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An Electrostatics Approach to the Determination of Extremal Measures

JEAN MEINGUET
Université Catholique de Louvain, Institut Mathématique, Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgium. e-mail: meinguet@anma.ucl.ac.be

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Abstract. One of the most important aspects of the minimal energy (or induced equilibrium) problem in the presence of an external field – sometimes referred to as the Gauss variation problem – is the determination of the support of its solution (the so-called extremal measure associated with the field). A simple electrostatic interpretation is presented here, which is apparently new and anyway suggests a novel, rather systematic approach to the solution. By way of illustration, the classical results for Jacobi, Laguerre and Freud weights are explicitly recovered by this alternative method.

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1. Introduction

Mathematicians (and physicists!) generally ‘know’ Dirichlet’s principle. They are likely less familiar with the related W. Thomson (Lord Kelvin) principle (in electrostatics) and its special case called the *Gauss variation problem* (or forced equilibrium problem), which is the problem of minimizing – in the presence of a given external field – the ‘energy’ associated with any sourceless (or solenoidal) vector field in the outer region bounded by a given closed set (the so-called ‘conductor’, supposed once for all to be ‘perfect’) over which a positive (electric) charge of prescribed amount is to be distributed so as to reach equilibrium (see, e.g., [5], pp. 43–44, 55–57, or [2], pp. 46, 51).

As a matter of fact, the underlying potential theory needed in the following is the theory of *logarithmic potentials with external fields,* whose interaction with approximation-theoretical techniques and problems in the complex plane or on the real line proved extremely fruitful in recent years. As is well known, a point charge in the plane is ‘equivalent’ to a uniformly distributed charge on a straight line – perpendicular to the plane – in $\mathbf{R}^3$, such (positive or negative) point charges repelling or attracting each other according to an inverse distance law (well known consequence of Coulomb's law). Gauss's variation problem then becomes that of minimizing the (weighted) *energy integral*

$$I_Q(\mu) := \int_\Sigma \int_\Sigma \log \frac{1}{|z-t|} \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(t) + 2\int_\Sigma Q\,\mathrm{d}\mu, \tag{1}$$

where the minimum is taken over all positive unit charge distributions (i.e., positive unit Borel measures) μ carried by the conductor Σ (i.e., $\mathrm{supp}(\mu) \subseteq \Sigma$) while Q (defined on Σ and real-valued) is the so-called external field (strictly speaking, such a scalar 'field' Q is a potential). It is known – see, e.g., [6], pp. 26–33, for the basic theorem and its detailed mathematical proof – that, under rather weak conditions of 'admissibility' on Q, there exists a unique solution μ_Q (called *equilibrium* or *extremal measure* associated with Q) of this optimization problem, which is such that the relation

$$U^\mu(z) := \int_\Sigma \log \frac{1}{|z-t|} \,\mathrm{d}\mu(t) = -Q(z) + F_Q, \quad z \in S_Q, \tag{2}$$

holds quasi-everywhere (i.e., possibly up to a set of zero logarithmic capacity), where $S_Q := \mathrm{supp}(\mu_Q)$ is compact of positive capacity and F_Q is the so-called *modified Robin constant* for Q. It should be stressed that a most glaring difference with the classical equilibrium problem (for which $Q = 0$) is that S_Q need not coincide with the outer boundary of Σ and, in fact, can be an arbitrary subset of Σ, possibly with positive area.

Determining S_Q is therefore one of the most important aspects of the energy problem (or minimization of (1)). To find the extremal measure, it then remains to solve Dirichlet problems (for the Laplace equation and the essential boundary conditions (2)) and to launch the classical recovery machinery (e.g., the Sokhotskyi–Plemelj formula for arcs and its integrated version known as the Stieltjes–Perron inversion formula of Cauchy transforms). As discovered by Mhaskar–Saff in the eighties, determining S_Q amounts to minimizing over the set of possible supports the (quasi-everywhere) constant value F_Q of the extremal potential. It is surprising that such an obviously hard problem can be solved explicitly under suitable convexity assumptions (satisfied by the important weights $w := \mathrm{e}^{-Q}$ of Jacobi, Laguerre, and Freud), S_Q being then an interval whose endpoints can be obtained by solving a (simple) integral equation.

The main goal of this paper is to present (in Section 2) a novel, rather systematic approach to the determination of S_Q. This mathematical method can be regarded as a modern example of 'physical mathematics' in the sense of Sommerfeld; it is indeed motivated by an apparently new electrostatic interpretation. By applying this alternative method, we will rediscover rather automatically the 'classical results' mentioned above (see Sections 3, 4 and 5).

2. A Physically Oriented Approach

By way of constructive illustration, we will consider here the simple, two-dimensional physical picture: a (perfect) conductor in vacuum, say the finite segment $\Sigma := [-1, 1]$ in the extended complex z-plane $\overline{\mathbf{C}}$, is subjected to the electrostatic field of potential

$$Q(z) := \lambda \log \frac{1}{|z-a|}, \quad z = x + iy, \tag{3}$$

due to an electric charge $\lambda > 0$ located at an exterior point, say $a > 1$.

• Suppose first that the conductor Σ is *grounded* (i.e., connected to earth), which means that it may acquire whatever charges are necessary to enable it to remain at the same potential (zero, by convention). The resulting potential thus created (that is, (3) in the presence of the grounded Σ) is classically – up to the proportionality factor λ – the Green function of the complement of Σ (the so-called cut plane) with pole at a, viz.,

$$g(z, a) = \log \left| \frac{1 - \overline{\phi(a)}\phi(z)}{\phi(z) - \phi(a)} \right|, \quad \phi(z) := z + \sqrt{z^2 - 1} \tag{4}$$

(see, e.g., [6], p. 110). It should be noted once for all that any expression like $\sqrt{z^2-1}$ is to be understood as the branch that behaves like z near infinity, so that $w = \phi(z)$ is simply the inverse of the well-known Joukowski conformal map $z = (1/2)(w + 1/w)$ of the exterior of the unit disk (in the ϕ-plane) onto the complement of Σ. It follows in particular that the circle (in the ϕ-plane):

$$\phi(z) = \phi(a)\mathrm{e}^{i\theta}, \qquad -\pi \leqslant \theta \leqslant \pi,$$

corresponds to the ellipse (in the z-plane):

$$z = a\cos\theta + i\sqrt{a^2-1}\sin\theta,$$

with foci at $z = \pm 1$, and semiaxes a, $\sqrt{a^2-1}$, whose polar representation can be written in the form

$$\rho = \frac{a^2-1}{a + \cos\Theta} = a - \cos\theta, \tag{5}$$

where ρ denotes the distance to the pole (of polar coordinates) $z = 1$, Θ is the 'true anomaly' and θ is the 'eccentric anomaly' (these terms are borrowed from celestial mechanics).

The distribution μ of the charge that is induced (by electrostatic *influence*) on the grounded conductor Σ by the point charge $\lambda > 0$ at $a > 1$ or, equivalently, $-\lambda$ times the so-called balayage measure of the Dirac point mass at a onto Σ (see, e.g., [6], pp. 81–82), is given by

$$\begin{aligned} \mathrm{d}\mu(x) &:= -\frac{\lambda}{\pi}\frac{\partial}{\partial n} g(x, a)\,\mathrm{d}x \\ &= -\frac{\lambda}{\pi}\left| \frac{\sqrt{a^2-1}}{(a-x)\sqrt{1-x^2}} \right| \mathrm{d}x, \quad x \in [-1, 1], \end{aligned} \tag{6a}$$

where dx is the *arc length on* Σ, and $\partial/\partial n$ denotes differentiation in the direction of the inner normal with respect to the complement of Σ (as a matter of fact, for obvious symmetry reasons, n may denote here either the upper or the lower normal). The concrete expression on the right in (6a) is most important for the following; it is found by taking the limit of the real part of $(-i\lambda/\pi)$ times the derivative of the analytic function

$$\log\frac{1-\phi(a)\phi(z)}{\phi(z)-\phi(a)}$$

as z tends to $x\in(-1,1)$ from the upper half-plane, while keeping in mind that (for continuation reasons) $\sqrt{1-x^2}$ is positive for $y=0+$ (resp. negative for $y=0-$) and that $\sqrt{a^2-1}$ is positive for $a>1$ (but negative for $a<-1$, see Section 3). With the change of variable $x=\cos\theta$, (6a) takes the simpler form

$$\mathrm{d}\mu(\cos\theta)=-\frac{\lambda}{2\pi}\left|\frac{\sqrt{a^2-1}}{a-\cos\theta}\right|\mathrm{d}\theta,\quad \theta\in[-\pi,\pi],\tag{6b}$$

where dθ denotes – throughout the whole paper – *arc measure on the unit circle*; in view of (5), the corresponding density (or *Radon–Nikodym derivative*) $\mathrm{d}\mu(\cos\theta)/\mathrm{d}\theta$ of the induced charge has a nice geometric interpretation. As is classically expected (see, e.g., [4], p. 230), for any grounded conductor occupying a bounded region in the presence of a point charge, the density of the induced charge will never change sign; more precisely, the total mass of the distribution μ is $-\lambda$ (this can be verified by explicit integration), while

$$\min_\theta\frac{\mathrm{d}\mu(\cos\theta)}{\mathrm{d}\theta}=-\frac{\lambda}{2\pi}\sqrt{\frac{a+1}{a-1}},\tag{7}$$

this minimal value being attained for $\theta=0$.

• Suppose now that the conductor Σ is *insulated* (i.e., imbedded in vacuum). If a positive unit charge is placed on it in the absence of any external field, then its equilibrium distribution μ_0 (i.e., the unique positive unit Borel measure minimizing the energy integral (1) where $Q=0$) is known to be the arcsine distribution

$$\mathrm{d}\mu_0(x)=\frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x,\quad x\in[-1,1],\tag{8a}$$

or, equivalently,

$$\mathrm{d}\mu_0(\cos\theta)=\frac{1}{2\pi}\,\mathrm{d}\theta,\quad \theta\in[-\pi,\pi],\tag{8b}$$

that is, the normalized arc measure. The constant value F_0 assumed by its (logarithmic) potential on Σ (the so-called *Robin constant*) is clearly

$$F_0:=\log\frac{1}{\mathrm{cap}(\Sigma)}=\log 2,\tag{9}$$

the logarithmic capacity $\mathrm{cap}(\Sigma)$ of a finite segment being notoriously equal to one-fourth its length.

• Suppose finally that the conductor Σ is *insulated in the field of potential* (3). In view of (7), it is clear that

$$\mathrm{d}\mu_Q := \mathrm{d}\mu + \lambda\sqrt{\frac{a+1}{a-1}}\,\mathrm{d}\mu_0 = \frac{\lambda}{2\pi}\left(-\frac{\sqrt{a^2-1}}{a-\cos\theta} + \sqrt{\frac{a+1}{a-1}}\right)\mathrm{d}\theta \tag{10}$$

is the unique (nonnegative and absolutely continuous with respect to θ) equilibrium distribution of charges over Σ that minimizes its potential; indeed, the definition (10) amounts simply to adding to the signed measure $\mathrm{d}\mu$ (whose logarithmic potential plus the external field has the constant value 0 on Σ) the smallest multiple $C\,\mathrm{d}\mu_0$ of the positive measure $\mathrm{d}\mu_0$ (whose logarithmic potential on Σ has the constant value F_0) that makes the resulting measure $\mathrm{d}\mu_Q$ nonnegative, its logarithmic potential on Σ, namely, the constant CF_0 with $C := \lambda\sqrt{(a+1)/(a-1)}$, being therefore as small as possible. Provided that

$$\lambda\left(-1 + \sqrt{\frac{a+1}{a-1}}\right) = 1, \tag{11}$$

which simply means that *the total mass of* (10) *over its support* Σ *is* 1 (or equivalently, that the charge placed on Σ is $\lambda + 1$), the distribution μ_Q is nothing but the extremal measure minimizing (1) for Q defined by (3) (after all, the potential in electricity and magnetism is identical with potential energy per unit charge, see, e.g., [4], p. 53). The (logarithmic) potential F_Q of Σ corresponding to the distribution (10) 'normalized' by (11) – that is, the modified Robin constant for Q (see [6], p. 27) – is thus clearly

$$F_Q = (\lambda + 1)\log 2. \tag{12}$$

As is easily verified (by an elementary computation detailed in [6], p. 46), the potentials (9) and (12) satisfy the important relation

$$F_Q = F_0 + \int_\Sigma Q\,\mathrm{d}\mu_0 \tag{13}$$

according to which $-F_Q$ is the so-called 'F-functional' of Mhaskar–Saff (see [6], Chap. IV) whose maximization (over the set of possible supports) is achieved by the support S_Q of the extremal measure μ_Q; an alternative proof of (13) follows from the successive relations implied by

$$Q(x) + U^\mu(x) = 0, \quad x \in [-1, 1], \int_\Sigma \mathrm{d}\mu = -\lambda,$$

where U^μ denotes the logarithmic potential of the induced measure (6a), viz.:

$$\begin{aligned}\int_\Sigma Q\,\mathrm{d}\mu_0 &= -\int_\Sigma U^\mu\,\mathrm{d}\mu_0 = -\int_\Sigma U^{\mu_0}\,\mathrm{d}\mu = -F_0\int_\Sigma \mathrm{d}\mu\\ &= \lambda F_0 = (1+\lambda)F_0 - F_0 = F_Q - F_0\end{aligned}$$

(the only nontrivial equality sign is the second one, which is justified by the Fubini–Tonelli Theorem).

• It is just a simple exercise to rewrite our results about the extremal measure in a more familiar form (see, e.g., [7], p. 773). As a matter of fact, we have only to change the interval $[-1, a]$ into $[-1, 1]$ via the affine transformation

$$\xi = \frac{2x + (1-a)}{1+a} \tag{14}$$

without modifying the ratio of the fixed charge of amount $\lambda > 0$ at the point $x = a > 1$ (resp. $\vartheta \in (0, 1)$ at $\xi = 1$) to the continuous charge of amount 1 (resp. $1 - \vartheta$) to be distributed on $[-1, 1]$ (resp. on its image by (14)) so as to reach equilibrium. The latter condition, viz.,

$$\lambda = \frac{\vartheta}{1-\vartheta}, \tag{15}$$

combined with the normalization relation (11), yields

$$a = \frac{1+\vartheta^2}{1-\vartheta^2},$$

so that the actual support of the continuous charge $1 - \vartheta$ on the ξ-axis (i.e., the image of $[-1, 1]$ by (14)) is

$$S_Q = [-1, \xi_0], \quad \xi_0 := \frac{3-a}{1+a} = 1 - 2\vartheta^2. \tag{16}$$

As to the distribution of this charge on S_Q, it readily follows from (10) – always normalized by (11) – by the affine transformation (14), owing to formulas (15), (16). It turns out that the associated Jacobian has a remarkable form, viz.,

$$\frac{d\xi}{d\theta} = \frac{1}{2}\sqrt{(\xi+1)(\xi_0-\xi)}, \quad -1 \leqslant \xi \leqslant \xi_0,$$

where the factor $1/2$ is due to the fact that we must integrate twice along cuts if we integrate once over the unit circle. Hence, the final result

$$d\mu_Q(\xi) = \frac{1}{\pi(1-\vartheta)} \frac{\sqrt{(\xi+1)(\xi_0-\xi)}}{1-\xi^2}\, d\xi, \quad -1 \leqslant \xi \leqslant \xi_0, \tag{17}$$

which concludes our alternative treatment of the simplest example of explicit determination of an extremal measure that is considered in [6] (see pp. 205–206, 243), that is, the application entitled 'Incomplete Polynomials of Lorentz' (note, however, that our ξ is to be identified with $-t$ in the last formula of Example 5.3 on p. 243 in [6]).

3. The Extremal Measure for Jacobi Weights

This quite natural generalization of the physical picture considered in Section 2 corresponds to the replacement of (3) by the electrostatic field of potential

$$Q(z) := \lambda_1 \log \frac{1}{|z-a_1|} + \lambda_2 \log \frac{1}{|z-a_2|}, \tag{18}$$

the electric charge $\lambda_1 > 0$ (resp. $\lambda_2 > 0$) being located at a point outside the conductor $\Sigma := [-1, 1]$, say $a_1 > 1$ (resp. $a_2 < -1$).

• Since the complement of Σ possesses an explicitly known Green function, namely (4), the potential of the total field created by the charges in (18) and the countercharges induced by influence on the conductor Σ supposed to be grounded is classically given by the associated *Green potential*, viz.,

$$V(z) := \lambda_1 g(z, a_1) + \lambda_2 g(z, a_2) \tag{19}$$

(see, e.g., [6], p. 124). In view of (6a), (6b) and (19), the distribution μ of the charge that is induced by (18) on the grounded conductor Σ is given by

$$\begin{aligned} \mathrm{d}\mu(\cos\theta) &= -\frac{1}{2\pi}\left(\lambda_1 \frac{\sqrt{a_1^2-1}}{a_1-\cos\theta} + \lambda_2 \frac{\left|\sqrt{a_2^2-1}\right|}{|a_2|+\cos\theta}\right)\mathrm{d}\theta, \\ &\theta \in [-\pi, \pi]. \end{aligned} \tag{20}$$

The total mass of this distribution is clearly $-\lambda_1 - \lambda_2$, while

$$C := -2\pi \min_\theta \frac{\mathrm{d}\mu(\cos\theta)}{\mathrm{d}\theta} = -2\pi \min\left(\frac{\mathrm{d}\mu}{\mathrm{d}\theta}(-1), \frac{\mathrm{d}\mu}{\mathrm{d}\theta}(1)\right) \tag{21}$$

immediately follows from the convexity with respect to the variable $\cos\theta$ of the parenthesized expression in (20) (its second derivative is indeed positive over $[-1, 1]$).

• Suppose now that the conductor Σ is insulated in the field of potential (18). It is clear that

$$\mathrm{d}\mu_Q := \mathrm{d}\mu + C\,\mathrm{d}\mu_0, \quad \text{with definitions (8b), (20) and (21),} \tag{22a}$$

is the unique (nonnegative and absolutely continuous with respect to θ) equilibrium distribution of charges over Σ that minimizes its potential. However, to have a chance to solve eventually the underlying Gauss variation problem or, equivalently, to minimize the potential F_Q of Σ corresponding to the extremal measure μ_Q (of total mass 1!) in the presence of the external field of potential (18), the points $a_1 > 1$ and $a_2 < -1$ must be such that the minimal value (21) is as great as possible. This requires of the two expressions on the right in (21) to be equal (they vary indeed in opposite directions as either $a_1 > 1$ or $a_2 < -1$ varies), their common value being necessarily

$$C = 1 + \lambda_1 + \lambda_2 \tag{22b}$$

(since the total mass of (22a) over Σ must be 1, while the total mass of (20) is $-\lambda_1 - \lambda_2$). In other words, the following equations must be satisfied:

$$\vartheta_1\sqrt{\frac{a_1-1}{a_1+1}} + \vartheta_2\sqrt{\frac{|a_2|+1}{|a_2|-1}} = 1, \tag{23a}$$

$$\vartheta_1\sqrt{\frac{a_1+1}{a_1-1}} + \vartheta_2\sqrt{\frac{|a_2|-1}{|a_2|+1}} = 1, \tag{23b}$$

where

$$\vartheta_1 := \frac{\lambda_1}{1+\lambda_1+\lambda_2}, \qquad \vartheta_2 := \frac{\lambda_2}{1+\lambda_1+\lambda_2}. \tag{23c}$$

This is equivalent to the apparently simpler nonlinear system for a_1, a_2:

$$\vartheta_1 = \frac{\sqrt{a_1^2-1}}{a_1-a_2}, \qquad \vartheta_2 = \frac{\left|\sqrt{a_2^2-1}\right|}{a_1-a_2}, \quad a_1 > 1, a_2 < -1, \tag{24a}$$

whose (unique) solution is

$$a_1 = \frac{1+\vartheta_1^2-\vartheta_2^2}{\sqrt{\Delta}}, \qquad a_2 = -\frac{1+\vartheta_2^2-\vartheta_1^2}{\sqrt{\Delta}}, \tag{24b}$$

where

$$\Delta := [1-(\vartheta_1+\vartheta_2)^2][1-(\vartheta_1-\vartheta_2)^2], \tag{24c}$$

as it can be shown by somewhat lengthy (though elementary) computations.

• To rewrite the extremal measure in a more familiar form (see, e.g., [7], pp. 772–774), it remains only to change the interval $[a_2, a_1]$ into $[-1, 1]$ via the affine transformation

$$\xi = \frac{2x-(a_1+a_2)}{a_1-a_2} \tag{25}$$

without modifying the ratios of the fixed charge of amount $\lambda_1 > 0$ at $x = a_1 > 1$ (resp. $\vartheta_1 \in (0, 1)$ at $\xi = 1$) and of the fixed charge of amount $\lambda_2 > 0$ at $x = a_2 < -1$ (resp. $\vartheta_2 \in (0, 1)$ at $\xi = -1$) to the continuous charge of amount 1 (resp. $1 - \vartheta_1 - \vartheta_2$) to be distributed on $[-1, 1]$ (resp. on its image by (25)) so as to reach equilibrium; in fact, the conditions

$$\lambda_1 = \frac{\vartheta_1}{1-\vartheta_1-\vartheta_2}, \qquad \lambda_2 = \frac{\vartheta_2}{1-\vartheta_1-\vartheta_2} \quad \text{and}$$
$$1+\lambda_1+\lambda_2 = \frac{1}{1-\vartheta_1-\vartheta_2} \tag{26}$$

are trivially equivalent to (23c). It follows that the actual support of the continuous charge $1 - \vartheta_1 - \vartheta_2$ on the ξ-axis (i.e., the image of $[-1, 1]$ by (25)) is

$$S_Q = [\xi_2, \xi_1], \tag{27a}$$

where

$$\xi_1 = \vartheta_2^2 - \vartheta_1^2 + \sqrt{\Delta}, \quad \xi_2 = \vartheta_2^2 - \vartheta_1^2 - \sqrt{\Delta}, \text{ with definition (24c).} \tag{27b}$$

As to the distribution of this continuous charge on S_Q, it follows from (22) by the affine transformation (25) – owing to formulas (23c), (24), (26), (27) – the final result being

$$\mathrm{d}\mu_Q(\xi) = \frac{1}{\pi(1 - \vartheta_1 - \vartheta_2)} \frac{\sqrt{(\xi - \xi_2)(\xi_1 - \xi)}}{1 - \xi^2}\, \mathrm{d}\xi, \quad \xi_2 \leqslant \xi \leqslant \xi_1, \tag{28}$$

in accordance with, e.g., [6] (see pp. 207 and 241).

4. The Extremal Measure for Laguerre Weights

A crucial step in the approach presented in this paper is the determination of the electrostatic potential outside the grounded conductor Σ (i.e., any given compact set of $\mathbf{C}$, of positive capacity) in the presence of the given external field. In the applications considered so far, this fundamental *influence* problem could be solved readily owing to the explicit knowledge of the Green function (of the outer domain relative to Σ). On the other hand, in the remaining applications, where the external field is defined directly (at least in part) rather than via given external charges, this crucial step actually requires solving explicitly a *Dirichlet boundary value problem.*

In the Laguerre case, the external field has for potential

$$Q(z) := \lambda\, \Re z + s \log \frac{1}{|z - a|}, \quad \lambda > 0, s \geqslant 0, a < -1. \tag{29}$$

Unlike the second term, which is of the type considered before (i.e., potential of a charge $s \geqslant 0$ located at a given point $a < -1$), the first term is not created by a charge but rather by a dipole at infinity (of axis $0x$ and of moment λ); though this 'physical' interpretation may prove interesting (see, e.g., [1], p. 35), we will not exploit it here, essentially because it does not hold for non-uniform fields such as the one considered in Section 5.

• Now suppose that the conductor $\Sigma := [-1, 1]$ is grounded and subjected to the field of potential (29). The potential of the total field thus created outside Σ is clearly the sum of three terms: the Green potential of the charge s located at the point a (i.e., s times the Green function (4), where $\phi(a) := a - |\sqrt{a^2 - 1}|$ since $a < -1$), the external field of potential λx, and the solution $h(z)$ of the *exterior*

Dirichlet problem:

$$\Delta h(z) = 0, \quad z \notin [-1, 1],$$
$$h(z) \text{ bounded as } |z| \to \infty \quad \text{(i.e., regularity at infinity)},$$
$$h(x) = -\lambda x, \quad x \in [-1, 1].$$

It turns out that the conformal transplant of h under the Joukowski mapping, viz.,

$$H(w) := h\left(\frac{1}{2}\left(w + \frac{1}{w}\right)\right), \quad w = |w|e^{i\theta} \text{ with } |w| \geqslant 1, \theta \in [-\pi, \pi] \tag{30}$$

can be obtained readily by separating the variables in the transplanted exterior Dirichlet problem:

$$\Delta H(w) = 0, \quad |w| \geqslant 1, \tag{31a}$$
$$H(w) \text{ bounded as } |w| \to \infty, \tag{31b}$$
$$H(e^{i\theta}) = -\lambda \cos\theta, \quad \theta \in [-\pi, \pi]. \tag{31c}$$

Indeed, if we transform ΔH to polar coordinates $|w|, \theta$, we get for the solution $H(w)$ of (31a) the general form

$$A_0 + B_0 \log|w| + \sum_{k \neq 0} \frac{A_k \cos k\theta + B_k \sin k\theta}{|w|^k};$$

now the condition (31b) of regularity at infinity (see, e.g., [4], p. 248) implies $B_0 = 0$ and $A_k = B_k = 0$ for all negative integers k; the Dirichlet condition (31c) thus reduces to

$$\sum_{k=0}^{\infty} (A_k \cos k\theta + B_k \sin k\theta) = -\lambda \cos\theta, \quad \theta \in [-\pi, \pi],$$

which finally yields

$$A_1 = -\lambda, \qquad B_1 = 0, \qquad A_k = B_k = 0 \quad \text{for } k \neq 1.$$

In view of (30), the required potential of the total field created outside the grounded conductor Σ by the external field of potential (29) is given by

$$V(z) = s \log\left|\frac{1 - \phi(a)\phi(z)}{\phi(z) - \phi(a)}\right| + \lambda\, \Re z - \lambda\, \Re \frac{1}{\phi(z)}. \tag{32}$$

According to the classical definition

$$d\mu(x) := -\frac{1}{\pi} \frac{\partial}{\partial n} V(x)\, dx, \quad x \in [-1, 1],$$

we get from (32) the explicit expression

$$d\mu(\cos\theta) = -\frac{1}{2\pi}\left(\frac{s|\sqrt{a^2 - 1}|}{|a| + \cos\theta} + \lambda \cos\theta\right) d\theta, \quad \theta \in [-\pi, \pi], \tag{33}$$

for the distribution μ of the charge induced by (29) on the grounded conductor Σ. The remarkable relation (21) holds again (the parenthesized function of $\cos\theta$ in (33) is indeed convex over $[-1, 1]$), so that we get explicitly

$$C = \max\left(s\sqrt{\frac{|a|+1}{|a|-1}} - \lambda, s\sqrt{\frac{|a|-1}{|a|+1}} + \lambda\right). \tag{34}$$

• Suppose now that the conductor Σ is insulated in the field of potential (29). It is clear that

$$d\mu_Q := d\mu + C\,d\mu_0, \quad \text{with definitions (8b), (33) and (34),} \tag{35}$$

is the unique (nonnegative and absolutely continuous with respect to θ) equilibrium distribution of charges over Σ that minimizes its potential, for any given values of the parameters $\lambda > 0, s \geqslant 0, a < -1$. It turns out that a further minimization of this potential is automatically achieved if the point $a < -1$ is such that the two expressions on the right in (34) are equal (they vary indeed in opposite directions as a varies); owing to this condition, which amounts to $s = \lambda|\sqrt{a^2-1}|$, (34) reduces to $C = \lambda|a| = \sqrt{\lambda^2+s^2}$. But the total mass of (35) over Σ must be 1, while the total mass of (33) is $-s$, so that necessarily $C = s+1$; all these relations finally imply

$$a = -\frac{s+1}{\sqrt{2s+1}}, \tag{36}$$

$$\lambda = \sqrt{2s+1}. \tag{37}$$

• To rewrite the extremal measure in a more familiar form (see, e.g., [6], pp. 208 and 243), it remains only to change the interval $[a, 1]$ into $[0, \xi_1]$ (where ξ_1 is any finite positive number) via the affine transformation

$$\xi = \xi_1 \frac{x-a}{1-a} \tag{38}$$

without modifying the ratio of the fixed charge of amount $s > 0$ at the point $x = a$ defined by (36) (resp. $\vartheta \in (0, 1)$ at $\xi = 0$) to the continuous charge of amount 1 (resp. $1-\vartheta$) to be distributed on $[-1, 1]$ (resp. its image S_Q by (38)) so as to reach equilibrium. It follows that the actual support of the continuous charge

$$1 - \vartheta := \frac{1}{s+1} \tag{39}$$

on the ξ-axis is

$$S_Q = [\xi_2, \xi_1], \tag{40a}$$

where

$$\frac{\xi_2}{s+1-\sqrt{2s+1}} = \frac{\xi_1}{s+1+\sqrt{2s+1}} =: \frac{1}{\Lambda}. \tag{40b}$$

It should be noticed that, up to an additive (unimportant!) constant, the potential (29) on the positive ξ-axis has the simple expression $\Lambda\xi - s\log\xi$, which depends on two independent parameters $\Lambda > 0$, $s \geqslant 0$ (remember that λ was eliminated by (37), for the sake of normalization). As to the distribution of the continuous charge (39) on S_Q, it readily follows from (35) – normalized by (36), (37) – by the affine transformation (38) and formulas (39), (40), the final result being

$$\mathrm{d}\mu_Q(\xi) = \frac{\Lambda}{\pi}\frac{\sqrt{(\xi-\xi_2)(\xi_1-\xi)}}{\xi}\,\mathrm{d}\xi, \quad \xi_2 \leqslant \xi \leqslant \xi_1, \tag{41}$$

in accordance with [6] (see p. 243).

5. The Extremal Measure for Freud Weights

The external field – to which the standard conductor $\Sigma := [-1, 1]$ is subjected – has now for potential

$$Q(z) := c|x|^\lambda, \quad c > 0, \lambda > 0 \text{ (and } x := \Re z). \tag{42}$$

Unlike the ‘physical’ fields considered before, it is thus directly defined by its mathematical expression rather than via given external electric charges (or dipoles).

• If the conductor Σ is grounded, the potential of the total field thus created outside Σ is naturally obtained by adding to (42) the solution $h(z)$ of the exterior Dirichlet problem:

$$\Delta h(z) = 0, \quad z \notin [-1, 1],$$
$$h(z) \text{ bounded as } |z| \to \infty \quad \text{(i.e., regularity at infinity)},$$
$$h(x) = -c|x|^\lambda, \quad x \in [-1, 1].$$

Here again, the conformal transplant H of h under the Joukowski mapping, which is the function defined by (30), can be obtained readily by separating the variables in the transplanted exterior Dirichlet problem:

$$\Delta H(w) = 0, \quad |w| \geqslant 1, \tag{43a}$$
$$H(w) \text{ bounded as } |w| \to \infty, \tag{43b}$$
$$H(\mathrm{e}^{i\theta}) = -c|\cos\theta\,|^\lambda, \quad \theta \in [-\pi, \pi]. \tag{43c}$$

Indeed, if we transform ΔH to polar coordinates $|w|, \theta$, we get for any solution of (43a, b) the Fourier series representation

$$H(w) = -c\sum_{k=0}^{\infty}{}' A_{2k}\frac{\cos 2k\theta}{|w|^{2k}},$$

the Dirichlet boundary condition (43c) reducing to

$$\sum_{k=0}^{\infty}{}' A_{2k}\cos 2k\theta = |\cos\theta\,|^\lambda, \quad \theta \in [-\pi, \pi] \tag{44a}$$

(where the prime affecting the summation symbol means that the first term is to be taken with half weight), or equivalently, to

$$A_{2k} := \frac{4}{\pi}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\lambda}\cos 2k\theta\,\mathrm{d}\theta = \frac{1}{2^{\lambda-1}}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda/2+k+1)\Gamma(\lambda/2-k+1)} \tag{44b}$$

(see, e.g., [8], p. 263, Example 40). Owing to the reflection formula of Euler for the gamma function, this expression of A_{2k} can be rewritten in the form

$$A_{2k} = \frac{\Gamma(\lambda+1)\sin(\pi\lambda/2)}{2^{\lambda-1}\pi}(-1)^{k+1}\frac{\Gamma(k-\lambda/2)}{\Gamma(k+\lambda/2)}\frac{1}{k+\lambda/2}, \tag{44c}$$

which yields (via Stirling's formula) the asymptotic formula

$$A_{2k} \sim \frac{\Gamma(\lambda+1)\sin(\pi\lambda/2)}{2^{\lambda-1}\pi}(-1)^{k+1}\frac{1}{k^{\lambda+1}} \quad \text{as } k\to\infty; \tag{44d}$$

Weierstrass's test is thus applicable, so that the Fourier series (44a) of $|\cos\theta|^{\lambda}$ converges uniformly and absolutely to its generating function.

Since the potential $V(z)$ of the total field created outside the grounded conductor Σ by the external field of potential (42) has for conformal transplant (under the Joukowski mapping)

$$\mathcal{V}(w) = -c\sum_{k=0}^{\infty}{}' A_{2k}\frac{\cos 2k\theta}{|w|^{2k}} + \frac{c}{2^{\lambda}}\left(|w|+\frac{1}{|w|}\right)^{\lambda}|\cos\theta|^{\lambda}, \quad |w|\geqslant 1,$$

the distribution μ of the charge induced on Σ is apparently given by

$$\mathrm{d}\mu(\cos\theta) = -\frac{1}{2\pi}\lim_{|w|\to 1+}\frac{\partial\mathcal{V}(w)}{\partial|w|}\,\mathrm{d}\theta = -\frac{c}{\pi}\sum_{k=1}^{\infty} kA_{2k}\cos 2k\theta\,\mathrm{d}\theta. \tag{45}$$

The total mass of μ is evidently 0 (since the lines of force of the field of potential (42) are parallel to the x-axis), while

$$C := -2\pi\min_{\theta}\frac{\mathrm{d}\mu(\cos\theta)}{\mathrm{d}\theta} = 2c\sum_{k=1}^{\infty} kA_{2k}, \tag{46}$$

this minimal value being attained for $\theta = 0 \bmod \pi$ – were it simply for 'physical' reasons (logical interpretation of the underlying problem of electrostatic influence) – whereas

$$\max_{\theta}\frac{\mathrm{d}\mu(\cos\theta)}{\mathrm{d}\theta} = -\frac{c}{\pi}\sum_{k=1}^{\infty}(-1)^k kA_{2k} > 0$$

is attained for $\theta = \pi/2 \bmod \pi$ and is finite or not according as $\lambda > 1$ or not (this follows from the properties of A_{2k} mentioned above).

It should be stressed that the trigonometric series in (45) is actually an *Abel sum;* however, by virtue of classical tests (substantially due to Abel) exploiting the properties (44c), (44d) of the A_{2k}'s, this series is convergent (except for $\theta = \pm\pi/2 \bmod 2\pi$, whenever $\lambda \leqslant 1$), necessarily to its Abel sum. It turns out that the sum of the series in (46) can be found as an Abel sum by an explicit (but lengthy) computation, the final result being

$$\sum_{k=1}^{\infty} kA_{2k} = \frac{\lambda}{\pi}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\lambda}\,\mathrm{d}\theta = \frac{\Gamma(\lambda/2+1/2)}{\Gamma(\lambda/2)\Gamma(1/2)}; \tag{47}$$

the last expression is simply (44b) for $k = 0$, rewritten by means of Legendre's duplication formula (see, e.g., [8], p. 240). Rather than give complementary details, we deem it preferable to describe briefly an alternative approach to (47), based on the modern theory of *generalized functions* or *distributions.* Consider the classical Fourier series expansion

$$\sum_{k=1}^{\infty} \frac{\cos k\theta}{k} = -\log|2\sin(\theta/2)| \tag{48}$$

whose generating function goes out of bound at $\theta = 0 \bmod 2\pi$, while being integrable in the Lebesgue sense over the fundamental period interval $(-\pi, \pi)$. It is easily proved (see [3], p. 30) that the Fourier series in (48) converges in the sense of generalized functions to the function on the right-hand side, so that it may be differentiated term-by-term (in the distributional sense) any number of times, which yields in particular the distributional result:

$$\sum_{k=1}^{\infty} k\cos\theta = (\log|2\sin(\theta/2)|)''; \tag{49a}$$

by techniques that are standard in the theory of distributions (see, e.g., [3], p. 65, for similar results), we are led to concrete definitions of the second distributional derivative – denoted by the symbol $''$ – in (49a), viz.,

$$\begin{aligned}
&\langle(\log|2\sin(\theta/2)|)'', \Phi(\theta)\rangle\\
&\quad= -\int_0^{\pi} \frac{\cos(\theta/2)}{2\sin(\theta/2)}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(\theta) + \frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(-\theta)\right]\mathrm{d}\theta\\
&\quad= -\int_0^{\pi} \frac{1}{4\sin^2(\theta/2)}[\Phi(\theta) + \Phi(-\theta) - 2\Phi(0)]\,\mathrm{d}\theta,
\end{aligned} \tag{49b}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket between the dual topological vector spaces of periodic test functions $\Phi(\theta)$ (i.e., infinitely differentiable functions of period 2π) and periodic distributions (of period 2π). The result (49b) can be extended by continuity to the function $|\cos(\theta/2)|^{\lambda}$, which indeed can be regarded as the limit of a uniformly convergent sequence of test functions; it is readily verified that the values taken on this function by the two forms of the accordingly extended distribution in (49b) are simply π times the first two expressions in (47), which identity is again rigorously established.

• Suppose now that the conductor Σ is insulated in the field of potential (42). It is clear that

$$\mathrm{d}\mu_Q := \mathrm{d}\mu + C\,\mathrm{d}\mu_0, \quad \text{with definitions (8b), (45) and (46),} \tag{50}$$

is the unique (nonnegative and absolutely continuous with respect to θ) equilibrium distribution of charges over Σ that minimizes its potential, for any given values of the parameters $c > 0, \lambda > 0$. But the total mass of (50) over Σ must be 1, while the total mass of (45) is 0, so that necessarily

$$C = 1, \quad \text{or equivalently,} \quad 1/c = 2\sum_{k=1}^{\infty} kA_{2k}. \tag{51}$$

• To rewrite these results in a more familiar form (see [6], pp. 204 and 238), it remains only to change the interval $[-1, 1]$ into

$$S_Q = [-a, a], \quad a > 0, \tag{52a}$$

via the linear substitution $\xi = ax$. S_Q is the support of the extremal measure μ_Q relative to the external potential

$$\gamma\,|\xi|^{\lambda}, \quad \gamma > 0, \tag{52b}$$

if and only if

$$a = \gamma^{-1/\lambda} c^{1/\lambda}, \quad \text{where } c := \frac{\sqrt{\pi}\,\Gamma(\lambda/2)}{2\,\Gamma(\lambda/2+1/2)}, \tag{52c}$$

as it follows from (51) in view of (47).

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