Asymptotics and super asymptotics for best rational approximation error norms to the exponential function (the '1/9' problem) by the Carathéodory-Fejér method.

Alphonse P. Magnus
Institut Mathématique, Université Catholique de Louvain
Chemin du Cyclotron 2
B-1348 Louvain-la-Neuve
Belgium
E-mail: magnus@anma.ucl.ac.be
web: http://www.math.ucl.ac.be/~magnus/

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Abstract. Let E_n be the error norm of the best L_∞ rational approximation of degree n to the exponential function $\exp(-t)$ on $[0,\infty)$. Grounds are given for setting the conjectured limit $E_n/q^n \to 2q^{1/2}$ when $n \to \infty$, where q is the known constant '1/9'= 1/9.28902549192081891875544943595174506103169486 77..., based on the singular values and functions of the relevant Hankel operator (Carathéodory-Fejér's method). Moreover, hints are given according to which a valuable asymptotic expansion of E_n should also contain n^{th} powers of new constants $q_1 = \text{'}1/56\text{'}$, $q_2 = \text{'}1/240\text{'}$, etc.
Keywords: Rational approximation, Carathéodory Fejér's method, '1/9' problem.

1. Introduction

Best rational approximation to $\exp(-t)$ on $0 \le t < \infty$ has been much worked since [5]: as recalled in [3] and [26] chap. 2, the best L_∞ error norms

$$E_n = \min_{P_n,Q_n} \max_{0\le t<\infty} \left| e^{-t} - \frac{P_n(t)}{Q_n(t)} \right|,$$

where the minimum is taken on the real polynomials P_n and Q_n of degree $\le n$, have been found to decrease geometrically. The rate of decrease, once thought to be 1/9, later called '1/9', has been determined in [11]. This will be recalled briefly in section 2.
On the other hand, quasioptimal rational approximations based on the Carathéodory-Fejér's (CF) method have been shown in [24] to be close to the optimal ones (see section 3). Table I shows values of E_n (from [3]) and eigenvalues λ_n of the appropriate Hankel matrix (from [15] and [24]): the relative closeness of the $\sigma_n = |\lambda_n|$'s to the E_n's is amazing, a tentative explanation will be given in section 5. Following [15], more asymptotic spectral properties of the relevant Hankel matrix are studied here. More precisely, a description of the asymptotic behaviour of the eigenfunctions of a spectrally equivalent integral Hankel operator will be attempted here (in section 4).
The following conjecture will be presented:

Conjecture 1. *E_n/q^n and $|\lambda_n|/q^n \to 2q^{1/2}$ when $n \to \infty$, where $q = \text{'}1/9\text{'}$, the constant described in [11] (Theorem 2) and [15].*

2. Exponential behaviour of the error norm.

Let us first consider that the approximation P_n/Q_n interpolates $\exp(-t)$ at the positive numbers $t_1,\dots,t_{n+1}$. From the Hermite-Walsh remainder formula,

$$e^{-t} - \frac{P_n(t)}{Q_n(t)} = \frac{Z_{n+1}(t)}{Q_n(t)} \frac{1}{2\pi i} \int_C \frac{Q_n(\tau)}{Z_{n+1}(\tau)} e^{-\tau} \frac{d\tau}{\tau - t},$$

TABLE 1. Error norms and eigenvalues.

n	E_n	λ_n
0	5.000000000000000E-01	5.601715174207940E-01
1	6.683104216185045E-02	-6.680573308019967E-02
2	7.358670169580528E-03	7.355581867871742E-03
3	7.993806363356878E-04	-7.994517064498902E-04
4	8.652240695288851E-05	8.652095258749368E-05
5	9.345713153026646E-06	-9.345740936352446E-06
6	1.008454374899671E-06	1.008453857122026E-06
7	1.087497491375248E-07	-1.087497586430036E-07
8	1.172265211633491E-08	1.172265194363526E-08
9	1.263292483322314E-09	-1.263292486435714E-09
10	1.361120523345448E-10	1.361120522787731E-10
11	1.466311194937487E-11	-1.466311195036850E-11
12	1.579456837051239E-12	1.579456837033622E-12
13	1.701187076340353E-13	-1.701187076343463E-13
14	1.832174378254041E-14	1.832174378253495E-14
15	1.973138996612803E-15	-1.973138996612899E-15
16	2.124853710495224E-16	2.124853710495207E-16
17	2.288148563247892E-17	-2.288148563247895E-17
18	2.463915737765169E-18	2.463915737765169E-18
19	2.653114658063313E-19	-2.653114658063313E-19
20	2.856777383549094E-20	2.856777383549094E-20

where $Z_{n+1}(t) = (t-t_1)\dots(t-t_{n+1})$, and where the parabola-like contour C encloses t and $[0,\infty)$. One can then choose Q_n, Z_{n+1} and C (which may depend on n) such that the absolute value of the rational function Z_{n+1}/Q_n is much smaller on $[0,\infty)$ than on C (Zolotarev problem). A geometric decrease of the error norms can then already be exhibited ([7] chap.6), but happens to be larger than the observed value. This is explained by the numerous phase changes of Q_n/Z_{n+1} on C, making the integral much smaller that the integral of the absolute value. The proper way to go further is to take into account n more interpolation points $t_{n+2},\dots,t_{2n+1}$, this amounts to the property that Q_n is orthogonal to polynomials of degree $< n$ with respect to the complex weight $\exp(-\tau)/Z_{2n+1}(\tau)$ on C, where $Z_{2n+1}(t) = (t-t_1)\dots(t-t_{2n+1})$. We then have a more convenient remainder formula

$$e^{-t} - \frac{P_n(t)}{Q_n(t)} = \frac{Z_{2n+1}(t)}{Q_n^2(t)} \frac{1}{2\pi i} \int_C \frac{Q_n^2(\tau)}{Z_{2n+1}(\tau)} e^{-\tau} \frac{d\tau}{\tau - t}. \tag{1}$$

Using and extending results of Stahl ([21]; see [22]- [23] for other striking examples of Stahl's *savoir-faire*), Gončar and Rahmanov [11] showed that the interpolation points and the zeros of Q_n tend to be distributed for large n in such a way that $[Q_n^2(\tau)/Z_{2n+1}(\tau)]^{1/n}$ behaves essentially as $\exp(-2\mathcal{V}(\tau/n))$, where the real part of $\mathcal{V}$ is constant on $[0,\infty)$, the real part of $2\mathcal{V}(z)+z$ is another constant on a fixed cut F (which has to be found), and where the phase of $\exp(2\mathcal{V}(z)+z)$ takes opposite values on the two sides of F (from the Stahl-Gončar-Rahmanov symmetry property about the normal derivatives of the real part of $2\mathcal{V}+z$ and the Cauchy-Riemann conditions translating this as a property of the tangent derivatives of the imaginary part of $2\mathcal{V}+z$, i.e., the phase of the exponential). From the values on $[0,\infty)$ and F, the limit '1/9' of the n^{th} root of the norm of (1) could be established (Theorem 2 of [11]).

Here is another way to recover an approximate description of the error function, using a change of variables which will be useful later in connection with the CF method: $Z_{2n+1}(t)/Q_n^2(t)$ is very close to equioscillating on $[0,\infty)$, let $\tilde{Q}_n$ be the polynomial whose zeros are the square roots with positive real parts

TABLE 2. '1/9' related data.

$k = \sin\theta_1$	=	0.9089085575485414782361189087447935049010139693404 1
$k' = \cos\theta_1$	=	0.41699548440604205639041957807087776692610248051382
$1/q = \exp(\pi K'/K)$	=	9.28902549192081891875544943595174506103169486775012
K	=	2.32104973253042114734283739983633918849213061106173
$E = K/2$	=	1.16052486626521057367141869991816959424606530553086
K'	=	1.64669144431946837372958069030713103423036178930922
E'	=	1.50010688965199892576311071782207995131063998866470
$\xi_1, \xi_2 = \pi(k' \pm ik)/K$	=	0.564412701731271 $\pm$ 1.230228033100522 i

of $-n^{-1}$ times the zeros of Q_n, i.e., if $p_1, \ldots, p_n$ are the zeros of $\tilde{Q}_n$, $-np_1^2, \ldots, -np_n^2$ are the zeros of Q_n, and $\tilde{Q}_n(u)\tilde{Q}_n(-u) = Q_n(-nu^2)$. Then, $Z_{2n+1}(-nu^2)/Q_n^2(-nu^2) \sim [\tilde{Q}_n(u)/\tilde{Q}_n(-u)]^2 + [\tilde{Q}_n(-u)/\tilde{Q}_n(u)]^2$ is a good guess relating the distribution of the interpolation points on $u \in$ the imaginary axis ($t = -nu^2 \geq 0$) to the poles of the approximation. According to [11], the p_k's tend to be distributed on a fixed locus Γ: $\tilde{Q}_n(-u)/\tilde{Q}_n(u) \sim \exp(n\Phi(u))$, where

$$\Phi(u) = \int_\Gamma \log \frac{-u-p}{u-p}\, d\mu(p), \tag{2}$$

where μ is a (still unknown) positive measure of unit total weight on Γ (still unknown too). Remark that Φ is an odd function defined outside Γ and $-\Gamma$, that $\Phi(u)$ is a pure imaginary number when u is imaginary, because the distribution μ is symmetric about the real axis: $d\mu(p) = d\mu(\overline{p})$, that its expansion about 0 starts as $\Phi(u) = 2\int_\Gamma p^{-1} d\mu(p)\, u + \cdots$, and that $\Phi(\infty) = \pi i$ (as $\int_\Gamma d\mu(p) = 1$). Then, (1) becomes, with $t = -nu^2$ and $\tau = -n\xi^2$:

$$\begin{aligned} e^{nu^2} - \frac{P_n(-nu^2)}{Q_n(-nu^2)} &= \frac{Z_{2n+1}(-nu^2)}{Q_n^2(-nu^2)} \frac{1}{2\pi i} \int_\Gamma \frac{Q_n^2(-n\xi^2)}{Z_{2n+1}(-n\xi^2)} e^{n\xi^2} \frac{2\xi}{\xi^2 - u^2}\, d\xi \\ &\sim \left(e^{2n\Phi(u)} + e^{-2n\Phi(u)}\right) \frac{1}{2\pi i} \int_\Gamma \frac{e^{n\xi^2}}{e^{2n\Phi(\xi)} + e^{-2n\Phi(\xi)}} \frac{2\xi}{\xi^2 - u^2}\, d\xi. \end{aligned} \tag{3}$$

The connection with the function $\mathcal{V}$ of [11] is $\Phi(u) = -\mathcal{V}(-u^2) +$ constant.

The following expression of Φ as an elliptic integral will be needed:

$$\Phi(u) = \pi i + \int_{+i\infty}^{u} \left(\frac{u^2}{\xi^2} - 1\right) \left[\left(1 - \frac{\xi^2}{\xi_1^2}\right)\left(1 - \frac{\xi^2}{\xi_2^2}\right)\right]^{-1/2} d\xi, \tag{4}$$

where ξ_1 and ξ_2 are two complex conjugate numbers which will soon be determined, and where $\left[\left(1 - \frac{\xi^2}{\xi_1^2}\right)\left(1 - \frac{\xi^2}{\xi_2^2}\right)\right]^{-1/2} \sim |\xi_1|^2/\xi^2$ when ξ is large. In order to see that (4) has the desired properties, we first establish the link with the Jacobi notation for elliptic integrals through the change of variables ([17]) $\eta = 2i/[\xi/|\xi_1| - |\xi_1|/\xi]$, $v = 2i/[u/|\xi_1| - |\xi_1|/u]$, then, with $\xi_1 = |\xi_1|\exp(i\theta_1)$, $\xi_2 = |\xi_1|\exp(-i\theta_1)$, $k = \sin\theta_1$, $\xi = i|\xi_1|(1 + \sqrt{1-\eta^2})/\eta$ if $|\xi| \geq |\xi_1|$, $\xi = i|\xi_1|(1 - \sqrt{1-\eta^2})/\eta$ if $|\xi| \leq |\xi_1|$, $\left[\left(1 - \frac{\xi^2}{\xi_1^2}\right)\left(1 - \frac{\xi^2}{\xi_2^2}\right)\right]^{-1/2} d\xi = \pm\frac{i}{2}|\xi_1|(1 - k^2\eta^2)^{-1/2}(1-\eta^2)^{-1/2} d\eta$ (with the $+$ sign if $|\xi| \geq |\xi_1|$, with the minus sign otherwise). Then, $\Phi(0) = \pi i - \int_{+i\infty}^{0} \left[\left(1 - \frac{\xi^2}{\xi_1^2}\right)\left(1 - \frac{\xi^2}{\xi_2^2}\right)\right]^{-1/2} d\xi = \pi i - i|\xi_1|K$, where $K = \int_0^1 (1-\eta^2)^{-1/2}(1-k^2\eta^2)^{-1/2} d\eta$ is the complete elliptic integral of first kind of modulus k. The condition $\Phi(0) = 0$ yields the first condition $|\xi_1|K = \pi$. Next, we want $u - \Phi'(u)$ to take opposite values on the sides of Γ, in particular $\Phi'(u)/u = 1$ at ξ_1:
$u^{-1}\Phi'(u) = -i|\xi_1|^{-1} \int_0^v [1 + (1-\eta^2)^{1/2}]^{-2} \eta^2 (1-\eta^2)^{-1/2}(1-k^2\eta^2)^{-1/2} d\eta =$

$$-\frac{i}{|\xi_1|}\left[2(1-k^2v^2)^{1/2}\frac{1-(1-v^2)^{1/2}}{v}+K(v;k)-2E(v;k)\right],$$

where $K(v;k)=\int_0^v(1-\eta^2)^{-1/2}(1-k^2\eta^2)^{-1/2}d\eta$ and $E(v;k)=\int_0^v(1-\eta^2)^{-1/2}(1-k^2\eta^2)^{1/2}d\eta$ are the incomplete elliptic integrals of first and second kind (the results of [2] section 3.1 are used, but the notation is slightly different here [Akhiezer uses K as the argument of E]). At $u=\xi_1$, $\eta=1/k$, where one has $K(k^{-1};k)=K+iK'$ and $E(k^{-1};k)=E+i(EK'-\pi/2)/K$ (see bottom of [2] section 3.1), so that $\Phi'(\xi_1)/\xi_1=1-(i/\pi)(K-2E)(K+iK')$ whence the condition $K=2E$ discussed in various forms (going back to Halphen 1886!) in [11] [15] [26]. The condition $K=2E$ could be obtained more easily ([17]) by expressing $\Phi'(\xi_1)/\xi_1-\Phi'(\xi_2)/\xi_2=0$, but the present derivation showed that $\Phi'(\xi_1)/\xi_1=1$ as well. It is then clear that $\Phi'(u)-u$ takes opposite values on the sides of Γ, as $\Phi'(u)-u=$ $\pm 2u\int_{\xi_1}^{u}\xi^{-2}\left[\left(1-\frac{\xi^2}{\xi_1^2}\right)\left(1-\frac{\xi^2}{\xi_2^2}\right)\right]^{-1/2}d\xi$. If we call Φ_1 the analytic continuation of $\Phi=\Phi_0$ across Γ, we see that $\Phi_1'(u)=2u-\Phi'(u)$, therefore $\Phi_1(u)=u^2-\Phi(u)+$ constant. This constant is found from $\Phi_1(\xi_1)=\Phi(\xi_1)$, hence, by $\Phi(u)=\pi i+u\Phi'(u)/2-i|\xi_1|K(v;k)/2$: $\Phi(\xi_1)=\pi i+\xi_1^2/2-i|\xi_1|(K+iK')/2=\pi i/2+\xi_1^2/2+\pi K'/(2K)$, so

$$\Phi_1(u)=u^2-\Phi(u)+\pi i-\log q, \tag{5}$$

where $q=\exp(-\pi K'/K)$ is the famous '1/9' constant. Similarly, continuation of Φ across $-\Gamma$ yields

$$\Phi_2(u)=-u^2-\Phi(u)+\log q+\pi i. \tag{6}$$

The real part of $\xi^2-2\Phi(\xi)$ is the constant $\log q$ on the arc Γ, as it should. The numerical values are recalled in Table II. The point ξ_1 ($\sqrt{-a}$ in [11]) is marked on figure 2; the arc Γ is almost the rectilinear segment joining ξ_1 and $\xi_2=\overline{\xi_1}$ (see also fig.1 ($m=1$) of [15] where ξ_1 and ξ_2 are called X_2 and X_1). Interesting expansions of Φ are,

$$\text{near } 0: \Phi(u)=2u-(\xi_1^{-2}+\xi_2^{-2})u^3/3-(\xi_1^{-4}/4+\xi_1^{-2}\xi_2^{-2}/6+\xi_2^{-4}/4)u^5/5+\cdots \tag{7}$$

$$\text{near } +i\infty: \Phi(u)=\pi i+2|\xi_1|^2/(3u)+|\xi_1|^2(\xi_1^2+\xi_2^2)/(15u^3)+\cdots \tag{8}$$

3. Use of the Carathéodory-Fejér's method.

Carathéodory and Fejér (and also Schur and Takagi) studied several problems of estimation of Fourier and Laurent series and gave very constructive answers. Initially, the problem was: given a bounded complex sequence $g_0,g_1,\ldots$, estimate $\nu=\sup|\sum_0^\infty g_k\zeta^k|$ on $|\zeta|<1$. Looking at the norms of the partial sums $\|\sum_0^N g_k\zeta^k\|_\infty$ would be most clumsy and would suggest wrong answers in some cases. The method of Carathéodory and Fejér is the following: for each $N>1$, construct the expansion $\sum_0^{N-1}g_k\zeta^k+\sum_N^\infty g_k^{(N)}\zeta^k$ of *lowest possible* L_∞ norm ν_N. It is then shown that the ν_N's form an increasing sequence with limit ν, moreover, that ν_N is the largest singular value of a Hankel matrix constructed with $g_0,\ldots,g_{N-1}$ Other singular values become useful if n poles in $|\zeta|<1$ are allowed, by adding an expression of the form $\sum_N^\infty g_k^{(N)}\zeta^k/\sum_0^n e_k\zeta^{n-k}$.

The appropriate setting for approximating a given series $f_+(z)=\sum_1^\infty a_kz^k$ with real coefficients is to look for a function of the form

$$r_n(z)=\sum_{-\infty}^n d_kz^k/\sum_0^n e_kz^k=\sum_{-\infty}^0 d_{k+n}z^k/\sum_0^n e_{n-k}z^{-k},$$

where the numerator series converges in $|z|>1$, with n poles in $|z|>1$, and such that the norm of $\sum_1^N a_kz^k-r_n(z)$ is the lowest possible on $|z|=1$ ([24], section 1; the connection with the original CF problem is made by $z=1/\zeta, g_k=a_{N-k}$). If the given function is continuous on the unit circle, we may consider infinite Hankel matrices ($N\to\infty$) as compact operators ([1], [4], [16]). We then arrive at an error of the form

$$f_+(z)-r_n(z)=\lambda_n z\frac{u_1+u_2z+\ldots}{u_1+u_2z^{-1}+\ldots} \tag{9}$$

with real λ_n, $u_1, u_2, \ldots$, and where $u_1 + u_2 z + \cdots$ has exactly n zeros in $|z| < 1$, so that the error curve is a perfect circle about the origin of radius $\sigma_n = |\lambda_n|$ and winding number $2n+1$. Factorizing $u_1 + u_2 z^{-1} + \cdots$ as $\sum_0^n e_{n-k} z^{-k}$ times a second factor, which is another expansion in z^{-1}, we see that gathering the positive powers in the product of (9) by $u_1 + u_2 z^{-1} + \cdots$, we have $a_k u_1 + a_{k+1} u_2 + \cdots = \lambda_n u_k, k = 1, 2, \ldots$, i.e., the eigenproblem $H\mathbf{u} = \lambda_n \mathbf{u}$, where H is the infinite Hankel matrix $[a_{i+j+1}]_{i,j=0}^{\infty}$, and where λ_n is the $(n+1)^{\text{th}}$ eigenvalue of H in decreasing order of the absolute values: $|\lambda_0| \geq |\lambda_1| \geq \ldots$ This brief description supposes that $|\lambda_n|$ is not repeated: $|\lambda_{n-1}| > |\lambda|_n > |\lambda_{n+1}|$, see [1], [9], [12], [16] for a more complete discussion including complex functions and degenerate cases.

The expression $(a_0 + r_n(z) + r_n(1/z))/2$ yields then an approximation with n poles outside $[-1, 1]$ in the $x = (z + 1/z)/2$ plane to $F(x) = (a_0 + f_+(z) + f_+(1/z))/2 = a_0/2 + \sum_1^\infty a_k T_k(x)$, with an error function which equi-oscillates exactly at $2n+2$ points of $[-1, 1]$. Finally, a Chebyshev economization is performed on $(a_0 + r_n(z) + r_n(1/z))/2$ in order to get a rational function of x [24].

Rational approximations to $\exp(-t)$ on $[0, \infty)$ are first translated as rational functions of $x = (c-t)/(c+t)$, $c > 0$, so that we have to consider the Chebyshev expansion $F(x) = \exp(c(x-1)/(x+1))$, or the Laurent coefficients of $F((z+1/z)/2) = \exp(c(z-1)^2/(z+1)^2)$. Any positive value of c yields a spectrally equivalent Hankel matrix H, but the most explicit setting appears when one considers $c \to \infty$:

Theorem 1. *The eigenvalues of the Hankel matrix H are the same as those of the integral Hankel operator $\mathcal{H}$ given on $L_2(0, \infty)$ by*

$$(\mathcal{H}f)(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(x+y)^2} f(y)\, dy \tag{10}$$

Indeed, the meaningful part of the integral $a_k = (\pi i)^{-1} \int_{|z|=1} \exp(c(z-1)^2/(z+1)^2) z^{-k-1} dz$ is a neighbourhood of $z = 1$, as $c(z-1)^2/(z+1)^2$ has a strongly negative real part when $z \neq 1$ on the unit circle. Let $z = 1 + 2iuc^{-1/2}$: $a_k \sim 2\pi^{-1} c^{-1/2} \int_{-\infty}^\infty \exp(-u^2 - 2ikc^{-1/2}u) du = 2(\pi c)^{-1/2} \exp(-k^2/c)$ when $c \to \infty$. The elements

$$(H\mathbf{u})_k = \sum_{m=0}^\infty a_{k+m} u_m \sim 2(\pi c)^{-1/2} \sum_{m=0}^\infty \exp(-(k+m)^2/c) u_m$$

behave essentially as values of a function of the variable $kc^{-1/2}$: let $x = kc^{-1/2}$, $y = mc^{-1/2}$, $f(y) = u_m c^{1/4}$, (so that $\sum_0^\infty u_m^2 = c^{-1/2} \sum_0^\infty [f(mc^{-1/2})]^2 \sim \int_0^\infty (f(y))^2 dy$ remains constant), $g(x) = (H\mathbf{u})_k c^{1/4} \sim 2(\pi c)^{-1/2} \sum_{m=0}^\infty \exp(-(k+m)^2/c) c^{1/4} u_m \sim 2\pi^{-1/2} \int_0^\infty \exp(-(x+y)^2) f(y)\, dy$.
In particular, if $H\mathbf{u} = \lambda \mathbf{u}$, then $\mathcal{H}f = \lambda f$, with the same λ.

The eigenfunctions $f_0, \ldots, f_9$ corresponding to $\lambda_0, \ldots, \lambda_9$ of $\mathcal{H}$ are plotted in figure 1. The study of the asymptotic behaviour of f_n for large n should shed some light on the value of λ_n. If a uniform asymptotic formula $\tilde{f}_n$ could be found so that the L_2 norm $\|\mathcal{H}\tilde{f}_n - \tilde{\lambda}_n \tilde{f}_n\| < \varepsilon_n$ could hold, with $\|\tilde{f}_n\| = 1$ and $\varepsilon_n / |\tilde{\lambda}_n| \to 0$ when $n \to \infty$, then $\tilde{\lambda}_n$ would be a valuable asymptotic estimate of λ_n. Unfortunately, the subject is not yet so advanced.

4. Asymptotic behaviour of the eigenfunctions.

The typical oscillations of the eigenfunctions shown by figure 1 suggest that $f_n(x)$ contains the exponential of n times a function taking imaginary values (from 0 to πi) on a part of the real axis. Actually, as the transition point separating oscillatory behaviour from monotonously decreasing behaviour seems to increase like $n^{1/2}$, we expect an asymptotic formula involving the exponential of n times a fixed function of $x/n^{1/2}\ldots$ For a more accurate guess, we know that if the CF method is successful, the poles of the CF approximant are very close to the poles of the best rational approximation [24], so $u_1 + u_2 z^{-1} + \cdots$ and $\tilde{Q}_n(u)$ should have common factors, with $n^{1/2} u = c^{1/2}(z-1)/(z+1)$. When $c \to \infty$: $\sum_1^\infty u_k z^k \sim \sum_1^\infty f_n(kc^{-1/2})[1 + 2(n/c)^{1/2} u]^k \sim c^{1/2} \int_0^\infty f_n(\xi) \exp(-2n^{1/2} u\xi) d\xi$,i.e., the Laplace transform of f_n should contain at least the factor $\exp(-n\Phi(u))$. This asks for the inverse Laplace transform of $\exp(-n\Phi)$: $f_n(\xi)$ is expected to involve $\int_{Br} \exp[-n\Phi(u/(2n^{1/2})) + \xi u] du$ on some Bromwich contour. A saddlepoint analysis predicts this integral to behave like $\exp[-n\Phi(s/(2n^{1/2})) + \xi s]$, where s is a root of

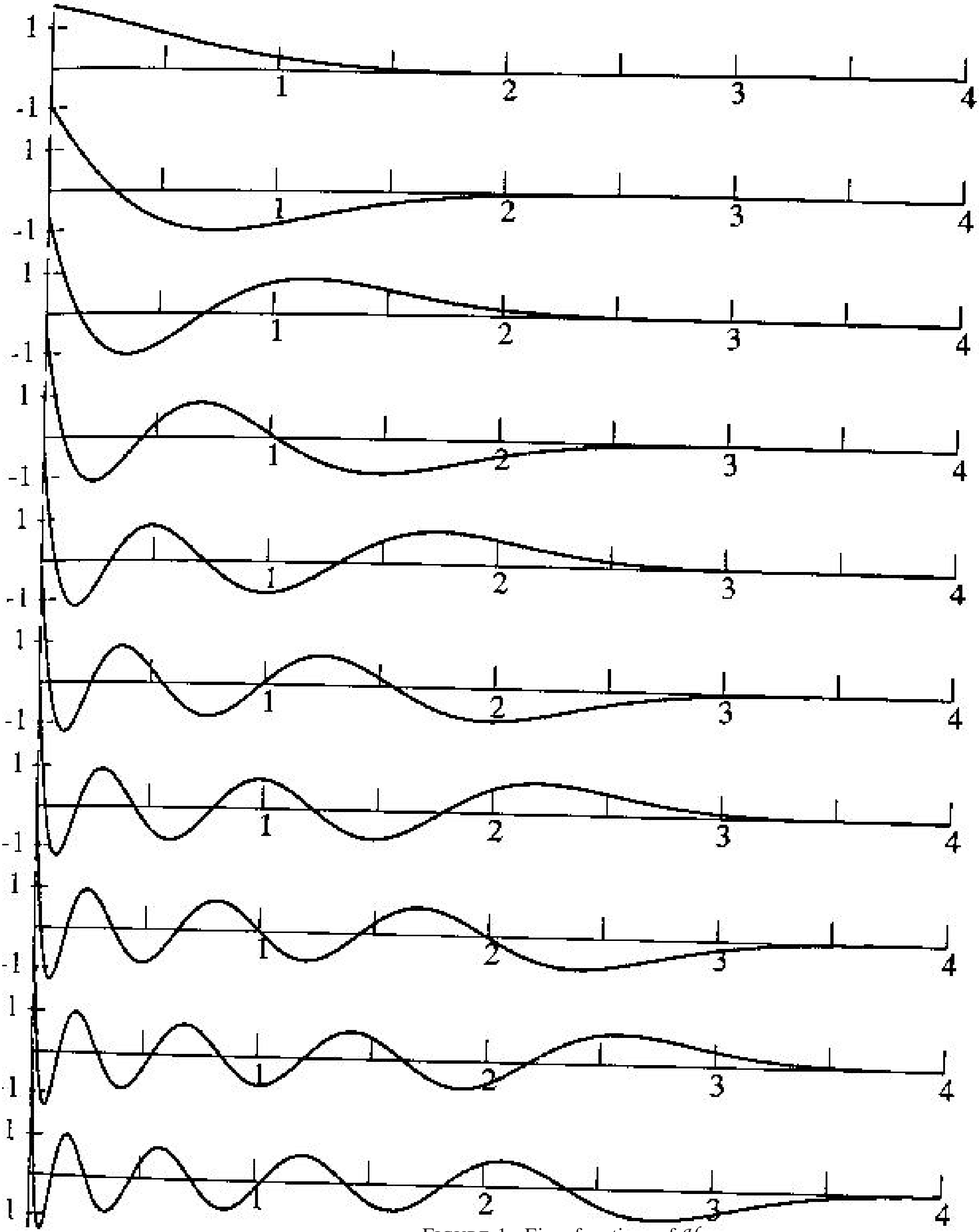

FIGURE 1. Eigenfunctions of $\mathcal{H}$.

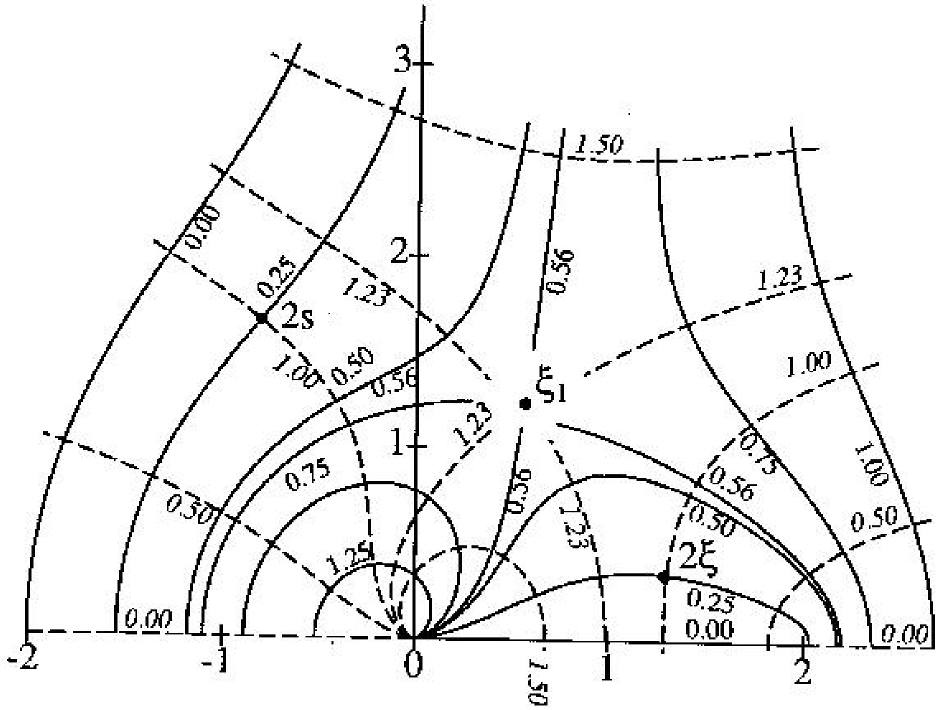

FIGURE 2. Image of the upper right quarter plane by Φ'

$\xi = (n^{1/2}/2)\Phi'(s/(2n^{1/2}))$. So, $s/n^{1/2}$ is a root x of the equation $2y = \Phi'(x/2)$ when $y = \xi/n^{1/2}$ is given. Let Ψ' be the inverse function of Φ', then $x = 2\Psi'(2y)$ and

$$\Phi(x/2) = \pi i + xy - \Psi(2y) \;\text{ if }\; 2y = \Phi'(x/2),\text{ i.e. } x = 2\Psi'(2y) \tag{11}$$

indeed, $\Phi(x/2) - \pi i = \int_\infty^{x/2} \Phi'(\zeta)d\zeta = \int_0^{2y} \eta d\Psi'(\eta) = 2y\Psi'(2y) - \Psi(2y)$, so that $-\Phi(s/(2n^{1/2})) + \xi s = \Psi(2\xi/n^{1/2})$.

Figure 2 shows the images of horizontal and vertical lines of the upper right quarter plane $\mathrm{Re}u \geq 0$ and $\mathrm{Im}u \geq 0$ by Φ' and its analytic continuation $\Phi'_1(u) = 2u - \Phi'(u)$ across the cut $\Gamma = [\xi_1, \overline{\xi}_1]$. The net of solid lines (images of verticals Reu=0,0.25,0.50,etc. by Φ') and dashed lines (images of horizontals Imu=0,0.25,0.50, etc. by Φ') cover a region extending up to -2 to the left ($-2 = \Phi'_1(0)$). This means that Ψ' maps this region to the upper right quarter plane. For instance, the point $1.311 + 0.344i$ marked 2ξ in figure 2 is at the intersection of the solid line of label 0.25 and the dashed line of label 1.00: this means that $1.311 + 0.344i = \Phi'(0.25 + i)$ and also $0.25 + i = \Psi'(1.311 + 0.344i)$.

From the expansions (7) and (8):

$$\text{near } 0: \Psi(v) = 2\left(-\frac{2|\xi_1|^2}{3}\right)^{1/2} v^{1/2} + \cdots, \tag{12}$$

$$\text{near } 2: \Psi(v) = \pi i + \frac{2}{3}\left(-\frac{3}{2(\xi_1^{-2} + \xi_2^{-2})}\right)^{1/2} (v-2)^{3/2} + \cdots, \tag{13}$$

So, 0 and 2 are singular points of Ψ which is pure imaginary in $[0, 2]$. This seems to explain the observed behaviour of the eigenfunctions f_n:

Conjecture 2. *The eigenvalues* $\lambda_0, \lambda_1, \ldots$ *of the Hankel integral operator (10), with* $|\lambda_0| \geq |\lambda_1| \geq \ldots$, *satisfy*

$$\lambda_n \sim 2(-1)^n q^{n+1/2} \text{ when } n \to \infty. \tag{14}$$

The corresponding eigenfunctions $f_0, f_1, \ldots$ of (10) satisfy

$$f_n(x) \sim A(x/\nu^{1/2})\exp[\nu\Psi(2x/\nu^{1/2})] + B(x/\nu^{1/2})\exp[-\nu\Psi(2x/\nu^{1/2})]\,, n \to \infty. \tag{15}$$

with $\nu = n + 1/2$, and where A and B are (still unknown) fixed functions, and where Ψ is such that Ψ' is the inverse function of Φ' (with Φ given by (2) and (4)).

To explore the validity of this conjecture, let us introduce the proposed formula of $f_n(\nu^{1/2}y)$ in (10):

$$(\mathcal{H}f_n)(\nu^{1/2}\xi) \sim (2\nu/\pi^{1/2})\int_0^\infty \exp(-\nu(\xi+\eta)^2)[A\exp(\nu\Psi(2\eta)) + B\exp(-\nu\Psi(2\eta))]d\eta.$$

We estimate these integrals by saddlepoint analysis:

$$(\mathcal{H}f_n)(\nu^{1/2}\xi) \sim 2A(s_1)(1-2\Psi''(2s_1))^{-1/2}\exp[-\nu((\xi+s_1)^2 - \Psi(2s_1)]+ \\ +2B(s_2)(1+2\Psi''(2s_2))^{-1/2}\exp[-\nu((\xi+s_2)^2+\Psi(2s_2)],$$

where s_1 and s_2 are roots of $\xi + s = \pm\Psi'(2s)$, found through Φ' and its continuations across Γ and $-\Gamma$: let 2ξ be in the range of Φ', say $2\xi = \Phi'(w)$. Then, as $\Phi'(w) + \Phi_1'(w) = 2w$ and $\Phi'(w) + \Phi_2'(w) = -2w$, $s_1 = \Phi_1'(w)/2$ and $s_2 = \Phi_2'(w)/2$ are valid solutions. For instance, figure 2 shows a point $2\xi = 1.311 + 0.344i$ corresponding to $w = 0.25 + i$. The point marked $2s$ is $\Phi_1'(w) = 2s_1 = -0.811 + 1.656i$ ($2s_2$ is not on figure 2).

Now, using (5), (6) and (11): $-(\xi+s_1)^2 + \Psi(2s_1) = -w^2 - \Phi_1(w) + 2s_1w + \pi i = -w^2 + \Phi(w) - w^2 + \log q - \pi i + 2s_1w + \pi i = -\Psi(2\xi) + \log q + \pi i$, $-(\xi+s_1)^2 - \Psi(2s_2) = -w^2 + \Phi_2(w) - 2s_1w - \pi i = -w^2 - \Phi(w) - w^2 + \log q + \pi i - 2s_2w - \pi i = \Psi(2\xi) + \log q - \pi i$, so, we see that the exponentials of $\pm\nu\Psi(2\xi)$ are recovered. Eigenfunctions should therefore satisfy

$$2A(s_1)[1-2\Psi''(2s_1)]^{-1/2}\exp(\nu\pi i)q^\nu = \lambda_n B(\xi)\,, \tag{16}$$

$$2B(s_2)[1+2\Psi''(2s_2)]^{-1/2}\exp(-\nu\pi i)q^\nu = \lambda_n A(\xi)\,, \tag{17}$$

where $s_1, s_2 = \Phi_{1,2}(w)/2 = -\xi \pm \Psi'(2\xi)$ if $2\xi = \Phi'(w)$.

If this is enough to guess that λ_n will behave like $(-1)^n q^n$, these equations do not give clear indications on what the functions A and B should be. Moreover, these functions are probably discontinuous (on Stokes lines), as a result of representing the entire function f_n by an asymptotic formula involving functions with branch-points. So, as $\Psi(\xi) \to +\infty$ when $\xi \to +\infty$, and as $f_n \in L_2$, one must have $A(\xi) \equiv 0$ in a region containing $[2,\infty)$, but $A(\xi) \not\equiv 0$ in $[0,2]$ where the two imaginary exponentials are needed in order to explain the oscillations of f_n. This state of things is common in differential equations discussions (JWKB-Liouville-Green-Steklov theory [6]), where one has connection formulas. Assuming similar tools to be valuable here, let us look for a model entire function sharing the properties of $f_n(\nu^{1/2}\xi)$ near $\xi = 1$: as the main behaviour is $\exp(\nu\Psi(2\xi)) \sim$ const. $\exp((\xi-1)^{3/2})$, let us choose the Airy function Ai which shares this behaviour, and is often found in asymptotic estimates ([6] [18] [19]): $f_n(\nu^{1/2}\xi) \sim$ const. $(-1)^n$ Ai$((3\nu(\Psi(2\xi) - \pi i)/2)^{3/2}) \sim$ const. $(-1)^n(\Psi(2\xi)-\pi i)^{-1/6}\sin(\nu(-\Psi(2\xi)/i+\pi)+\pi/4)$ near $\xi = 1$ when n is large. This formula (with $\nu = n+1/2$) agrees quite well with the numerical results, moreover, estimates of integrals in terms of Airy functions are known to be valid when two saddle points coalesce ([18], [19]), which is precisely true when $\xi = 1$ ($s_1 = s_2 = -1$). Near $\xi = 0$, $\Psi(2\xi) \sim$ const. $(-\xi)^{1/2}$ and a close model appears now to be the Bessel function const. $J_0(\nu\Psi(2\xi)/i) \sim$ const. $(\Psi(2\xi))^{-1/2}\cos(\nu\Psi(2\xi)/i - \pi/4)$. The satisfactory matching of the oscillating terms $(-1)^n\sin(\nu\Psi/i + \pi/4) = \cos(\nu\Psi/i - \pi/4)$ holds on the whole interval $\xi \in [0,1]$ if $\nu = n+1/2$. This suggests that the ratio $A(\xi)/B(\xi)$ is the constant $\exp(-i\pi/2) = -i$ in a region containing $[0,1]$. If this is true up to $\xi = \xi_1/2$, where $s_1 = \xi = \xi_1/2$, (and where $\Psi''(\xi_1) = 0$) (16) gives indeed $\lambda_n \sim -2i\exp((n+1/2)\pi i)q^{n+1/2}$, i.e., (14).

A more complete asymptotic expansion has been worked in [3], the most concise expression seems to be $E_n \sim 2q^{n+1/2}\exp[-1/(12(n+1/2)) + O(n^{-5})]$.

5. Super and hyper asymptotics: '1/56', etc.

The use of CF as quasi optimal rational approximation is justified by various results (inequalities in [9] [14]) showing how the E_n's can indeed be close to the σ_n's. But Table III shows that the matching is

TABLE 3. Differences between error norms and singular values.

n	$E_n - \sigma_n$	ratios	acceleration
0	-6.017152E-02	-16.619159	16.619159
1	2.530908E-05	-2377.467434	198.775020
2	3.088302E-06	8.195145	68.668900
3	-7.107011E-08	-43.454295	61.246073
4	1.454365E-09	-48.866753	58.541695
5	-2.778333E-11	-52.346699	57.460486
6	5.177776E-13	-53.658797	56.901326
7	-9.505479E-15	-54.471496	56.591766
8	1.726997E-16	-55.040521	56.417269
9	-3.113400E-18	-55.46979	56.32180
10	5.57717 E-20	-55.8240	56.2763
11	-9.9363E-22	-56.129	56.264
12	1.7617E-23	-56.402	56.275
13	-3.110E-25	-56.65	56.30
14	5.46E-27	-57.	56.3
15	-9.6E-29	-57.	56.
16	1.7E-30	-56.	56.
17	-3.E-32		

TABLE 4. Other rates of decrease: '1/56' etc.

k	s_k	$\Phi'(s_k)/s_k$	$1/q_k$
1	0.72878+ 1.48300i	0.33333	56.690353
2	0.86860+ 1.70178i	0.20000	240.251663
3	0.97785+ 1.87517i	0.14286	846.908936

quite dramatic. Trefethen and Gutknecht have studied classes of functions with Chebyshev coefficients decreasing like powers of ε, to show that if $E_n \sim \sigma_n$ behaves like ε^{2n+1}, $|E_n - \sigma_n|$ could decrease as fast as ε^{5n+3}! ([24], sec.2). Table III shows indeed that the $|E_n - \sigma_n| = |E_n - |\lambda_n||$'s decrease like powers of a number close to 1/56, definitely smaller that '1/9'. How can we explain this phenomenon?

If we consider that the oscillation of the best error function is completely explained by the $\exp(2n\Phi) + \exp(-2n\Phi)$ factor of (3), the amplitude is therefore given by the integral on Γ of $\exp(n\xi^2)[\exp(2n\Phi(\xi) + \exp(-2n\Phi(\xi)]^{-1} = \sum_0^\infty (-1)^k \exp[n(\xi^2 - 2(1+2k)\Phi(\xi))]$, as $\mathrm{Re}\Phi > 0$ on Γ. Of course, '1/9' is the constant modulus of $\exp(\xi^2 - 2\Phi(\xi))$ on Γ. Curiously enough, the integration of the other terms on Γ yield other decreasing exponentials of which the first appears indeed to be close to 1/56... (see Table IV: the exponential $\exp[n(\xi^2 - 2(1+2k)\Phi(\xi))]$ is evaluated at the saddlepoint s_k: $s_k - (1+2k)\Phi'(s_k) = 0$ for $k = 1, 2, \ldots$) It is not clear how to detect these new exponentially decreasing contributions from the numerical sequence $\{E_n\}$ alone, but the comparison of the E_n's and the σ_n's appears to be a lucky circumstance allowing to observe this new phenomenon. Can we find a formula for σ_n suggesting how it can be so close to E_n? The best starting point is probably this one: among other characterizations, σ_n is the smallest possible norm of a Hankel matrix of symbol $B_n\varphi$, where $\varphi(z)$ is here $\exp(c(z-1)^2/(z+1)^2)$ and $B_n(z)$ is a Blaschke product $\prod_1^n (z - a_k)/(1 - za_k)$ ([1], [16]; the a_k's are supposed to be symmetrically placed with respect to the real axis($|a_k| < 1$)). This norm involves only the Laurent coefficients with negative index of $B_n\varphi$: it may be estimated as $(2\pi i)^{-1} \sum_{p=1}^\infty \int_\gamma B_n(\zeta)\varphi(\zeta)\zeta^{p-1} z^{-p} d\zeta = (2\pi i)^{-1} \int_\gamma B_n(\zeta)\varphi(\zeta)(z - \zeta)^{-1} d\zeta$ on $|z| = 1$. By the change of variable $n^{1/2}\xi = c^{1/2}(\zeta - 1)/(\zeta + 1)$, we have an integral involving

$\prod_1^n(\xi - b_k)/(\xi + b_k)\exp(n\xi^2)$. This begins to look to $\exp(n\xi^2)\tilde{Q}_n(\xi)/\tilde{Q}_n(-\xi)$ in the notations of section 2. It should be possible to use the orthogonality arguments of [21] in order to exhibit the square of $\tilde{Q}_n(\xi)/\tilde{Q}_n(-\xi)$ instead. The study of the CF method of approximation of analytic functions would then be on the same level than the existing theory of rational approximation.

Finally, work is presently done on asymptotic expansions involving several exponentially decreasing term: see [18], [20] for super- and hyper-asymptotics. It would be interesting to see if these theories can cover the present phenomenon, in particular if *convergent* asymptotic expansions can be produced:

Conjecture 3. *The best rational approximation error norms E_n and the appropriate CF singular values σ_n have hyper-asymptotic (perhaps convergent) expansions*

$$E_n \sim \sum_{k=0}^{\infty} q_k^n \exp[S_k(n+1/2)], \tag{18}$$

and

$$\sigma_n \sim \sum_{k=0}^{\infty} q_k^n \exp[U_k(n+1/2)], \tag{19}$$

when $n \to \infty$, with $S_k(x) \sim \sum_{m=0}^{\infty} s_{k,m}x^{-2m-1}$, $U_k(x) \sim \sum_{m=0}^{\infty} u_{k,m}x^{-2m-1}$, $U_0(x) \equiv S_0(x)$, and where $q_k = \exp[s_k^2 - 2(1+2k)\Phi(s_k)]$, s_k being the root of $s_k - (1+2k)\Phi'(s_k) = 0$, $k = 0, 1, \ldots$

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The eigenfunctions of Figure 3 have been computed with the LAPACK library (program `DSPEVX`) installed on the Convex C3820 of the University.

Van al die niks te zeggen hebben
zijn die die zwijgen 't aangenaamst. [1]
(From the programme 'Antwerp,
Cultural Capital of Europe 1993').

Special thanks are also due A. Cuyt and L. Wuytack who organized the first Antwerp Conference on Nonlinear Numerical Methods and Rational Approximation in 1987, and all the people who then gave me warm and kind advices on the use of the CF method in the study of the '1/9' problem. At the very least, I should have supplied the missing proofs in [15] for these Proceedings of the Second Conference. I had more than 5 years for achieving that. Of course, I did nothing of the sort, and years passed by without disturbing the peace of my mind. Then, notice came that the abstracts for the Second Conference should be ready for April 1st (no joke) 1993. As all I had to do was to establish an asymptotic formula exhibiting an exponential behaviour with respect to n, I proceeded to fill the following template:

Theorem. *The required eigenfunctions behave like AX^n when n is large, where the functions A and X are $A =$* so and so, *and* $X =$ so and so.

But I did not know neither A nor X on April 1st. Fortunately, the contributors were allowed the new deadline May 1st, it seems that it was known that was in deep trouble. After painful weeks, I got the X function, it is the eerie function $\exp\Psi$ discussed in Section 4. Considering the technical difficulties, I decided to present this triumphal finding in a more cautious way:

Proposition. *The required eigenfunctions behave like AX^n when n is large, where the functions A and X are $A =$* so and so (still to be found) , *and* $X = \exp\Psi$.

(A theorem is a statement for which I think I have a proof, a proposition is a statement for which I have no proof, but pretend to have one).

On September 1st (the conference was to start on September 5th), I still did not know A, so I decided to be honest:

[1] Of all those who have nothing to say, the most agreeable are those who are silent.

Conjecture. *The required eigenfunctions behave like* etc. (basically, Conjecture 2 in Section 4).

I don't know if I shall still be *persona grata* on the Third Conference, but I can promise at most one **Theorem**, perhaps several **Propositions**, quite a number of **Conjectures** and a lot of **Problems**.

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