix, easily explainable by the addition of a masspoint to an elementary \( L^2 \) or Sobolev measure, and we invert \( G_n = D + \mathbf{uv}' \) by Sherman-Morrison formula \([15, p.3]\): \[ G_n^{-1} = D^{-1} - \frac{D^{-1}\mathbf{uv}'D^{-1}}{1 + \mathbf{v}'D^{-1}\mathbf{u}} \] and... hey! this is not sparse! Indeed, \([11]\) is not a rank one perturbation to a diagonal matrix \( D' \) from the nondiagonal elements of \( G_n \), \( \alpha_m\overline{\alpha_k}(z^0, z^0) = u_m v_k \), (with \( \alpha_0 = 1 \)), so \( u_m = \gamma \alpha_m \), \( v_m = \gamma^{-1}(z^0, z^0) \alpha_m \). But then the first diagonal element should be \( (z^0, z^0) = d_0 + u_0 v_0 = d_0 + (z^0, z^0) \), or \( d_0 = 0 \), aargh.

Well, there is nothing wrong with \( d_0 = 0 \), such as in
\[ (f, g) = \int_0^{2\pi} f'(e^{i\theta}) \overline{g'(e^{i\theta})} \frac{d\theta}{2\pi} + f(e^{is})\overline{g(e^{is})}, \]
so that \([11]\) holds with \( \alpha_m = r^m e^{im\theta}, m = 0, 1, \ldots \).

Remark that, if \( r > 1 \), all the zeros but one of \( \Phi_m \) are outside the convex hull of the support of the measures involved in the scalar product, a known feature of Sobolev orthogonal polynomials. Actually, I do not know if these zeros are useful in any respect.

\[ z^m - \alpha_m e^{im\theta} \]

1.2.3. Here is an example where the \( (z^k, z^k) \) satisfy special relations which make orthogonal polynomials immediately appear:
let \( \alpha \) be an irrational positive number, and the scalar product be \( (f, g) = \frac{4}{\pi^2} \sum_{m=-\infty}^{\infty} \frac{1}{m^2} f(e^{i\alpha m})\overline{g(e^{i\alpha m})}, \)
so that \( (z^k, z^k) = \frac{8}{\pi^2} \sum_{m=1 \text{ odd}}^{\infty} \frac{1}{m^2} \cos[(k - \ell)m\alpha] = F((k - \ell)\alpha), \) where \( F \) is the even periodic function of period 2, with value \( F(u) = 1 - 2|u| \) on \([-1, 1]\). Remark that \( F(u + k) = (-1)^k F(u) \) if \( k \in \mathbb{Z} \).
Now, let \( m'/n' \) be the best rational approximant to \( \alpha \) in the sense that \( 1 \leq n' \leq n \) and \( |n'\alpha - m'| < \) any other \( p\alpha - q \) with \( 1 \leq p \leq n \); \( m'/n'' \) the second best rational approximant to \( \alpha \). Then, \( \Phi_n(z) = z^n + \) a combination of \( z^n \) and \( z^{n-1} \). Indeed, 

\[
(F_n, z^{n-k}) = F(k\alpha) + the same combination of F((k-n')\alpha) and F((k-n'')\alpha).
\]

more on “second best approx.”

About this second best rational approximation which most of us would ignore completely, if it were not available in B. Casselman’s home page

\[
F((k-n')\alpha) = (-1)^m F(k\alpha - (n'\alpha - m')), \quad F((k-n'')\alpha) = (-1)^m F(k\alpha - (n''\alpha - m')).
\]

As long as \( k\alpha, (n'\alpha - m'), \) and \( (n''\alpha - m'') \) have the same integer (floor) part:

\[
\frac{F(k\alpha) - F(k\alpha - (n'\alpha - m'))}{n'\alpha - m'} - \frac{F(k\alpha) - F(k\alpha - (n''\alpha - m''))}{n''\alpha - m''} = 0,
\]

which is interpreted as \( (\Phi_n, z^{n-k}) = 0 \), where 

\[
\Phi_n(z) = \frac{z^n - z^{n-1}}{(-1)^m(n'\alpha - m')} - \frac{z^n - z^{n-1}}{(-1)^m(n''\alpha - m'')}
\]

Check with the program used in p. 12:

```{Foucoeff(k)=
    -(-1)^(floor(k*sqrt(2)))*( 2*frac(k*sqrt(2))-1 )
}
```

... 

Script V1.1 session started Mon Sep 20 17:54:35 1999
1.3. Cholesky factors.

Considering all the monic orthogonal polynomials $\Phi_k(z) = z^k + \xi_{k-1}z^{k-1} + \cdots + \xi_0$, $k = 0, \ldots, n$:

$$G_n = \begin{bmatrix}
\xi_{0,0} & 1 & \cdots & \xi_{0,n} \\
\xi_{1,1} & 1 & \cdots & \xi_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{n,n} & 1 & \cdots & 1
\end{bmatrix}
= \begin{bmatrix}
(\Phi_0, z^0) = ||\Phi_0||^2 \\
(\Phi_0, z^1) \\
\vdots \\
(\Phi_0, z^n)
\end{bmatrix}
= \begin{bmatrix}
(\Phi_1, z^1) = ||\Phi_1||^2 \\
(\Phi_1, z^2) \\
\vdots \\
(\Phi_1, z^n)
\end{bmatrix}
= \begin{bmatrix}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{bmatrix}
= \begin{bmatrix}
(\Phi_n, z^n) = ||\Phi_n||^2
\end{bmatrix}
$$

i.e., the Cholesky factorization of $G_n$ is $LL^T$, with $L_{k,\ell} = (\varphi_\ell, z^k)$, where

$$\varphi_\ell = \frac{\Phi_\ell}{||\Phi_\ell||}$$

is the $\ell$th degree orthonormal polynomial.

Check (Riesz-Fisher- generalized Parseval):

$$(z^\ell, z^k) = \sum_{m=0}^{\min(\ell, k)} (z^\ell, \varphi_m)(\varphi_m, z^k).$$

And, as $L_{k,\ell} = (z^\ell, \varphi_k)$ is the coefficient of $z^\ell$ in the $\varphi$–basis: $[\varphi_0, \ldots, \varphi_n] = [\Phi_0, \ldots, \Phi_n]L^T$, we also confirm that $(L^{-1})^T$ contains the coefficients of the $\varphi$'s, and that $\xi_{k,\ell} = (L^{-1})^T_{k,\ell}/||\varphi_\ell||$:

$$G_n = LL^T = \begin{bmatrix}
(\varphi_0, z^0) & (\varphi_0, z^1) & \cdots \\
(\varphi_1, z^1) & (\varphi_1, z^2) & \cdots \\
\vdots & \vdots & \ddots \\
(\varphi_n, z^n) & (\varphi_n, z^1) & \cdots
\end{bmatrix}
= \begin{bmatrix}
(\varphi_0, z^0) & (\varphi_0, z^1) & \cdots & (\varphi_0, z^n) \\
(\varphi_1, z^1) & (\varphi_1, z^2) & \cdots & (\varphi_1, z^n) \\
\vdots & \vdots & \ddots & \vdots \\
(\varphi_n, z^n) & (\varphi_n, z^1) & \cdots & (\varphi_n, z^n)
\end{bmatrix}.
$$

$$(G_n)^{-1} = (L^{-1})^T L^{-1} = \begin{bmatrix}
\xi_{0,0} & \xi_{0,1} & \cdots & \xi_{0,n} \\
\xi_{1,1} & \xi_{1,1} & \cdots & \xi_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{n,n} & \xi_{1,n} & \cdots & \xi_{n,n}
\end{bmatrix}
= \begin{bmatrix}
||\Phi_0||^{-2} \\
||\Phi_1||^{-2} \\
\cdots \\
||\Phi_n||^{-2}
\end{bmatrix}
= \begin{bmatrix}
\xi_{0,0} & \xi_{0,1} & \cdots & \xi_{0,n} \\
\xi_{1,1} & \xi_{1,1} & \cdots & \xi_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{n,n} & \xi_{1,n} & \cdots & \xi_{n,n}
\end{bmatrix}.
$$

In particular, the last column of $(G_n)^{-1}$ is

$$\frac{1}{||\Phi_n||^2} \begin{bmatrix}
\xi_{0,n} \\
\xi_{1,n} \\
\vdots \\
\xi_{n,n}
\end{bmatrix}^T.$$

(2)
Example: Jacobi $(\alpha, 0)$ and Gegenbauer polynomials on the unit circle. There are not so many explicit scalar products allowing a complete description of the Cholesky factors. Most known cases are related to combinatorial identities (Knuth? Wilf?). Here is such a case (Delsarte & Genin):

The scalar product is $\langle f, g \rangle = \frac{1}{2\pi} \int_T f(\theta) g(\theta) \sin \theta /2 |^{2\alpha} d\theta$ on the unit circle $T$, $\alpha > -1/2$. The elements of $G_n$ are $c_{k-n}$, with $c_k = (-1)^k \frac{\Gamma(\alpha + 1)\Gamma(\alpha + 1/2)}{\sqrt{\pi} \Gamma(k+1)\Gamma(\alpha - k + 1)} = (-1)^k \Gamma(2\alpha + 1)$ (see later on, §2.1 for unit circle orthogonality)

One finds $\|\Phi_n\|^2 = \frac{n! \Gamma(2\alpha + n + 1)}{4^{n+1} \Gamma(\alpha + n + 1)^2}$.

$$L_{\ell,k} = \|\Phi_n\| \langle -1 \rangle^{k-\ell} \frac{k}{\ell} \frac{\Gamma(\alpha + 1)\Gamma(\alpha + \ell + 1)}{\Gamma(\alpha - k + \ell + 1)\Gamma(\alpha + k + 1)} = \frac{\Gamma(\alpha + m - \ell)\Gamma(\alpha + \ell + 1)}{\Gamma(\alpha + m + \ell)\Gamma(\alpha + 1)}.$$

Gegenbauer: sieved polynomials $\ldots, \Phi_n(z^2), z\Phi_n(z^2), \ldots$

Indeed, first, the product of the two latter matrices is, for $k \geq \ell$,

$$\sum_{m=0}^{\min(k,\ell)} \|\Phi_n\| \langle -1 \rangle^{k-m} \frac{k-m}{m} \frac{\Gamma(\alpha + 1)\Gamma(\alpha + m + 1)}{\Gamma(\alpha - k + m + 1)\Gamma(\alpha + k + 1)} = \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + m + \ell)\Gamma(\alpha + \ell + 1)} \frac{\Gamma(\alpha + m - \ell)\Gamma(\alpha + \ell + 1)}{\Gamma(\alpha + m + \ell)\Gamma(\alpha + \ell + 1)}.$$

Next, the $(k, \ell)^{th}$ element of $L^T$ is

$$\sum_{m=0}^{\min(k,\ell)} \|\Phi_n\| \langle -1 \rangle^{k-m} \frac{k-m}{m} \frac{\Gamma(\alpha + 1)\Gamma(\alpha + m + 1)}{\Gamma(\alpha - k + m + 1)\Gamma(\alpha + k + 1)} = \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + m + \ell)\Gamma(\alpha + \ell + 1)}.$$

The sum must be $(\alpha + k) \cdots (\alpha + 1) (\alpha + \ell + 1)$, and it is indeed a polynomial of degree $k + \ell$. The polynomial vanishes at $\alpha = -1, \ldots, -k$, as it contains then the $(-2\alpha - 1)^{th}$ difference of $(k - m - 1) (\ell - m - \alpha) - 1 (\ell - m + 1)$, which has degree $-2\alpha - 2$ in $m$,

It is current practice to accept that the scalar product may fail to be positive definite on polynomials of exact degree $N$, i.e., that some nonzero polynomials of degree $N$, but not $N - 1$, may have zero norm: $\|\Psi_{N-1}\| > 0$, $\|\Psi_N\| = 0$, as $\|\Psi_N\|$ achieves least possible norm. If this happens, $\Phi_N$, which is still orthogonal to any polynomial of degree $< N$, is the only monic polynomial of degree $N$ with zero norm: any other monic polynomial of degree $N$ is $\Phi_N + \Psi_{N-1}$, and $\|\Phi_N + \Psi_{N-1}\|^2 = \|\Phi_N\|^2 + 2\Re(\Phi_N, \Psi_{N-1}) = \|\Psi_{N-1}\|^2 > 0$. See for instance [7].

What is stated above is still true, the positive semidefinite $G_n$ still has a Cholesky factorization.

1.4. Decreasing sequences. In the semidefinite case just seen the sequence of norms $\{\|\Phi_n\|\}$ stops at $\|\Phi_N\| = 0$. Can we measure such a situation by producing a positive decreasing sequence up to the index $N - 1$? The sequence of norms $\{\|\Phi_n\|\}$ is generally not decreasing (it is in the unit circle case...), but an interesting decreasing sequence is certainly the sequence of smallest eigenvalues of $G_n$, from the
minimizing property:

\[ \lambda_{\text{min}}(G_n) = \min_{\|\eta\| = 1} \eta^T G_n \eta, \]

where \( \|\eta\| \) is the Euclidian norm on \( \mathbb{C}^n \).

Any item constructed as a minimum on larger and larger subspaces will do, such as norms of kernel polynomials:

\[ \min \|p\|, \text{ on } p \in \mathcal{P}_{\mathfrak{a}}, \text{ with } p(z_0) = 1. \]

Solution [6]:

\[ \frac{K_n(z_0)}{K_n(z_0, z)} = \frac{\sum_{k=0}^{n} \frac{\Phi_k(z_0) \Phi_k(z)}{\sum_{k=0}^{n} |\Phi_k(z_0)|^2}}{n}, \]

and the norm is \( \frac{1}{\sqrt{\sum_{k=0}^{n} |\Phi_k(z_0)|^2}} \), \( n < N \).

A very interesting feature of the coefficients of the kernel polynomial \( K_n(z) := K_n(z, 0) \) is that they build the first column of \( G_n^{-1} \):

\[ G_n \begin{bmatrix} K_n(0) \\ K'_n(0) \\ \vdots \\ K_{n,n} \end{bmatrix} = \begin{bmatrix} (K_n, z^0) \\ (K_n, z) \\ \vdots \\ (K_n, z^n) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (4) \]

from the reproducing property \( (K_n, f) = f(0) \) for all \( f \) of degree \( \leq n \).

A hint of Szegő’s theory: the sequence \( \{K_n(0)\} \) is of course increasing. Should it be bounded, the functions \( K_n \) make a Cauchy sequence: if \( m \leq n, \|K_n - K_m\|^2 = \|K_n\|^2 - 2 \text{ Re } (K_n, K_m) + \|K_m\|^2 = K_n(0) - K_m(0) \). What can \( \lim_{n \to \infty} K_n \) be?

1.5. Most general scalar product. on polynomials is

\[ (\sum a_k z^k, \sum b_l z^l) = \begin{bmatrix} a_0 & b_1 & \cdots & b_L \end{bmatrix} G_T^{(K,L)} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_L \end{bmatrix} \]

with any positive definite Hermitian tableau \( G \), whether it can be interpreted as a functional integral or not.

If one does not go further than the degree \( N \), one may always write \( (f, g) \) through functional values:

\[ (f, g) = \sum_{k,l=0}^{N} h_{k,l} f(a_k) g(a_l), \text{ where } h_{k,l} = (\ell, \ell_k) \text{ (Lagrange interpolation).} \]

A kind of “true support” could be deduced as the set of \( z_0 \) such that the sequence \( \{K_n(z_0, z_0)\} \) is bounded or mildly increasing, for what the meaning of this may be.

1.6. \( \int S f(t) \overline{g(t)} d\mu(t) \) scalar product.

Here, \( d\mu \) is a positive measure supported on a closed complex set \( S \) containing at least \( N \) points.

The two arguments scalar product form is now a simple linear form with the single argument \( f \overline{g} \), written (Maroni, Marcellan):

\[ (f, g) = u(f \overline{g}). \quad (5) \]

Theorem. (Fejér) [6] All the zeros of \( \Phi_n \) are in the convex closure of the support \( S \).
Indeed, let $z$ be a zero of $\Phi_n$, and $\Psi(z) = \frac{\Phi_n(z)}{z-z_k}$. As $\Psi$ has degree $n-1$,

$$0 = \int_S \Phi_n(t)^2 \Psi(t) d\mu(t) = \int_S (t-z_k) \Phi_n(t)^2 d\mu(t) \Rightarrow z_k = \frac{\int_S t |\Psi(t)|^2 d\mu(t)}{\int_S |\Psi(t)|^2 d\mu(t)}$$

is therefore in the convex closure of $S$, as center of mass of the non-negative distribution $|\Psi|^2 d\mu$ on $S$. □

A very interesting example of orthogonal polynomials on the unit circle are the Rogers-Szegő polynomials \[21\]

$$\Phi_n(z) := \sum_{j=0}^{n} \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n-j+1})}{(1-q)(1-q^2) \cdots (1-q^j)} z^j.$$ 

Zeros have strange properties if $|q| = 1$ and the argument of $q$ is a rational multiple of $\pi$ \[21\]...

2. $\int_T f(t) g(t) \frac{d\mu(t)}{2\pi}$ scalar product on the unit circle $\mathbb{T}$.

2.1. Toeplitz matrix and Fourier series.

Now, the Gram matrix is the Toeplitz matrix

$$G_n = \begin{bmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix},$$

with

$$c_k = \frac{1}{2\pi} \int_T t^{-k} d\mu(t), \tag{6}$$

$k \in \mathbb{Z}$ (or $k = -N, -N+1, \cdots, N-1, N$).

We shall only consider measures $d\mu$ without singularly continuous part, then $d\mu(t) = w(\theta) d\theta$, where $w$ is $2\pi-$periodic positive integrable, together with a countable set of positive Dirac masses. Then, $c_k$ is the Fourier coefficient of $w$:

$$w(\theta) = \sum_{-\infty}^{\infty} c_k e^{i k \theta}.$$ 

Simplest Fourier series is the square wave

\[\theta\]

The Fourier coefficients are $c_0 = 1/2$, $c_k = (-1)^{(k-1)/2}/(\pi k)$ for odd $k$, $c_k = 0$ for even $k \neq 0$.

Another well known expansion is $w(\theta) = |\theta|$ on $[-\pi, \pi]$:

\[\theta\]

with $c_0 = \pi/2$, $c_k = -2/(\pi k^2)$ for odd $k$, $c_k = 0$ for even $k \neq 0$. Will these simple Fourier series yield easy orthogonal polynomials?
2.2. Recurrence relation.

An Hermitian Toeplitz matrix turns into its conjugate when elements are moved left-to-right and top-to-bottom

\[ \overline{G_n} = SG_nS. \]

So, \( G_n^{-1} = S G_n^{-1} S \).

Remember from (4) that the first column is made of the coefficients of \( K_n \). It is therefore the same than the conjugate upside-down last column:

\[ \begin{bmatrix} K_n(0) \\ \\ \\ K_n,n \end{bmatrix} = \frac{1}{\|\Phi_n\|^2} \begin{bmatrix} 1 \\ \xi_{n-1} \\ \vdots \\ \xi_0 = \Phi_n(0) \end{bmatrix} \quad \text{(from (2)), i.e.,} \]

\[ K_n(z) := \sum_{k=0}^{n} \frac{\Phi_k(0)}{\|\Phi_k\|^2} \Phi_k(z) \chi(z) := \frac{\Phi_n(1)}{\|\Phi_n\|^2} \frac{\Phi_n(z)}{\|\Phi_n\|^2}. \]

(7)

As \( \Phi_n \) leads to the knowledge of \( K_n = \text{constant} \Phi_n^* \), we use the obvious recurrence for the \( K \)'s:

\[ \frac{\Phi_n^*}{\|\Phi_n\|^2} - \frac{\Phi_{n-1}^*}{\|\Phi_{n-1}\|^2} = \frac{\Phi_n(0)\Phi_n}{\|\Phi_n\|^2}, \]

which is the \( ^* \) operator applied to

\[ \Phi_n(z) = \frac{\|\Phi_n\|^2}{\|\Phi_{n-1}\|^2} \Phi_{n-1}^*(z) = \Phi_n(0)\Phi_n^*(z), \]

(8)

and combine the two latter equations:

\[ \Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_n^*(z), \]

(9)

together with an interesting identity

\[ \frac{\|\Phi_n\|^2}{\|\Phi_{n-1}\|^2} = 1 - |\Phi_n(0)|^2. \]

(10)

Of course, \( \Phi_n(0) \) is not yet known when we try to deduce \( \Phi_n \) from \( \Phi_{n-1} \! \). 

In matrix-vector form, (2) tells that

\[ \begin{bmatrix} \Phi_n(0) \\ \\ \\ \Phi_n(n) \end{bmatrix} = \begin{bmatrix} 0 \\ \Phi_{n-1}(0) \\ 1 \end{bmatrix} + \Phi_n(0) \begin{bmatrix} 1 \\ 0 \\ \xi_{n-1} \end{bmatrix}, \]

which is not surprising, as a left-multiplication by \( G_n \) shows

\[ \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \Phi_n(0) + \frac{\eta_{n-1}}{\|\Phi_n\|^2} \begin{bmatrix} \eta_{n-1} \\ 0 \\ 0 \end{bmatrix}, \]

where \( \eta_{n-1} \) can be computed from \( \Phi_{n-1} \):

\[ \eta_{n-1} = c_{-n} \Phi_{n-1}(0) + c_{-2} \xi_{1,n-1} + \cdots + c_{-n+1} \xi_{n-2,n-1} + c_{-n} = (z\Phi_{n-1}(z), z^n). \]

Let us look at more rows of the matrix-vector relation:
and, as 1st column = 2nd column + $\Phi_{n+1}(0)$ times 3rd column,

\[
u(z^2\Phi_{n}(z)) = u(z\Phi_{n+1}(z)) - \Phi_{n+1}(0)u(z^{-1}\Phi_{n}(z))
\]

\[
u(z^{n-2}\Phi_{n+1}(z)) = u(z^{n-1}\Phi_{n}(z)) + \Phi_{n+1}(0)\left[u(z\Phi_{n+1}(z)) - \Phi_{n+1}(0)u(z^{-1}\Phi_{n}(z))\right],
\]

finally:

\[
u(z\Phi_{n}(z)) = \eta_n = -\Phi_{n+1}(0)||\Phi_n||^2,
\]

\[
u(z^{-n-1}\Phi_{n}(z)) = -(\Phi_{0}(0)\Phi_{1}(0) + \Phi_{1}(0)\Phi_{2}(0) + \cdots + \Phi_{n}(0)\Phi_{n+1}(0)||\Phi_n||^2,
\]

Other forms of $\Phi$:

\[
\begin{bmatrix}
\Phi_{n+1}(z) \\
\Phi_{n+1}^*(z)
\end{bmatrix} = \begin{bmatrix}
z & \Phi_{n+1}(0) \\
\frac{z}{\Phi_{n+1}(0)} & 1
\end{bmatrix} \begin{bmatrix}
\Phi_{n}(z) \\
\Phi_{n}^*(z)
\end{bmatrix},
\]

\[
\begin{bmatrix}
\Phi_{n}(z) \\
\Phi_{n}^*(z)
\end{bmatrix} = \begin{bmatrix}
1 & \frac{-\Phi_{n+1}(0)z^{-1}}{\Phi_{n+1}(0)} \\
\frac{1}{1-||\Phi_{n+1}(0)||^2} & 1
\end{bmatrix} \begin{bmatrix}
\Phi_{n+1}(z) \\
\Phi_{n+1}^*(z)
\end{bmatrix}
\]

(13)

Also,

\[
\begin{bmatrix}
\Phi_{n+r}(z) \\
\Phi_{n+r}^*(z)
\end{bmatrix} = \begin{bmatrix}
zU_{r-1}(z;n) & V_{r-1}(z;n) \\
V_{r-1}^*(z;n) & U_{r-1}^*(z;n)
\end{bmatrix} \begin{bmatrix}
\Phi_{n}(z) \\
\Phi_{n}^*(z)
\end{bmatrix}
\]

(14)

where $U_{r-1}$ and $V_{r-1}$ are polynomials of degrees $\leq r-1$:

\[
\begin{array}{c|c|c}
& U_r(z;n) & V_r(z;n) \\
\hline
0 & \frac{z}{\Phi_{n+1}(0)} & \frac{x_{n+1}}{\Phi_{n+1}(0)} \\
1 & \frac{z^2 + x_n + 2x_{n+1}}{x_{n+1} + x_{n+2}} & \frac{x_{n+1}z + x_{n+2}}{x_{n+1} + x_{n+2}} \\
2 & \frac{z^2 + x_{n+1} + x_{n+2} + x_{n+3}z}{x_{n+2} + x_{n+3}z} & \frac{x_{n+1}z + x_{n+2} + x_{n+3}z}{x_{n+2} + x_{n+3}z}
\end{array}
\]

where $x_n = \Phi_n(0)$, etc.

One has $U_r(z;n) = zU_{r-1}(z;n) + x_{n+r-1}V_{r-1}(z;n), V_r(z;n) = zV_{r-1}(z;n) + x_{n+r-1}U_{r-1}(z;n)$.

The Cholesky factorization becomes

\[
G_n = TL^\top \Rightarrow G_n^{-1} = (SL^\top S)^{-1},
\]

(15)

which is the Cholesky factorization of $G_n^{-1}$!

### 2.3. Multiplication operator: Toeplitz matrices in Grenander and Szegö’s sense.

A deep remark (Bultheel, Marcellan): the operator $\mathcal{M}$ of multiplication by $z$ is represented in any

polynomial basis by a Hessenberg matrix

\[
\begin{bmatrix}
x & x & x & x & x & x & \cdots \\
x & x & x & x & x & x & \cdots \\
x & x & x & x & x & x & \cdots \\
x & x & x & x & x & x & \cdots \\
x & x & x & x & x & x & \cdots \\
x & x & x & x & x & x & \cdots \\
\end{bmatrix}
\]

. On the real interval, the operator is symmetric, hence the matrix representation is tridiagonal $\Rightarrow$ the three terms recurrence relation; on the circle, the operator is unitary.
On curves $|P(z)| = \text{constant}$ (lemniscates), or $\text{Re} P(z) = \text{constant}$, one has to consider the polynomial $P(\mathcal{M}(z))$ (Vigil, Marcellán).

Let $\{\phi_0, \phi_1, \ldots\}$ be a complete orthonormal sequence in a Hilbert function space $X$, then

$\mathcal{M}(f) = \begin{bmatrix}
\langle \phi_0 f, \phi_0 \rangle & \langle \phi_1 f, \phi_0 \rangle & \langle \phi_2 f, \phi_0 \rangle & \cdots \\
\langle \phi_0 f, \phi_1 \rangle & \langle \phi_1 f, \phi_1 \rangle & \langle \phi_2 f, \phi_1 \rangle & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}$

(16)

represents the operator of multiplication by $f$ if an element $F$ of $X$ is represented by its sequence of coefficients $\{c_k(F)\}$ in the expansion $F = \sum_0^\infty c_k(F)\phi_k$. Indeed $\mathcal{M}(f)$ is the transpose of the complex conjugate of $\mathcal{M}$, and is also $\mathcal{M}^{-1}$: $\mathcal{M}$ is a unitary matrix.

One also have $\mathcal{M}(f)\mathcal{M}(g) = \mathcal{M}(fg)$.

The $\mathcal{M}$ of above is simply $\mathcal{M}(z)$. On the circle, we have $\mathcal{M}_{n,m} = \int_0^1 z^m\phi_n(z)\phi_{n-m}(z)\,du$, so that $\mathcal{M}(z^{-1})$ is the transpose of the complex conjugate of $\mathcal{M}$, and is also $\mathcal{M}^{-1}$: $\mathcal{M}$ is a unitary matrix.

The $n^{th}$ column (starting at 0) of $\mathcal{M}$ is made with the coefficients of $z\phi_n(z)$. From the recurrence relation (9),

\[
z\phi_n(z) = \frac{z\Phi_n(z)}{\|\Phi_n\|} = \frac{\Phi_{n+1}(z) - \Phi_{n+1}(0)\Phi_n'(z)}{\|\Phi_n\|} = \frac{\|\Phi_{n+1}\|}{\|\Phi_n\|} \phi_{n+1}(z) - \Phi_{n+1}(0)\|\Phi_n\| \sum_{k=0}^n \frac{\Phi_k(0)}{\|\Phi_k\|} \phi_k(z)
\]

from (7). Whence

$\mathcal{M}_{n+1,n} = \frac{\|\Phi_{n+1}\|}{\|\Phi_n\|}, \mathcal{M}_{k,n} = -\frac{\|\Phi_{n+1}\|}{\|\Phi_k\|} \frac{\Phi_{n+1}(0)\Phi_k(0)}{\|\Phi_k\|}, k \leq n$.

found in B. Simon’s preprint [29], with reference to Nevai [1] for some identities involving the $\Phi_n$’s.

2.4. First and last columns of $G_n^{-1}$.

2.4.1. First column. We already saw that the first column of $G_n^{-1}$ is made of the coefficients of the kernel polynomial $K_n$, which is here $\Phi_n/\|\Phi_n\|^2$, so is

$\begin{bmatrix}
\xi_{n,n}, \xi_{n-1,n}, \ldots, \xi_{1,n}, \xi_{0,n}
\end{bmatrix}^T / \|\Phi_n\|^2$.

2.4.2. Second column. Let us try $z\Phi_n^{-1}(z)$, i.e.,

$\begin{bmatrix}
c_0 & c_{-1} & \cdots & c_{-n} \\
c_1 & c_0 & \cdots & c_{-n+1} \\
\vdots & \vdots & \ddots & \vdots \\
c_n & c_{n-1} & \cdots & c_0
\end{bmatrix}
\begin{bmatrix}
0 \\
\xi_{n-1,n-1} \\
\xi_{n-2,n-1} \\
\vdots \\
\xi_{1,n} \\
\xi_{0,n}
\end{bmatrix}$.

Thanks to the Toeplitz structure, the $n$ last elements of the product are $\|\Phi_{n-1}\|^2, 0, \ldots, 0$, and the first element of the product is $u(z\Phi_n^{-1}(z)) = u(z^{-n}\Phi_n^{-1}(z))$, known from (11). So, second column is

$\begin{bmatrix}
0, \xi_{n-1,n-1}, \xi_{n-2,n-1}, \ldots, \xi_{1,n-1}, \xi_{0,n-1}
\end{bmatrix}^T - u(z^{-n}\Phi_n^{-1}(z))\begin{bmatrix}
\xi_{n,n}, \xi_{n-1,n}, \ldots, \xi_{1,n}, \xi_{0,n}
\end{bmatrix}^T / \|\Phi_n\|^2 / \|\Phi_{n-1}\|^2$.  


2.4.3. Last column. Known to be
\[
[\xi_0, n; \xi_1, n; \ldots; \xi_n, n] / \| \Phi_n \|^2.
\]

2.4.4. First before last column.
\[
\{ [\xi_{n-1}, n-1; \xi_n, n-1; \ldots; \xi_1, n; \xi_0, n] \} = n(1 - n) / \| \Phi_{n-1} \|^2 / \| \Phi_n \|^2.
\]

2.5. Behaviour of reflection coefficients.

With the program PARI [3].

```plaintext
\{ \\
  \// reflec.gp : launch gp and make \r reflec \\
  \// Reflection coefficients given Fourier coefficients \\
  \// N=22;
  \}
\// vector of Fourier coeff. c_k = c[k+N+1]
\{  Fcoeff(k)=
    if(k==0, 1.0/2,
        (I^k-(-I)^k)/(2*Pi*I*k))
  }
\{  c=vector(2*N+1,k,Fcoeff(k-N-1));
  Phin=x^0;normPhin2=c[N+1];
  for(n=1,N-1,
      print(n-1, " ",polcoeff(Phin,0)," ",normPhin2);
      scalPhizm1 = sum(k=1,n, c[N+1-k]*polcoeff(Phin,k-1) ) ;
      \print(scalPhizml);
      Phin = x*Phin -(scalPhizml/normPhin2)* polrecip(conj(Phin));
      normPhin2 -= scalPhizml*conj(scalPhizml)/normPhin2;
  )
\}
```

Script V1.1 session started Wed Aug 18 11:19:38 1999

C:\calc\pari>gp

GP/PARI CALCULATOR Version 2.0.12 (alpha)
ix86 running emx (ix86 kernel) 32-bit version
(readline enabled, extended help not available)

Copyright (C) 1989-1998 by

Type ? for help, \q to quit.
Type ?12 for how to get moral (and possibly technical) support.
realprecision = 28 significant digits
seriesprecision = 16 significant terms
format = g0.28
parisize = 4000000, primelimit = 500000

One finds more and more complicated expressions,
\[ \Phi_1(0) = -\frac{2}{\pi}, \Phi_2(0) = \frac{4}{\pi^2 - 4}, \Phi_3(0) = -\frac{2}{3\pi} \frac{16 - \pi^2}{\pi^2 - 8}, \]

etc. which we can perhaps explore further through numerical values:

<table>
<thead>
<tr>
<th>n</th>
<th>(\Phi_n(0))</th>
<th>(|\Phi_n|^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.57079632679489661923132169</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.4052847345693510857755178528</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.1965382464879410164306311434</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1551854679951381617834429861</td>
<td></td>
</tr>
</tbody>
</table>

Good bye!

And the \(\theta\) example:

```pari
{Foucoeff(k)=
  if(k==0, Pi/2,
      ((-1)ˆk-(1)ˆk)/(Pi*kˆ2))
}
```

Script completed Wed Aug 18 11:20:08 1999
The description and understanding of the behaviour of these coefficients \( \Phi_n(0) \) is the main subject of the present lecture. The following completely explicit cases are known:

- Special relations, as in p. 3, also in [36, 37],
- Rogers-Szegő polynomials, p. 8,
- Bernstein-Szegő, § 2.8.1, p. 17,
- Bernstein-Szegő pol. on a circular arc [11],
- Gegenbauer, p. 6, and more generally:
  - Jacobi, § 2.7,
  - and others from known interval cases, from § 2.6, as in [36].

2.6. Circle versus interval.

Among other feats, Szegő achieved the description of the connection between orthogonal polynomials on \([-1,1]\) and orthogonal polynomials on the unit circle [31, § 11.5]. The connection is usually interpreted as a reduction of interval polynomials to the "more basic" circle polynomials, but wait.

Ah, let \( P_n \) be the \( n \)th degree monic polynomial orthogonal with respect to a measure \( d\nu \) on \([-1,1]\), i.e., a (real) scalar product \( \langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, d\nu(x) \). We express orthogonality to lower degree polynomials through Chebyshev polynomials of first kind: let

\[
P_n(x) = \frac{d_0}{2} + d_1 T_1(x) + \cdots + d_n T_n(x),
\]

which must be \( \langle , \rangle \)-orthogonal to \( T_0, \ldots, T_{n-1} \):

\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
\langle P_n, T_n \rangle
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{2} \langle T_0, T_0 \rangle & \cdots & \langle T_n, T_0 \rangle \\
\vdots & \ddots & \vdots \\
\frac{1}{2} \langle T_0, T_{n-1} \rangle & \cdots & \langle T_n, T_{n-1} \rangle \\
\frac{1}{2} \langle T_0, T_n \rangle & \cdots & \langle T_n, T_n \rangle
\end{bmatrix}
\begin{bmatrix}
d_0 \\
\vdots \\
d_{n-1}
\end{bmatrix}.
\]

As \( \langle T_k, T_k \rangle = \langle T_k, T_k, 1 \rangle \), and \( T_k/T_k = \frac{T_{k+1} + T_{k-1}}{2} \), let \( c_k = \langle T_k, 1 \rangle \). Remark that \( c_{-k} = c_k \), also that \( c_k = \int_{-1}^{1} T_k(x) \, d\nu(x) = \int_0^\pi \cos(k\theta) \, d\nu(\cos \theta) = \frac{1}{2} \int_0^\pi e^{-i\theta k} \, d\nu(\cos \theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta k} \, d\mu(e^\theta) \), with...
\[ d\mu(e^\theta) = -\pi d\nu(\cos \theta). \] Then, the equations above may be rearranged as

\[
\begin{bmatrix}
2\langle P_n, T_n \rangle \\
0 \\
\vdots \\
0
\end{bmatrix} = \begin{bmatrix}
c_0 & \cdots & c_{1-n} & c_{-n} & c_{-n-1} & \cdots & c_{-2n} \\
c_1 & \cdots & c_{2-n} & c_{1-n} & c_{-n} & \cdots & c_{-2n+1} \\
\vdots \\
c_n & \cdots & c_1 & c_0 & c_{-1} & \cdots & c_n \\
\vdots \\
c_{2n-1} & \cdots & c_n & c_{n-1} & c_{n-2} & \cdots & c_{-1} \\
c_{2n} & \cdots & c_{n+1} & c_n & c_{n-1} & \cdots & c_0
\end{bmatrix} \begin{bmatrix}
d_n \\
d_{n-1} \\
\vdots \\
d_2 \\
d_1 \\
d_0
\end{bmatrix},
\]

which is exactly a Toeplitz system! The solution is \( \langle P_n, T_n \rangle \) times the sum of the first and the last columns of \( G_{2n}^{-1} \), so, \( 2^nP_n(x) = d_{2n}z^{2n} + \cdots + d_{2n+1}z^{2n+1} + \cdots + d_n = 2\langle P_n, T_n \rangle \left( K_{2n}(z) + \frac{\Phi_{2n}(z)}{||\Phi_{2n}||^2} \right) = 2\left( \frac{\langle P_n, T_n \rangle}{||\Phi_{2n}||^2} (\Phi_{2n}(z) + \Phi_{2n}(z)) \right), \]

with \( d_n = 2^{1-n} \Rightarrow ||\Phi_{2n}||^2 = 2^n \langle P_n, T_n \rangle (1 + \Phi_{2n}(0)) \). (N.B., \( \Phi_{2n}(0) \) is real).

**Conversely,** can we recover the \( \Phi \)'s from the \( P \)'s?? We need the difference of the first and last columns of \( G_{2n} \ldots \). Let \( Q_{n-1} \) be the monic orthogonal polynomial of degree \( n-1 \) with respect to some scalar product \( \langle , , \rangle \) on \([-1, 1]\). We use the basis of Chebyshev polynomials of second kind: \( Q_{n-1} = \sum_{k=0}^{n-1} c_k U_k \).

Now, \( U_l(\cos \theta)U_k(\cos \theta) = \frac{\sin((l+1)\theta) \sin((k+1)\theta)}{\sin^2 \theta} = \frac{\cos(l-k)\theta - \cos(l+k+2)\theta}{1-x^2} \)

and we recover the \( c_k \)'s if \( \langle f(x), 1 \rangle = \langle (1-x^2)f(x), 1 \rangle \), or \( \{ f, g \} = \int_{-1}^{1} f(x)g(x)(1-x^2)d\nu(x) \). The convenient rearrangement of the equations is

\[
\begin{bmatrix}
2\langle Q_{n-1}, U_{n-1} \rangle \\
0 \\
\vdots \\
0
\end{bmatrix} = \begin{bmatrix}
c_0 & \cdots & c_{1-n} & c_{-n} & c_{-n-1} & \cdots & c_{-2n} \\
c_1 & \cdots & c_{2-n} & c_{1-n} & c_{-n} & \cdots & c_{-2n+1} \\
\vdots \\
c_n & \cdots & c_1 & c_0 & c_{-1} & \cdots & c_n \\
\vdots \\
c_{2n-1} & \cdots & c_n & c_{n-1} & c_{n-2} & \cdots & c_{-1} \\
c_{2n} & \cdots & c_{n+1} & c_n & c_{n-1} & \cdots & c_0
\end{bmatrix} \begin{bmatrix}
e_{n-1} \\
e_{n-2} \\
\vdots \\
e_2 \\
e_1 \\
e_0
\end{bmatrix},
\]

so, \(-z^n(z^{-1})Q_{n-1}(x) = e_{n-1} + \cdots + e_0z^{n-1} + 0z^n - e_0z^n - \cdots - e_{n-1}z^{2n} = 2\langle Q_{n-1}, U_{n-1} \rangle \left( K_{2n}(z) - \frac{\Phi_{2n}(z)}{||\Phi_{2n}||^2} \right) = 2\left( \frac{\langle Q_{n-1}, U_{n-1} \rangle}{||\Phi_{2n}||^2} (\Phi_{2n}(z) + \Phi_{2n}(z)) \right), \)

with \( e_{n-1} = 2^{1-n} \Rightarrow ||\Phi_{2n}||^2 = 2^n \langle Q_{n-1}, U_{n-1} \rangle (1 - \Phi_{2n}(0)) \).
Finally,
\[ \Phi_{2n}(z) = \frac{\|\Phi_{2n}\|^2}{2(P_n,T_n)} z^n P_n(x) + \frac{\|\Phi_{2n}\|^2}{4\{Q_{n-1},U_{n-1}\}} z^n (z - z^{-1}) Q_{n-1}(x), \]
\[ \Phi_{2n}^*(z) = \frac{\|\Phi_{2n}\|^2}{2(P_n,T_n)} z^n P_n(x) - \frac{\|\Phi_{2n}\|^2}{4\{Q_{n-1},U_{n-1}\}} z^n (z - z^{-1}) Q_{n-1}(x), \]
\[ \Phi_{2n} \text{ monic } \Rightarrow \|\Phi_{2n}\|^2 = \frac{1}{\frac{1}{2}(P_n,T_n) + \{Q_{n-1},U_{n-1}\}}. \]
\[ \Phi_{2n}(0) = \{Q_{n-1},U_{n-1}\} - \{P_n,T_n\}, \]
\[ \|\Phi_{2n}\|^2 = 2^{-n-1} \{Q_{n-1},U_{n-1}\} \{P_n,T_n\}, \]
\[ \|\Phi_{2n}^\ast\|^2 = 2^{-n-1} \{Q_{n-1},U_{n-1}\} \{P_n,T_n\}. \]

See also \[36\]

2.7. **Jacobi polynomials on the unit circle.**

Here, \(d\mu(e^{i\theta}) = \left(\frac{\cos \theta}{2}\right)^\beta \left|\sin \frac{\theta}{2}\right|^{2\alpha} d\theta\) \((-\pi < \theta < \pi, \alpha \text{ and } \beta > -1/2).\)

\[d\nu(\cos \theta) = (-1/\pi)d\mu(e^{i\theta}), \text{ so } d\nu(x) = (1/\pi) \left(\frac{1+x}{2}\right)^\beta \left(\frac{1-x}{2}\right)^\alpha (1-x^2)^{-1/2} dx,\]
\[P_n = P_{n,\text{monic}}^{\alpha-1/2,\beta-1/2} = x^n + \frac{n(\alpha - \beta)}{2n + \alpha + \beta - 1} x^{n-1} + \cdots, \quad Q_n = Q_{n,\text{monic}}^{\alpha+1/2,\beta+1/2} = x^n + \frac{(n-1)(\alpha - \beta)}{2n + \alpha + \beta - 1} x^{n-2} + \cdots,\]
\[\{P_n,T_n\} = 2^{-n-1} \{P_n,P_n\}, \quad \{Q_{n-1},U_{n-1}\} = 2^{-n-1} \{Q_{n-1},Q_{n-1}\}.\]
\[\Phi_{2n}(0) = \frac{\alpha + \beta}{2n + \alpha + \beta}. \]
\[\Phi_{2n-1}(z) = 2^{-n-1} z^{n-1} \left[ \frac{z + z^{-1}}{2} \right]^n + \frac{n(\alpha - \beta)}{2n + \alpha + \beta - 1} \left(\frac{z + z^{-1}}{2}\right)^{n-1} + \frac{z - z^{-1}}{2} \left(\frac{z + z^{-1}}{2}\right)^n + \frac{z - z^{-1}}{2} \left(\frac{z + z^{-1}}{2}\right)^{n-1} \right].\]

so,
\[\Phi_n(0) = \frac{\alpha + (-1)^n\beta}{n + \alpha + \beta}. \quad (17)\]

2.8. **The Szegő-Geronimus theory.**

Szegő \[31\] investigated the cases where the increasing sequence \(\{K_n(0)\}\) remains bounded, the sequence of functions
\[K_n(z;0) = \sum_{k=0}^n \Phi_k(0) \Phi_k(z) = \sum_{k=0}^n \Phi_k(0) \Phi_k(z)/\|\Phi_k\|^2\]
is a Cauchy sequence. What about the limit?
2.8.1. The Bernstein-Szegö polynomials. If \( p(z) = p_0 z^{-d} + \cdots + p_d z^d \) is a Laurent polynomial, with real values on the unit circle, \( p_k = \overline{p_{d-k}} \). \( p_{2d-k} = (2\pi)^{-1} \int_T p(e^{i\theta})e^{i(d-k)\theta} d\theta \) is the complex conjugate of \( p_k = (2\pi)^{-1} \int_T p(e^{i\theta})e^{i(d-k)\theta} d\theta \). The 2d zeros of \( p \) come in inverse pairs: if \( p(\zeta) = 0 \), \( p(1/\zeta) = \sum_k p_k (1/\zeta)^{d-k} = \sum_k p_{d-k} \zeta^{d-k} = 0 \).

So, \( p(z) = q(z)r(z) \) (Fejér), where \( q(z) = q_0 + q_1 z^{-1} + \cdots + q_d z^{-d} \) has its zeros in the unit disk (in the open unit disk if \( p \) does not vanish on the unit circle), and \( r(z) = r_0 + \cdots + r_d z^d \) has its zeros outside the unit disk.

When \( w(\theta) = 1/p(e^{i\theta}) \), where \( p \) is a positive Laurent polynomial of degree \( d \),

\[
\Phi_n(z) = \frac{1}{q_0} q(z) z^n
\]

from \( n = d \) onwards.

Indeed, \((\Phi_n(z), \zeta^k) = (1/(2\pi i q_0)) \int_T q(z) z^{n-k} \frac{1}{q(z) r(z)} \frac{dz}{z} = 0 \) for \( k = 0, \ldots, n-1 \), as there is no residue.

When \( k = n \), we find the residue \( ||\Phi_n||^2 = 1/(q_0 r_n) \).

And, when \( n \geq d \), \( K_n \) is the polynomial

\[
K_n(z) = \frac{\Phi_n(z)}{||\Phi_n||^2} = q_0 r(z).
\]

2.8.2. The Szegö theory.

A necessary condition to get a generalization of the factorization of \( w = \log w \in L^1 \), so that \( \log w \) has a Fourier series, and we take

\[
\log w = \cdots + \lambda_{-2} z^{-2} + \lambda_{-1} z^{-1} + \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots,
\]

and we expect \( \Phi_n^\ast(z) \rightarrow r(z)/\exp(\lambda_0) \) when \( n \rightarrow \infty \), for \( z \in \mathbb{D} \). For the coefficients:

\[
\Phi_n^\ast(z) = 1 + \xi_{-n-1} z^2 + \xi_{-n-2} z^4 + \cdots \rightarrow \exp(\lambda_1 z + \lambda_2 z^2 + \cdots).
\]

Proofs and extensions: [31], [33] etc.]

2.8.3. Some theorems by Geronimus. cf. [13]

The following statements are equivalent

- \( \sum_{n=0}^\infty |\Phi_n(0)|^2 < \infty \).
- \( \sum_{n=0}^\infty |\phi_n(z)|^2 < \infty \) for at least one point \( z \in \mathbb{D} \).
- there is a subsequence \( n_k \) such that \( \phi_{n_k}(z) \) converges at least for one point \( z \in \mathbb{D} \).
- \( \lim_{n \rightarrow \infty} \phi_n(z) = S(z) \) uniformly inside \( \mathbb{D} \).

2.9. Formal unit circle orthogonal polynomials: non hermitian Toeplitz matrices.

3. Semi-classical orthogonal polynomials on unit circle.

3.1. Definition, forms, and differential equation for weight function.

Cf. [11] and references therein

We can not guess easily the behaviour, especially the asymptotic behaviour of the reflection coefficients \(-\Phi_n(0)\) from the entries \( c_k \) with the recurrence relation [22], this relation is a universal relation for all
unit circle orthogonal polynomials. We only need $\Phi_n(0) = \xi_{0,n}$, but $\xi$ compels us to get all the two-dimensional tableau of the $\xi_{k,n}$’s.

It is very useful to select classes of orthogonal polynomials whose coefficients $\Phi_n(0)$ satisfy a kind of one-dimensional recurrence relation. This will happen with the class defined hereafter, where such recurrence relations will be associated to differential relations.

Speaking of differential relations, the famous Sonine-Hahn characterization of classical orthogonal polynomials as having orthogonal derivatives falls short on the unit circle, as it only works for $z^n$.

In the language of forms defined on Laurent polynomials, one defines the product of a function and a form $f u$ as the form such that $\forall p, (f u)(p) = u(f p)$; and the derivative $Du$ of $u$ as the form such that $\forall p, (Du)(p) = -iu(t p'(t))$. This latter strange definition corresponds actually to a simple relation for the weight functions: let $d\mu(t = e^{i\theta}) = w(\theta)d\theta$, where $w$ has at most a finite number of Dirac masses, and let $P$ be a Laurent polynomial vanishing at these masses, then if $u(p) = \frac{1}{2\pi} \int_{0}^{2\pi} p(e^{i\theta})d\mu(e^{i\theta})$, $\forall p : (Pu)(p) = \frac{1}{2\pi} \int_{0}^{2\pi} P(e^{i\theta})p(e^{i\theta})w(\theta)d\theta$;

$$(D(Pu))(p) = -i(Pu)(tp'(t)) = \frac{1}{2\pi} \int_{0}^{2\pi} P(e^{i\theta})e^{i\theta}dP(e^{i\theta})w(\theta)d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} P(e^{i\theta})dP(e^{i\theta})w(\theta)d\theta,$$

i.e., $D(Pu)$ has the integral representation involving $d[P(e^{i\theta})w(\theta)]/d\theta$. Remark that, in order to get rid of annoying boundary terms in the integration by parts, $P$ must vanish at all Dirac points and other discontinuities of $w$.

**Definition. A semi-classical form** on the circle satisfies

$$D(A(z)u) = B(z)u,$$

with Laurent polynomials $A$ and $B$, which means, if $u$ has an integral representation involving $w(\theta)d\theta$, that

$$\frac{d|A(e^{i\theta})w(\theta)|}{d\theta} = B(e^{i\theta})w(\theta),$$

or

$$\frac{dw(\theta)/d\theta}{w(\theta)} = \frac{B(e^{i\theta}) - dA(e^{i\theta})/d\theta}{A(e^{i\theta})} = \frac{B(e^{i\theta}) - ie^{i\theta}dA(e^{i\theta})/de^{i\theta}}{A(e^{i\theta})}$$

with $A(e^{i\theta}) = 0$ at the singular points of $w$.

Jacobi:

$$u(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(e^{i\theta}) \left( \cos \frac{\theta}{2} \right)^{2\beta} \left| \sin \frac{\theta}{2} \right|^{2\alpha} d\theta,$$

$$\frac{dw(\theta)/w(\theta)}{w(\theta)} = \alpha \cot \frac{\theta}{2} - \beta \tan \frac{\theta}{2} = i\alpha \frac{z + z^{-1}}{z - z^{-1}} + i\beta \frac{z - 2 + z^{-1}}{z - z^{-1}}, (z = e^{i\theta}),$$

so, $A(z) = z - z^{-1}, B(z) = A(z)(dw(\theta)/d\theta)/w(\theta) + izdA(z)/dz = i(\alpha + \beta + 1)z + 2i(\alpha - \beta) + i(\alpha + \beta - 1)z^{-1}$, or, if we prefer polynomials without negative powers,

$$A(z) = z^2 - 1; B(z) = i((\alpha + \beta + 2)z^2 + 2(\alpha - \beta)z + \alpha + \beta)$$

**Exercise.** Show that the square wave exemple is semi-classical, but that $w(\theta) = |\theta|$ is not semi-classical.
3.2. Recurrence relation for the moments.

From (18) (p. 3), the moment \( c_k \) is just \( u(t^{-k}) \) when \( u \) has the integral representation \( u(p) = (2\pi)^{-1} \int_{\mathbb{R}} p(e^{ib})d\mu \).

So, we define \( c_k = u(t^{-k}), k \in \mathbb{Z} \) for any form defined on Laurent polynomials.

Let us apply (18) to monomials \( t^{-k}, k \in \mathbb{Z} \):

\[
\bar{u}(A(t)k t^{-k}) = u(B(t) t^{-k}), \quad k \in \mathbb{Z}.
\]

If \( A(t) = \sum_{p=0}^{d} a_p t^{p-p_0} \) and \( B(t) = \sum_{p=0}^{d} b_p t^{p-p_0} \), then

\[
\sum_{p=0}^{d} (i\bar{a}_p b_p) c_{k-p-p_0} = 0, \quad k \in \mathbb{Z}.
\]  (19)

A linear recurrence relation of the form (19) is another way to recognize a semi-classical functional.

Without more information than (19), we may find differential equations for the generating functions

\[
G_{\pm(z)} = \sum_{m=0}^{\infty} c_{\pm m} z^m,
\]

and contour integral representations for \( c_k \)

to be continued

3.3. Differential relations for the orthogonal polynomials and recurrence relations for the reflection coefficients.

Let \( A \) and \( B \) be (plain, i.e., without negative powers) polynomials of degree \( \leq d \). Then, the product \( A(z) d\Phi_n(z) / dz \) is a remarkably short combination of the \( \Phi \) and the \( \Phi^* \)'s.

The matrix \( G_{n+d-1} \) times the vector \( [A(0)\Phi'_n(0), A(0)\Phi''_n(0) + A'(0)\Phi'_n(0), \ldots, (n-1)A_d\xi_{n-1,n} + nA_{d-1}, nA_d]^T \)

of the coefficients of \( A\Phi'_n \) is the vector of

\[
(A(z)\Phi'_n(z), z^k) = u(A(z)\Phi'_n(z)z^{-1}) = u(zA(z)[\Phi_n(z)z^{-1}]) - u(zA(z)\Phi_n(z)[z^{-1}])
\]

\[
= i(D(Au))(\Phi_n(z)z^{-1}) + (k+1)u(\Phi_n(z)A(z)z^{-1}) = iu(B(z)\Phi_n(z)z^{-1}) + (k+1)u(A(z)\Phi_n(z)z^{-1})
\]

\[
= u \left( \Phi_n(z) \left( iB(z) + (k+1)A(z) \right) \right) z^{k+1}
\]

for \( k = 0, \ldots, n + d - 1 \). The latter form vanishes as soon as \( (ib(z) + (k+1)A(z))z^{-1} \) contains only nonpositive powers, down to \( z^{-n+1} \), which makes \( k = d - 1, \ldots, n - 2 \). Therefore, \( A\Phi'_n \) is a combination of \( \Phi_{n+d-1}, \ldots, \Phi_{n-1}, \) and \( \Phi^*_{n+d-1}, \ldots, \Phi^*_{n-2}, \ldots, \Phi^*_{n+1} \).

A more detailed look:

\[
G_{n+d-1} = \begin{bmatrix}
A(0)\Phi'_n(0) \\
A(0)\Phi''_n(0) + A'(0)\Phi'_n(0) \\
\vdots \\
(n-1)A_d\xi_{n-1,n} + nA_{d-1} \\
nA_d 
\end{bmatrix} = \begin{bmatrix}
u((ib_d + (d-1)A_d)\Phi_n(z)) & \cdots \\
0 & \ddots & \cdots \\
0 & \ddots & 0 \\
v((ib(0) + nA(0))\Phi_n(z)/z^n) & \cdots \\
\end{bmatrix}
\]

where we already encountered \( \eta_n = -\Phi_{n+1}(0)\|\Phi_n\|^2 \).
Manifesto.

As all the polynomials $\Phi_n$ depend, from the recurrence relations (2), on the reflection coefficients $-\Phi_n(0)$, the semi-classical identities will yield equations for these reflection coefficients.

How $\Phi_n$ depends on the $\Phi_m(0)$: with $x_n := \Phi_m(0)$,

$$\Phi_n(z) = z^n + \xi_{n-1,n}z^{n-1} + \cdots + \Phi'_n(0)z + \Phi_n(0)$$

with $\xi_{n-1,n} = x_1x_0 + x_2x_1 + \cdots + x_{n}x_{n-1},$

$$\Phi'_n(0) = \Phi_{n-1}(0) + \Phi_n(0)\xi_{n-2,n-1}.$$ 

Simplest case: Jacobi $(\alpha, 0)$, $d = 1$, $A(z) = z - 1$, $B(z) = i(\alpha + 1)z + i\alpha, iB(0) + nA(0) = -(n + \alpha)$, one finds, from (3)

$$(z - 1)\Phi'_n(z) = n\Phi_n(z) - \frac{n(n + 2\alpha)}{n + \alpha} \Phi_{n-1}(z).$$

remember (10): $\|\Phi_n\|^2/\|\Phi_{n-1}\|^2 = 1 - |\Phi_n(0)|^2$.

3.4. More Jacobi polynomials on the unit circle.

Let us see how the features of the polynomials $\Phi_n$, in particular the reflection coefficients $-\Phi_n(0)$, can be recovered (and, why not, discovered) from the semi-classical identities:

$d = 2$, we know $A(z) = z^2 - 1, B(z) = i(\alpha + \beta + 2)z^2 + 2i(\alpha - \beta)z + i(\alpha + \beta)$.

$$\left(z^2 - 1\right)\Phi'_n(z) = n_z^{n+1} + (n - 1)\xi_{n-1,n}z^n + \left[(n - 2)\xi_{n-2,n} - n\right]z^{n-1} + \cdots = X_n\Phi_{n+1}^*(z) + Y_n\Phi_{n-1}^*(z),$$

$$W_n = -(n + \alpha + \beta)\|\Phi_n\|^2/\|\Phi_{n-1}\|^2 = -(n + \alpha + \beta)(1 - |\Phi_n(0)|^2).$$

Comparing the terms in

- $z^{n+1}$: $X_n\Phi_{n+1}(0) + Y_n = n,$
- $z^n$: $X_n\Phi_{n+1}^*(0) + Y_n\xi_{n-1,n} + Z_n = (n - 1)\xi_{n-1,n},$
- $z^{n-1}$: $X_n\Phi_{n-1}^*(0) + Y_n\xi_{n-1,n} + Z_n = (n - 2)\xi_{n-2,n} - n.$

The two first equations merely yield expressions for $Y_n$ and $Z_n$, and the third equation is actually an equation for the $\Phi_m(0), \ldots$ after all the $\xi$s have been expanded!

We avoid to have to bother with the $\xi_{n,n}$’s by

$$Z_n\|\Phi_n\|^2 + W_nu(z^{-n}\Phi_{n-1}(z)) = \left(iB(0) + (n + 1)A(0)\right)u(z^{-n-1}\Phi_n(z)) + \left(iB'(0) + nA'(0)\right)|\Phi_n|^2$$

$x_n := \Phi_n(0):$

$$(n + \alpha + \beta + 1)x_{n+1} = (n + \alpha + \beta - 1)x_{n-1} + \frac{2x_n\Im(x_nx_1 + x_1x_2 + \cdots + x_{n-1}x_n)}{1 - |x_n|^2}$$

if $x_1$ is real, so are the next $x_n$’s, and one recovers (17) what if $x_1$ is not real?

3.5. The second order differential equation.

The most attractive feature for most people is obviously the linear second order differential equation

$$zA(z)A'(z)\Phi_n(z) = R_n(z)\Phi'_n(z) + S_n(z)\Phi_n(z)$$

with $\Theta_n$ of degree $\leq d, R_n$ and $S_n$.

Indeed, we put (14) in (20), to have $A\Phi'_n = \text{polynomials times } \Phi_n$ and $\Phi'_n$, and also a constant times $\Phi_{n-1}$, which is eliminated from the second equation of (13), the net result is

$$zA(z)\Phi'_n(z) = A_n(z)\Phi_n(z) + B_n(z)\Phi'_n(z)$$

(22)
with $\mathfrak{A}_n$ and $\mathfrak{B}_n$ of degree $\leq d$. Considering the contribution of $\Phi_{n-1}$ in (20), one has $\mathfrak{A}_n(0) = iB(0) + nA(0)$ and $\mathfrak{B}_n(0) = -\Phi_n(0)[iB(0) + nA(0)]$.

With
\[ zA^r(z)(\Phi^r_n(z))' = -\mathfrak{B}_n(z)\Phi_n(z) + (nA^r(z) - \mathfrak{A}_n(z))\Phi^r_n(z), \tag{23} \]
using $\mathfrak{B}_n(z) = n\Phi_n(z) - z(\Phi^r_n(z))'$, we have a linear differential system of the first order for $[\Phi_n, \Phi^r_n]$. And we eliminate $\Phi^r_n$ between (22)–(23).

### 3.6. Linear differential equation for Jacobi polynomials on the unit circle: a bigger flop.

Of course we want to see the differential equation for Jacobi polynomials on the unit circle, to compare with the differential equation for plain Jacobi polynomials. What a thrill.

... to be completed

So, a unit circle Jacobi polynomial is a more composite object than a plain Jacobi polynomial. That’s why I do not believe that reduction to unit circle is always the simplest thing to do.

### 4. Generalized Jacobi polynomials on the unit circle, with two singular points.

#### 4.1. The recurrence relation for the $\Phi_n(0)$'s.

We must have $\frac{dw/d\theta}{w} = \frac{az^2 + bz + c}{(z - e^{i\theta_1})(z - e^{i\theta_2})}$, real on the unit circle. As denominator $(z \exp(i(\theta_1 + \theta_2)/2))$ is real, $(az + b + cz^{-1})/\exp(i(\theta_1 + \theta_2)/2)$ must remain real for $z$ on the unit circle $\Rightarrow |a| = |c|, ac \exp(-i(\theta_1 + \theta_2)) > 0$ and $b \exp(-i(\theta_1 + \theta_2)/2)$ real.

We solve now for $w$:
\[ \frac{dw/dz}{w} = \frac{dw/d\theta}{i\omega} = \frac{az + b + cz^{-1}}{i(z - e^{i\theta_1})(z - e^{i\theta_2})} = \frac{2\alpha}{z - e^{i\theta_1}} + \frac{2\beta}{z - e^{i\theta_2}} + \gamma, \]
with the residues
\[ 2\alpha = \frac{ae^{i\theta_1} + b + ce^{-i\theta_1}}{i(e^{i\theta_1} - e^{-i\theta_1})} = \frac{ae^{i\theta_2} + b + ce^{-i\theta_2}}{i(e^{i\theta_1} - e^{-i\theta_1})}, \]
\[ 2\beta = \frac{ae^{i\theta_1} + b + ce^{-i\theta_1}}{i(e^{i\theta_1} - e^{-i\theta_1})}, \]
so, $w = constant e^{i\gamma} \sin((\theta - \theta_1)/2)z^{2\alpha} e^{i\beta} = constant \exp(i(\gamma + \alpha + \beta)\theta) \sin((\theta - \theta_1)/2)z^{2\alpha} \sin((\theta - \theta_2)/2)z^{2\beta},$ with two constants on the two arcs of endpoints of $i(\theta_1)$ and $i(\theta_2)$.

$\alpha$ and $\beta$ are real, $\gamma + \alpha + \beta = \frac{d}{2i} + \frac{c}{2i(e^{i(\theta_1)} + e^{i(\theta_2)})}$ is a pure imaginary number. So,
\[ d = 2\alpha, \quad A(z) = (z - e^{i\theta_1})(z - e^{i\theta_2}) = z^2 - (e^{i\theta_1} + e^{i\theta_2})z + e^{i(\theta_1 + \theta_2)}, \]
\[ B(z) = izdA/dz + A(dw/d\theta)/w = izdA/dz + Ai((dw/d\theta)/w) = iz(2z - e^{i\theta_1} - e^{i\theta_2}) + i[2az(z - e^{i\theta_1}) + 2\beta(z - e^{i\theta_2})], \]

$\gamma = (\alpha + \beta + \gamma)\theta + (\alpha + \beta - \gamma)\theta = (\alpha + \beta)\theta$.

The building of the equations proceeds as before.

$A(z)\Phi_n(z) = X_n\Phi_{n+1}(z) + Y_n\Phi_{n+1}(z) + Z_n\Phi_n(z) + W_n\Phi_{n+1}(z) = X'_n\Phi_{n+1}(z) + n\Phi_{n+1}(z) + Z_n\Phi_n(z) + W_n\Phi_{n+1}(z),$

$X'_n\Phi_{n+1}(z) = X_n(1 - x_{n+1}^2) = iB_2 + A_2 = (-\alpha + \beta + 1 + i\gamma,$

---

3 If $\Phi(z) = az^m + bz^{m-1} + \cdots + cz + w$, $\Phi^r(z) = \Phi(z) = az^m + bz^{m-1} + \cdots + cz + w$, $\Phi^r(z) = az^m + (a - 1)bz^{m-2} + \cdots + w$. (z$\Phi^r(z)$ = $\Phi(z)$. (z$\Phi^r(z)$ = $\Phi(z)$.
\[ W_n = (iB(0) + nA(0))(1 - |x_n|^2) = (n + \alpha + \beta - i\gamma)e^{i(\theta_1 + \theta_2)}(1 - |x_n|^2), \]
and we look at the coefficients of \( A^n_k \):
\[ z^n: (n + 1) \xi_{\alpha - 1, n} - n(e^{\theta_1} + e^{\theta_2}) = X'_{\alpha - 1, n} x_n + n \xi_{\alpha - 1, n + 1} + Z_n, \]
\[ \xi^n_k: A(0) \Phi^n_k(0) = X'_{\alpha} x_n + n x_n + Z_n x_n + W_n x_n, \]
elimination of \( Z_n \):
\[ \xi^n_{\alpha - 1, n} = (n + X'_{\alpha} x_n + (n + 1) \xi_{\alpha - 1, n - 1} - n(e^{\theta_1} + e^{\theta_2}) x_n - X'_{\alpha} x_n x_n + 1 - n x_n \xi_{\alpha - 1, n + 1} + \]
\[ W_n x_n, \]
which is a linear equation for \( x_{n+1} \) if we know all the \( x_m \) with \( m \leq n \).
\[ (n + X'_{\alpha})(1 - |x_n|^2) x_{n+1} = [A(0) \xi_{\alpha - 1, n} + \xi_{\alpha - 1, n} + n(e^{\theta_1} + e^{\theta_2}) x_n + A(0)(1 - |x_n|^2) x_n - W_n x_n, \]
\[ \xi_{\alpha - 1, n} x_n = x_n x_n + x_n x_{n+1} + \cdots + x_n x_{n+1}. \]

4.2. Numerical experiments with the asymptotics of (24).

4.2.1. We now proceed to extract the last asymptotic secrets of the \( \Phi_n(0) = x_n \)'s of (24).

One obviously expects \( \Phi_n(0) \) to behave like a combination of \( e^{i\theta_1} \) and \( e^{i\theta_2} \) with slowly varying coefficients. What works is
\[ \Phi_n(0) \sim A_1 n^\kappa_1 e^{i\theta_1} + A_2 n^\kappa_2 e^{i\theta_2} \]
as guessed from [24][25][26] etc., also [23]. The powers of \( n \) are deduced from (24) looking as a linear recurrence relation when \( |x_n|^2 \) is neglected, and where \( \xi_{\alpha - 1, n} \) is replaced by its limit, say \( \xi \). Then each particular solution \( A_1 n^\kappa_1 e^{i\theta_1} \) satisfies \( \frac{x_{n+1}}{x_n} \sim (1 + \kappa_j/n)^{e^{i\theta_1}} \), so (24) becomes \( [1 + (1 + \alpha + \beta + i\gamma)/n](1 + \kappa_j/n)e^{i\theta_1} \) or \( [1 + (1 + \alpha + \beta + i\gamma)/n]e^{i(\theta_1 + \theta_2)} \) \( n^{-1} \) contributions:
\[ (1 + \alpha + \beta + i\gamma + \kappa_j) e^{i\theta_1} = e^{i(\theta_1 + \theta_2)} \xi + (1 - \alpha - \beta + i\gamma + \kappa_j) e^{-i\theta_1}, \]
or
\[ \kappa_1 \text{ and } \kappa_2 = -1 - i\delta, \text{ with } i\delta = (1 + \beta) \frac{e^{i\theta_1} + e^{i\theta_2}}{e^{i\theta_1} - e^{i\theta_2}} + \frac{e^{-i\theta_2} - e^{-i\theta_1}}{e^{i\theta_1} - e^{i\theta_2}}. \]

What can \( A_1 \) and \( A_2 \) be? We perform numerical explorations. First with known cases: when \( \theta_2 - \theta_1 = \pi \), and when \( w(\theta)/|\sin(\theta - \theta_1)/2|^{2\alpha} |\sin(\theta - \theta_2)/2|^{2\beta} \) is the same constant on the whole circle, one must have \( A_1 = \alpha, A_2 = \beta \), and \( \delta = 0 \).

When \( \gamma = 0 \), and where is no jump, more experiments lead to
\[ A_1 = \alpha \exp (i\beta (\pi - \theta_2 + \theta_1)), A_2 = \beta \exp (-i\alpha (\pi - \theta_2 + \theta_1)), \]
(25)
where \( \pi - \theta_2 + \theta_1 \) must be understood modulo 2\pi, with a value between \( -\pi \) and \( \pi \).

4.3. Final (so far) experiments.

4.3.1. Discontinuous case. Now, \( w(\theta)/|\sin(\theta - \theta_1)/2|^{2\alpha} |\sin(\theta - \theta_2)/2|^{2\beta} \) is discontinuous, i.e.:
\[ w(\theta) = \begin{cases} r_1 e^{-\gamma\theta} |\sin(\theta - \theta_1)/2|^{2\alpha} |\sin(\theta - \theta_2)/2|^{2\beta}, & \theta_1 < \theta < \theta_2 \\ r_2 e^{-\gamma\theta} |\sin(\theta - \theta_1)/2|^{2\alpha} |\sin(\theta - \theta_2)/2|^{2\beta}, & \theta_2 - 2\pi < \theta < \theta_1 \end{cases} \]
\[ (26) \]
where \( r_1 \) and \( r_2 \) are positive, possibly different. Of course, it’s the ratio \( r_2/r_1 \) which is important.

The multiplicative jump is \( r_1 / r_2 \) at \( \theta_1 \), and \( r_2 / r_1 \) at \( \theta_2 \).

We proceed with examining the output of (24), but where some tricks must be explained: (24) needs a complex number \( x_1 = \Phi_1(0) \) which is not so easy to get, as it is the ratio \( -c_1 / c_0 \) of the two Fourier coefficients \( \int_{-\pi}^{\pi} w(\theta) \exp(i\theta) d\theta / \int_{-\pi}^{\pi} w(\theta) d\theta. \)
However, from the already encountered differential equation for $w = \text{const}$:

$$\frac{dw}{dz} w = \sum_{k=0}^{\infty} k c_k z^{k-1} = \frac{\tilde{\gamma}}{z} + \frac{2\alpha}{z-e^{\theta_1}} + \frac{2\beta}{z-e^{\theta_2}}.$$

(almost) everywhere on $(0,2\pi)$, with $\tilde{\gamma} = i\gamma - \alpha - \beta$: 

$$\sum_{k=0}^{\infty} k c_k z^k (z^2 - (e^{\theta_1} + e^{\theta_2}) z + e^{i(\theta_1 + \theta_2)}) =$$

$$\sum_{k=0}^{\infty} c_k z^k ([i\gamma + \alpha + \beta] z^2 + [i(\beta - \alpha)(e^{\theta_1} - e^{\theta_2}) - i\gamma(e^{\theta_2} + e^{\theta_1})] z + (i\gamma - \alpha - \beta) e^{i(\theta_1 + \theta_2)})$$

whence the linear recurrence relation for the $c_k$'s (see also [19], p. 19).

At $k = 0$, knowing that $c_1 = e^{-i\gamma}$, $(\alpha + \beta + 1 + i\gamma)x_1 - (\beta - \alpha)(e^{\theta_1} - e^{\theta_2}) + i\gamma(e^{\theta_2} + e^{\theta_1}) - (\alpha + \beta + 1 - i\gamma)e^{i(\theta_1 + \theta_2)} = 0$, or $(\alpha + \beta + 1 + i\gamma) \frac{x_1}{e^{\theta_2} - e^{\theta_1}} + \alpha - \beta + \gamma \cot((\theta_2 - \theta_1)/2) - (\alpha + \beta + 1 - i\gamma) \frac{x_1}{e^{-\theta_1} - e^{-\theta_2}}$

amounting to

$$x_1 = (e^{\theta_2} - e^{\theta_1}) \left[ \frac{\alpha - \beta + \gamma \cot((\theta_2 - \theta_1)/2)}{2(\alpha + \beta + 1 + i\gamma)} + \gamma + (\alpha + \beta + 1)i \right], \quad t \in \mathbb{R}$$

so that we only have to experiment with various (real) values of $t$, which correspond to various real ratios $r_2/r_1$, which we recover in the Christoffel function-aided weight function reconstruction.

Check that $|x_1| < 1$ when $t = 0$: $|x_1| = \frac{|(\alpha - \beta) \sin((\theta_2 - \theta_1)/2) + \gamma \cos((\theta_2 - \theta_1)/2)|}{|\alpha + \beta + 1 + i\gamma|}, \text{OK if } |\alpha - \beta| < |\alpha + \beta + 1|.$

Simple check with $\alpha = \beta = \gamma = 0$: we immediately have $x_1 = \Phi(0) = -c_1/c_0 = i(e^{\theta_2} - e^{\theta_1})(r_1 - r_2) / [r_1(\theta_2 - \theta_1) + r_2(2\pi - \theta_2 + \theta_1)].$ When $\gamma \neq 0$, $x_1 = \frac{(i - \gamma)^{-1} \left[ r_1(e^{i(-\gamma)(\theta_2 - \theta_1)} + r_2(e^{i(-\gamma)(\theta_1 - 2\pi)}) \right]}{\gamma^{-1} \left[ r_1(e^{-\gamma\theta_2} - e^{-\gamma\theta_1}) + r_2(e^{-\gamma\theta_1} - e^{-\gamma(\theta_1 - 2\pi)}) \right]} = e^{i(\theta_1 + \theta_2)/2} \left[ \frac{\gamma}{i - \gamma} \cos((\theta_2 - \theta_1)/2) + i \sin((\theta_2 - \theta_1)/2) \right] \frac{r_1(e^{-\gamma\theta_2} + e^{-\gamma\theta_1}) - r_2(e^{-\gamma\theta_1} + e^{-\gamma(\theta_1 - 2\pi)})}{r_1(e^{-\gamma\theta_2} - e^{-\gamma\theta_1}) + r_2(e^{-\gamma\theta_1} - e^{-\gamma(\theta_1 - 2\pi)})} - 2(\gamma^{-1} + \gamma)i$.

Check with $r_1/r_2 = \exp(2\pi \rho)$ with $\rho = 0.25$.

reflecj2 unit circle gen Jacobi Sat Mar 13 2013:26

alpha=0 beta=0 , gamma=0.50000; theta1,2/pi=-0.35000 0.25000
Phi(0)=-0.063355 - 0.27785*I
A1=-0.0089576 - 0.24984*I A2=-0.0089729 - 0.24984*I
xi=-0.045993 - 0.29027*I i delta=- 0.000022164*I
r1=0.79468 r2=0.16520 rho=0.25000
A1/(alpha-rho i)=0.99936 - 0.035830*I A2/(beta+(rho-gamma) i)=0.99936 - 0.035892*I
abs: 1.0000 1.0000
For general $\alpha$ and $\beta$, we could use that, as $c_0$ and $c_{-1}$ are linear in $r_1$ and $r_2$, $x_1 = \frac{X + Y(r_2/r_1)}{1 + Z(r_2/r_1)}$, so that we could extract $X$, $Y$, and $Z$ from three different trials, without having to compute $X$, $Y$, and $Z$, which are complicated integrals, and arrive at a full algorithm for computing long sequences of reflection coefficients for (26), given $\alpha$, $\beta$, $\theta_1$, $\theta_2$, $r_1$, and $r_2$, but we just finish now asymptotic experiments.

In the recurrence (24), we know that $x_n \to 0$, and even that $\xi_{n-1, n} \to n \to \infty \lambda_1$, from Szegő-Geronimus theory, where

$$\log w = \log q(z) + \log r(z) = \log q(z) + \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots$$

We recover $A_1$ and $A_2$ from numerical runs and guess:

$$A_1 = (\alpha - \rho_1 i)e^{i(\pi - \theta_2 + \theta_1)} + i\psi_1, \quad A_2 = (\beta - \rho_2 i)e^{-i(\pi - \theta_2 + \theta_1) + i\psi_2},$$

with $\rho_1 = \frac{\log(r_1/r_2)}{2\pi}$ and $\rho_2 = \gamma - \rho_1$, the jumps of $(2\pi)^{-1}\log w$ at $\theta_1$ and $\theta_2$, as seen above.

The absolute values of $A_1$ and $A_2$ are probably correct, as they agree with what is needed in the Hartwig-Fisher formula of next section; the phases $\psi_1$ and $\psi_2$ are still unknown, they will be examined later on.

4.3.2. Relation with Fisher-Hartwig determinants.
We may conjecture that the main behaviour of \( \Phi_n(0) \) for a weight which is smooth except at \( \theta_1, \ldots, \theta_p \), where it behaves like \( r_{k,1} |\theta - \theta_k|^2 \alpha_k \) for \( \theta \to \theta_k \), \( \theta < \theta_k \), and like \( r_{k,2} |\theta - \theta_k|^2 \alpha_k \) for \( \theta \to \theta_k \), \( \theta > \theta_k \), will be

\[
x_n = \Phi_n(0) = \sum_{k=1}^p \frac{A_k r_{k,1}}{n} e^{i \theta_k} + o(1/n),
\]

with \( |A_k|^2 = \alpha_k^2 + \rho_k^2 \), where \( \rho_k = \frac{\log(r_{k,1}/r_{k,2})}{2\pi} \). From (10),

\[
||\Phi_n||^2 = \prod_{m=1}^n (1 - |\Phi_m(0)|^2) = \text{const} \left( 1 + \sum_{k=1}^p |A_k|^2 + \cdots \right),
\]

as the only non-oscillating terms in the expansion of \( |\Phi_m(0)|^2 = \Phi_m(0) \Phi_m(0) = \sum_i \sum_j A_i A_j m(\delta_i - \delta_j) \exp(i m(\theta_k - \theta_j))/m^2 \) are the \( |A_k|^2/m^2 \) terms. The constant is actually very well known from the Szegő theory, as it must be \( \exp(2\lambda_0) \).

Finally, the product of these square norms yields the determinant of the Gram matrix (here, a Toeplitz matrix) \( \det G_n = \prod_{m=1}^n ||\Phi_m||^2 = \text{const} n^{\sum_{i=1}^n |A_i|^2} \exp(2n\lambda_0) \), (with another constant). This formula is exactly the Hartwig-Fisher asymptotic formula for such Toeplitz determinants, see [4] for details and full history!

4.3.3. Last last calculations, Spring 2012 and March 2013.

We look at the influence of the jump \( \rho = \rho_1 \) in \( A_1 = (\alpha - \rho_1 i) e^{i \beta (\pi - \theta_2 + \theta_1)} \psi_1 \), \( A_2 = (\beta - \rho_2 i) e^{-i \alpha (\pi - \theta_2 + \theta_1)} \psi_2 \), when \( \alpha = \beta = \gamma = 0 \) and \( \theta_1, \theta_2 = \pm \pi/2 \).

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \psi_1 )</th>
<th>( d\psi_1/d\rho )</th>
<th>( d^2\psi_1/d\rho^2 )</th>
<th>( 2 \arg \Gamma(1+i\rho) - \rho \log \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.10347</td>
<td>0.26208</td>
<td>-0.2533</td>
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<td>0.26201</td>
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<tr>
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</tbody>
</table>

Various values of \( t \), so of \( x_1 \) are entered in the recurrence [24]. \( A_1 \) and \( A_2 \) are extracted from \( x_n \), with large \( n \), and \( \rho \) from weight reconstruction. One finds \( \psi_2 = -\psi_1 \), and we look at \( \psi_1 \) as a continuous function of \( \rho \), which may ask for adding integer multiples of \( 2\pi \).

The second order divided difference on almost uniformly spaced sets of three values of \( \rho \) yield a good estimate of (half the) second derivative at the middle point. The formula \( d^2 \psi_1 / d\rho^2 \sim 2/\rho \) seems to hold.

This means a main behaviour \( 2\rho \log \rho \) for the function itself. The simplest special function with this behaviour is the Gamma function. More precisely, an exploration of the pages of Abramowitz & Stegun’s masterpiece, where even the pages with tables are wonderfully inspiring, shows that the imaginary part of \( \log \Gamma(1+i\rho) \) is the needed odd continuous function of \( \rho \). From Stirling formula, Abr 6.1.44, the behaviour for large positive \( \rho \) is \( \log \rho - \rho + \pi/4 + o(1) \). So the first and second derivatives are close to \( \log \rho + 1/\rho \).

The derivative at \( \rho = 0 \) of two times the argument of the Gamma function of \( 1+i\rho \)
is \(-1.15443 = -2\) times the Euler constant. The derivative of our function there is estimated to be \(-2.533\). The difference is \(-1.379\), close enough to \(-\log 4\). GOT IT! (Spring 2012). Music, at last!

For general \(\theta_1\) and \(\theta_2\), we find readily the term \(-2p\log\sin((\theta_2 - \theta_1)/2)\) to be present n \(\psi_1\).

### Gamma Function and Related Functions

**Table 6.7**

<table>
<thead>
<tr>
<th>(y)</th>
<th>(y \ln y)</th>
<th>(y^2 \ln y)</th>
</tr>
</thead>
<tbody>
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<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.0000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0063</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0087</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

We now must consider the influence of \(\alpha\), \(\beta\), and \(\gamma\).
4.3.4. The final conjecture. Consider the special weight on the unit circle

\[ w(\theta) = \begin{cases} r_1 e^{-\theta} \sin(\theta - \theta_1)/2 & \text{if } \theta_1 < \theta < \theta_2, \\ r_2 e^{-\theta} \sin(\theta - \theta_1)/2 & \text{if } \theta_2 - 2\pi < \theta < \theta_1, \end{cases} \]

where \( \theta_1 < \theta_2 < \theta_1 + 2\pi, r_1 \) and \( r_2 \) are positive, \( \alpha, \beta, \) and \( \gamma \) real, \( \alpha < \beta \) and \( \beta > -1/2. \) Then

\[ \Phi_n(0) = A_1 n^{\kappa_1} e^{i\theta_1} + A_2 n^{\kappa_2} e^{i\theta_2} + o(n^{-1}), \]

with \( \kappa_1 \) and \( \kappa_2 = -1 - i\delta, \) where \( i\delta = -\left(\alpha + \beta\right)/e^{i\theta_1} - e^{-i\theta_2} + e^{-i\theta_1} - 1, \) \( e^{-i\theta_2} \) being the coefficient of \( e^{-i\theta} \) in the Fourier expansion \( \log w(\theta) = \cdots + \xi e^{-i\theta} + \lambda_0 + \xi e^{i\theta} + \cdots, \) \( \xi \) is also the limit when \( n \to \infty \) of \( \xi_n \) in \( \Phi_n(z) = e^{i\theta} + \xi_n e^{-i\theta} + \cdots. \) And the big conjecture is

\[ A_1 = (\alpha - \rho_1) i \exp[2i \arg(1 + \alpha + \rho_1 i) + 2(\gamma - \rho_1) \log(2 \sin((\theta_2 - \theta_1)/2))], \]
\[ A_2 = (\beta - \rho_2) i \exp[2i \arg(1 + \beta + \rho_2 i) + 2(\gamma - \rho_2) \log(2 \sin((\theta_2 - \theta_1)/2))], \]

where \( \rho_1 = \frac{\log(r_1/r_2)}{2\pi} \) and \( \rho_2 = \gamma - \rho_1, \) the jumps of \( (2\pi)^{-1} \log w(\theta_1) \) and \( \theta_2. \)

4.3.5. The GP-PARI program. Given \( \alpha, \beta, \gamma, \theta_1, \) and \( \theta_2, \) the program computes \( x_1 = \Phi_1(0) = \left( e^{i\theta_1} - e^{i\theta_2} \right) \left[ -\frac{\alpha - \beta + \gamma \cot((\theta_2 - \theta_1)/2)}{2(\alpha + \beta + 1 + i\gamma)} + (\gamma + (\alpha + \beta + 1)i)t \right], \) \( t \in \mathbb{R}, \) with various values of \( t \) which will allow experiments with various values of \( \rho. \) Then, several thousands of the next \( x_n, \) are computed with \( \Phi_n(0). \) At each power of two, \( A_1 \) and \( A_2 \) are estimated from \( x_n \sim A_1 n^{\kappa_1} e^{i\theta_1} + A_2 n^{\kappa_2} e^{i\theta_2}, \)

\[ x_{n+1} \sim A_1 (n+1)^{\kappa_1} e^{i(\theta_1+1)} + A_2 (n+1)^{\kappa_2} e^{i(\theta_2+1)} \Rightarrow n^{\kappa_1} e^{i\theta_1} x_n - (n+1)^{\kappa_1} x_{n+1} \sim [n^{\kappa_1} - n^{\kappa_2}] e^{i\theta_1} A_1 + \text{etc.} \]

A further simple acceleration is performed with \( A_1 \) replaced by \( 2A_1 \) minus the former \( A_1 \) from the step \( n/2. \) Several thousands of \( \Phi_n(0) \)'s allow an accurate reconstruction of the weight function through Christoffel function (P.Nevari, the big 1986 paper, p. 26) \( w(\theta) = \lim_{n \to \infty} n \omega_n(e^{i\theta}) = \frac{n}{\sum_{n=0}^{\infty} |\Phi_n(e^{i\theta})|^2}, \)

from which \( \rho = \rho_1 = (2\pi)^{-1} \log(r_1/r_2) \) is found, and the formulas for \( A_1 \) and \( A_2 \) are checked.
Chris(z) = 
  locp=1;locps=1;locsom=1;
  for(n=1,Nr,locp=z*locp+ref[n+1]*locps;
      locps=(1-(abs(ref[n+1]))^2)*locps+conj(ref[n+1])*locp;
      locsom=locsom+(abs(locp))^2/norm2[n+1])
  1/locsom
}

ref=vector(Nr+1,k,0);
\casalpha=beta=0
\rr=exp(2*Pi*0.25);t=-( (rr-1)*exp(-ga*Pi*tp1)+(rr-exp(2*ga*Pi))*exp(-ga*Pi*tp2) )/(2*(ga+1/ga)*
  (1-rr)*exp(-ga*Pi*tp2));
t=0.1;ref[2]=(be-al-ga*ct1t2/st1t2)/(2*(albe+1+ga*I))+(ga+(albe+1)*I)*t;
\ vector of reflection coeff.

ref[1]=xi=ref[2];

\vector of reflection coeff.
ref[1]=1;ref[2]=(et2-et1)*( (be-al-ga*ct1t2/st1t2)/(2*(albe+1+ga*I)) +
  (ga+(albe+1)*I)*t );
\ vector of reflection coeff.

for(n=1,Nr-1,
  if( abs(ref[n])>1 , stop=1; return );
  refn=ref[n+1];refn1=ref[n+1];refn1o=refn1;cr1o=0;cr2o=0;
  for(n=2,Nr-1,
    \ asymptotic behaviour cr1 et1'n''/n' (1+i gamma +i delta) + cr2 et2'n''/n'' (1+i gamma +i delta)
    if(n==p2,dn=n;dn1=n+1;
      cr1[dn]=dn1*refn1*et1n''+cr1o;
      cr2[dn]=dn1*refn1*et2n''+cr2o;
      print(n," A1=",cr1[dn]," A2=",cr2[dn],
    \ asymptotic behaviour cr1 et1'n''/n' (1+i gamma +i delta) + cr2 et2'n''/n'' (1+i gamma +i delta)
    if(n==p2,dn=n;dn1=n+1;
      cr1[dn]=dn1*refn1*et1n''+cr1o;
      cr2[dn]=dn1*refn1*et2n''+cr2o;
      print(n," A1=",cr1[dn]," A2=",cr2[dn],
      \ asymptotic behaviour cr1 et1'n''/n' (1+i gamma +i delta) + cr2 et2'n''/n'' (1+i gamma +i delta)
    if(n==p2,dn=n;dn1=n+1;
      cr1[dn]=dn1*refn1*et1n''+cr1o;
      cr2[dn]=dn1*refn1*et2n''+cr2o;
      print(n," A1=",cr1[dn]," A2=",cr2[dn],
      \ \ end if p2
      cr1o=cr1[dn];cr2o=cr2[dn];
    \ end for n
    if(stop,return);
    cr1=cr1o;cr2=cr2o;
  \ end for
if(stop,return);

\ vector(Nr+1,k,0);norm2[1]=1;
for(n=2,Nr+1,
  norm2[n]=(1-(abs(ref[n]))^2)*norm2[n-1]);
if(stop,return);
\ \ B/A, al, be:
\ dw/d theta =w[ alpha cot((theta-theta1)/2)+beta cot((theta-theta2)/2) -gamma ]
  th1=Pi*(tp1+tp2/2)/w1=NR*Chris(exp(I*th1));
  tans=tan( Pi*(tp1+tp2)/4 )/4 ; cots=1/tans;
  dw1=NR*( Chris(exp[I*(th1+0.0001)])-Chris(exp[I*(th1-0.0001)]) )/(0.0002*w1)+ga;
  al1=(dw1*tans+albe)/2bel=-(dw1*tans+albe)/2;
  print(1" check alpha, beta= ",al1," ",bel);
  th2=Pi*(tp2+tp1/2)/w2=NR*Chris(exp[I*th2]);
  dw2=NR*( Chris(exp[I*(th2+0.0001)])-Chris(exp[I*(th2-0.0001)]) )/(0.0002*w2) +ga;
  al2=(dw2*cots+albe)/2be2=(dw2*cots+albe)/2;
  print(1" albe= ",al2," ",be2);
  \ all=albe-(abs(cr2')^2-abs(cr1')/2)/albe';
  \ all1=albe+((abs(cr2')^2-abs(cr1')/2)/albe';
  print(1" xi= ",xi," i delta= ",id1);
  r1=wi1/( abs(sin((th1-Pi*tp1)/2)) )^(2*a1) *
abs(sin((th1-Pi*tp2)/2)) *(2*be) * exp(-ga*th1) );

r2=w2/( abs(sin((th2-Pi*tp1)/2)) *(2*al) * abs(sin((th2-Pi*tp2)/2)) *(2*be) * exp(-ga*th2) );

rho1=log(r1/r2)/(2*Pi);rho2=ga-rho1;errho=r1/r2;

Astart=cr1/(al-rho1*I);Bstart=cr2/(be-rho2*I);


psi1=-rho1*L12+2* arg(gamma(1+al+rho1*I));psi11=be*Pi*(1-tp2+tp1)+psi1+ga*L12;

psi11=psi11-2*Pi*round(0.5*psi11/Pi);

psi2=-rho2*L12+2* arg(gamma(1+be+rho2*I));psi12= -al*Pi*(1-tp2+tp1)+psi2+ga*L12;

psi12=psi12-2*Pi*round(0.5*psi12/Pi);

print(" logarithmes: ",log(Astart)," abs: ",abs(Astart)," abs: ",abs(Bstart)," abs: ",abs(Bstart)");

plot(th=Pi*tp2-2*Pi,Pi*tp2,Nr*Chris(exp(I*th)));

GP/PARI CALCULATOR Version 2.2.8 (development CHANGES-1.887)
i686 running cygwin (ix86 kernel) 32-bit version
compiled: Jan 13 2004, gcc-3.3.1 (cygming special)
(readline v4.3 enabled, extended help available)

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Type ? for help, \q to quit.

Type ?12 for how to get moral (and possibly technical) support.

realsize = 28 significant digits
seriesprecision = 16 significant terms
format = g0.28

parisize = 4000000, primelimit = 500000

reflecj2 unit circle gen Jacobi Tue 19 Mar 2013 15:15
alpha=1.6000 beta=0.90000 , gamma=2; theta1,2/pi=-0.15000 0.27500
Phi1(0)= -0.63105 - 0.31312*I , t=0.100000000000000000000000000000

1 A1=-5.6514 + 2.8024*I A2=-2.7805 - 1.20871 relerr=10^-28.20871
2 A1=-1.6032 - 3.7685*I A2= 0.72380 - 1.1202*I relerr=10^-31.3150
4 A1=-2.7805 - 2.5577*I A2= 0.72380 - 1.1202*I relerr=10^-31.3150
8 A1=-2.7833 - 2.5577*I A2=-1.4613 + 0.20793*I relerr=10^-32.3209
16 A1=-2.7643 - 2.9268*I A2=-0.58543 - 0.23763*I relerr=10^-30.5384
32 A1=-2.0948 - 2.0138*I A2=-0.68737 - 0.78442*I relerr=10^-30.93469
64 A1=-1.5325 - 2.5385*I A2=-0.42672 - 0.90166*I relerr=10^-30.5613
128 A1=-1.0397 - 2.8077*I A2=-0.31776 - 0.98557*I relerr=10^-30.56055
256 A1=-0.78972 - 2.8517*I A2=-0.22935 - 0.98711*I relerr=10^-30.64465
512 A1=-0.66794 - 2.8593*I A2=-0.18537 - 0.99123*I relerr=10^-30.80112
1024 A1=-0.61219 - 2.8605*I A2=-0.16559 - 0.99087*I relerr=10^-31.5224
2048 A1=-0.58561 - 2.8612*I A2=-0.15649 - 0.99045*I relerr=10^-31.7537
4096 A1=-0.57276 - 2.8622*I A2=-0.15201 - 0.99067*I relerr=10^-32.0813
8192 A1=-0.56663 - 2.8629*I A2=-0.14989 - 0.99080*I relerr=10^-32.3460
16384 A1=-0.55652 - 2.8633*I A2=-0.14885 - 0.99090*I relerr=10^-32.6060
32768 A1=-0.56213 - 2.8635*I A2=-0.14833 - 0.99096*I relerr=10^-32.4234

check alpha, beta= 1.6000 0.89998 ; 1.6000 0.90002
xi=-0.56702 - 1.9303*I i delta=4.1709 E-29 - 2.8812*I
r1=2934.6 r2=0.000064375 rho=2.4402 A1/(alpha-rho i)=0.92629 - 0.37698*I A2/(beta-(gamma-rho) i)=-0.56760 - 0.82341*I

5.1. The problem.

"3. The following Toeplitz matrix arises in several applications. Define for \( i \neq j, A_{i,j}(\alpha) = \frac{\sin \pi \alpha (i-j)}{\pi (i-j)} \) and set \( A_{i,i} = \alpha \). Conjecture: the matrix \( M = (I-A)^{-1} \) has positive entries. A proof is known for \( 1/2 \leq \alpha < 1 \). Can one extend this to \( 0 < \alpha < 1 \)? Submitted by Alberto Grünbaum, November 3, 1992. (grunbaum@math.berkeley.edu)" [17].

\( I - A \) is the \( G_N \) of the weight \( w = 1 \) on the circular arc shown left. For all the entries of all the \( G_N^{-1} \) to be positive, it is necessary that all the reflection coefficients \( \Phi_n(0) > 0, n = 1, \ldots, N \), and the condition is sufficient: from [6], all the \( \Phi_n \)'s have positive coefficients, so does \( L^{-1} \), and \( G_N^{-1} \), from [15] [7] p.645.

In [7], Delsarte & al. study the robustness of a signal recovery procedure amounting to find the polynomial \( p = p_0 + \cdots + p_N z^N \) minimizing the integral of \( |f(\theta) - p(e^{i\theta})|^2 \) on the circular arc shown above. This elementary least-squares problem involves the Gram matrix \( G_N \), and the stability of the recovery procedure is related to the size of the smallest eigenvalue of the matrix. The corresponding eigenvector is shown to have elements of the same sign. The theory of this eigenvalue-eigenvector pair could be more complete if
it could be shown that $G_n^{-1}$ has only positive elements, for any $N = 1, 2, \ldots$, and any $\alpha \in (0, 1)$. It is also reported in [2] p. 644] that Grünbaum stated this conjecture as early as 1981.

If $\alpha \geq 1/2$, all the zeros of $\Phi_n$ have negative real part (Fejér), so $\Phi_n(0) = (-1)^n$ times the product of all the zeros must be $> 0$ (conjugate pairs have no influence on the sign, and the number of real zeros is $n$—an even number).

From continuity of the zeros with respect to $\alpha$, we are trying to show that the real zeros of $\Phi_n$ all remain negative for all $0 < \alpha < 1$. Most zeros are close to the support anyhow, and there are probably only a small number of real zeros which are not close to $-1$.

In order to remove the insufferable tension, here is some numerical evidence, where the relevant $\Phi_n(0)$’s are computed either with the all-purpose algorithm of p. 12 or from the formula (24) which will be further worked in [27].

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\Phi_1(0)$</th>
<th>$\Phi_2(0)$</th>
<th>$\Phi_3(0)$</th>
<th>$\Phi_4(0)$</th>
<th>$\Phi_5(0)$</th>
<th>$\Phi_6(0)$</th>
<th>$\Phi_7(0)$</th>
<th>$\Phi_8(0)$</th>
<th>$\Phi_9(0)$</th>
<th>$\Phi_{10}(0)$</th>
<th>$\Phi_1$</th>
</tr>
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<td>0.109292</td>
<td>0.117289</td>
<td>0.124056</td>
<td>0.131824</td>
<td>0.141343</td>
<td>0.143889</td>
<td>0.145959</td>
<td>0.147640</td>
<td>0.149008</td>
<td>0.149366</td>
<td>0.150301</td>
</tr>
<tr>
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<td>0.233872</td>
<td>0.258015</td>
<td>0.285306</td>
<td>0.303299</td>
<td>0.304249</td>
<td>0.304953</td>
<td>0.305715</td>
<td>0.306616</td>
<td>0.307000</td>
<td>0.307076</td>
<td>0.307700</td>
</tr>
<tr>
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<td>0.406603</td>
<td>0.447202</td>
<td>0.487098</td>
<td>0.503272</td>
<td>0.511433</td>
<td>0.517111</td>
<td>0.521149</td>
<td>0.523953</td>
<td>0.525167</td>
<td>0.525556</td>
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<tr>
<td>0.4</td>
<td>0.504551</td>
<td>0.550672</td>
<td>0.59686</td>
<td>0.636619</td>
<td>0.681477</td>
<td>0.703373</td>
<td>0.705258</td>
<td>0.707000</td>
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<td>0.709716</td>
<td>0.710707</td>
</tr>
<tr>
<td>0.5</td>
<td>0.636619</td>
<td>0.681477</td>
<td>0.703373</td>
<td>0.705258</td>
<td>0.707000</td>
<td>0.708352</td>
<td>0.709716</td>
<td>0.710707</td>
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<td>0.712067</td>
<td>0.712313</td>
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<td>0.756826</td>
<td>0.793314</td>
<td>0.822464</td>
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<td>0.870503</td>
<td>0.890173</td>
<td>0.890853</td>
<td>0.891529</td>
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<td>0.892716</td>
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<td>0.886798</td>
<td>0.899138</td>
<td>0.908296</td>
<td>0.909192</td>
<td>0.909410</td>
<td>0.909551</td>
<td>0.909647</td>
<td>0.890716</td>
<td>0.890930</td>
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<tr>
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<td>0.947570</td>
<td>0.949601</td>
<td>0.950254</td>
<td>0.950547</td>
<td>0.950704</td>
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<td>0.950959</td>
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<tr>
<td>0.9</td>
<td>0.983631</td>
<td>0.986853</td>
<td>0.987333</td>
<td>0.987491</td>
<td>0.987563</td>
<td>0.987601</td>
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<td>0.987639</td>
<td>0.987650</td>
<td>0.987657</td>
<td>0.987662</td>
</tr>
</tbody>
</table>

? quit
Good bye!

C:\calc\pari> exit Script completed Wed Oct 13 14:09:47 1999

All the $\Phi_n(0)$’s seem indeed to be positive. Moreover, it is known that $\Phi_n(0) \to \sin(\pi \alpha/2)$ when $n \to \infty$ [10][11][12] etc.]. N.B.: the capacity of the support is $\cos(\pi \alpha/2)$ [32]. A possible strategy, although not extremely elegant, is to look for a more accurate asymptotic behaviour, so to prove the positivity for $n \geq$ some finite $n_0$, and to show positivity of the finite number of remaining $\Phi_n(0)$, $n \leq n_0$… But perhaps we may learn something from a (nonlinear) recurrence relation between the $x_n = \Phi_n(0)$:

5.1.1. A recurrence relation for the $\Phi_n(0)$’s .

(24) becomes

$$(n + 1)x_{n+1} = \frac{2(x_1x_0 + \cdots + x_nx_{n-1}) + 2n \cos(\pi \alpha)}{1 - x_n^2} x_n - (n - 1)x_{n-1}, \quad (27)$$

with $x_0 = 1$ and $x_1 = \frac{\sin(\pi \alpha)}{(1 - \alpha) \pi}$.

Question: are all the $x_n$’s positive??

The most elegant proof should establish that the sequence of the $\Phi_n(0)$’s is increasing, as suggested by the numerical tests, but how to achieve that?

OK, it will be achieved in section 5.1.10 p. 44

5.1.2. Trying to solve (27).

From numerical runs of (27), the following empiric asymptotic formula:

$$x_n = \sin(\pi \alpha/2) - \frac{\cos^2(\pi \alpha/2)}{8n^2 \sin(\pi \alpha/2)} + O(n^{-4}).$$
quite in agreement with formula (56) of Golinskii, Nevai, and Van Assche stating that
\[ \Phi_n(0) = \sin(\pi\alpha/2) + (-1)^n \frac{6\cos^2(\pi\alpha/2)}{2n} \]
\[ \cos^3(\pi\alpha/2) \left[ 1 + \delta^2 \sin^2(\pi\alpha/2) - 4\gamma + 2(\delta - 2\gamma - \sin(\pi\alpha/2)) \right] + O(n^{-3}) \]
for the weight function \((\cos \pi\alpha - \cos \theta)^3(\cos \theta/2)^5 \sin \theta/2 \) on \((\pi\alpha, 2\pi - \pi\alpha)\). The formula is based on values of Jacobi polynomials, as the problem is reduced to the weight \((\cos \pi\alpha - x)^{m+1}/2n\) on the real interval \((-1, \cos \pi\alpha)\). Unfortunately, the same technique would lead here to \((1 - x)^{-1/2}\), which is not related to known real line orthogonal polynomials. Actually, we are struggling here like mad to make these polynomials known!!!

What can be shown from (27) are -probably useless- expansions in powers of \(\alpha\):
\[ x_n = \alpha + n\alpha^2 - n^2(\pi^2 - 6)\alpha^3/6 - [2(\pi^2/9) n^3 + \pi^2 n]\alpha^4/18 + \cdots \]

5.1.3. First exploration of the solutions. Making some numerical runs of (27) with a definite precision leads to troubles in the long run:

```
ubasic
  20 ' grunbaum
  40 A=0.25
  10 0.379705895 0.379706107321706677649351283092675216787995515802880861153
  15 0.38140294 0.3814126886484914173147263608761372570575076621489739705
  20 0.381507291 0.38151666473125727973974461115525299704926866848914173147263608761372570575076621489739705
  25 0.381610932 0.3816195509053152879915126969792416349998488803889425801828065619215663469351283092675216787995515802880861153
  ... 0.379705895 0.3797061073217066762769091921569654142523516196833649351283092675216787995515802880861153
  ... 0.38140294 0.3814126886484914173147263608761372570575076621489739705
  ... 0.381507291 0.38151666473125727973974461115525299704926866848914173147263608761372570575076621489739705
  20.381507291 0.38151666473125727973974461115525299704926866848914173147263608761372570575076621489739705
```

What can be shown from (27) are -probably useless- expansions in powers of \(\alpha\):
\[ x_n = \alpha + n\alpha^2 - n^2(\pi^2 - 6)\alpha^3/6 - [2(\pi^2/9) n^3 + \pi^2 n]\alpha^4/18 + \cdots \]
That happens because the general solution of \( \Phi(\alpha) \), keeping only \( x = 1 \), is related to the piecewise constant weight function \( w = B \) for \( \alpha < \pi \), and \( w = B \) for \( \alpha > \pi \). The overwhelming majority of solutions have \( A \) and \( B \neq 0 \), are related to a Szegő weight if \( A \) and \( B > 0 \), and have therefore \( x_n \to 0 \) when \( n \to \infty \). Taking \( x_1 \) just a trifle above or below the ideal value \( \sin(\pi \alpha) / (1 - \pi \alpha) \) will therefore end up with unsatisfactory values of \( x_n \) for large \( n \). More precisely, \( x_1 \) is the \( \Phi(0) \) related to the \( (A,B) \) weight function:

\[
x_1 = \frac{\mu_1}{\mu_0} = \frac{A}{A} \int_0^{\pi} \cos \theta \, d\theta + \frac{B}{B} \int_0^{\pi} \cos \theta \, d\theta = \left( 1 - \frac{B}{A} \right) \sin \pi \alpha / \left( 1 - \pi \alpha + (B/A) \pi \right).
\]

Values of \( x_1 \) smaller than \( \sin \pi \alpha / (1 - \pi \alpha) \) correspond indeed to a Szegő weight (as long as \( x_1 > -\sin \pi \alpha / (\pi \alpha) \)); larger values of \( x_1 \) does not even correspond to positive weights and will have some \( x_n \not\in (-1, 1) \).}

These numerical difficulties show that a direct study of the recurrence relation \( (27) \) trying to get an induction of the form “if some inequality is valid for \( x_n \), it is also valid for \( x_{n+1} \)”, will lead nowhere.

Script V1.1 session started Tue Mar 18 14:45:45 2003

type grunbg gp

/*
grunbg gp : launch gp and make \$r grunbg

Reflection coefficients for Grunbaum- Delsarte et al. problem.

weight= (cos pi a - cos theta)^(beta) on pi a < theta < 2pi -pi a

default(format,"g2.7");

for(bigN=50,50,print("N= ",bigN);
for(ia=1,1,a=0.25*ia;
 \ a=2*asin(a)/Pi;
 print1("alpha="a, " "); sa=sin(Pi*a/2);
 print1(" sigma=sin pi a/2= ",sa);
 ca=cos(a*Pi);
 print(" cos pi a= ",ca);
 \ vector of reflection coeff.
 v=vector(bigN+2,k,0);
 print("\psline(0,-5)(0,0)(10,0)(0,0)(0,5)(10,5)(10,-5)(0,-5)\"");
 v[1]=1;
 for(iba=1,25,
 \ ba=10*(iba-20);
 if(iba<=20,ba=0.25*(iba-10));
 if(iba<10,ba=0.01*(iba-5));
 v[2]=(1-ba)*sin(a*Pi)/(Pi*(1-a+ba*a)); \print(v[2]);
 print("\psline(0,",5*v[2],")(-0.25,",5*v[2],"");
 print("\uput[180](-0.25,",5*v[2],$",ba,"$)\"; "\psline%;
 som=0; kpr=1;
 for(n=1,bigN,
 \ som=som+v[n+1]*v[n];
 v[n+2]=( 2*(som*n*ca)*v[n+1]/(1-v[n+1]^2) - (n-1)*v[n]/n+1);
 if(kpr==1,print1("(",0.2*n," ",5*v[n+1],")\"));
 if(divrem(n,5)[2]==0,print("%"));
 if(abs(v[n+1])>1.2,kpr=0);
 )\});
 )
)

C:\calc\pari>gp-sta
GP/PARI CALCULATOR Version 2.1.3 (released)
Copyright (C) 2000 The PARI Group PARI/GP
\r grunbg
N= 50
 alpha=0.250000003 sigma=sin pi a/2= 0.382683438 cos pi a= 0.707106791
Check that $x_n \to \sigma \neq 0$ in (27) yields $\sigma = \pm \sin(\pi \alpha/2)$. There is also an alternating solution $(-1)^n x_n \to \sigma$ which yields $\sigma = \pm \cos(\pi \alpha/2)$, corresponding to the arc of circle joining $e^{i \pi \alpha}$ to $e^{-i \pi \alpha}$ and containing +1. These two exceptional solutions correspond to $B = 0$ and $A = 0$. So, we are looking for the only solution of (27) satisfying $x_n \to \sin(\pi \alpha/2)$ when $n \to \infty$, and we are wondering if all the $x_n$'s are positive.

Of course, each $x_n$ is a rational function of $x_1$:

$$x_2 = \frac{x_1(\cos(\pi \alpha + x_1))}{1-x_1^2}, x_3 = \frac{x_1(4\cos^2(\pi \alpha) - 1 + 6\cos\pi \alpha x_1 + (4 - \cos^2(\pi \alpha))x_1^2)}{3(1 - (2 + \cos^2(\pi \alpha))x_1^2 - 2\cos(\pi \alpha)^2)}, \ldots$$

Here is an information on the variation of each $x_n$ with respect to $x_1$, while $x_n$ remains positive: We look at the influence of $x_1$ on $x_n$, i.e., at $\partial x_n/\partial x_1$, which we write $\dot{x}_n$.

5.1.4. Influence of $x_1$ on $x_n$. **Proposition.** If $x_1, x_2, \ldots, x_n \in (0, 1)$, then $\dot{x}_n > 0$.

Derivating (27) for $i = 1, 2, \ldots, n - 1$:

$$(i+1)\dot{x}_{i+1} = \frac{2x_i}{1-x_i^2} \left[ \dot{x}_1 + \sum_{j=1}^{i-1} (x_j \dot{x}_{j+1} + x_{j+1} \dot{x}_j) \right] + \frac{\text{num.}}{(1-x_i^2)(1-x_{i-1}^2)} (1+x_i^2)\dot{x}_i - (i-1)\dot{x}_{i-1},$$

where “num.” is the numerator in the right-hand side of (27). We now use precisely this equation (27) to replace “2num.”/(1 - $x_i^2$) by ((i + 1)$x_{i+1}$ + (i - 1)$x_{i-1}$)/$x_i$:

$$(i+1)\dot{x}_{i+1} = \frac{2x_i}{1-x_i^2} \left[ \dot{x}_1 + \sum_{j=1}^{i-1} (x_j \dot{x}_{j+1} + x_{j+1} \dot{x}_j) \right] + \frac{(i+1)x_{i+1} + (i-1)x_{i-1}}{x_i(1-x_i^2)} (1+x_i^2)\dot{x}_i - (i-1)\dot{x}_{i-1},$$
All the $x_i$’s are positive: if true up to $i$,

$$(i + 1)\dot{x}_{i+1} > \frac{(i + 1)x_{i+1} + (i - 1)x_{i-1}}{x_i} \dot{x}_i - (i - 1)\dot{x}_{i-1},$$

$$(i + 1)x_{i+1} \left[\frac{\dot{x}_{i+1}}{x_{i+1}} - \frac{\dot{x}_i}{x_i}\right] > (i - 1)x_{i-1} \left[\frac{\dot{x}_i}{x_i} - \frac{\dot{x}_{i-1}}{x_{i-1}}\right].$$

whence indeed $\dot{x}_{i+1} > 0$ too.

Incidentally, we also have $\dot{x}_i/x_i$ increasing with $i$, as also $i(i - 1)[x_{i-1}x_i - x_i\dot{x}_{i-1}]$.

Remark that, if $x_1$ is very small,

$$x_n \approx \frac{\sin n\pi\alpha}{n\sin\pi\alpha} x_1,$$

going to 0 as $n\alpha < 1$ if $x_1$ is a small positive number. And when $x_1$ increases, the length of the initial sequence of positive $x_n$’s must increase too, and we hope to have the whole infinite sequence at $x_1 = \sin\alpha\pi/((1 - \alpha)\pi)$.

If this is true, what happens when $x_1$ still increases a little bit, may we ask? According to the Proposition above, all the $x_n$’s are positive, so that all the $x_n$’s will increase by an infinitesimal amount and still keep a valid value < 1. What about the unicity of the positive solution? Answer: for any small but nonzero increase of $x_1$, there is an $n$ large enough where $x_n$ will increase so much as to be $> 1$. A strong hint is the fast exponential increase of $\dot{x}_n$ with $n$:

ubasic

20 ' grunbaum
25 point=print*$grunbaum.1*
30 point 21:word 74
40 A=0.25
201 X0=1:X1=\sin(A*#pi)/{(1-A)*#pi}:Dx0=0:Dx1=1:C=\cos(A*#pi):N=100
202 ' recurrence x1 and dx1/x1
205 X1=X1:Dx1=1:for M=1 to N-1:Xp1=(2*(X1+M*C)*X1/(1-X1^2)-(M-1)*X0)/(M+1)
206 Dxp1=((2*Dx1*X1+(M+1)*Xp1+(M-1)*X0)/(1-X1^2)-(M-1)*Dx0)/(M+1)
207 Xi=X1+X1*Xp1:Dxi=1:for M=1 to N-1:Xp1=(2*(Xi+M*C)*X1/(1-X1^2)-(M-1)*X0)/(M+1)
208 print M;";";X1;";";Dx1
210 X0=X1:X1=\sin(A*#pi)/{(1-A)*#pi}:Dx0=0:Dx1=1:C=\cos(A*#pi):N=100

The lesson here is that the above Proposition is useful only with respect to a finite sequence.

More on the nonpositivity of the Szegő’s solutions:

When $A$ and $B > 0$, we know that $x_n \to 0$ and $\xi_n$ has a bounded limit, hence (27) “looks like”

$$(n + 1)x_{n+1} = 2n\cos(\pi\alpha)x_n - (n - 1)x_{n-1},$$
and we expect a $\sin(n\pi\alpha)$ behaviour as seen before. What is true is that in

$$\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = A_n \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix},$$

the matrix $A_n$ is close to $A_\infty := \begin{bmatrix} 0 & 1 \\ -1 & 2\cos\pi\alpha \end{bmatrix}$ for large $n$, so that, for a given $p$,

$$\begin{bmatrix} x_{n+p} \\ x_{n+p+1} \end{bmatrix} = \begin{bmatrix} A_{n+p} & A_{n+p-1} & \cdots & A_{n+1} \\ 0 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix},$$

with $A_{n+p}A_{n+p-1}\cdots A_{n+1}$ close to $A_\infty^p = \begin{bmatrix} 1 & \sin(p\pi\alpha) \\ -\sin(p\pi\alpha) & \sin(p+1)\pi\alpha \end{bmatrix}$, so that $x_{n+p}$ is close to $C_n\sin(p\pi\alpha + \xi_n)$.

We shall need a finite, non infinitesimal, version of the Proposition:

5.1.5. Difference of two solutions. Proposition. If $x'_1, x'_2, \ldots$ and $x''_1, x''_2, \ldots$ are two solutions of (27), with $0 < x'_i < 1$ and $0 < x''_i < 1$ for $i = 1, 2, \ldots, n$, then $x'_i = x''_i$ or all the $x''_i - x'_i$ have the same sign for $i = 1, 2, \ldots, n$.
Indeed, let us write (28) as

\[(i+1)x_{i+1} = C_i x_i - (i-1)x_{i-1}, i = 1, 2, \ldots \]

where \( C_i = 2x_1 + x_2 x_1 + \cdots + x_i x_{i-1} + i \cos \pi \alpha. \)

Suppose \( x''_i > x'_i \), and that \( x''_j > x'_j \) for \( j = 1, 2, \ldots, i. \)

\[
(i+1)[x''_{i+1} - x'_{i+1}] = C''_i [x''_i - x'_i] + [C''_i - C'_i] x'_i - (i-1)[x''_{i-1} - x'_{i-1}]
\]

\[
> C''_i [x''_i - x'_i] - (i-1)[x''_{i-1} - x'_{i-1}] \quad \text{as} \quad C''_i > C'_i
\]

\[
> (i+1)x''_{i+1} - (i-1)[x''_{i-1} - x'_{i-1}] \quad \text{from (28)}
\]

and \( x''_{i+1} > x'_{i+1} \) follows.

As we still do not know if the \( \Phi_n(0)'s \) are all positive, we shall try to build a positive solution of (27).

CHICO: Go to the house next door.

GROUCHO: That’s great. Suppose there isn’t any house next door.

CHICO: Well, then of course we gotta build one!

from Animal Crackers

5.1.6. \textit{It seems that a smart idea} is to look at (27) as a relation between positive sequences. We use (27) to extract \( x_n \) through the only positive root of

\[
\frac{x^{-1}_n - x_n}{2} = \frac{x_1 x_0 + \cdots + x_n x_{n-1} + n \cos \pi \alpha}{(n+1)x_{n+1} + (n-1)x_{n-1}}, n = 1, 2, \ldots, \ x_0 = 1,
\]

recomputing new estimates of \( x_1, x_2, \ldots, x_N \) from old ones in the right-hand side of (29), with \( x_0 = 1 \) and \( x_N = \lim \sin(\pi \alpha/2) \). The first estimate of the sequence \( \{x_n\} \) may simply be \( x_n = \sin(\pi \alpha/2) \) for all \( n > 0 \). Remark that, as a bonus, \( x_n < 1 \) if all the \( x_i's \) on the right-hand side are positive.

It works!! Here is a test with \( \alpha = 0.3 \), applying first (27) directly with \( x_1 = \sin(0.3 \pi)/(1 - 0.3 \pi) = 0.367883098 \ldots \) and using several iterations of (29) with \( x_n = \sin(0.3 \pi/2) = 0.453990607 \ldots, n = 1, 2, \ldots \) as starting sequence:

\[
\]

\[
/*
reflecjg.gp : launch gp and make \r reflecjg

Reflection coefficients for Delsarte et al. and Grunbaum problem
*/
default(realprecision,50);
flone=1.0;pr=precision(flone);ep=10^-pr;
default(format,"g1.6");
N=20;
for(ia=3,3,alpha=0.1*ia;print1(alpha," ");ca=cos(alpha*Pi);
 \ \ vector of reflection coeff.
ref=vector(N+1,k,0);
ref[1]=1;ref[2]=sin(alpha*Pi)/(Pi*(1-alpha));
xi=ref[2];
for(n=1,N-1,

4considered by J. Adamson, in Groucho, Harpo, Chico and sometimes Zeppo, Simon & Schuster 1973 = Pocket Book 1976, as an overestimated sample of the Marx Brothers humor. Perhaps a better instance is the much repeated scene where a handshake produces a flood of knives falling from Harpo sleeves... when one should have expected mere spoons.
print1(ref[n+1]," ");
ref[n+2]= ( ref[n+1]*(2*xi+2*n*ca)/(1-(ref[n+1])**2) 
- (n-1)*ref[n] )/(n+1);
xi=xi+ref[n+1]*ref[n+2] 
);
print(" ");

sa=sin(Pi*alpha/2); print(" sin pi alpha/2= ",sa,
\ par iteration de suites
newref=vector(N+1,k,0);
for(n=2,N,ref[n]=sa);ref[1]=1;
for(ns=1,99, newref[1]=1; residm=0; xi=0; print(ns," ");
for(n=2,N-1,print1(ref[n]," ");
xi=xi+ref[n-1]*ref[n];
resid=
ref[n+1]-(ref[n]*(2*xi+2*(n-1)*ca)/(1-(ref[n])**2) 
- (n-2)*ref[n-1] )/n ;
residm=max(residm,abs(resid));
newnref[n]=
(xi+(n-1)*ca) / ( n*ref[n+1]+(n-2)*ref[n-1] ) ;
newnref[n]= 1/( sqrt( 1+newnref[n]**2 )+newnref[n] ) ;
} ; print(" ",residm);
for(n=2,N-1,ref[n]=newnref[n]);
}
);

Script Vi.1 session started Thu Dec 19 11:38:53 2002
C:\calc\pari>gp-2-1
GP/PARI CALCULATOR Version 2.1.0 (released)
i686 running Windows 3.2 (i86 kernel) 32-bit version
(readline v4.0 enabled, extended help not available)

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comes WITHOUT ANY WARRANTY WHATSOEVER.

? \r reflectjc
alpha= 0.3000000 sin pi alpha/2= 0.453990607

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<th>\Phi_3(0)</th>
<th>\Phi_4(0)</th>
<th>\Phi_5(0)</th>
<th>\Phi_6(0)</th>
<th>\Phi_7(0)</th>
<th>\Phi_8(0)</th>
<th>\Phi_9(0)</th>
</tr>
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<td>0.406603001</td>
<td>0.427202438</td>
<td>0.438032120</td>
<td>0.443863125</td>
<td>0.447146049</td>
<td>0.449098425</td>
<td>0.450327477</td>
<td>0.451143664</td>
</tr>
</tbody>
</table>

<table>
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<td>0.450327477</td>
<td>0.451143664</td>
</tr>
</tbody>
</table>

...
Hmm, we should now show that (29) leads to a contraction of positive sequences, so to allow a proof of a unique fixed point. We can see (29) as

\[ x = F(x), \]

acting on positive sequences \( x = \{x_1, x_2, \ldots \} \), with \( F_n(x) = \sqrt{A_n(x)^2 + 1} - A_n(x) \), and where \( A_n(x) \) is the right-hand side of (29).

We estimate the contraction in the \( \| \cdot \|_\infty \) norm by looking at the sum of all the \( |\partial F_n/\partial x_i| \). For \( i = 1, 2, \ldots, n-2 \) and \( i = n \), one finds \( \frac{\partial A_n}{\partial x_i} = \frac{x_{i-1} + x_{i+1}}{(n+1)x_{n+1} + (n-1)x_{n-1}} \). For \( i = n \pm 1 \), one must subtract \((n \pm 1)|((n+1)x_{n+1} + (n-1)x_{n-1})|^2\), where “num” is the numerator of the right-hand side of (29). The sum of absolute values is bounded by

\[
1 + 2x_1 + 2x_2 + \cdots + 2x_{n-2} + x_{n-1} + x_n + n(F_n^{-1} - F_n),
\]

having introduced \( F_n^{-1} - F_n \) from the left-hand side of (29). For the derivatives of \( F_n \), we multiply by \( dF_n/dA_n = -F_n/\sqrt{A_n^2 + 1} = -2F_n/(F_n^{-1} + F_n) \). The sum of the \( |\partial F_n/\partial x_i| \) is therefore bounded by

\[
2F_n \left( \frac{1 + 2x_1 + 2x_2 + \cdots + 2x_{n-2} + x_{n-1} + x_n + n(F_n^{-1} - F_n)}{(F_n^{-1} + F_n)((n+1)x_{n+1} + (n-1)x_{n-1})} \right).
\]

For large \( n \), when most of the \( x_i \)'s and \( F_n \) itself are close to the limit \( \sigma \), we find asymptotically \( 2\sigma \frac{2n\sigma + n(\sigma^{-1} - \sigma)}{(\sigma^{-1} + \sigma)2\sigma} \), which is...1!

If the positive sequence \( x = \{x_n\} \) has a limit \( \sigma \) with \( \sigma^2 + \cos \pi \alpha > 0 \), then \( A_n(x) \to \frac{\sigma^2 + \cos \pi \alpha}{2\sigma} \), and

\[ F_n(x) \to f(\sigma) := \sqrt{\frac{\sigma}{2} + \frac{\cos \pi \alpha}{2\sigma} \right)^2 + 1 - \frac{\sigma}{2} - \frac{\cos \pi \alpha}{2\sigma}, \]

which has \( \sin(\pi \alpha/2) \) as attractive fixed point, as \( f'(\sigma) = \frac{3\cos \pi \alpha - 1}{3 - \cos \pi \alpha} \) at the fixed point \( \sigma = \sin(\pi \alpha/2) \).

\[ \begin{array}{c}
\cos \pi \alpha < 0 \\
\sigma
\end{array} \quad \begin{array}{c}
0 < \cos \pi \alpha < 1/3 \\
\sigma
\end{array} \quad \begin{array}{c}
1/3 < \cos \pi \alpha < 1 \\
\sigma
\end{array} \]

However, contraction cannot be established on the set of all the positive sequences. A convenient subset must be found. Let us try the already suspected set of the increasing positive sequences. Does \( F \) transforms an increasing sequence into an increasing sequence? Alas no, some iterated sequences may show decreasing episodes:

alpha= 0.4000  sin pi alpha/2= 0.587785261  cos pi alpha= 0.309016998
x1  x23 x24 x25 x26 x27 x28 x29 x30
5.1.7. Anticlimax. Moreover, some intermediate \( x_i \)'s are larger than the limit \( \sigma = \sin \pi \alpha /2 \), whereas all the final values appear to be \(< \sigma \). Finally, asymptotic behaviour of intermediate \( x_n \)'s is not as expected:

\[
F_n(x) = \frac{\xi_n + n \cos \pi \alpha}{2} = \frac{\sigma + (n-1)\sigma^2 + n \cos \pi \alpha}{2n\sigma} \sim \frac{1 - \sigma}{2} + \frac{1}{2n}, \text{ whence } F_n(x) = \sigma + O(n^{-1}), \text{ whereas } O(n^{-2}) \text{ is expected.}
\]

Obviously, we have a problem with \( \xi_n = x_1x_0 + \cdots + x_nx_{n-1} \) which is numerically found to be \( n\sigma^2 + O(n^{-1}) \), a feature which is impossible to catch.

5.1.8. More smart ideas. Aha! In order to be sure of the sought feature for \( \xi_n \), I intend to compute \( x_1, x_2, \ldots, x_N \) with boundary values \( x_0 = 1 \) and \( x_{N+1} = \sigma \), and where \( \xi_n \) is approximated by \( N\sigma^2 - x_{n+1}x_n - x_{n+2}x_{n+1} - \cdots - x_Nx_{N-1} \), which has the expected right behaviour.

So, the iteration is \( x = F(x) \), with

\[
F_n(x) = \frac{\xi_n + n \cos \pi \alpha}{2} = \frac{\sigma + (n-1)\sigma^2 + n \cos \pi \alpha}{2n\sigma} \sim \frac{1 - \sigma}{2} + \frac{1}{2n}.
\]

for \( n = 1, 2, \ldots, N \), with \( x_0 = 1, x_{N+1} = \sigma \).

Let us try a run, I can’t wait:

```plaintext
/*
 grunb2.gp : launch gp and make \r grunb2

Reflection coefficients for Grunbaum- Delsarte et al. problem

*/

default(realprecision,75);
flone=1.0;pr=precision(flone);ep=10^{-pr};
default(format,"g1.7");
N=50;
a=0.2;
print1(a," "); sa=sin(Pi*a/2);
print(" \sigma=sin pi a/2= ",sa);
ca=cos(a*Pi);
print(" \cos pi a= ",ca);
\ vector of reflection coeff.
ref=vector(N+2,k,0);
newref=vector(N+2,k,0);
print(" \ iteration");
for(n=2,N+2,ref[n]=sa);ref[1]=1;
for(ns=1,5, newref[1]=1; xi=N*sa^2; print(ns," ");
for(n=1,N, newref[n]=\xi_n; xi=x(N+1-ni); n=N+1-ni;```
MAPA3xxxA 2002-03 – Semi-classical orth. pol. on unit circle. 5 – Grunbaum problem.

\begin{align*}
\text{An} & = \left( x_i + n^* c_a \right) / \left( (n+1)^* \text{ref}[n+1] + (n-1)^* \text{ref}[n] \right) ; \\
\text{newref}[n+1] & = 1/\sqrt{1 + \text{An}^2} \times \text{An} ; \\
\text{xi} & = \text{xi} - \text{ref}[n+1] \times \text{ref}[n] ; \\
\text{print} & (" \text{final xi=}", \text{xi}) ; \\
\text{resid} & = 0 ; \text{for} (n=2, N+1, \text{print1(\text{ref}[n]," ")}) ; \\
\text{resid} & = \text{max(\text{resid,abs(newref}[n]-\text{ref}[n])}; \text{\text{ref}[n]=newref}[n]) ; \\
\text{print} & (" \text{resid=}", \text{resid}) ; \\
\end{align*}

... 

This is ridiculous! Indeed, \( x_n = \sigma \) for all \( n \geq 1 \) is a solution of (30). The mistake was to input the wrong value \( \xi_N = N \sigma^2 \) and, even as the error is small when \( N \) is large, it induced a big error on \( \xi_0 = 0 \).

I now try \( \xi_N = N \sigma^2 + \varepsilon \), with a small value of \( \varepsilon \):

\[
F_n(x) = 2 \frac{\xi_n + n \cos \pi \alpha}{(n+1)x_{n+1} + (n-1)x_{n-1}}
\]

for \( n = 1, 2, \ldots, N \), with \( x_0 = 1 \), and \( x_{N+1} \) given in \((0, \sigma]\).

... 

\begin{align*}
\text{epsN} & = 0.01 ; \\
\text{for} & (n=2, N+2, \text{ref}[n]=\text{sa}); \text{ref}[1]=1 ; \\
\text{for} & (n=1, 99, \text{newref}[1]=1; \text{xi}=N \times \text{sa}^2 + \text{epsN}); \text{\text{\text{print}\{ns," ")}} ; \\
\end{align*}

... 

Script V1.1 session started Mon Jan 13 10:15:37 2003

C:\calc\pari>gp-2-1

<table>
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<tr>
<th>iteration</th>
<th>x1</th>
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<th>x3</th>
<th>x4</th>
<th>x5</th>
<th>x6</th>
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<td>0.309016998</td>
<td>0.309016998</td>
<td>0.309016998</td>
<td>0.309016998</td>
<td>0.309016998</td>
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</table>

\begin{align*}
\text{final xi} & = -0.213525497
\end{align*}

Script completed Mon Jan 13 10:25:40 2003
Now, we get a slightly better estimate, and we try other values of \( \epsilon \), trying to have a vanishing \( \xi_0 \). As the iteration has become so slow, only the 50\(^{th} \) and the 75\(^{th} \) are shown:

### Script V1.1 session started Mon Jan 13 10:51:51 2003

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<th>it.</th>
<th>( x_0 )</th>
<th>( x_1 )</th>
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<th>( x_3 )</th>
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Script completed Mon Jan 13 11:00:21 2003

Still further detail, keeping now 100\(^{th} \) and the 200\(^{th} \) iterations:

### Script V1.1 session started Mon Jan 13 12:05:34 2003

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<th>res.</th>
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<tr>
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<td>100</td>
<td>0.0003769518</td>
<td>0.011763364</td>
<td>0.230001143</td>
<td>0.255153580</td>
<td>0.272376095</td>
<td>0.283848606</td>
<td>0.291140979</td>
<td>0.296405822</td>
<td>0.300974</td>
<td></td>
</tr>
<tr>
<td>0.0050</td>
<td>100</td>
<td>0.000386006</td>
<td>0.015604763</td>
<td>0.228701077</td>
<td>0.254186601</td>
<td>0.271680013</td>
<td>0.283535050</td>
<td>0.291056790</td>
<td>0.296148811</td>
<td>0.299550</td>
<td></td>
</tr>
</tbody>
</table>

Scripts completed Mon Jan 13 13:54:13 2003

? quit  Good bye! C:\calc\pari>exit
where “res.” is the residue norm \( \max_n |(x_n^{-1} - x_n)/2 - A_n(x)| \). It seems indeed that we get the correct sequence when \( \xi_0 = 0 \), if one reminds that one should find

\[
x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10}
\]

\[
0.233872324 \quad 0.258015427 \quad 0.274428042 \quad 0.285306081 \quad 0.292452074 \quad 0.297160853 \quad 0.300299896 \quad 0.302429553 \quad 0.303905700 \quad 0.304953308
\]

\[
0.200000002 \quad 0.200000002 \quad \sigma = \sin \pi \alpha/2 = 0.309016998 \quad \cos \pi \alpha = 0.809017006
\]

The iteration based on (31) is numerically rather poor, but has very interesting monotony properties:

5.1.9. **Lemma.** For any \( \alpha \in (0, 1) \), the map \( F \) acting on \( \mathbb{R}^N \) through (31) has the properties:

1. For any \( \varepsilon \geq 0 \), a positive sequence bounded by \( \sigma = \sin \pi \alpha/2 \) is transformed by \( F \) in a sequence with the same properties.
2. If \( x \) is a positive sequence bounded by \( \sigma \), and if \( F(x) \leq x \), then \( F(F(x)) \leq F(x) \), where \( x \leq y \) means \( x_n \leq y_n, n = 1, 2, \ldots, N \).
3. \( \xi_0 \) is an increasing function of \( \varepsilon \) when one starts (31) with the constant sequence \( x_n = \sigma, n = 1, \ldots, N \).
4. \( \xi_0 = \sigma^2 - \sigma < 0 \) if \( \varepsilon = 0 \) and \( x_{N+1} = \sigma; \xi_0 \geq \varepsilon + \sigma^2 - \sigma \). There is therefore exactly one \( \varepsilon > 0 \) such that \( \xi_0 = 0 \) if \( x_{N+1} = \sigma \).

Indeed,

1. If all the \( x_n \)'s are positive and \( \leq \sigma = \sin \pi \alpha/2 \), the numerator of \( A_n \) in (31) is \( \geq n\sigma^2 + n\cos \pi \alpha = n\cos^2(\pi \alpha/2) \), so that \( A_n \geq \cos^2(\pi \alpha/2)/(2\sin \pi \alpha/2) = (\sigma^{-1} - \sigma)/2 \), and 0 \( < F_n(x) \leq \sigma \).
2. We show \( y \leq x \Rightarrow F(y) \leq F(x) \) on positive sequences such that \( A_n \) is positive. Indeed: \( A_n(y) \geq \sigma \).
3. At each iteration, \( x'_n \geq x''_n \) if \( x'_n \) and \( x''_n \) correspond to \( \varepsilon' \) and \( \varepsilon'' \), with \( \varepsilon' < \varepsilon'' \). This is true for the first step; if true at some iteration, \( A'_n \leq A''_n \), and \( x'_n \geq x''_n \) again at the next iteration. Finally, \( \xi_0 = N\sigma^2 + \varepsilon - x_1 - x_2 - \cdots - x_N = \sigma^2 - \sigma < 0 \) at \( \varepsilon = 0 \) (as all the \( x_i \)'s = \( \sigma \)); 2) when \( \varepsilon \) is large, the \( x_n \)'s are very small, so that \( \xi_0 \) is close to \( N\sigma^2 + \varepsilon \).

That there is always an \( \varepsilon_N > 0 \) ensuring \( \xi_0 = 0 \) follows from 1) that \( \xi_0 = N\sigma^2 - x_1 - x_2 - \cdots - x_{N-1} x_N = \sigma^2 - \sigma < 0 \) at \( \varepsilon = 0 \) (as all the \( x_i \)'s = \( \sigma \)); 2) when \( \varepsilon \) is large, the \( x_n \)'s are very small, so that \( \xi_0 \) is close to \( N\sigma^2 + \varepsilon \).

See here a computer run with \( N = 50 \) and \( \sigma = 0.309 \), so that \( \xi_0 \) is expected near 4.774 + \( \varepsilon \):

Script V1.1 session started Wed Jan 15 13:51:32 2003

\[
\begin{align*}
\text{epsN=} & \quad 1 \\
\text{epsN=} & \quad 2 \\
\text{epsN=} & \quad 3
\end{align*}
\]
5.1.10. **Theorem.** For any $\alpha \in (0, 1)$, the positive solution -known to be unique- of \( (27) \) with the sole boundary condition $x_0 = 1$ is the limit (in the $\ell_\infty$ norm) when $N \to \infty$ of the unique positive solution of the system of $N$ equations from $\Phi_i(x)$ with $n = 1, 2, \ldots, N$, where $x_{N+1}, x_{N+2}, \ldots$ are replaced by $\sigma = \sin(\pi \alpha/2)$. One also has $\Phi_n(0) < \sigma = \sin(\pi \alpha/2)$, $n = 1, 2, \ldots$

\[ \Phi_1(0) + \Phi_2(0) + \cdots + \Phi_n(0) \Phi_{n-1}(0) > n \sigma^2, \quad n = 1, 2, \ldots \]

Of course, all this means that the sequence $\{\Phi_n(0)\}$ is positive.

I would be more at ease with a lower bound of $\xi_0 = 0$: then, $x_1 + x_2 x_3 + x_2 x_3 + \cdots + x_{n-1} x_n > n \sigma^2$ and all the $x_n$'s $< \sigma \Rightarrow x_1 > \sigma^2$.

Now, the part “limit when $N \to \infty$”:

**The step $N \to N+1$.**

To show: $x_i^{(N+1)} < x_i^{(N)}$, $i = 1, 2, \ldots, N+1$.

\[ \{x_1^{(N+1)}, \ldots, x_N^{(N+1)}\} \]

is the unique positive solution of the $N$ first equations of $\Phi_i(x)$ when $x_0 = 1$ and $x_{N+1} = \sigma$ are given.

Let $\{x_1^{(N+1)}, \ldots, x_N^{(N+1)}\}$, $i = 1, \ldots, N+1$, be as above with $x_{N+2} = \sigma$. We know that $x_{N+1}^{(N+1)} < \sigma$, as all the other $x_i^{(N+1)}$ for $i = 1, \ldots, N$.

A first point is that $x_i^{(N+1)}$ for $i = 1, 2, \ldots, N$ is another positive solution of the $N$ first equations of $\Phi_i(x)$, with the boundary conditions $x_0^{(N+1)} = 1$ and $x_{N+1}^{(N+1)} < \sigma$.

Then, $x_i^{(N+1)} < x_i^{(N)}$ for $i = 1, 2, \ldots, N+1$ from Proposition 5.1.5, p. 36.

5.1.11. **Extended conjecture.** Another generalized Jacobi weight giving rise to real $\Phi_n(0)$'s is

\[ 2^{\beta} \left| \sin \frac{\theta - \theta_1}{2} \right|^{2\beta} \left| \sin \frac{\theta + \theta_1}{2} \right| = (\cos \theta_1 - \cos \theta)^{2\beta} \] on the same arc $\theta_1 < \theta < 2\pi - \theta_1$ as above.

Then we apply $\Phi_i$ with $\alpha = \beta$ (the $\alpha$ of $\Phi_i$, of course), and $\theta_1 = \pi \alpha$:

\[ (n + 1 + 2\beta)x_n = 2\xi_n\xi_{n-1} + 2n \cos \pi \alpha \sum_{i=1}^{n-1} x_i^2 - (n + 1 + 2\beta)x_{n-1}, \tag{32} \]

with $\xi_n = x_1 x_2 + x_2 x_3 + \cdots + x_n x_{n-1}$.

Everything should follow as previously, at least when $\beta \geq 0$. If $-1/2 < \beta < 0$, $x_1$ may be negative. Of course: when $\alpha \to 0$, $x_n = \Phi_n(0) \to \frac{2\beta}{n + 2\beta + 1}$, from the Jacobi case $\Phi_1$. Check that it is a solution of $\Phi_2$, and that $\xi_n \to \frac{2n}{n + 2\beta}$.

The algorithm based on $\Phi_i$ still works, but monotonicity is lost if $\beta < 0$.

---

C: \calc\pari>gp-2-1
?
$\text{r grunb2}$
beta= -0.4000
$a=0.200$ sigma=sin pi a/2= 0.309016998 cos pi a= 0.809017006

epsN it. res. x10 x1 x2 x3 x4 x5 x6 x7
-0.7300 1 1.870135 -0.943525519 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998 0.309016998
6. Generalized Jacobi polynomials on the unit circle, with > 2 singular points.

7. References.


2002-03 – Semi-classical orth. pol. on unit circle. 7 – References. –


[34] M. Vanlessen, Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight, rep. KULeuven, 2002.

