‘1/9’, summary & afterthoughts.

Nov. 2005

Asymptotic convergence rates of rational interpolation to exponential functions.

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1.1. Algebra of rational interpolation, orthogonality.

Rational interpolant p/q of degrees m/n of f at $m+n+1$ points $z_0, \ldots, z_{m+n} \in E$: if f analytic in a domain containing E,

$$q(z)f(z) - p(z) = \frac{1}{2\pi i}\int_{C_n} \frac{(z-z_0)\cdots(z-z_{m+n})}{(t-z_0)\cdots(t-z_{m+n})}\frac{q(t)f(t)}{t-z}\,dt \qquad \text{(Hermite, Walsh)}, \qquad (1)$$

where C_n is a contour containing $z_0, \ldots, z_{m+n}$, and z in its interior. This leaves a numerator

$$p(z) = \frac{1}{2\pi i}\int_{C_n} \frac{(t-z_0)\cdots(t-z_{m+n}) - (z-z_0)\cdots(z-z_{m+n})}{t-z}\frac{f(t)}{(t-z_0)\cdots(t-z_{m+n})}q(t)\,dt$$

which is of degree $m+n$ unless q is 'orthogonal' with respect to the 'weight' $w_n(t) := f(t)/[(t-z_0)\cdots(t-z_{m+n})]$: $\displaystyle\int_{C_n} q(t)t^k w_n(t)\,dt = 0$, for $k = 0, \cdots, n-1$. Indeed, the big polynomial in z and t above contains terms $t^a z^b$, with $a+b \leqslant m+n$. If only the t^a with $a \geqslant n$ have to be considered, then only z^b with $b \leqslant m$ are left.

Using orthogonality, (1) is left unchanged when one susbtracts from $(t-z)^{-1}$ its interpolant at the zeros of q, so that $\dfrac{q(t)}{(t-z)q(z)}$ is left, and

$$f(z) - \frac{p(z)}{q(z)} = \frac{1}{2\pi i}\int_{C_n} \frac{(z-z_0)\cdots(z-z_{m+n})q^2(t)}{(t-z_0)\cdots(t-z_{m+n})q^2(z)}\frac{f(t)}{t-z}\,dt. \qquad (2)$$

Remark also that the 'scalar' product of two functions u and v is

$$\langle u, v\rangle = \frac{1}{2\pi i}\int_{C_n} \frac{u(t)v(t)}{(t-z_0)\cdots(t-z_{m+n})}f(t)\,dt \qquad (3)$$

is the ***divided difference*** of uvf at $z_0, \ldots, z_{m+n}$.

1.2. Distributions of interpolation points, poles, and their potentials.

Let $\mu_{n,i}$ and $\mu_{n,p}$ the distributions of interpolations points on E, and poles on C_n, with unit total weight, i.e., such that

$$\int_E F(t)\,d\mu_{n,i}(t) = \frac{1}{m+n+1}\sum_{k=0}^{m+n} F(z_k), \qquad \int_{C_n} F(t)\,d\mu_{n,p}(t) = \frac{1}{n}\sum_{k=1}^{n} F(p_k).$$

These distributions can be seen as staircase functions, but they will receive smoother approximations.

(Complex) logarithmic potentials: $\mathcal{V}(z) = \int \log(z-t)\,d\mu(t)$. Then,

$$f(z) - \frac{p(z)}{q(z)} = \frac{1}{2\pi i}\exp((m+n+1)\mathcal{V}_{n,i}(z) - 2n\mathcal{V}_{n,p}(z))\int_{C_n} \exp(2n\mathcal{V}_{n,p}(t)) - (m+n+1)\mathcal{V}_{n,i}(t))\frac{f(t)}{t-z}\,dt. \qquad (4)$$

When $m \sim n$, everything depends on $\mathcal{V}_n(z) := \mathcal{V}_{n,i}(z) - \mathcal{V}_{n,p}(z)$.

Let c_n be the largest absolute value of $f(t)\exp(-2n\mathcal{V}_n(t))$ on C_n, then the error bound of (**??**) is dominated by

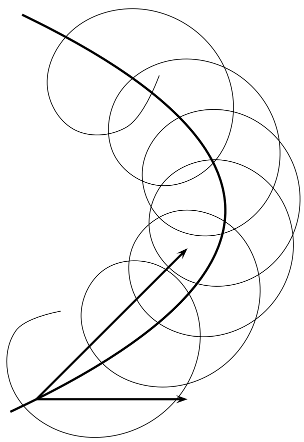
$$\begin{cases} c_n \exp(2n\,\mathrm{Re}\mathcal{V}_n(z)) \text{ if } z \text{ is inside } C_n, \\ \max\,[c_n \exp(2n\,\mathrm{Re}\mathcal{V}_n(z)), |f(z)|] \text{ if } z \text{ is outside } C_n, \end{cases} \qquad (5)$$

as one must take into account the residue at $t = z$ in the latter case.

$\mu_{n,i}$ and $\mathcal{V}_{n,i}$ are known if one interpolates on a given set of points. $\mathcal{V}_{n,p}$ has to be determined from a theory of orthogonal polynomials.

Simple example: interpolation concentrated on a single point (Padé), say 0, and f analytic outside the real interval $[a,b]$, $0 \notin [a,b]$. Then C_n may be deformed up to $[a,b]$ used twice, first with limit values from above $f_+(t)$ of f, and next with $-f_-(t)$. If $f_+ - f_-$ has a constant phase and basically an integrable logarithm, then $\mathcal{V}_{n,p}(z)$ outside $[a,b]$ is close to $\log[d(z-c) + (1-d)\sqrt{(z-a)(z-b)}]$, where $c = 2/(a^{-1}+b^{-1})$ and $d = 1/(1+|c|/\sqrt{ab})$ (Szegő[1]). Then, $\mathcal{V}_n(z)$ is close to $-\log[d(1-c/z) + (1-d)\sqrt{(1-a/z)(1-b/z)}]$, whose real part is the constant $\log[(b+a)/(d(b-a))]$ on $[a,b]$, and is less than this constant everywhere else the square root must be taken accordingly). Also, $c_n \sim [d(b-a)/(b+a)]^{2n}$. For small z, the error behaves like $\left[\dfrac{d(b-a)}{b+a}\dfrac{z}{2dc}\right]^{2n} = \left[\dfrac{(b-a)z}{4ab}\right]^{2n}$ ($(b-a)/(4ab)$ is the logarithmic capacity of $[b^{-1}, a^{-1}]$).

1.3. **Orthogonal polynomials behaviour.**



Artist's (?) view of a typical $w_n q^2$ along its support.

What we'd like to see.

Remark that, as c_n is the maximum of $|f \exp(-2n\mathcal{V}_n)|$ on a contour C_n which may be deformed, we can as well look for the contour yielding the smallest maximum: the smallest extimate will be the most realistic. The point where the maximum occurs (actually, it will be a whole subarc) is a saddle-point of $|f \exp(-2n\mathcal{V}_n = w_n q^2|$. This is exactly what happens with true orthogonal polynomials with respect to positive measures, for the L^2 norm on C_n, and we have a theory giving $\mathcal{V}_{n,p}$ in that case (Szegő, Widom). So, for a given C_n, the true (monic) orthogonal polynomal has the smallest possible L^2 norm on C_n, $w_n q^2$ has often an almost constant absolute value ("envelope") on a subarc Δ_n, but probably a fast varying phase there. As a consequence, the integrals we need, involving $w_n q^2$, without the absolute value, will be much smaller than L^2 norms, and we will not get valuable estimates.

Suppose that, among all the possible C_n's, a miraculous one is such that $w_n q^2$ happens to have only a slowly varying phase on the subarc where the absolute value is close to its maximum c_n. Then, $q\sqrt{w_n}$ is almost real, we do not need complex conjugation in the scalar products any more, and the kind of orthogonal polynomials needed in rational interpolation look like righteous L^2-orthogonal polynomials.

1.3.1. *L^2-orthogonal polynomials.* Szegő-Widom theory: $w_n q^2$ behaves essentially outside $\Delta_n \bigcup E$ as $c_n \Phi^{2n}$, where Φ maps the exterior of Δ_n on $|\Phi| > 1$ and behaves near E as dictated by $c_n \Phi^{2n} = f \exp(-2n\mathcal{V}_{n,i} + 2n\mathcal{V}_{n,p})$.

For instance, if all the interpolation points are concentrated on z_0, $c_n^{1/(2n)}\Phi$ must have a pole with unit residue at z_0. The unit residue allows to compute c_n. So, in the example above, $\Phi(z) = [2ab/z - a - b + 2\sqrt{ab(1-a/z)(1-b/z)}]/(b-a)$ maps

[1] Actually, $t^n q(1/t)$ is orthogonal with respect to a fixed weight on $[b^{-1}, a^{-1}]$, so that $z^{-n}q(z)$ involves mainly the logarithm of $z^{-1} - (a^{-1}+b^{-1})/2 + \sqrt{(z^{-1}-a^{-1})(z^{-1}-b^{-1})}$, which is the "usual" potential related to an interval.

indeed[2] the exterior of $[a,b]$ on $|\Phi|>1$, and has a pole at $z=0$ with residue $4ab/(b-a)$, whence $c_n=[(b-a)/(4ab)]^{2n}$ as already found.

Remark also that $\log\Phi$ is the complex Green function of Δ_n with a singularity at a given point.

"Essentially" means that only n^{th} powers are considered for the moment.

If the interpolation points are spread on an arc E with a known distribution, $\log\Phi$ is a sum of Green fuctions

$$\log\Phi(z)=\frac{1}{2n}\sum_{k=0}^{2n}\log\Phi(z;z_i)\to\int_E\log\Phi(z;t)\,d\mu_{n,i}(t). \tag{6}$$

$$\frac{1}{2n}\log c_n\to-\int_E\log(\text{res.}\,\Phi(z;t))\,d\mu_{n,i}(t). \tag{7}$$

In the example above,

$\Phi(z;t)=\dfrac{(2t-a-b)(z-t)+2(t-a)(t-b)+2\sqrt{(t-a)(t-b)(z-a)(z-b)}}{(b-a)(z-t)}$, with the square root such that $|\Phi|>1$ outside $[a,b]$ (there can be no doubt: the other possibility is $1/\Phi$). The residue at the pole $z=t$ is $4(t-a)(t-b)/(b-a)$.

The $z-$derivative is $\dfrac{d\Phi(z;t)/dz}{\Phi(z;t)}=-\dfrac{1}{z-t}\sqrt{\dfrac{(t-a)(t-b)}{(z-a)(z-b)}}$.

Let E be another interval $[c,d]$ with a uniform distribution $d\mu_{n,i}(t)=dt/(d-c)$ (here is where point distributions are replaced by easier smooth distributions). So we have $\log\Phi(z)=\int_c^d\log\Phi(z;t)\,dt/(d-c)$, with the eerie $\Phi(z;t)$ just above. But use

$\Phi(z;t)+\dfrac{1}{\Phi(z;t)}=2\dfrac{(2t-a-b)(z-t)+2(t-a)(t-b)}{(b-a)(z-t)}=2\dfrac{a+b-2z}{b-a}+4\dfrac{(z-a)(z-b)}{(b-a)(z-t)}$, so

$t=t(\Phi)=z-\dfrac{4(z-a)(z-b)}{(b-a)(\Phi+\Phi^{-1})-2(a+b)+4z}$, with poles $\Phi=\Phi^{\pm1}(z,\infty)$ and residues $\pm\Phi^{\pm1}\sqrt{(z-a)(z-b)}$, and

$$\begin{aligned}\log\Phi(z)&=\frac{1}{d-c}\int_c^d\log\Phi(z;t)\,dt=\frac{1}{d-c}\int_{\Phi(z;c)}^{\Phi(z;d)}\log\Phi\,dt(\Phi)\\&=\frac{d\log\Phi(z;d)-c\log\Phi(z;c)}{d-c}-\frac{1}{d-c}\int_{\Phi(z;c)}^{\Phi(z;d)}\frac{t(\Phi)}{\Phi}\,d\Phi\end{aligned}$$

turning as $\dfrac{(d-z)\log\Phi(z;d)-(c-z)\log\Phi(z;c)}{d-c}$

$-\dfrac{\sqrt{(z-a)(z-b)}}{d-c}\log\dfrac{(\Phi(z;d)-\Phi(z;\infty))(\Phi(z;c)-1/\Phi(z;\infty))}{(\Phi(z;d)-1/\Phi(z;\infty))(\Phi(z;c)-\Phi(z;\infty))}$ Awful. From the derivative above:

$$\frac{\Phi'(z)}{\Phi(z)}=-\frac{1}{d-c}\int_c^d\frac{1}{z-t}\sqrt{\frac{(t-a)(t-b)}{(z-a)(z-b)}}\,dt$$

A more interesting interpolation points distribution is the Chebyshev distribution on $[c,d]$: $d\mu_{n,i}(t)=\pi^{-1}[(t-c)(d-t)]^{-1/2}\,dt$. Then, $\log\Phi(z)$ is the constant term of the Chebyshev expansion of $\log\Phi(z;t)$. The $z-$derivative is

$$\frac{\Phi'(z)}{\Phi(z)}=-\frac{1}{\pi\sqrt{(z-a)(z-b)}}\int_c^d\frac{1}{z-t}\sqrt{\frac{(t-a)(t-b)}{(t-c)(d-t)}}\,dt,$$

a typical complete elliptic integral of the third kind. "Simple examples" do not seem much easier than the "big" example. But suppose that $[c,d]$ is far from $[a,b]$. Then, for z near $[c,d]$, Φ'/Φ is

[2]See that $\Phi+1/\Phi=2(2ab/z-a-b)$.

not far from $-\pi^{-1}\int_c^d (z-t)^{-1}[(t-c)(d-t)]^{-1/2}dt = -[(z-c)(z-d)]^{-1/2}$, which is pure imaginary on $[c,d]$, so the rational interpolant is close to the best rational approximation. And the error norm, from (7), is about $c_n \approx \left[\dfrac{b-a}{(c+d-2a)(c+d-2b)}\right]^{2n}$.

1.3.2. *Experiment with several poles supports.* So far, with real intervals $[a,b]$, $[c,d]$, we discussed actual orthogonal polynomials as true denominators of rational interpolants. But let us keep $[c,d]$ real, and try several arcs joining two fixed nonreal points, say ia and $-ia$.

to be continued

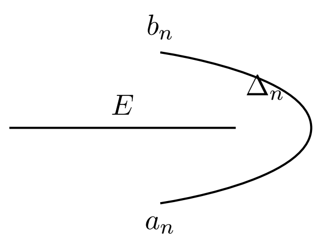
2. '1/9', again and again, ad nauseam.

OK, back to '1/9'. Now, according to Trefethen *et al.* [8, 10] recent work, I stick to rational approximation to e^z on $(-\infty, 0]$.

There are still things to find! Did anybody see that the denominators in Carpenter *et al.* [2] look like $\exp(-0.712z)$ (after $z \leftrightarrow -z$)[3], and, of course, the numerators look like $\exp(0.288z)$. What can these numbers be??

We try to go further in investigating the distributions of poles and interpolation points.

2.1. **The complex potential.**



2.1.1. *Conditions.* As we suspect the poles to be distributed on an single arc Δ_n joining a_n and b_n(still unknown), $\mathcal{V}_n$ is a function with branch-points such that

1. its derivative $\mathcal{V}_n'$ takes opposite pure imaginary values on the two sides of the negative real axis $= E$,
2. $\mathcal{V}_n' - f'/(2nf) = \mathcal{V}_n' - 1/(2n)$ takes opposite values on the two sides of Δ_n, and vanishes at the endpoints a_n and b_n.
3. for $\mathcal{V}_n$ itself, $\mathcal{V}_n(-\infty+0i) - \mathcal{V}_n(-\infty-0i) = 2\pi i$.

2.1.2. *First integral formula.* The second condition means that $[\mathcal{V}_n'(z) - 1/(2n)]/\sqrt{(z-a_n)(z-b_n)}$ has no more branchpoints at a_n and b_n, and can be recovered at any $z \notin E$ through a Cauchy integral on a contour allowed to stretch up to the two sides of E. Same experiment with a further multiplication by $\sqrt{z}$:

$$\sqrt{\frac{z}{(z-a_n)(z-b_n)}}\left(\mathcal{V}_n'(z) - \frac{1}{2n}\right) = \frac{1}{2\pi n}\int_{-\infty}^{0}\sqrt{\frac{-t}{(t-a_n)(t-b_n)}}\frac{dt}{z-t} \qquad (8)$$

Explanation: there should be a numerator $\mathcal{V}_n'(t) - 1/(2n)$ in the integral, but $\sqrt{t}\mathcal{V}_n'(t)$ has no branchpoint at 0, and its contributions from the two sides of E cancel, only $-1/(2n)$ remains, whose equal contributions on the two sides are added.

Signs of the square roots: if the square root in z at the left is positive for positive z, square root inside integral is positive. $\sqrt{z}\mathcal{V}_n'(z)$ must be positive if z is a small positive number.

[3] Coefficient of z in denominators of [2] behave like $0.712 + 0.18/n$.

2.1.3. *Transformation of* (8). : the derivative of the left-hand side of (8) is

$$\frac{a_nb_n - z^2}{2\sqrt{z(z-a_n)^3(z-b_n)^3}}\left(\mathcal{V}_n'(z) - \frac{1}{2n}\right) + \sqrt{\frac{z}{(z-a_n)(z-b_n)}}\mathcal{V}_n''(z),$$

and we integrate by parts (in t) the $z-$derivative of the right-hand side to get

$$\frac{1}{2\pi n}\int_{-\infty}^{0}\frac{(t^2-a_nb_n)dt}{2(z-t)\sqrt{-t(t-a_n)^3(t-b_n)^3}} = \frac{1}{2\pi n}\int_{-\infty}^{0}\sqrt{\frac{-t}{(t-a_n)(t-b_n)}}\frac{a_nb_n-t^2}{2(z-t)z(t-a_n)(t-b_n)}\,dt$$

(must decrease faster than $|z|^{-1}$ for large z, remark also that $t^2 - a_nb_n$
over the big $\sqrt{}$ is the derivative of a function vanishing at 0 and ∞
so, replace $1/(z-t)$ by $1/(z-t) - 1/z = t/[z(z-t)]$)

$$= \frac{1}{2\pi n}\int_{-\infty}^{0}\sqrt{\frac{-t}{(t-a_n)(t-b_n)}}\frac{a_nb_n-z^2}{2(z-t)z(z-a_n)(z-b_n)}\,dt$$

$$+\frac{1}{2\pi n}\int_{-\infty}^{0}\sqrt{\frac{-t}{(t-a_n)(t-b_n)}}\frac{[2a_nb_n-(a_n+b_n)t]z+2a_nb_nt-a_nb_n(a_n+b_n)}{2z(z-a_n)(z-b_n)(t-a_n)(t-b_n)}\,dt$$

which is

$$\frac{a_nb_n - z^2}{2\sqrt{z(z-a_n)^3(z-b_n)^3}}\left(\mathcal{V}_n'(z) - \frac{1}{2n}\right) + \frac{\text{a polynomial of degree } \leqslant 1 \text{ in } z}{z(z-a_n)(z-b_n)}$$

and what remains is

$$\mathcal{V}_n''(z) = \frac{\text{this polynomial}}{\sqrt{z^3(z-a_n)(z-b_n)}}$$

but, as $\mathcal{V}_n$ is the potential of the sum of two opposite charges, $\mathcal{V}_n''(z)$ must decrease faster than $|z|^{-2}$ for large z, this implies a first condition on a_n and b_n

$$2a_nb_n\int_{-\infty}^{0}\sqrt{\frac{-t}{(t-a_n)^3(t-b_n)^3}}\,dt = (a_n+b_n)\int_{-\infty}^{0}\sqrt{\frac{-t}{(t-a_n)^3(t-b_n)^3}}\,t\,dt, \tag{9}$$

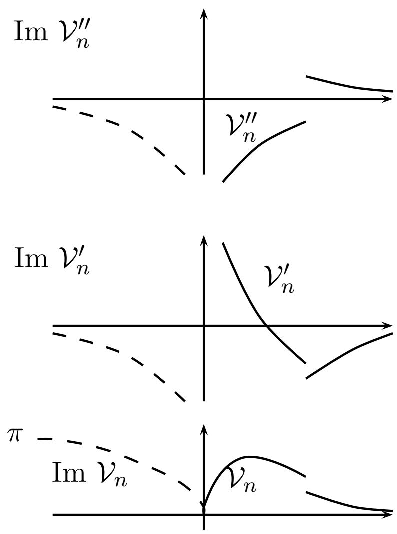
leading to (in?)famous elliptic integrals (to do: look at Carlson's forms). Elementary change of variable $t = -u\sqrt{a_nb_n}$ and $\dfrac{a_n+b_n}{2\sqrt{a_nb_n}} = -\cos\theta$ leads to

$$0 = \int_0^{\infty}\sqrt{\frac{u}{(1-2u\cos\theta+u^2)^3}}\,(1-u\cos\theta)\,du = -\int_0^1\frac{\cos\theta(1+u^2)-2u}{\sqrt{u(1-2u\cos\theta+u^2)^3}}\,du \tag{10}$$

(put $u \leftrightarrow 1/u$ in the integral from 1 to ∞). Apply a crude integration formula:

$\cos\theta$	0	0.25	0.5	0.6	0.65	0.7	0.75	1
(10)	0.8472	0.5919	0.2656	0.0998	0.0045	-0.102	-0.224	$-\infty$

Root is $\cos\theta = 0.6522295\ldots$ (computed through elliptic integrals [5]).



We have now

$$\mathcal{V}_n''(z) = \frac{A_n}{\sqrt{z^3(z-a_n)(z-b_n)}}, \tag{11}$$

with some (still unknown[4]) constant A_n, as already stated by Gonchar and Rakhmanov [4]. When one crosses the line of poles, $\mathcal{V}_n''$ is replaced by its opposite. The picture at left shows $\mathcal{V}_n''(z)$ for positive z, and its imaginary part on the upper side of $(-\infty, 0)$.

$$\mathcal{V}_n'(z) = \int_\infty^z \frac{A_n\,dt}{\sqrt{t^3(t-a_n)(t-b_n)}} \tag{12}$$

to be sure that $\mathcal{V}'(\infty) = 0$. The path of integration in (12) joins ∞ to z by avoiding the cuts[5] $(-\infty, 0)$ and (a_n, b_n). The continuation from small positive z to large z would exhibit $-1/n$ as limit. However one switches to $1/n - \mathcal{V}_n'$ by crossing the line of poles. We also have $\mathcal{V}_n'(a_n) = \mathcal{V}_n'(b_n) = 1/(2n)$, allowing a first connection between A_n, a_n, and b_n:

$$\frac{-1}{2n} = A_n \int_\infty^{a_n} \frac{dt}{\sqrt{t^3(t-a_n)(t-b_n)}} = A_n (a_n b_n)^{-3/4} i \int_{\exp(i\theta)}^\infty \frac{du}{\sqrt{u^3(u^2 - 2u\cos\theta + 1)}},$$

or $A_n = -X(a_n b_n)^{3/4}/n$, where $X = 0.369\ldots$ is a computable[6] constant, as θ is known.

$$\mathcal{V}_n(z) = A_n \int_\infty^z \frac{(z-t)\,dt}{\sqrt{t^3(t-a_n)(t-b_n)}} = z\mathcal{V}_n'(z) - A_n \int_\infty^z \frac{dt}{\sqrt{t(t-a_n)(t-b_n)}}, \tag{13}$$

where the imaginary part depends on the integration contour, as the periods around $(-\infty, 0)$ and (a_n, b_n) are $2\pi i$ and $-2\pi i$ ($\mathcal{V}_n$ looks like log around the first cut (negative unit charge), and $-$log around the second cut (positive unit charge)). These periods values allow at last the full determination of a_n and b_n:

$$\pi i = -A_n \int_{-\infty}^0 \frac{dt}{\sqrt{t(t-a_n)(t-b_n)}} = -A_n (a_n b_n)^{-1/4} i \int_0^\infty \frac{du}{\sqrt{u(u^2 - 2u\cos\theta + 1)}}, \tag{14}$$

or $A_n = -(a_n b_n)^{1/4} Y$, with another computable constant $Y = 0.677\ldots$. Then, $\sqrt{a_n b_n} = nY/X$ remains, as well as $A_n = -\sqrt{XY\, a_n b_n/n}$. Funny thing is that the product XY is exactly $1/4$, I have a proof[7] hidden in [5], but not a fast one.

2.1.4. *Some constants.* a_n/n and $b_n/n = -1.19489931555068 \mp 1.38871265581533\,i$, $\sqrt{a_n b_n}/n = 1.83202271130168$, $\theta = \arg(-a_n) = 0.86027434674909$, $\sin\theta = 0.75802152847146$,
$\cos\theta = 0.65222953196998$, $\cos\theta/2 = k = 0.90890855754855$, $\mathsf{K}(k) = 2.32104973253061$, $c_n^{1/n} =$ '$1/9$' $= 0.10765391922651 = \exp(-\pi\mathsf{K}(\sqrt{1-k^2})/\mathsf{K}(k) = -2.22883364871411)$.

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[4] actually, related to a_n and b_n by an integral formula, but a simpler one will be considered further.

[5] However, the path may accumulate any number of tours around the cuts: the periods about $(-\infty, 0)$ (there is a nasty pole on this one) and (a_n, b_n) do vanish. From this latter cut, an interesting variant of the condition on $\cos\theta$ follows: choose the circular arc $t = \sqrt{a_n b_n}\exp i\varphi$, $\theta - \pi \leqslant \varphi \leqslant \pi - \theta$. Then, $0 = \displaystyle\int_{a_n}^{b_n} \frac{dt}{\sqrt{t^3(t-a_n)(t-b_n)}} =$ const. $\displaystyle\int_{\theta-\pi}^{\pi-\theta} \frac{\cos\varphi\, d\varphi}{\sqrt{\cos\varphi + \cos\theta}}$.

[6] Of course related to elliptic integrals, see later on

[7] That $X = \mathsf{K}(\cos\theta/2)/(2\pi)$ and $Y = \pi/(2\mathsf{K}(\cos\theta/2))$.

2.1.5. *A littlebit AGM.*

Let us consider transformations of the two integrals

$$F(z;a,b)=\int_\infty^z \frac{dt}{\sqrt{t(t-a)(t-b)}},\qquad G(z;a,b)=\int_\infty^z \frac{dt}{\sqrt{t^3(t-a)(t-b)}},$$

where we put $u=\dfrac{t-2\sqrt{ab}+\dfrac{ab}{t}}{4}$. Remark that $(t-a)(t-b)=4t(u-a')$, with $a'=(a+b)/4-\sqrt{ab}/2$. $t=2u+\sqrt{ab}+\sqrt{4u^2+4u\sqrt{ab}}$, $dt/t=du/\sqrt{u(u-b')}$, with $b'=-\sqrt{ab}$.

$$F(z;a,b)=F(z';a',b'),$$

with $z'=\dfrac{z-2\sqrt{ab}+\dfrac{ab}{z}}{4}$. This transformation is convenient when $a+b<0$ and $ab>0$. Starting with the a and b above, fast convergence to a common limit occurs: from $a,b=-1.19489931555068\pm1.38871265581533i$,

$-\sqrt{ab}$	-1.83202271130168	-1.66514111992540	-1.66704022256651	-1.66704049330862
$(a-2\sqrt{ab}+b)/4$	-1.51346101342618	-1.66894149114466	-1.66704076405077	-1.66704049330863

and when $a=b$, $F(z;a,a)=\dfrac{1}{\sqrt{a}}\log\dfrac{\sqrt{z}-\sqrt{a}}{\sqrt{z}+\sqrt{a}}$

$$\begin{aligned} G(z;a,b) &= \int_\infty^{z'} \frac{2u+\sqrt{ab}-\sqrt{4u^2+4u\sqrt{ab}}}{ab\sqrt{u(u-b')}}\frac{du}{\sqrt{4(u-a')}} \\ &= \int_\infty^{z'}\left\{\frac{2}{ab}\frac{d}{du}\left[\sqrt{\frac{(u-a')(u-b')}{u}}-\sqrt{u-a'}\right]+\frac{1/(2\sqrt{ab})+(a'b')/(abu)}{\sqrt{u(u-a')(u-b')}}\right\}du \\ &= \frac{2}{ab}\left[\sqrt{\frac{(z'-a')(z'-b')}{z'}}-\sqrt{z'-a'}\right]+\frac{F(z';a',b')}{2\sqrt{ab}}+\frac{a'b'}{ab}G(z';a',b') \end{aligned}$$

using $\dfrac{d}{du}\sqrt{\dfrac{(u-a')(u-b')}{u}}=\dfrac{d}{du}\sqrt{u-a'-b'+a'b'/u}=\dfrac{u-a'b'/u}{2\sqrt{u(u-a')(u-b')}}$

2.1.6. *Playing with Legendre expansions.* :

Whenever $|t=-u\sqrt{a_nb_n}|\leqslant$ or $\geqslant\sqrt{a_nb_n}$,

$$\frac{1}{\sqrt{(1-t/a_n)(1-t/b_n)}}=\frac{1}{\sqrt{1-2u\cos\theta+u^2}}=\sum_0^\infty P_m(\cos\theta)\,u^m=\sum_0^\infty P_m(\cos\theta)\,u^{-m-1}$$

$$\sqrt{n}\mathcal{V}_n'(z)=\text{const.}+\sum_0^\infty\frac{P_m(\cos\theta)z^{m-1/2}}{(1-2m)(-\sqrt{a_nb_n})^m}=\sum_0^\infty\frac{P_m(\cos\theta)(-\sqrt{a_nb_n})^{m+1}z^{-m-3/2}}{3+2m}\qquad(15)$$

The constant vanishes, as $\mathcal{V}_n'$ has opposite (imaginary) values on the two sides of $(-\infty,0)$.

Check that $\mathcal{V}_n'(a_n)=\mathcal{V}_n'(b_n)=1/(2n)$: the two slowly convergent series at $z=-\sqrt{a_nb_n}\exp(\pm i\theta)$:

```
nineleg.m

ab4=sqrt(1.8320227113);th=0.86027434675;c=cos(th);
sqrtz=i*exp(-i*th/2); Vp=1/(ab4*sqrtz);Vp2=-1/(3*ab4*sqrtz^3);
P0=1;P1=c;sm=-1;
for m=1:1000,Vp=Vp+sm*P1*sqrtz^(2*m-1)/(ab4*(1-2*m));
```

```
            Vp2=Vp2-sm*P1*sqrtz^(-2*m-3)/(ab4*(3+2*m))
    P2=((2*m+1)*c*P1-m*P0)/(m+1);P0=P1;P1=P2;sm=-sm;
    if mod(m,100)==0,[m/100,Vp,Vp2],end;
end;
```

m	1	10	100	200	500	1000
	0.5090 - 0.2335i	0.5078 - 0.1084i	0.5026 - 0.0336i	0.5018 - 0.0239i	0.5011 - 0.0151i	0.5008 - 0.0107i
	0.3173 - 0.0153i	0.3984 - 0.0080i	0.4666 - 0.0024i	0.4762 - 0.0018i	0.4849 - 0.0011i	0.4893 - 0.0008i

$$\sqrt{n}\mathcal{V}_n(z) = \sum_0^\infty \frac{2P_m(\cos\theta) z^{m+1/2}}{(1-4m^2)(-\sqrt{a_n b_n})^m} = -\sum_0^\infty \frac{2P_m(\cos\theta)(-\sqrt{a_n b_n})^{m+1} z^{-m-1/2}}{(2m+1)(2m+3)} \tag{16}$$

choosing $\mathcal{V}_n(0) = 0$.

At $z = -\sqrt{a_n b_n}\exp(-i\theta)$,

```
...V=2*ab4*sqrtz; ...  V=V+sm*2*P1*sqrtz^(2*m+1)*ab4/(1-4*m*m);
   V2=2*ab4/(3*sqrtz);     V2=V2+sm*2*P1*sqrtz^(-2*m-1)*ab4/((2*m+1)*(2*m+3));
```

value of $\mathcal{V}_n(z) - z/(2n)$ is found to be $1.1144... + 1.5708...i = -(\log c_n)/(2n) + \pi i/2$ with the first series; $-(\log c_n)/(2n) - \pi i/2$ with the second one.

The poles cut Δ_n is the locus where the real part of $\mathcal{V}_n(z) - z/(2n)$ is the constant $-(\log c_n)/(2n)$.

Reversion of the first series (16)

$$Y := \mathcal{V}_n(z) - \frac{z}{2n} = 2\sqrt{\frac{z}{n}} - \frac{z}{2n} + \frac{2\cos\theta}{3\sqrt{a_1 b_1}}\left(\frac{z}{n}\right)^{3/2} :$$

$$\sqrt{\frac{z}{n}} = \frac{Y}{2} + \frac{Y^2}{16} + \left(\frac{\cos\theta}{24\sqrt{a_1 b_1}} - \frac{1}{64}\right)Y^3 + \cdots$$

```
%nineleg.m
ab2=1.8320227113;ab4=sqrt(ab2);th=0.86027434675;c=cos(th);
sqrtz=i*exp(-i*th/2); Vp=1/(ab4*sqrtz);V=2*ab4*sqrtz;
dirser(1)=2;dirser(2)=-1/2;    % direct series for V
P0=1;P1=c;sm=-1;
for m=1:30,Vp=Vp+sm*P1*sqrtz^(2*m-1)/(ab4*(1-2*m));
         dirser(2*m+1)=2*P1*sm/((1-4*m^2)*ab2^m);dirser(2*m+2)=0;
             V=V+sm*2*P1*sqrtz^(2*m+1)*ab4/(1-4*m*m);
    P2=((2*m+1)*c*P1-m*P0)/(m+1);P0=P1;P1=P2;sm=-sm;
    if mod(m,10)==0,[m/100,Vp,V+ab4*ab4*exp(-i*th)/2],end;
end;
remser=dirser; dirserp=dirser;
% reverse series
for m=2:25,
    dirserp=conv(dirserp,dirser);
    dirserp=dirserp(1:62);
    invser(m)=-remser(m)/dirserp(1);
    remser(m:62)=remser(m:62)+invser(m)*dirserp(1:63-m);
end;
invser(1)=1;invser=invser/2;


>> invser'

   0.50000000000000
   0.06250000000000
   0.00079099694365
  -0.00438843941022
  -0.00175241217516
  -0.00017749173926
```

```
  0.00017594354142
  0.00011197044512
  0.00002432567762
 -0.00000832085472
 -0.00000889811069
 -0.00000296374323
  0.00000025219279
  0.00000074799463
  0.00000034599104
  0.00000002544551
 -0.00000006162914
 -0.00000003920805
 -0.00000000772542
  0.00000000458961
  0.00000000429681
  0.00000000133248
 -0.00000000025095
 -0.00000000045047
 -0.00000000019381
```

Nice "sine wave" (Henrici), these Taylor coefficients behave like real parts of powers of about $e^{\pi i/3}/10^{1/3} \approx 0.3 + 0.4i$. It figures: the direct series of $Y = \mathcal{V}_1(z) - z/2$ has singularities at a_1 and b_1 with behaviour $-\log(c_1)/2 \pm \pi i/2 + A_1(z - a_1, b_1)^{3/2} + \cdots = 1.1144... \pm 1.5708...i+$ const. $(z-a_1, b_1)^{3/2}+\cdots$ whence for the inverse function $z = a_1$ or b_1+ const $(Y-(\log c_1 \pm \pi i)/2)^{2/3}+\cdots$ near a_1 or b_1, and coefficients behaviour as $n^{-5/3}$ times a combination of n^{th} powers of $2/(\log c_1 \pm \pi i) = 0.300... \mp 0.423...i$ (Darboux).

Locus of poles Δ_n/n is the image of $[1.1144.. - \pi i/2, 1.1144.. + \pi i/2]$. With 100 terms:

```
>> yy=1.1144168...+(0:0.05:0.5)*pi*i

yy = 1.1144168 , 1.1144168+0.0157796...i,  1.1144168 + 0.31415926535898i ,  ... 1.1144168 + 1.57079632679490i


>> (yy.^100.*polyval(invser,1./yy)).^2

  0.39243973943344
  0.38283919896697 + 0.11948758440998i
  0.35378081869954 + 0.23932622697562i
  0.30445719600034 + 0.35990563615929i
  0.23338515636343 + 0.48170394036478i
  0.13814111186681 + 0.60536640116891i
  0.01482293663859 + 0.73185191424500i
 -0.14313215474693 + 0.86275232571528i
 -0.34802443454987 + 1.00114603711627i
 -0.62904340404614 + 1.15483608035005i
 -1.15073795280994 + 1.34847500068745i
```

The last item should have been $-1.194899... + 1.3887...i$

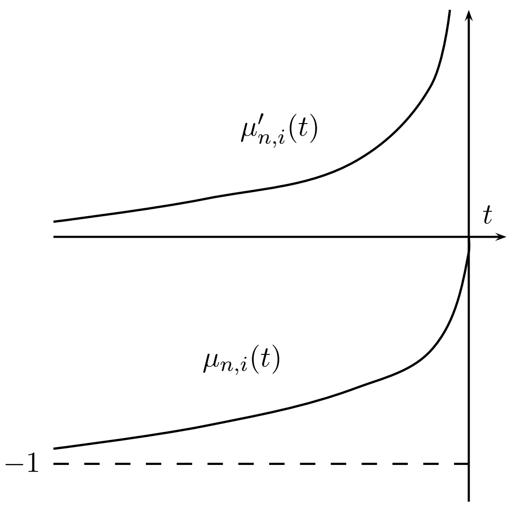
2.2. **Distributions of interpolation points and poles.**

2.2.1. *Interpolation points.* As the second term of $\mathcal{V}_n'(z) := \mathcal{V}_{n,i}'(z) - \mathcal{V}_{n,p}'(z) = \displaystyle\int_E \frac{d\mu_{n,i}(t)}{z-t} - \int_{\Delta_n} \frac{d\mu_{n,p}(t)}{z-t}$ is real on the two sides of $E = (-\infty, 0)$ (the distributions are symmetric with respect to the real axis), we immediately have

$$\mathcal{V}_n'(z \pm 0i) = \mp \pi i \mu_{n,i}'(z), \qquad z < 0$$
(***Sokhotskyi-Plemelj*** formulas). This means that, for any reasonable f,

$$\frac{1}{m+n+1}\sum_0^{m+n} f(x_j) \to_{m\sim n\to\infty} \frac{1}{2\pi\sqrt{n}}\int_{-\infty}^0 f(t)\int_{-\infty}^t \frac{du}{\sqrt{-u^3(1-u/a_n)(1-u/b_n)}}\,dt$$
$$\sim -\frac{1}{2\pi\sqrt{n}}\int_{-\infty}^0 \frac{F(t)dt}{\sqrt{-t^3(1-t/a_n)(1-t/b_n)}} \qquad (17)$$

where $F(t) = \int_0^t f(u)du$. Check with $f(t) \equiv 1$: use (14), knowing $A_n = -\sqrt{a_n b_n}/2$. No many other elementary examples: with $f(t) = t$ in order to discuss $(x_0 + \cdots + x_{m+n})/(m+n+1)$, the integral is divergent (result is of order n^2, see below).



For large (negative) t, $\mu''_{n,i}(t) = \dfrac{1/(2\pi\sqrt{n})}{\sqrt{-t^3(1-t/a_n)(1-t/b_n)}}$ is about $(\sqrt{a_1 b_1 n}/(2\pi))(-t)^{-5/2}$, so $\mu'_{n,i}(t) \sim (\sqrt{a_1 b_1 n}/(3\pi))(-t)^{-3/2}$, and $\mu_{n,i}(t) \sim -1 + (2\sqrt{a_1 b_1 n}/(3\pi))(-t)^{-1/2}$. This means that the most negative interpolation points are in the n^3 range. Indeed, $\mu_{n,i}(t) = -1 + k/(2n) \Rightarrow t \sim -(16 a_1 b_1/9\pi^2)(-n^3/k^2)$. Near the origin, $\mu''_{n,i}(t) = (1/(2\pi\sqrt{n}))(-t)^{-3/2} + O(t^{-1/2})$, $\mu'_{n,i}(t) \sim (1/(\pi\sqrt{n}))(-t)^{-1/2}$, $\mu_{n,i}(t) \sim -(2/(\pi\sqrt{n}))(-t)^{1/2}$. Corresponding interpolation points are at about $\mu_{n,i}(x_j) \approx j/(2n) \Rightarrow x_j \approx -j^2\pi^2/(16n)$. Here are samples of smallest and largest interpolation points for best approximants of degrees $(n-1)/n$:

n	x_0	x_1	x_2	x_{2n-3}	x_{2n-2}	x_{2n-1}
2	−0.062	−0.574	−1.891	−0.574	−1.891	−5.751
3	−0.043	−0.402	−1.185	−2.612	−5.359	−14.906
4	−0.034	−0.311	−0.892	−5.874	−10.905	−29.745
5	−0.028	−0.254	−0.715	−10.432	−18.993	−53.292

The smallest points happen to be about $1, 9, 25, \ldots$ times $-\pi^2/(64n+32)$: a better formula is $x_j \sim -(j+1/2)^2\pi^2/(16n+8)$, corresponding to $\mu_{n,i}(x_j) \sim -(j+1/2)/(2n+1)$.
x_{2n-1} is about $-0.4n^3$, and x_{2n-2} and x_{2n-3} about 3 and 5 times smaller, I hope that no strongly accurate estimate will be needed. However the expected value of x_n is about $-16a_1b_1n^3/(9\pi^2 n) \approx -0.6045\ldots n$, whereas the formula for the small x's predicts $x_n \sim -n^2\pi^2/(16n) \approx -0.6185\ldots n$.

2.2.2. *The potential function of the distribution of the interpolation points.* When $f(t) = \log(z-t)$, $F(t) = (t-z)\log(z-t) - t + z\log z$, ouch, I try differential equations for $\mathcal{V}_{n,i}$, as above in section 2.1.3 for $\mathcal{V}_n$:

$$\mathcal{V}'_{n,i}(z) = \int_{-\infty}^0 \frac{\mu'_{n,i}(t)\,dt}{z-t},$$
$$\mathcal{V}''_{n,i}(z) = -\int_{-\infty}^0 \frac{\mu'_{n,i}(t)\,dt}{(z-t)^2} = \lim_{\varepsilon\to 0}\left[-\frac{\mu'_{n,i}(\varepsilon)}{z-\varepsilon} + \int_{-\infty}^{\varepsilon}\frac{\mu''_{n,i}(t)\,dt}{z-t}\right]$$

hmm, multiply by $z = z - t + t$:

$$z\mathcal{V}''_{n,i}(z) = \lim_{\varepsilon\to 0} -\frac{\varepsilon\mu'_{n,i}(\varepsilon)}{z-\varepsilon} - \frac{1}{2\pi\sqrt{n}}\int_{-\infty}^0 \frac{dt}{(z-t)\sqrt{-t(1-t/a_n)(1-t/b_n)}}$$

this begins to look like (8), leading to

$$\sqrt{n}\sqrt{\frac{z}{(1-z/a_n)(1-z/b_n)}}\left(\mathcal{V}_n'(z)-\frac{1}{2n}\right)=1+z^2\mathcal{V}_{n,i}''(z)$$

$$\mathcal{V}_{n,i}''(z)=-z^{-2}-\mathcal{V}_n''(z)(2n\mathcal{V}_n'(z)-1)$$

$$\mathcal{V}_{n,i}'(z)=z^{-1}+\mathcal{V}_n'(z)-n(\mathcal{V}_n'(z))^2 \tag{18}$$

Wow! Remark that $\mathcal{V}_n'-n(\mathcal{V}_n')^2=n\mathcal{V}_n'(1/n-\mathcal{V}_n')$ is precisely the product (i.e., a symmetric function) of the two determinations of $\mathcal{V}_n'$ near the cut (a_n,b_n): there is no more any branchpoints there.

Series: from (15),

$$\sqrt{n}\mathcal{V}_{n,i}'(z)=\sum_0^\infty\frac{P_m(\cos\theta)z^{m-1/2}}{(1-2m)(-\sqrt{a_nb_n})^m}-\sqrt{n}\sum_{m=1}^\infty\frac{\sum_{k=0}^m\dfrac{P_k(\cos\theta)P_{m-k}(\cos\theta)}{(1-2k)(1-2m+2k)}}{(-\sqrt{a_nb_n})^m}z^{m-1}$$

$$=\frac{\sqrt{n}}{z}+\sum_0^\infty\frac{P_m(\cos\theta)(-\sqrt{a_nb_n})^{m+1}}{(3+2m)z^{m+3/2}}-\sqrt{n}\sum_{m=0}^\infty\frac{\sum_{k=0}^m\dfrac{P_k(\cos\theta)P_{m-k}(\cos\theta)}{(3+2k)(3+2m-2k)}}{z^{m+3}}(-\sqrt{a_nb_n})^{m+2}$$

the series are not easier than before, integer powers of z are added to the series of (15). And the convergence radius is not changed. Only the singular points on the second sheet $\mathrm{Re}\sqrt{z}<0$ are still there. Also, the series for $|z|<\sqrt{ab}$ and $|z|>\sqrt{ab}$ must be the perfect continuation of each other.

$z/\sqrt{a_nb_n}$	3/5	4/5	1	1	6/5	7/5
$n\mathcal{V}_n'(z)$	1.3050	1.2222	1.1715	-0.1715	-0.1377	-0.1138
$n\mathcal{V}_{n,i}'(z)$	0.5117	0.4108	0.3449	0.3449	0.2982	0.2632

How to decide the constants in the series for the integral $\mathcal{V}_{n,i}$?

$$\mathcal{V}_{n,i}(z)=\text{const.}+2(z/n)^{1/2}-2\cos\theta(z/n)/\sqrt{a_1b_1}+2\cos\theta(z/n)^{3/2}/\sqrt{a_1b_1}+\cdots$$
$$=\text{const.}+\log(z/n)+2\sqrt{a_1b_1}(n/z)^{1/2}/3-2a_1b_1(n/z)^{3/2}/15+a_1b_1(n/z)^2/18+\cdots$$

if we drop the last constant, so as to have a potential with $\lim[\mathcal{V}_{n,i}(z)-\log(z/n)]=0$ for large z, we find that 0.3946 must be subtracted from the first series, so

$z/\sqrt{a_nb_n}$	0	3/5	4/5	1	1	6/5	7/5
$\mathcal{V}_{n,i}(z)$	-0.3946	1.1347	1.3022	1.4399	1.4399	1.5573	1.6599

2.2.3. *The distributions of poles.*

And of course, from $\mathcal{V}_n=\mathcal{V}_{n,i}-\mathcal{V}_{n,p}$:

$$\mathcal{V}_{n,p}'(z)=z^{-1}-n(\mathcal{V}_n'(z))^2 \tag{19}$$

Must indeed be near z^{-1} for large z.

Check near the origin: $z^{-1}-[z^{-1/2}+z^{1/2}(\cos\theta)/(n\sqrt{a_1b_1})-z^{3/2}(3\cos^2\theta-1)/(6n^2a_1b_1)+\cdots]^2=-2\cos\theta/(n\sqrt{a_1b_1})+z/(3n^2a_1b_1)+\cdots$

Denominator $=\prod(1-z/\text{poles})\sim\exp(n(\mathcal{V}_{n,p}(z)-\mathcal{V}_{n,p}(0))=\exp(-2z\cos\theta/\sqrt{a_1b_1}+z^2/(6na_1b_1)+\cdots)$, has a fixed limit when $n\to\infty$. Moreover, $\exp(-2z\cos\theta/\sqrt{a_1b_1})=\exp(-0.71203...z)$ fits with tables from [2]

3. A family of '1/9' rational interpolants.

3.1. Trefethen's problem.

3.1.1. *Problem. Show that the* **best** *rational approximations* $\hat{r}_{m,n}$ *of degrees* m *and* n *(*$m \leqslant n$ *and* $m \sim n$*) to* $\exp z$ *on* $(-\infty, 0]$ *satisfy*

$$\limsup_{n\to\infty} \|e^z - \hat{r}_{m,n}(z)\|_{\infty,K}^{1/n} \leqslant \text{`}1/9\text{'} \tag{20}$$

for any compact set $K \subset \mathbb{C}$. (Trefethen, 2005 [8]).

3.1.2. *Strategy.* Current asymptotics [4] consider only weak limits of distributions, one could have errand poles visiting sometimes any bounded set (but avoiding the negative real axis).

Also, Aptekarev's near-best approximant [1] has a most decent behaviour, but there is no solid proof that the actual best approximant is equally well behaved.

I intend to study a family of rational functions, containing the best approximant, interpolating e^z at points close to be equidistributed with respect to $\mu_{n,i}$. Of course, 'close' will have to receive an accurate description.

First thing is to be sure of the denominator.

If denominator q is innocuous, we consider q and $q(x)e^x - p(x)$, which is the polynomial interpolation error

$$q(x)e^x - p(x) = [x_0, \dots, x_{m+n}, x]_{q(x)\exp(x)}(x - x_0)\cdots(x - x_{m+n}). \tag{21}$$

The product of the $x - x_i$'s behaves like $\exp(n\mathcal{V}_{n,i}(x))$, and the divided difference will be explored right now.

3.2. Retrieving the denominator.

3.2.1. *Scalar product.*

Denominator q is the orthogonal polynomial of degree n with respect to the scalar product

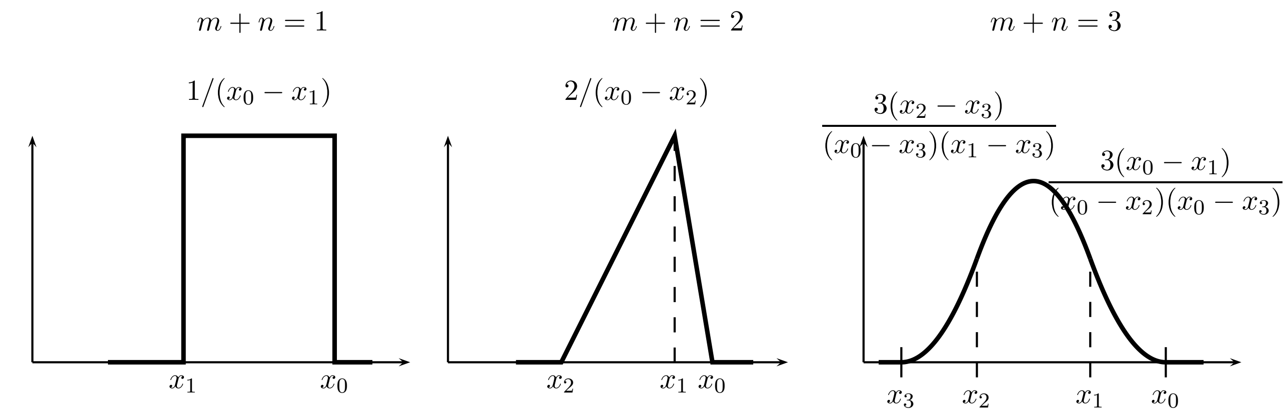
$$\begin{aligned}\langle f, g\rangle_n &= [x_0, \dots, x_{m+n}]_{f(x)g(x)\exp(x)} \\ &= \sum_{j=0}^{m+n} \frac{f(x_j)g(x_j)\exp(x_j)}{\prod\limits_{m\neq j}(x_j - x_m)} \\ &= \frac{1}{2\pi i}\int_{C_n} \frac{f(t)g(t)\exp(t)\,dt}{(t - x_0)\cdots(t - x_{m+n})}\end{aligned} \tag{22}$$

as seen in (3). Is there any chance to get accurate estimates of such things? First elementary fact is of course that the divided difference $= 1$ for x^{m+n}, suggesting an order $O(1/(m+n)!$ for the simplest scalar products. Probably not wrong, but no easy correction coming from $e^x = \dots + x^{m+n}/(m+n)! + x^{m+n+1}/(m+n+1)! + \cdots$, yielding the useless $1/(m+n)! + (x_0 + \cdots + x_{m+n})/(m+n+1)!$. Useless because e^x is so small at the most negative x_j's. The divided difference is also a particular value of the $(m+n)^{\text{th}}$ derivative divided by $(m+n)!$, and this derivative involves the exponential of a presumed strongly negative number. Ah, there is also the B-spline formula

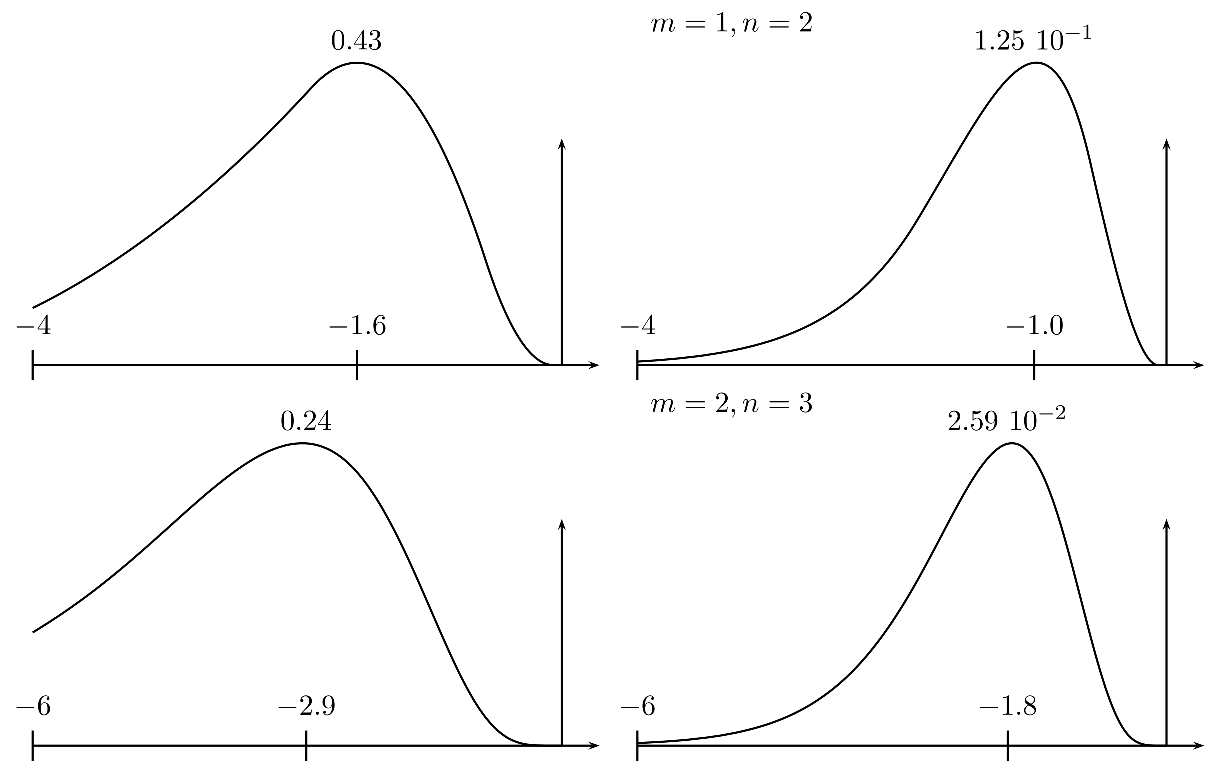
$$\langle f, g\rangle_n = \int_{x_{m+n}}^{x_0} \frac{B(x)}{(m+n)!}\frac{d^{m+n}}{dx^{m+n}}[f(x)g(x)e^x]\,dx, \tag{23}$$

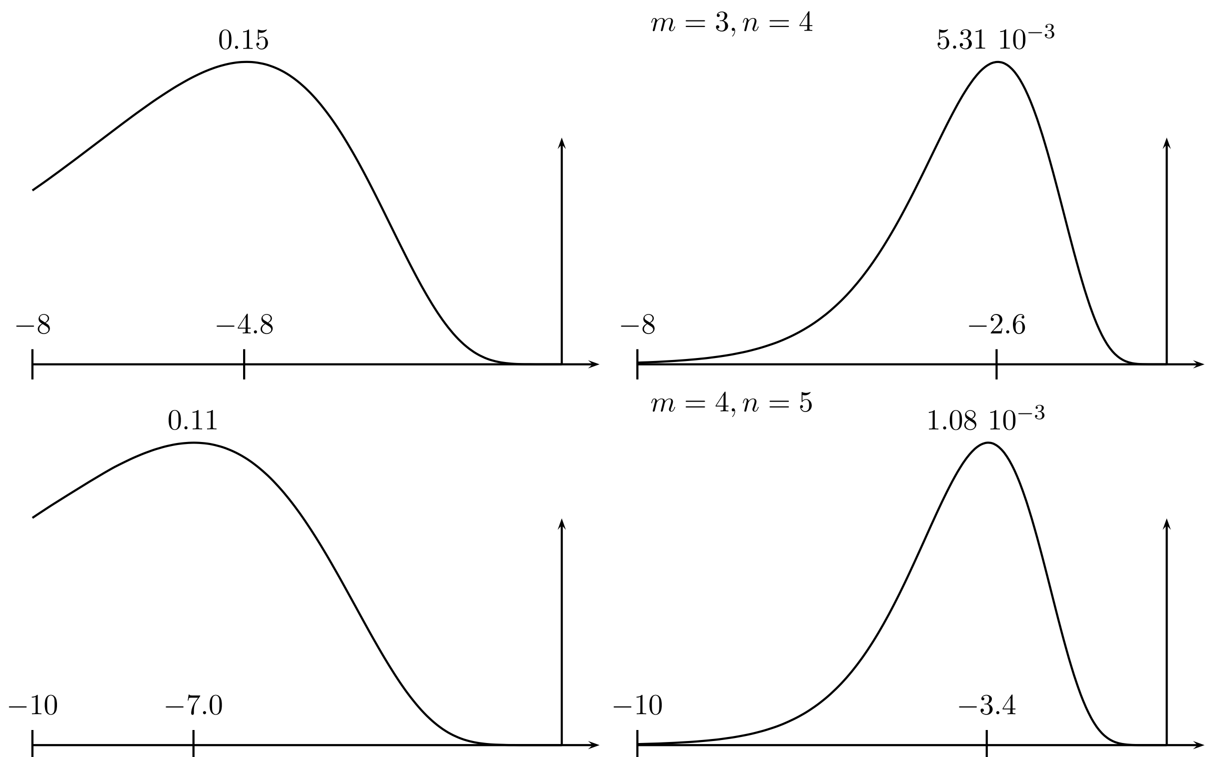
where $B(x)$ is actually (deBoor [3])

$$
\begin{aligned}
B(x) &= (m+n)[x_0, \ldots, x_{m+n}]_{(.-t)_+^{m+n-1}} \\
&= M(x; x_{m+n}, \ldots, x_0) \\
&= (m+n)\frac{B(x; x_{m+n}, \ldots, x_0)}{(x_0 - x_{m+n})}.
\end{aligned}
$$



3.2.2. *The shape of things to come.* Here are some instances of $B(x)$ and $B(x)e^x$ on the x_i's of best approximants, $m = n - 1$:





It seems that the scaled $B(x)e^x$ tends towards a fixed function.

3.2.3. *B−spline towards Gaussian.* Well-known and linked to the central limit theorem, but has only been worked for cardinal (equidistant points) B-splines [11].

Let us look at the moments of a B−spline defined on a set of real points $t_0, \dots, t_N$: apply $[t_0, \dots, t_N]_f = \left.\dfrac{f(t) - \text{interp. of} f \text{ at } t_0, \dots, t_{N-1}}{(t-t_0)\cdots(t-t_{N-1})}\right|_{\text{at } t=t_N} = \displaystyle\int_{-\infty}^{\infty} \frac{f^{(N)}(x)}{N!} M_N(x)\, dx$:

$[t_0, \dots, t_N]_{t^{N+r}} = \left.\dfrac{t^{N+r} - \text{interp. at } t_0, \dots, t_{N-1}}{(t-t_0)\cdots(t-t_{N-1})}\right|_{\text{at } t=t_N} = \dfrac{(N+r)!}{N!r!} \displaystyle\int_{-\infty}^{\infty} x^r M_N(x)\, dx$. The numerator is $(t-t_0)\cdots(t-t_{N-1})$ times a polynomial of degree r such that the product has no term in $t^{N+r-1}, \dots, t^N$. In other words, what remains is the singular part of the Laurent expansion of $t^{N+r}/((t-t_0)\cdots(t-t_{N-1}))$ at ∞. This expansion involves the complete homogeneous symmetric functions of $t_0, \dots, t_{N-1}$, and the result is the same function h_r for $t_0, \dots, t_N$, so, for large N, $\displaystyle\int_{-\infty}^{\infty} x^r M_N(x)\, dx \sim r! N^{-r} h_r(t_0, \dots, t_N)$. Fourier moments: $\displaystyle\int_{-\infty}^{\infty} e^{i\xi x} M_N(x)\, dx \sim \sum_0^{\infty} i^r \xi^r N^{-r} h_r(t_0, \dots, t_N)$ $= \displaystyle\prod_0^N (1 - i\xi t_k/N)^{-1}$. If the t_k's are regularly distributed on an interval (a, b) with respect to a measure $d\mu$ with finite moments, $\displaystyle\int_{-\infty}^{\infty} e^{i\xi x} M_N(x)\, dx \sim \exp[-N \int_a^b \log(1 - i\xi t/N)\, d\mu(t)]$, involving only the first moments of $d\mu$ when N is large, whence the Gaussian look.

3.2.4. *Moments and recurrence relations.* With the distribution $\mu_{n,i}$ of the interpolation points, $\displaystyle\int_{-\infty}^{\infty} e^{i\xi x} M_{m+n+1}(x)\, dx \sim \exp[-(m+n+1) \int_{-\infty}^{0} \log(1 - i\xi t/(m+n+1))\, d\mu_{n,i}(t)]$ $= \exp[-(m+n+1)\log(-i\xi/(m+n+1)) - (m+n+1)\mathcal{V}_{n,i}(-i(m+n+1)/\xi)]$. Polynomial moments of $e^x M_{m+n+1}(x)$ ask for derivatives at $\xi = -i$:

$$\int_{-\infty}^{\infty} e^x M_{m+n+1}(x)\,dx \sim ((-1)^{m+n+1}/(m+n+1)^{m+n+1}) \exp[-(m+n+1)\mathcal{V}_{n,i}(m+n+1)],$$
$$\int_{-\infty}^{\infty} xe^x M_{m+n+1}(x)\,dx \sim \text{etc.}$$

Now, as $\mathcal{V}_{n,i}(z) \sim$ a fixed function $\mathcal{V}_i(z/n)$, $\langle 1,1\rangle_n \sim (2n)^{-2n}\exp[-2n\mathcal{V}_i(2)]$ ... and $\mathcal{V}_i(2) = 1.495$ according to series calculations made above.

Let us explore some moments computed with interpolation points of actual best approximants, still with $m = n-1$:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\langle 1,1\rangle_n$	$5.44883\ 10^{-1}$	$3.12788\ 10^{-2}$	$4.56673\ 10^{-4}$	$2.8112\ 10^{-6}$	$9.3567\ 10^{-9}$

an approximate exponential pattern appears with

n	$(2n-1)^{2n-1}\langle 1,1\rangle_n$	$(2n-1)^{2n-1}\langle x,1\rangle_n$	$(2n-1)^{2n-1}\langle x^2,1\rangle_n$	$(2n-1)^{2n-1}\langle x^3,1\rangle_n$	$(2n-1)^{2n-1}\langle x^4,1\rangle_n$	$(2n-1)^{2n-}$
1	0.5448824	0.2432842				
2	0.8445255	1.3814579	0.1258217	-1.9972810		
3	1.4280243	3.9964734	5.3448847	-4.7087498	-10.2860880	5
4	2.3184672	9.1527609	23.0989445	17.2386863	-69.4132043	
5	3.6289523	18.4766184	68.1524072	140.5080478	-47.5999317	
6	5.5149784	34.3675234	166.4234039	547.7093760	734.8526838	

the $\langle 1,1\rangle_n$'s behave like powers of about 1.6, the $\langle x,1\rangle_n$'s are about n times larger, etc.

??

Much more accurate estimates will be needed in order to discuss the ***3-term recurrence relation*** of intermediate polynomials q_k amounting to the building of the denominator polynomial q_n (for each n, the whole set of intermediate polynomials is to be computed again, they should receive two indexes, but what follows is for a fixed n).

Keeping $q_k(0) = 1$, the recurrence relation is

$$q_{k+1}(x) = (1-\gamma_k-\delta_k x)q_k(x) + \gamma_k q_{k-1}(x)$$

with $\gamma_0 = 0$. Then,

$$\frac{1-\gamma_k}{\delta_k} = \frac{\langle xq_k, q_k\rangle_n}{\langle q_k, q_k\rangle_n}, \qquad \frac{\gamma_k}{\delta_k} = \frac{\langle xq_{k-1}, q_k\rangle_n = -\langle q_k, q_k\rangle_n/\delta_{k-1}}{\langle q_{k-1}, q_{k-1}\rangle_n},$$

or

$$\frac{1}{\delta_k} = \frac{\langle xq_k, q_k\rangle_n}{\langle q_k, q_k\rangle_n} - \frac{\langle q_k, q_k\rangle_n}{\delta_{k-1}\langle q_{k-1}, q_{k-1}\rangle_n},$$

allowing progressive calculation of the γ's and the δ's

Here is how they look:

n	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
2	1.6410					
3	0.7197	2.3551				
4	0.4521	1.2316	2.7843			
5	0.3280	0.8223	1.6250	3.0808		
6	0.2569	0.6141	1.1340	1.9385	3.2983	
7	0.2109	0.4890	0.8670	1.4018	2.1990	3.4767

n	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6
1	2.2397						
2	0.6113	1.0623					
3	0.3573	0.4925	0.7769				
4	0.2533	0.3175	0.4213	0.6138			
5	0.1964	0.2338	0.2873	0.3694	0.5099		
6	0.1605	0.1849	0.2175	0.2627	0.3295	0.4369	
7	0.1357	0.1530	0.1750	0.2037	0.2427	0.2985	0.3839

$$q_k - q_{k-1} = -\delta_{k-1} x q_{k-1} - \gamma_{k-1}(q_{k-1} - q_{k-2}), \ldots,$$

$$q_k(x) = 1 - x \sum_0^{k-1} (1 - \gamma_{j+1} + \gamma_{j+1}\gamma_{j+2} - \cdots + (-1)^{k-1-j}\gamma_{j+1}\cdots\gamma_{k-1})\delta_j q_j(x)$$

?

There must be a representation problem, as the "plain" writing of the successive q_k's is very smooth:

with $m = 6, n = 7$,

$$q_0(x) = 1.0000$$
$$q_1(x) = -0.1357x + 1.0000$$
$$q_2(x) = 0.0208x^2 - 0.2601x + 1.0000$$
$$q_3(x) = -0.0036x^3 + 0.0561x^2 - 0.3743x + 1.0000$$
$$q_4(x) = 0.0007x^4 - 0.0119x^3 + 0.1017x^2 - 0.4790x + 1.0000$$
$$q_5(x) = -0.0002x^5 + 0.0026x^4 - 0.0250x^3 + 0.1540x^2 - 0.5749x + 1.0000$$
$$q_6(x) = 0.0001x^6 - 0.0006x^5 + 0.0060x^4 - 0.0422x^3 + 0.2105x^2 - 0.6624x + 1.0000$$
$$q_7(x) = -0.0000x^7 + 0.0001x^6 - 0.0015x^5 + 0.0104x^4 - 0.0631x^3 + 0.2686x^2 - 0.7419x + 1.0000$$

3.3. **Error function behaviour.**

Asks now for a discussion of (21)

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