Seminar series on
Rational approximations and systems theory.
February-March 2002

Asymptotic convergence rates of rational interpolation to exponential functions.

The present slides file is
For more, see
http://www.math.ucl.ac.be/~magnus/num3/m3xxx00.pdf and ps
and references therein.

Alphonse Magnus,
Institut de Mathématique Pure et Appliquée,
Université Catholique de Louvain,
Chemin du Cyclotron,2,
B-1348 Louvain-la-Neuve
magnus@anma.ucl.ac.be, http://www.math.ucl.ac.be/~magnus/

Complex rational approx. 1 – Taylor & polynomial & rational. – 2

1. Taylor, polynomial, and rational interpolation.

1.1. Taylor expansions.

The Taylor series expansion of a function with finite convergence domain shows “typically” almost circular level lines of equal approximation, explained by a convenient representation of the error

\[ f(z) - \sum_{k=0}^{n} c_k z^k = \frac{1}{2\pi i} \int_{|t|=\varepsilon} f(t) \frac{z^{n+1}}{t^{n+1}(t-z)} \, dt, \]  
\[ |z| < |Z| = R, \]  
(1)

\[ f(z) - \sum_{k=0}^{n} c_k z^k = \frac{1}{2\pi i} \int_{C} f(t) \frac{z^{n+1}}{t^{n+1}(t-z)} \, dt, \]

Complex rational approx. 1 – Taylor & polynomial & rational. – 3

1.2. General rational interpolation with given poles. Condenser capacity.

…, tout entier à une idée qui lui était venue sur les potentiels.
Alphonse Allais (from Madrigal manquée)

The error at \( z \) is

\[ f(z) - \frac{p(z)}{q(z)} = \frac{\prod (z-z_j)}{2\pi i q(z)} \int_{C} \frac{q(t)f(t)}{t-z} \, dt \]

(2)

involving mainly \( \left( \frac{\Phi(z)}{\Phi(Z)} \right)^n \), with \( \Phi(z) = \exp \mathcal{V}(z) = \exp[\mathcal{V}_p(z) - \mathcal{V}_i(z)] \), where \( \mathcal{V}_p \) and \( \mathcal{V}_i \) are the (complex) potentials of the distributions of the poles and interpolation points:

\[ \mathcal{V}_p,i(z) := \int_C \log \frac{1}{z-t} \, d\mu_{p,i}(t). \]

\[ \frac{|\Phi(z)|}{|\Phi(Z)|} = \exp[\text{Re } (\mathcal{V}(z) - (\text{Re } \mathcal{V} \text{ on } L_p)) = \exp \left( \frac{-1}{\text{cap } (L, L_p)} \right). \]

Complex rational approx. 2 – Exponential function. – 4

1.3. The problem of rational interpolation at \( m+n+1 \) points, orthogonal polynomials.

Numerator interpolates \( q_n f \) at \( m+n+1 \) points: \( p_m(z) = \)

\[ \frac{1}{2\pi i} \int_{C_f} \frac{q_n(t)}{t-z} \sum_{j=0}^{m+n} L_j(z_j) f(t) \, dt = \frac{1}{2\pi i} \int_{C_f} q_n(t) \left[ \frac{1}{t-z} - \frac{\prod_{0}^{m+n}(z-j)}{(t-z) \prod_{0}^{m+n}(t-z_j)} \right] f(t) \, dt \]

So, \( p_m \) is only \( O(z^m) \) as it should if \( \int_{C_f} \frac{q_n(t)f(t)}{t-z} \prod_{0}^{m+n}(t-z_j) \, dt \) is \( O(z^{m-1}) \),
so, as \( (t-z)^{-1} = -z^{-1} - t z^{-2} - \cdots + t^m z^{-m} (z-t)^{-1} \), if

\[ \int_{C_f} q_n(t) t^j w_n(t) \, dt = 0, \quad j = 0, \ldots, n-1, \]

(3)

where \( w_n(t) = \frac{f(t)}{\prod_{0}^{m+n}(t-z_j)} \): formal orthogonality!
2. Known rational interpolations to the exponential function.

2.1. Padé.

For the error $e^z$ approximant, we have the $n^{th}$ powers of

\[
\begin{align*}
1 + \sqrt{1 + z^2/(4n^2)} & \quad e^z/n - 2 \sqrt{1 + z^2/(4n^2)} \\
1 - \sqrt{1 + z^2/(4n^2)} & \quad e^z/n + 2 \sqrt{1 + z^2/(4n^2)}
\end{align*}
\]

\[
\begin{align*}
1 + \sqrt{1 + z^2/(4n^2)} & \quad e^{5(0.250z)} \\
1 - \sqrt{1 + z^2/(4n^2)} & \quad e^{5(1.00z)} \\
1 + \sqrt{1 + z^2/(4n^2)} & \quad e^{5(2.00z)} \\
1 - \sqrt{1 + z^2/(4n^2)} & \quad e^{5(3.00z)}
\end{align*}
\]

\[
\begin{align*}
3.84 & \quad -0.0897 \\
1.325 & \quad -1.325 \\
2i & \quad e^z/n
\end{align*}
\]


2.2. Rational interpolation at equidistant points (Iserles). [3]

Interpolation of $\exp(Az)$ at $z_0, z_0 + h, \ldots, z_0 + (m+n)h$:

Rough asymptotics.

If $m-n$, one finds that the numerator, denominator, and the error behave like the $n^{th}$ powers of

\[
\exp \left[ \zeta \log \left( e^{Ah/2} \frac{\gamma \zeta - \sqrt{\sigma^2 \zeta^2 + 1}}{\zeta - 1} \right) \right] + \frac{Ah}{2} + \log(\gamma + \sqrt{\sigma^2 \zeta^2 + 1})
\]

\[
\exp \left[ \zeta \log \left( e^{-Ah/2} \frac{\gamma \zeta - \sqrt{\sigma^2 \zeta^2 + 1}}{\zeta - 1} \right) \right] - \frac{Ah}{2} + \log(\gamma + \sqrt{\sigma^2 \zeta^2 + 1})
\]

\[
\exp \left[ \zeta \log \left( e^{Ah/2} \frac{\gamma \zeta + \sqrt{\sigma^2 \zeta^2 + 1}}{\gamma \zeta + \sqrt{\sigma^2 \zeta^2 + 1}} \right) \right] + Ah + \log \left( \gamma + \sqrt{\sigma^2 \zeta^2 + 1} \right)
\]

where

\[\zeta = [2(z-z_0)/((m+n)h)] - 1, \gamma = \cosh Ah/2, \sigma = \sinh Ah/2.\]

3. Asymptotic features of rational interpolation.

3.1. According to Gončar-Rahmanov-Stahl (a sloppy rendering).

Interpolation to $f_n(z) = \int_{C_f} \frac{q_0(t)q_n(t)}{z-t} dt$ at $z_0, \ldots, z_{m+n}$ by $p_m/q_n$ yields

\[f_n(z) - \frac{p_m(z)}{q_n(z)} = \prod_{j=0}^{m+n} q_n^2(z) \int_{C_f} \frac{q_n^2(t)}{\prod_{j=0}^{m+n} (t-z_j)} \frac{q_0(t)q_n(t)}{z-t} dt,\]

where $q_n$ is (formally) orthogonal with respect to $w_n(t) := \frac{q_0(t)q_n(t)}{\prod_{j=0}^{m+n} (t-z_j)}$ on $C_f$, as in (3).

Well, we expect that most of the poles of $q_n$ will tend to a set $S \subseteq C_f$. 

We look at the performance of some examples of the region of good approximation in the complex plane, coloured in light gray:

\[
\begin{align*}
\exp(5(0.250z)) & \quad -12.6 \\
\exp(5(1.00z)) & \quad -0.0897 \\
\exp(5(2.00z)) & \quad 3.84 \\
\exp(5(3.00z)) & \quad 6.57
\end{align*}
\]
On the support of $\mu_p$, $q_n$ is almost a Szegő orthogonal polynomial! which means that $\pm q_n(t) \sqrt{w_n(t)}$ has slowly varying phase and absolute value there.

Let $g_0/2 + \sum g_n T_n$ be the expansion of $g$. $g_0 = 0, \quad g_1 = \frac{4}{\beta - \alpha}$.

\[
 q^\rho(z) = \sum_{n=1}^\infty g_n \rho^n.
\]

\[
 \frac{\rho + \rho^{-1}}{2} =\frac{2z - \alpha - \beta}{\beta - \alpha},
\]

4. Rational interpolation to $\exp(n B_1 z + n B_2 z^2)$.

This very interesting rational interpolation appears in special nonlinear Schrödinger problems ([6, 8] and remarks by J. Nuttall in [4]).

4.1. The single arc case.

\[
 \rho_k + \rho_k^{-1} = \frac{2I_k - \alpha - \beta}{\beta - \alpha}, k = 1, 2,
\]

\[
 q_k(z) = \frac{2}{I_2 - I_1} [(z - I_1) \log(1 - \rho_1 \rho) - (z - I_2) \log(1 - \rho_2 \rho)]
\]

\[
 \frac{\rho_1 \rho_2 + 1 \rho^2}{\rho_1 \rho_2 - 1} = \log \rho.
\]
4.2. First caustic.

5. Best rational approximation to $e^{-(An+B)z}$ on a real interval

Best rational approximation to $\exp(-z)$ on a given real interval, say $[0,c]$ has a strict equioscillating error function, as seen here with $e^{z} - p_n(z)/q_n(z)$ on $[0,1]$, for $n = 1,2$: 0.00158

5.1. Root asymptotics.

Finally, the error decreases like $\rho^n$, with

$$\log \frac{1}{\rho} = \pi \frac{\alpha g(K - E)(K' - E') - EE'}{(\alpha g - 1)E(K - E)}$$

(10)

5.2. Strong asymptotics.

Consider rational approximants to functions $f^n g$, and suppose that the Hermite-Walsh error formula can already be written as

$$f^n(z)g(z) = \frac{p_n(z)}{q_n(z)} \sim e^{\mathcal{W}_n(z)} \frac{1}{2\pi i} \int_C f^n(t)g(t)e^{-\mathcal{W}_n(t)} \frac{dt}{z-t}.$$

Aptekarev [1] established in some cases a more accurate picture $\mathcal{W}_n = 2n \mathcal{V} + \tilde{\mathcal{V}} + o(1)$ (strong asymptotics, also called first order asymptotics by Nuttall). I give here a probably very sloppy account of Aptekarev’s wonderful results (to be available soon):

*Also sprache Aptekarev: $\tilde{\mathcal{V}}$ is (multivalued) analytic outside $E \cup F$, with a period $2\pi i$ about $F$, and $-2\pi i$ about $E$, corresponding to a positive unit charge on $F$, and a negative unit charge on $E$, with $\mathcal{V}_+ + \mathcal{V}_-$ constant on $E$, $\mathcal{V}(z)_+ + \mathcal{V}(z)_- + 2 \log g(z) = -$ another constant on $F$, and finally $\mathcal{V}(z) \equiv \text{const.} + o(1)$ when $z \to \infty$ (if $E$ and $F$ are bounded).*
Moreover, the error norm is $E_n \sim 2 \rho^a \rho$, where $2 \log \rho = \Re \{ (\tilde{\nu}_+^l (z) + \tilde{\nu}_-^l (z))_E - [\tilde{\nu}_+^l (z) + \tilde{\nu}_-^l (z) + 2 \log g(z)]_F \}.

$$\rho_0 = \exp \left( -\frac{\pi K^l}{2} \right).$$ And for any $B$, $\nu_B^l = \frac{2B}{A'} \nu^l + \left( 1 - \frac{2B}{A} \right) \nu_0$ does the trick, see Meinguet [7] for such relations. So,

$$\rho_B = \rho_0^{B/A} \rho_0^{(1-2B/A)}.$$ and we just have to get $\rho_0 = \exp(-1/C)$, where $C$ is the plain condenser capacity of $(E,F)$.

Bibliography


