The elliptic functions and integrals of the '1/9' problem

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We describe the functions needed in the determination of the rate of convergence of best L^∞ rational approximation to $\exp(-x)$ on $[0,\infty)$ when the degree n of the approximation tends to ∞ ('1/9' problem).

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The Ninth ... like a lightning, an earthquake, a flood!
Kurt Pahlen

1. Introduction.

The '1/9' problem is the determination of the rate of convergence of best L^∞ rational approximation to $\exp(-x)$ on $[0,\infty)$. The error E_n has first been found numerically to decrease with the degree n of the approximation asymptotically as the n^{th} power of $\rho = 1/9.289025491920818918755449435951745061031694867 7\ldots$ ([4,19], see [20] for the history). The value of ρ, and the first proof that $\lim_{n\to\infty} E_n^{1/n} = \rho$, are in [7], where it is shown how to build a potential function V vanishing on $E = [0,\infty)$, with $2V + x =$ constant $= -\log\rho$ on an arc F of the complex plane, with equal normal derivatives on the two sides of F.

1.1. Rational interpolation of analytic functions.

Let f be holomorphic in a domain D of the extended complex plane $\overline{\mathbb{C}}$, and E be a closed subset of $\overline{D}$. Our subject-matter will be $D = \mathbb{C}$, $f(z) = e^{-z}$, and $E = [0,\infty]$.

As f is real on the real set E, the best (real) rational approximation R_n of degree n to f on E is known to interpolate normally f at $2n+1$ points of E, say $z_1, z_2,\dots z_{2n+1}$. The error of interpolation of analytic functions is usually discussed through Hermite-Walsh contour integral formula [21,22].

Let us consider first an approximation S_{2n} of degree $2n$, interpolating f at these $2n+1$ points, and with $2n$ poles $p_1, p_2,\dots p_{2n}$. Then, the Hermite-Walsh formula is

$$f(z) - S_{2n}(z) = \frac{1}{2\pi i}\frac{(z-z_1)\dots(z-z_{2n+1})}{(z-p_1)\dots(z-p_{2n})}\int_{C_n}\frac{(t-p_1)\dots(t-p_{2n})}{(t-z_1)\dots(t-z_{2n+1})}\,\frac{f(t)\,dt}{t-z}\ , \qquad (1)$$

where C_n is a contour containing $z_1, z_2,\dots z_{2n+1}$ and z in its interior.

In order to ensure that S_{2n} is the approximant R_n of degree n, and not $2n$, we must have only n true poles, say, $p_1,\dots, p_n$, so, vanishing residues at $p_{n+1},\dots, p_{2n}$, which amounts to the *formal orthogonality conditions*

$$\int_{C_n}\frac{(t-p_1)\dots(t-p_n)\ Q(t)}{(t-z_1)\dots(t-z_{2n+1})}\,f(t)\,dt\ = 0 \qquad (2)$$

for all polynomials Q of degree $< n$.

Then, with the *complex potential*

$$\mathcal{V}_n(z) = \frac{1}{2n}\log\left[\frac{(z-z_1)\dots(z-z_{2n})}{(z-p_1)\dots(z-p_{2n})}\right] \qquad (3)$$

we estimate the exponential behaviour of the error norm of (1) on E as

$$\|f-R_n\|_\infty = \|f-S_{2n}\|_\infty = \\ K_n \exp 2n\left[\max_{z\in E} V_n(z) - \min_{t\in C_n}\left(V_n(t) - \mathrm{Re}\,(\log f(t))/(2n)\right)\right], \qquad (4)$$

where $V_n = \mathrm{Re}\,\mathcal{V}_n$, and keeping in K_n what we hope will remain unessential factors.

Let E_n be the part of E where V_n reaches its maximum, and F_n be the (still unknown[1]) part of C_n where $V_n(t) - \mathrm{Re}\,(\log f(t))/(2n)$ reaches its minimum.

The problem is now to study a harmonic function V_n satisfying the *Dirichlet conditions* $V_n =$ a constant on E_n, and $V_n =$ constant $+\mathrm{Re}\,(\log f(t))/(2n)$ on F_n. Such a Dirichlet problem makes sense if E_n and F_n are a set of arcs in the complex plane. A mild consequence of the form (3) is that the arcs should be part of rational lemniscates. A much more important consequence of (3) is that, if we imagine V_n to be the potential

[1] C_n is also an unknown of the problem. While (1) and (2) do not depend on the choice of C_n, (4) does, if we want (4) to yield a realistic estimate.

of *electric charges* spread on E_n and F_n, we must have a positive *unit* charge on F_n and a negative unit charge on E_n.

Then, the constant values taken by V_n on E_n, and by $V_n - \mathrm{Re}\,(\log f)/(2n)$ on F_n, simply mean that the charges, free on E_n, and submitted to the action of the external potential $-\mathrm{Re}\,(\log f)/(2n)$ on F_n, are in *equilibrium*. Indeed, any part of the charged matter is subject to a force directed along the (opposite of the) gradient of the potential, and we just managed to have this gradient orthogonal to the sets E_n and F_n...

Finally, if the locus F_n is moved in the complex plane, one has to bring in the system an energy needed to move the forces acting on the charges. The total energy, which is the potential difference

$$\delta_n := \Big(V_n - \mathrm{Re}\,(\log f)/(2n)\Big)_{F_n} - \Big(V_n\Big)_{E_n}$$

we have to maximize, will indeed reach its largest possible value when all the forces acting on the system do vanish. We then expect the best approximation error norm to satisfy

$$\lim_{n\to\infty} \|f - R_n\|^{1/n} = \exp\left(-2 \lim_{n\to\infty} \delta_n\right). \tag{5}$$

It is a very remarkable fact that the constraints (2) will not change the validity of the discussion just made, as shown by Stahl [18,17]: if f has branchpoints limiting the extension of F_n, the interpolation points and the poles have (weak) limit distributions amounting to the correctness of (5). However, when f is entire, one must consider a variable scaling factor ensuring the existence of the limit of the external term $(\log f(z))/(2n)$. For our problem with $f(z) = \exp(-z)$, we will of course work with the variable z/n, and study the limits

$$V(z) = \lim_{n\to\infty} V_n(nz)\,,\, \mathcal{V}(z) = \lim_{n\to\infty} \mathcal{V}_n(nz)\,,\, F = \lim_{n\to\infty} \frac{F_n}{n}.$$

A thorough study of the existence of these limits is done in sections 1 and 3 of [7], where the special case of the exponential function is also treated (section 2 of [7]).

We proceed with this case in more detail.

2. Equation of the potential.

We look for a distribution $d\lambda$ on $E = [0, \infty]$ and a set F such that the corresponding potential

$$V(z) = \int_{E \bigcup F} \log \frac{1}{|z - t|}\, d\lambda(t)$$

satisfies

1. $V = 0$ on $E = [0, \infty]$,
2. $V = -x/2+$ constant on F, (this constant is $-(\log\rho)/2$),
3. $V + x/2 >$ this constant on a continuation of F
4. E carries a negative unit charge, and F carries a positive unit charge:
   $\int_E d\lambda(t) = -1$; $\int_F d\lambda(t) = 1$.
5. Symmetry property of F:
   $\partial(V + x/2)/\partial n_+ = \partial(V + x/2)/\partial n_-$ on F.

It is possible to show that F will be made of a finite number of analytic arcs, actually a single arc in the $f(z) = \exp(-z)$ case [7]. As F is the (scaled) limit of the set of poles of real rational functions, we know that F will be symmetric with respect to the real axis.

2.1. *Solution through Sokhotskyi-Plemelj formulas.*

Let us introduce the multivalued complex potential

$$\mathcal{V}(z) = \int_{E \bigcup F} \log \frac{1}{z-t}\, d\lambda(t) \tag{6}$$

whose real part is the sought potential V. An elementary consequence of the Cauchy-Riemann equations is that the two components of the gradient of V are the real part and the opposite of the imaginary part of the derivative

$$\mathcal{V}'(z) = -\int_{E \bigcup F} \frac{1}{z-t}\, d\lambda(t) \tag{7}$$

of $\mathcal{V}$. This function $\mathcal{V}'$ is therefore analytic and single-valued outside $E \bigcup F$. The behaviour of such an integral (7) when z comes close to one of the cuts is given by the very important ***Sokhotskyi-Plemelj*** formulas [9], chap. 14, [13]:

$$\lim \mathcal{V}'(z) = -⨍_{E \bigcup F} \frac{1}{z-t}\, d\lambda(t) \pm \pi i \lambda'(z), \;\; z \in E \bigcup F, \tag{8}$$

where the limit is taken when one approaches z from a definite side of the cut, the $\pm$ sign depends on the chosen side, and where $⨍$ means principal value.

Remark that, as $d\lambda(z)$ is real, the complex conjugates $\overline{\pm i\lambda'(z)} = \mp i d\lambda(z) dz/|dz|^2$ are directed along the normal to E or F at z, and will therefore be involved in the computation of the *normal derivatives* of V on the two sides of E or F.

We now have to work out the conditions satisfied by V on the two cuts E and F, in order to determine V,... and the still unknown cut F!

On F, we must consider $\mathcal{V}(z)+z/2$ instead of $\mathcal{V}(z)$, so, the value of the principal value in (8) must be $1/2$ for any $z \in F$.

We now transform the problem as a problem on F alone: the condition on E is classical and is easily satisfied by a suitable combination of (complex) *Green functions* of $\mathbb{C}\setminus E$

$$\mathcal{G}_E(z;t) = \log \frac{\sqrt{-\overline{t}}+\sqrt{-z}}{\sqrt{-t}-\sqrt{-z}}\,,$$

where $\sqrt{-z}$ is defined for $z \notin E$ as the square root of $-z$ which has a positive real part (see, for instance [5] § 8.11). The real part of $\mathcal{G}_E(z;t)$ vanishes indeed on E.

We now sum such Green functions on the remaining singular locus $t \in F$ as

$$\mathcal{V}(z) = \int_F \log \frac{\sqrt{-\overline{t}}+\sqrt{-z}}{\sqrt{-t}-\sqrt{-z}}\, d\mu(t). \tag{9}$$

We only have to look again at the Sokhotskyi-Plemelj formulas for (9):

$$\mathcal{V}'(z) = -\frac{1}{2\sqrt{-z}} \fint_F \left[\frac{1}{\sqrt{-\overline{t}}+\sqrt{-z}} + \frac{1}{\sqrt{-t}-\sqrt{-z}}\right] d\mu(t) \pm \pi i \mu'(z) \;\; ,z \in F. \tag{10}$$

As $\mathcal{V}(z)+z/2$ must have opposite derivatives on the two sides of F, we have

$$\frac{1}{2\sqrt{-z}} \fint_F \left[\frac{1}{\sqrt{-\overline{t}}+\sqrt{-z}} + \frac{1}{\sqrt{-t}-\sqrt{-z}}\right] d\mu(t) = \frac{1}{2}\,, z \in F,$$

which is an integral equation for $d\mu$, but it is easier to solve immediately for $\mathcal{V}$, actually for $\mathcal{V}'$, knowing that the values on the two sides of F are

$$\mathcal{V}'(z) = -\frac{1}{2} \pm \pi i \mu'(z) \;\; ,z \in F. \tag{11}$$

Remark already that, when z is an endpoint a or $\overline{a}$ of F, the integral in (10) is a plain integral, and $\mu' = 0$ at these points:

$$\mathcal{V}'(a) = \mathcal{V}'(\overline{a}) = -\frac{1}{2}\,. \tag{12}$$

We now go further in the study of a formula for $\mathcal{V}'$:

1. Comparing (8) and (10), knowing that the principal values vanish on F, we see that $\mu' = \lambda'$ on F. Therefore, as the total charge on F $\int_F d\lambda(t) = 1$, (9) shows that, for large z, $\mathcal{V}(z) = \pm\pi i + \gamma_1/\sqrt{-z} + \cdots$, with $\gamma_1 = \int_F 2\,\mathrm{Re}\,\sqrt{-t}\, d\lambda(t)$. Here, the $\pm\pi i$ is a result of the multivalued character of $\mathcal{V}$.

Also, $\mathcal{V}'(z) = \gamma_1/(2(-z)^{3/2}) + \cdots$ for large z, $\mathcal{V}''(z) = 3\gamma_1/(4(-z)^{5/2}) + \cdots$
For small z, $\mathcal{V}(z) = \gamma_2(-z)^{1/2} + \cdots$, $\mathcal{V}'(z) = -\gamma_2/(2(-z)^{1/2}) + \cdots$, $\mathcal{V}''(z) = -\gamma_2/(4(-z)^{3/2}) + \cdots$, with $\gamma_2 = \int_F 2\,\mathrm{Re}\,(1/\sqrt{-t})\,d\lambda(t)$.

2. Now, let $\sqrt{(z-a)(z-\overline{a})}$ be the square root of $(z-a)(z-\overline{a})$ which is continuous outside F and such that $\sqrt{(z-a)(z-\overline{a})} - z$ remains bounded when $z \to \infty$. As $\sqrt{(z-a)(z-\overline{a})}\,\mathcal{V}'(z)$ is an analytic function of $\sqrt{-z}$ outside F, vanishing at infinity, with a pole of residue -$|a|\gamma_2/2$ at 0, we have the Cauchy integral formula

$$\sqrt{(z-a)(z-\overline{a})}\,\mathcal{V}'(z) = -\frac{|a|\gamma_2}{2\sqrt{-z}} + \frac{1}{2\pi i}\oint_C \frac{\sqrt{(t-a)(t-\overline{a})}\,\mathcal{V}'(t)}{\sqrt{-z}-\sqrt{-t}}\,d\sqrt{-t},$$

where F is inside C, z and the origin are *outside* C. Letting C shrink to the two sides of F, we have an integral on F of the jump of $\sqrt{(t-a)(t-\overline{a})}\,\mathcal{V}'(t)$ across F. But as $\sqrt{(t-a)(t-\overline{a})}$ takes *opposite* values on the two sides of F (that's the trick of these Sokhotskyi-Plemelj-Privalov-Riemann-Hilbert problems [9], chap. 14), (11) tells that this jump is $\sqrt{(t-a)(t-\overline{a})}$, hence,

$$\sqrt{(z-a)(z-\overline{a})}\,\mathcal{V}'(z) = \frac{|a|\gamma_2}{2\sqrt{-z}} + \frac{1}{2\pi i}\int_a^{\overline{a}} \frac{\sqrt{(t-a)(t-\overline{a})}}{\sqrt{-z}-\sqrt{-t}}\,d\sqrt{-t}. \tag{13}$$

Now, everything is known up to a small number of constants!
We could have performed a *division* by $\sqrt{(z-a)(z-\overline{a})}$:

$$\frac{\mathcal{V}'(z)}{\sqrt{(z-a)(z-\overline{a})}} = \frac{\gamma_2}{2|a|\sqrt{-z}} + \frac{1}{2\pi i}\int_a^{\overline{a}} \frac{d\sqrt{-t}}{\sqrt{(t-a)(t-\overline{a})}[\sqrt{-z}-\sqrt{-t}]}\,, \tag{14}$$

and

$$\frac{z^m\mathcal{V}'(z)}{\sqrt{(z-a)(z-\overline{a})}} = \frac{1}{2\pi i}\int_a^{\overline{a}} \frac{t^m d\sqrt{-t}}{\sqrt{(t-a)(t-\overline{a})}[\sqrt{-z}-\sqrt{-t}]}\,, \quad m = 1,2 \tag{15}$$

may be useful.

3. We now look for a differential equation for $\mathcal{V}'$. The derivative of (13) times $\sqrt{-z}$ is

$$\frac{d}{dz}\left[\sqrt{-z(z-a)(z-\overline{a})}\,\mathcal{V}'(z)\right] = \frac{1}{4\pi i\sqrt{-z}}\int_a^{\overline{a}} \frac{\sqrt{-t(t-a)(t-\overline{a})}}{(\sqrt{-z}-\sqrt{-t})^2}\,d\sqrt{-t},$$

which we integrate by parts:

$$\frac{d}{dz}\left[\sqrt{-z(z-a)(z-\overline{a})}\,\mathcal{V}'(z)\right] =$$
$$\frac{-1}{4\pi i\sqrt{-z}}\int_a^{\overline{a}} \frac{1}{\sqrt{-z}-\sqrt{-t}}\,\frac{d}{d\sqrt{-t}}[\sqrt{-t(t-a)(t-\overline{a})}]\,d\sqrt{-t}$$

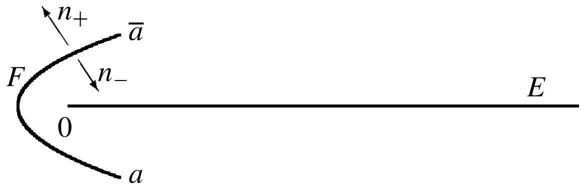


Figure 1. The sets E and F.

$$= \frac{-1}{4\pi i\sqrt{-z}} \int_a^{\overline{a}} \frac{1}{\sqrt{-z}-\sqrt{-t}} \left[\sqrt{(t-a)(t-\overline{a})} + 2t\frac{2t-a-\overline{a}}{2\sqrt{(t-a)(t-\overline{a})}} \right] d\sqrt{-t}$$

$$= -\frac{1}{2\sqrt{-z}} \left[\sqrt{(z-a)(z-\overline{a})}\,\mathcal{V}'(z) - \frac{|a|\gamma_2}{2\sqrt{-z}} \right] + z(2z-a-\overline{a})\frac{\mathcal{V}'(z)}{2\sqrt{-z(z-a)(z-\overline{a})}}$$

from (13) and (15). The differential equation for $\mathcal{V}'$ collapses into the surprisingly simple

$$\mathcal{V}''(z) = -\frac{|a|\gamma_2}{4\sqrt{-z^3(z-a)(z-\overline{a})}} \tag{16}$$

which is to be found in Gonchar & Rakhmanov [7] § 2 eq. (13) , and could probably be derived in a shorter and more elegant way as just done here...

4. Integration of (16) yields

$$\mathcal{V}'(z) = -\frac{|a|\gamma_2}{4}\int_\infty^z \frac{dt}{t\sqrt{-t(t-a)(t-\overline{a})}}, \tag{17}$$

on any path joining ∞ to z avoiding the cuts F and E. As the function $\mathcal{V}'$ is single-valued in $\mathbb{C}\setminus(E\cup F)$ (remember the short discussion after equation (7)), it must have a vanishing period about F:

$$\int_{\overline{a}}^a \frac{dt}{t\sqrt{-t(t-a)(t-\overline{a})}} = 0, \tag{18}$$

which we could also have deduced from (12): $\mathcal{V}'(a) - \mathcal{V}'(\overline{a}) = 0$.

$$\mathcal{V}(z) = (2k+1)\pi i - \frac{|a|\gamma_2}{4}\int_\infty^z \frac{(z-t)dt}{t\sqrt{-t(t-a)(t-\overline{a})}} \tag{19}$$

The condition (18) and the period $2\pi i$ of $\mathcal{V}$ in (19) will allow the complete determination of the solution of the potential problem. Elliptic functions and integrals will be needed in the following sections.

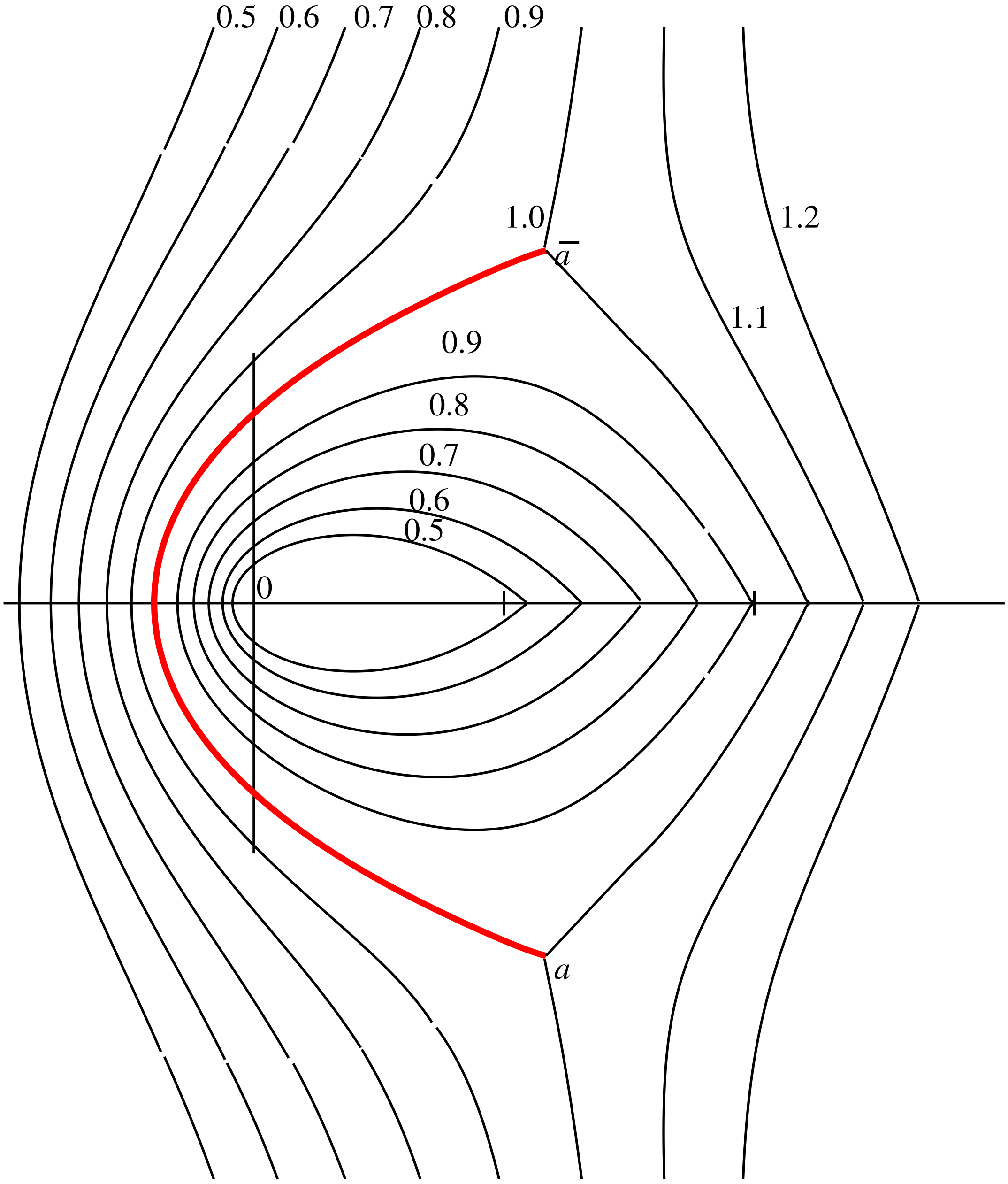


Figure 2. Level lines of $V(z) + x/2$ (in units of the value at a).

3. The Weierstrass connection.

Integrals like (13)-(19) involving the square root of a polynomial P of degree three or four are called *elliptic integrals*. They are multivalued functions, as their outcome depends on the path of integration. Even if the path is not allowed to cross given cuts (here: E and F), contour integrals around cuts leave nonzero values called *periods*.

An elliptic *function* $\mathcal{E}$ satisfies $\dfrac{d\mathcal{E}(u)}{du} = \sqrt{P(\mathcal{E}(u))}$, so that the change of variable $t = \mathcal{E}(v)$ in an integral involving $\sqrt{P(t)}$ will leave an integral free of square roots. The existence and the properties of single-valued elliptic functions is a deep subject pioneered mainly by Abel, Jacobi, Riemann, and Weierstrass.

3.1. Use of the Weierstrass $\wp$ function.

The Weierstrass function $\wp$ is the best suited in the study of (13)-(19), as it satisfies

$$\left(\frac{d\wp(u)}{du}\right)^2 = P(\wp(u)) = 4(\wp(u)-e)(\wp(u)-e')(\wp(u)-\overline{e'}),$$

where e is real, and e' is complex, with $e+e'+\overline{e'}=0$. As our (13)-(19) deal with a polynomial of zeros 0, $a=\alpha-i\beta$ (with $\beta>0$), and $\overline{a}$, we make the obvious translation of $2\alpha/3$ and use

$$z = \frac{2\alpha}{3}+\wp(u)\ ,t = \frac{2\alpha}{3}+\wp(v)\ , \tag{20}$$

with the Weierstrass function associated to

$$e = -\frac{2\alpha}{3}\ ,\ \text{and}\ e' = \frac{\alpha}{3}-i\beta\ . \tag{21}$$

It is especially important to see that the *discriminant*

$$\Delta = 16(e-\overline{e'})^2(\overline{e'}-e')^2(e'-e)^2 = -64\beta^2(\alpha^2+\beta^2)^2$$

is negative.

In order to proceed, we need more data on the Weierstrass $\wp$ function. Remember (see e.g. [1], p. 11) that the basic Weierstrass function

$$\wp(u) := \frac{1}{u^2}+\sum_{(m,m')\neq(0,0)}\left\{\frac{1}{(u-2m\omega-2m'\omega')^2}-\frac{1}{(2m\omega+2m'\omega')^2}\right\}\ , \tag{22a}$$

where 2ω and $2\omega'$ are a pair of primitive periods such that

$$\operatorname{Im}\tau > 0\ ,\qquad \tau := \frac{\omega'}{\omega}\ , \tag{22b}$$

is an (even) elliptic function, whose only poles (all of order 2) are the points $2m\omega + 2m'\omega'$ (for all integers m and m'). Since here the coefficients of P are real and such that $\Delta < 0$, it is known (see e.g. [6], p. 339) that there exists a pair of conjugate complex primitive periods $2\omega', 2\overline{\omega'}$ giving a rhombic fundamental parallelogram, whose translated diagonals mod $(\omega' \pm \overline{\omega'})$ are the only lines on which $\wp(u)$ is real. However, we will rather make use of the unimodular transformation

$$\begin{pmatrix} \omega' \\ \omega \end{pmatrix} = \begin{pmatrix} 1\,0 \\ 1\,1 \end{pmatrix} \begin{pmatrix} \omega' \\ \overline{\omega'} \end{pmatrix} ,$$

which anyway leaves unchanged the point-lattice of all periods, to legitimate the assumption adopted hereafter, viz.,

$$\omega = 2\,\mathrm{Re}\,\omega' > 0\,, \quad \mathrm{Im}\,\omega' > 0. \tag{23}$$

As a matter of fact, provided only that $\Delta \neq 0$, the customary definitions

$$e := \wp(\omega)\,, \qquad e' := \wp(\omega')\,, \tag{24a}$$

can be "inverted" to find a pair of primitive periods $2\omega, 2\omega'$ corresponding to given numbers e and e' (see e.g. [1], p. 30 and p. 33). Moreover, $\wp(u)$ satisfies the differential equation

$$\wp' = \sqrt{4(\wp - e)(\wp - e')(\wp - \overline{e'})} \tag{24b}$$

in the region $G \setminus L$ (L is any simple arc lying in $G = \mathbb{C} \setminus E = \mathbb{C} \setminus [0,\infty)$ and joining the points $a, \overline{a}$). It should be noted once for all that the square root in (24b) is to be understood as the branch that is positive along the upper side of the cut E; this is clearly a coherent choice, in view of the general principle of the permanence of functional equations, since $\wp(u)$ increases from e to ∞ as u increases from ω to 2ω (for an alternative convention, see e.g. [6], p. 334).

These classical results naturally suggest using the change of variable (20) in the integrals (13)-(19) to "uniformize" the integrand by eliminating the square root; on the other hand, determining a suitable range of integration apparently requires some familiarity with the underlying conformal map interpretation (which does not seem to be widely known when $\Delta < 0$!). It turns out (see also [16], p. 630 and p. 642) that the (open) rectangle in the $u-$plane with vertices $0, 2\omega, \omega' + 3\omega/2$ and $\omega' - \omega/2$,– to which we must add two adjacent sides in order to cover the entire plane by its translations mod $(2\omega, i\mathrm{Im}\,\omega')$– is a *fundamental region* for the mapping $z = \wp(u) - e$; in other words, the interior of this set is mapped in a one-to-one manner onto the whole extended $z-$plane, except for cuts, to wit:

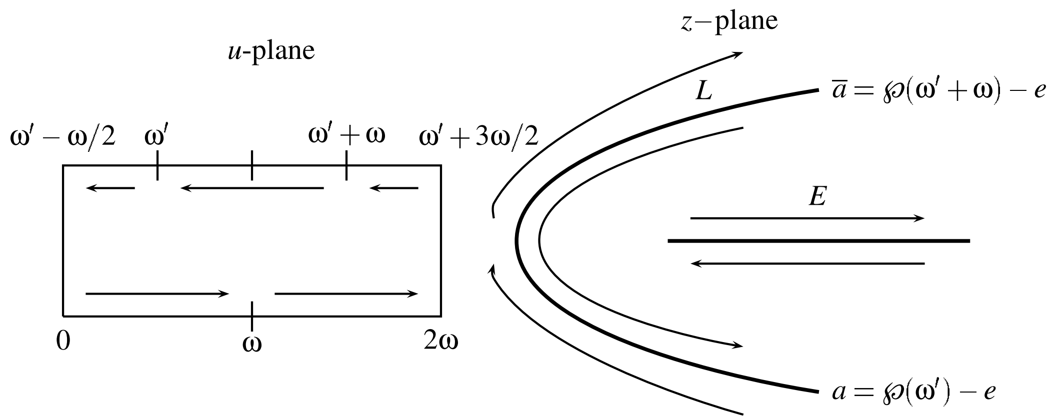


Figure 3. A fundamental region for $z = \wp(u) - e$.

- the real interval $[0, \infty]$ (the edges of this cut correspond to the line-segments $[0, \omega]$ and $[\omega, 2\omega]$, respectively);
- a simple arc L in G, symmetric with respect to $\mathbb{C}$ and joining the points $e' - e = a$ and $\overline{e'} - e = \overline{a}$ (the edges of this cut[2] correspond to the line-segment $[\omega' + \omega, \omega']$ and $[\omega', \omega' - \omega/2] \cup [\omega' + 3\omega/2, \omega' + \omega]$, respectively).

A somewhat less detailed information is conveyed by saying simply that the closed $u-$rectangle of vertices $0, \omega, \omega' + \omega/2, \omega' - \omega/2$ (resp. its translation by ω) is mapped by $z = \wp(u) - e$ onto the half-plane $\operatorname{Im} z \leq 0$ (resp. $\operatorname{Im} z \geq 0$).

The `GP-Pari` software [2] allows to compute the $\wp$ function. Some values are given in table 1.

3.2. *The potential in terms of the $\wp$ and zeta functions.*

We come now to the evaluation of the integrals (17)-(19): $\sqrt{-t(t-a)(t-\overline{a})}$ is negative real when t is real at the left of L; positive real when t is real between L and 0; negative imaginary when t is real on the upper side of E; and positive imaginary when t is real on the lower side of E. We have therefore, from (24b)

$$\frac{dt}{\sqrt{-t(t-a)(t-\overline{a})}} = 2i dv \qquad \text{when } t = \wp(v) - e,$$

and, from (17)-(19),

$$\mathcal{V}'(z) = -\frac{i|a|\gamma_2}{2} \int_0^u \frac{dv}{\wp(v) - e}, \mathcal{V}(z) = -\frac{i|a|\gamma_2}{2} \int_0^u \left(\frac{z}{\wp(v) - e} - 1 \right) dv.$$

[2] which need not to be the arc F, see Fig. 4 for a modified fundamental region.

u	$z = \wp(u) - e$
$0.05 + 0.01i$	$355.8254 - 147.9292i$
$\omega/2 + 0.01i$	$1.831555 - 0.04135211i$
$\omega + 0.01i$	-0.0003356574
$3\omega/2 + 0.01i$	$1.831555 + 0.04135211i$
$2\omega - 0.05 + 0.01i$	$355.8254 + 147.9292i$
$2\omega - 0.01 + i\,\mathrm{Im}\,\omega'/2$	$-9.953617 + 0.7107980i$
$2\omega - 0.01 + i\,\mathrm{Im}\,\omega' - 0.01i$	$-1.922076 + 0.0946833i$
$3\omega/2 + i\,\mathrm{Im}\,\omega' - 0.01i$	$1.195284 + 1.388380i$
$\omega + i\,\mathrm{Im}\,\omega' - 0.01i$	-1.744042
$\omega/2 + i\,\mathrm{Im}\,\omega' - 0.01i$	$1.195284 - 1.388380i$
$0.01 + i\,\mathrm{Im}\,\omega' - 0.01i$	$-1.922076 - 0.09468325i$
$0.01 + i\,\mathrm{Im}\,\omega'/2$	$-1.829805 - 0.09010485i$

Table 1
Boundary of the fundamental region chosen for the Weierstrass connection.

Now,

$$\frac{1}{\wp(v) - e} = \frac{\wp(v+\omega) - e}{H^2}, \qquad H^2 := (e - e')(e - \overline{e'}) = |a|^2,$$

in view of the *addition theorem* for $\wp$; on the other hand

$$\wp(u) = -\zeta'(u), \qquad \lim_{u \to 0} \left\{ \zeta(u) - \frac{1}{u} \right\} = 0,$$

where the (odd) Weierstrass zeta function ζ is meromorphic with only simple poles at the points $2m\omega + 2m'\omega'$, and quasi-periodic in the sense that

$$\zeta(u + 2\omega) = \zeta(u) + 2\eta, \qquad \zeta(u + 2\omega') = \zeta(u) + 2\eta',$$

so that the (half) quasi-periods are

$$\eta := \zeta(\omega), \qquad \eta' := \zeta(\omega').$$

Hence

$$\mathcal{V}'(z) = \frac{i\gamma_2}{2|a|}(-\zeta(u+\omega) - eu + \eta), \; \mathcal{V}(z) = \frac{i\gamma_2}{2|a|}[z(-\zeta(u+\omega) - eu + \eta) - |a|^2 u] \pm \pi i. \tag{25}$$

We now go on with conditions for the remaining unknowns:

As $\mathcal{V}'$ must be single-valued (as seen earlier), a contour integral of (17) about any arc joining a to $\overline{a}$ must leave a zero period. This means by (25) that in the $u-$plane, $-\zeta(u+\omega) - eu$ is left unchanged when u is replaced by $u + 2\omega$, as $z = \wp(u) - e$ is itself left unchanged. As the increase of the zeta function is 2η, we have the important condition

$$\eta + e\omega = 0. \tag{26}$$

The equation can also be deduced from (12) $\mathcal{V}'(a) - \mathcal{V}'(\overline{a}) = 0$, as $z = a$ and $z = \overline{a}$ correspond to $u = \omega'$ and $u = \omega' + \omega$.

The equation (26) already gives a and $\overline{a}$ up to a real scaling factor. Indeed, if one insists to write all the arguments involved in the computation of elliptic functions, one knows that $\wp(u) = \wp(u|\omega,\omega')$ is a homogeneous function of degree -2 of its *three* arguments, and $\zeta(u|\omega,\omega')$ is homogeneous of degree -1 (see [6], p. 331). Then, after division by ω of the three arguments of $\eta = \zeta(\omega|\omega,\omega') = \omega^{-1}\zeta(1|1,\omega'/\omega)$ and $e = \wp(\omega|\omega,\omega') = \omega^{-2}\wp(1|1,\omega'/\omega)$, (26) becomes

$$\zeta(1|1,\tau) + \wp(1|1,\tau) = 0, \tag{27}$$

where $\tau = \omega'/\omega$. This equation in τ can be solved numerically, see next subsection. One has

$$\tau = 0.5 + 0.35472989252243129027 2 * i\,.$$

We then find the phase of a (and $\overline{a}$):

$$a = \wp(\omega'|\omega,\omega') - \wp(\omega|\omega,\omega') = \omega^{-2}[\wp(\tau|1,\tau) - \wp(1|1,\tau)]$$

$$= \omega^{-2}[3.5137378044162934178 23 - 4.0836680502754215244 05i] = |a|\exp(-i\varphi),$$

$$\text{with } \varphi = 0.86027434674914619207 02, \quad \text{or } 49.29008929210000708037 \text{ degrees.}$$

3.3. Use of theta functions.

Other equivalent formulations of the key condition (18) follow, via (26), from the well known connection between the (odd) Weierstrass sigma function σ, which is defined by

$$\frac{\sigma'(u)}{\sigma(u)} = \zeta(u)\,, \qquad \lim_{u\to 0}\frac{\sigma(u)}{u} = 1,$$

and the Jacobi theta function ϑ_1. Remember (see e.g. [1], p. 37 and p. 53) that $\sigma(u)$ is an entire transcendental function having only simple zeros at the points $2m\omega + 2m'\omega'$. It turns out that

$$\sigma(u) = \frac{2\omega e^{\eta u^2/2\omega}\vartheta_1(v)}{\vartheta_1'(0)}\,, \qquad v = \frac{u}{2\omega},$$

which easily implies that

$$\wp(u) = -\frac{\eta}{\omega} - \frac{1}{4\omega^2}\left[\log\vartheta_1(v)\right]'' , \qquad v = \frac{u}{2\omega}. \tag{28}$$

Now $\vartheta_1(v)$ can be defined (see e.g. [6], § 13.19), either as a “rapidly convergent” simple series

$$\vartheta_1(v) = i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2} e^{\pi i(2n-1)v} , \qquad q = e^{\pi i\tau} \tag{29a}$$

or as an infinite product (via Jacobi’s triple product identity)

$$\vartheta_1(v) = 2q_0 q^{1/4}\sin\pi v\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2\pi v + q^{4n}) , \; q_0 = \prod_{n=1}^{\infty}(1-q^{2n}). \tag{29b}$$

By substituting in (28) these explicit definitions for $\vartheta_1(v)$ and making $v = 1/2$ (which corresponds to $u = \omega$), we get eventually

$$e = -\frac{\eta}{\omega} - \frac{1}{4\omega^2}\left[\frac{\vartheta_1''(v)}{\vartheta_1(v)}\right]_{v=1/2} ,$$

since $\vartheta_1'(1/2) = 0$ (easy verification), and

$$e = -\frac{\eta}{\omega} + \frac{\pi^2}{\omega^2}\left\{\frac{1}{4} + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2}\right\} .$$

Hence the key condition (18), reformulated as (26), becomes

$$-\vartheta_1''(1/2)/(2\pi^2) = \sum_{n=0}^{\infty}(2n+1)^2 q^{(n+1/2)^2} = 0 \tag{30}$$

from (29a), or

$$\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2} = -\frac{1}{8} \tag{31a}$$

from (29b). The Lambert series equation (31a), in turn, can be rewritten as

$$\sum_{n=1}^{\infty}\frac{(-1)^n n q^{2n}}{1-q^{2n}} = \frac{1}{8} \tag{31b}$$

by making use of Weierstrass’ theorem on double series (see e.g. [11], p. 430 and p. 450), or equivalently (for $x := -q^2$),

$$\frac{x}{1+x} + 2\frac{x^2}{1-x^2} + 3\frac{x^3}{1+x^3} + \cdots = \frac{1}{8} .$$

It is remarkable that the latter equation was exploited by Halphen in his book [8] (1886!) to prove elegantly the existence of a unique root x in $(0,1)$: each of the terms of this series is indeed an increasing function in $(0,1)$, so that the series itself increases actually from 0 to ∞; for the numerical determination of this root x, the formulation (30), or equivalently,

$$1-9x-25x^3+49x^6+81x^{10}+\cdots+(2n+1)^2(-x)^{n(n+1)/2}+\cdots=0$$

is, however, more advantageous (this remark is also due to Halphen [8], p. 287). For *very* fast solution of problems related to elliptic functions and integrals, AGM methods [3] must be considered too, see next section.

As for the number theoretic formulation privileged by Gonchar and Rakhmanov (see [7], Theorem 2, p. 321), viz.,

$$\sum_{n=1}^{\infty} a_n x^n = 1/8 \text{ , where } a_n = \left|\sum_{d|n}(-1)^d d\right| \ , \tag{32}$$

it easily follows from (31a) as sketched hereafter:

$$\begin{aligned}
-\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2} &= -\textstyle\sum_{n=1}^{\infty} q^{2n}\sum_{k=0}^{\infty}(-1)^k(k+1)q^{2nk} \\
&= \textstyle\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^k k(-x)^{nk} \ , \text{ with } x := -q^2 \ , \\
&= \textstyle\sum_{m=1}^{\infty} x^m\left\{(-1)^m\sum_{d|m}(-1)^d d\right\} \ ,
\end{aligned}$$

which actually equals the left-hand side of (32).

We now have a full proof of the existence of

$$x=-q^2=\rho=\text{‘}1/9\text{'}=0.10765391922648457661 5\ldots=1/9.28902549192081891875\ldots$$

3.4. *Complete determination of remaining parameters.*

Final determination of ω, a, etc. is done by using remaining equations, as (12) which yields, thanks to (25):

$$\frac{i\gamma_2}{2|a|}(-\zeta(\omega'+\omega)-e\omega'+\eta)=-\frac{i\gamma_2}{2|a|}(\eta'+e\omega')=-\frac{1}{2},$$

which, from the Legendre relation $\eta'\omega-\eta\omega'=\pi/(2i)$ ([1], § 12) and (26), becomes

$$\eta'+e\omega'=-\frac{\pi i}{2\omega} \ ; \qquad \gamma_2=\frac{2|a|\omega}{\pi}. \tag{33}$$

The positive unit charge of F means that $\mathcal{V}$ increases by $2\pi i$ when ones achieves a clockwise contour about any arc joining a to $\overline{a}$. One finds a valid trajectory by following the curved arrows in figure 3, which corresponds to a decrease of the u variable by the amount 2ω (the three upper arrows[3] in the left part of fig. 3). Now, by (25),

$$\mathcal{V}(u-2\omega)-\mathcal{V}(u)=z[\mathcal{V}'(u-2\omega)-\mathcal{V}'(u)]-\frac{i\gamma_2|a|}{2}[-2\omega]=i\gamma_2|a|\omega,$$

as z and $\mathcal{V}'$ are unaffected, therefore, $\gamma_2|a|\omega=2\pi$, and, from (33),

$$|a|\omega=\pi\,;\qquad \gamma_2=2, \tag{34}$$

and, as $|a|=|\wp(\omega')-\wp(\omega)|=|\wp(\omega'|\omega,\omega')-\wp(\omega|\omega,\omega')|=\omega^{-2}|\wp(\tau|1,\tau)-\wp(1|1,\tau)|$, finally:

$$\omega=\frac{|\wp(\tau|1,\tau)-\wp(1|1,\tau)|}{\pi}\,;\qquad |a|=\frac{\pi^2}{|\wp(\tau|1,\tau)-\wp(1|1,\tau)|}. \tag{35}$$

$\omega=$ 1.71482189287770440634,
$\omega'=\tau\omega=$ 0.85741094643885220317 + 0.60829858575562026735 i
$e=$ -0.79659954370052498591, $\eta=$ 1.36602633739404986733
$a=$ 1.19489931555078747886 - 1.38871265581562671229 i

Finally, the locus F is the arc joining a to $\overline{a}$ where $V+x/2=\mathrm{Re}\ (\mathcal{V}+z/2)$ has a constant value $-(\log\rho)/2$. Looking for the corresponding locus in the $u-$plane, we find a sine-like curve (whose equation is still a small enigma) which makes a convenient new upper boundary of the fundamental region, see Fig. 4.

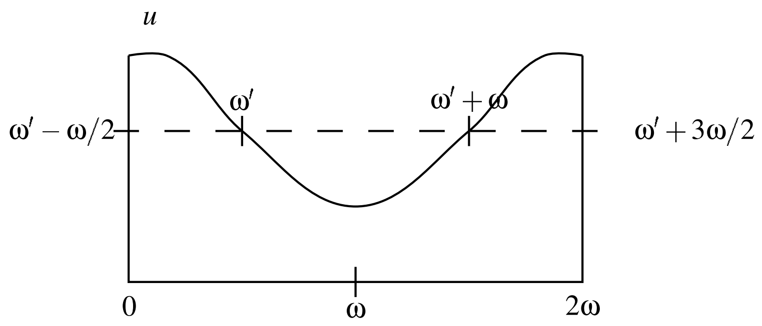


Figure 4. A modified fundamental region for $z=\wp(u)-e$, whose upper boundary corresponds to the two sides of F.

[3] see also that a clockwise contour around E correspond to an *increase* of u by an amount of 2ω (negative unit charge).

4. The Jacobi connection.

As is well known, the Weierstrass viewpoint is almost always preferable for theoretical investigations (higher symmetry), whereas the Jacobi viewpoint is more suitable for various computations (higher standardization).

The Jacobi elliptic integrals involve the square root of $(1-k^2x^2)(1-x^2)$. The *complete* elliptic integrals of first and second kind are

$$\mathbf{K}(k) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad \text{and} \qquad \mathbf{E}(k) := \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx.$$

The ***periods*** are $4\mathbf{K}(k)$ and $2i\mathbf{K}(k')$, with $k'^2 := 1-k^2$. The connection with the preceding section is therefore given by

$$\tau = \frac{\omega'}{\omega} = \frac{i\mathbf{K}(k')}{2\mathbf{K}(k)}.$$

The most convenient (and most tabulated) elliptic integrals correspond to k (and k') $\in (0,1)$, which supposes τ to be a pure imaginary number. We will show how to construct a pure imaginary $\dot{\tau} = \ddot{\tau} - 1 = 2\tau - 1$

A fundamental region for the conformal mapping of the *modular function*

$$\lambda : \tau \mapsto k^2 = \lambda(\tau)\,, \qquad \operatorname{Im}\tau > 0$$

(see e.g. [12], p. 197) is classically the (open) curvilinear quadrilateral

$$\{|\operatorname{Re}\tau| < 1 \text{ and } |2\tau \pm 1| > 1\,, \qquad \operatorname{Im}\tau > 0\}\,, \tag{36}$$

the right (resp. left) half of which is mapped one-to-one onto the upper (resp. lower) half of the k^2−plane cut along the line-segment $(-\infty, 0]$ and $[1,\infty)$; this mapping extends continuously to the boundary in such a way that $\tau = 0, \pm 1, \infty$ correspond to $k^2 = 1, \infty, 0$ (which are the only lacunary values of the modular function).

It turns out that the (unique) argument φ of any point $\overline{a} = \exp i\varphi = \alpha + i\beta(\beta > 0)$ in the region $G = \mathbb{C} \setminus [0,\infty)$ satisfying the Gonchar-Rakhmanov condition (18) belongs to $(0, \pi/2)$ (as a matter of fact, as seen above, $\varphi \approx 49°19'$).
In view of the cut along the positive real axis, the argument of the point $a = \alpha - i\beta$ can only be $2\pi - \varphi$, and not $-\varphi$; it follows that the complementary Jacobi modulus $k' := (e - \overline{e'})^{1/2}/(e-e')^{1/2}$ (see e.g. [6], p. 342) is necessarily

$$k' := (\overline{a}/a)^{1/2} = e^{i(\varphi - \pi)}\,, \qquad 0 < \varphi < \pi/2.$$

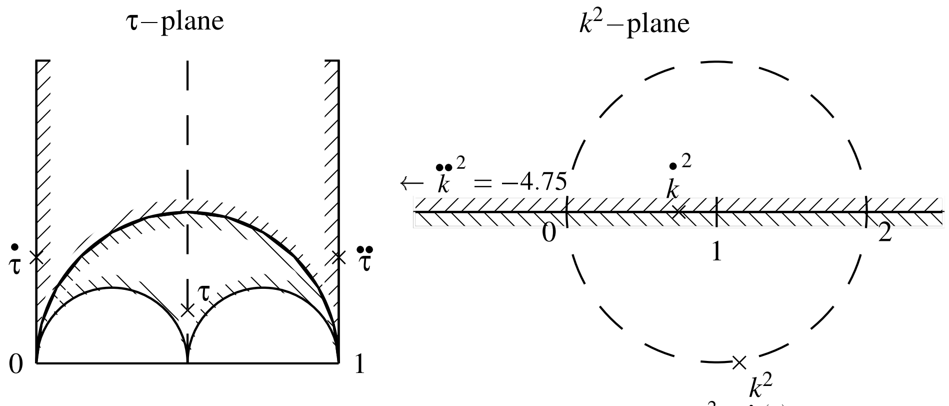


Figure 5. A fundamental region for the modular function $k^2 = \lambda(\tau)$.

On the other hand, the squared Jacobi modulus $k^2 := 1 - k'^2$ belongs to the *lower* half-plane, whereas the semiline $\text{Re}\ \tau = 1/2(\text{Im}\ \tau > 1/2)$ is mapped onto the *upper* semicircle of radius 1 centered at $k^2 = 1$. This apparent paradox can be easily removed, however, by adopting for fundamental region the right half of (36) together with its *Schwarz reflection with respect to the right-hand semicircle* (rather than its reflection with respect to the imaginary axis, see [1], p. 63).

The passage from the line-segment $\{\tau : \text{Re}\ \tau = 1/2, 0 < \text{Im}\ \tau < 1/2\}$ (which contains the unique τ satisfying to (23) and corresponding to k') to the *standard* line-segment $(0, i\infty)$ (on which $q := e^{\pi i \tau}$, and therefore k, belong to the standard interval $(0,1)$) is relatively easy. We are actually concerned here with the *modular transformation* of the second order

$$\tau \mapsto \overset{\bullet}{\tau} = 2\tau - 1 , \qquad q \mapsto \overset{\bullet}{q} = -q^2; ,$$

which can be achieved in two steps as summarized hereafter (see e.g [6], p. 319 and p. 342), namely:

- a ***Landen transformation***

$$\overset{\bullet\bullet}{\tau} = 2\tau , \qquad \overset{\bullet\bullet}{q} = q^2 ,$$

$$\overset{\bullet\bullet}{k} = \frac{1-k'}{1+k'} = i \cot\frac{\varphi}{2} , \qquad \overset{\bullet\bullet}{k}{}'^2 = 1/\sin^2\frac{\varphi}{2} ,$$

$$\overset{\bullet\bullet}{\mathbf{K}} = \frac{1+k'}{2}\mathbf{K} = \frac{1}{1+\overset{\bullet\bullet}{k}}\mathbf{K} , \overset{\bullet\bullet}{\mathbf{E}} = \frac{\mathbf{E}+k'\mathbf{K}}{1+k'} .$$

But $\mathbf{E} = (\eta + e\omega)/(e-e')^{1/2}$ must vanish in view of (26), so that the last equation can be rewritten as $\ddot{\mathbf{E}} = (1-\ddot{k})\mathbf{K}/2$, which strongly suggests eliminating $\mathbf{K}$ to get

$$\ddot{\mathbf{K}} = (2/\ddot{k}'^2)\,\ddot{\mathbf{E}}\ .$$

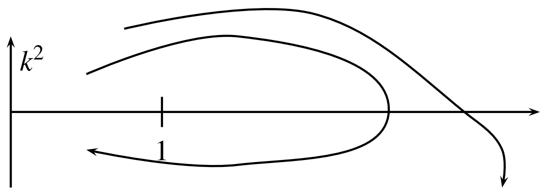
- a ***Jacobi transformation***

$$\dot{\tau} = \ddot{\tau} - 1\,, \qquad \dot{q} = -\ddot{q}\,,$$

$$\dot{k}^2 = -\frac{\ddot{k}^2}{\ddot{k}'^2} = \cos^2\frac{\varphi}{2}\,,$$

$$\dot{\mathbf{K}} = \ddot{k}'\,\ddot{\mathbf{K}}\,, \qquad \dot{\mathbf{E}} = (1/\ddot{k}')\,\ddot{\mathbf{E}}\ .$$

That $\mathbf{E} = 0$ makes us look for a quite unexpected zero of the function $k \mapsto \mathbf{E}(k)$. There are not many tables and graphs of this function in the complex plane. Jahnke and Emde ([10], p. 65) show level lines of $\mathbf{E}(k)$ in the k^2-plane cut by $[1,\infty)$ and one finds no zero. However, if we follow an analytic continuation across $[1,\infty)$,



the zero of $\mathbf{E}$ cannot be missed (Fig. 6).

In *conclusion*, the Gonchar-Rakhmanov condition (18) can be rewritten in the particularly nice form

$$\dot{\mathbf{K}} = 2\dot{\mathbf{E}}\,; \tag{37a}$$

this is indeed a real equation having only one root $\dot{k}$ in $(0,1)$, namely,

$$\dot{k} = \cos\frac{\varphi}{2}\,, \qquad 0 < \varphi < \pi\,, \tag{37b}$$

which determines the required point a (up to a normalization provided in subsection 3.4). It should also be realized that the half-periods $\dot{\omega} = \omega$ and $\dot{\omega}' = 2\omega' - \omega$ are respectively real and purely imaginary, and that

$$x = \dot{q} \approx 1/9.289 = 0.1077\ldots$$

is the famous '$1/9'$ constant.

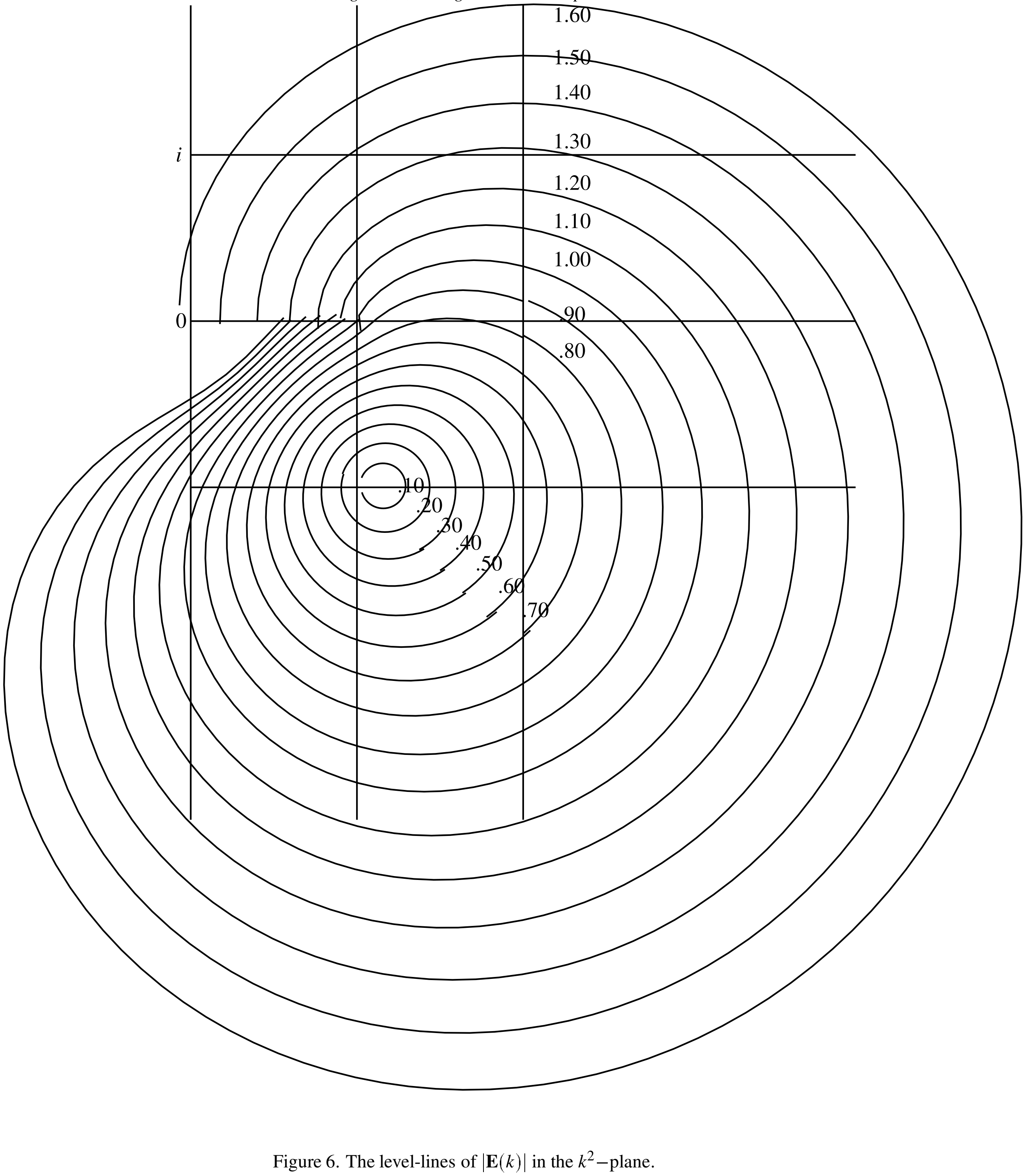


Figure 6. The level-lines of $|\mathbf{E}(k)|$ in the k^2-plane.

There are very efficient algorithms to compute Jacobi complete elliptic integrals through *arithmetic-geometric* mean (AGM, [3], chap. 1): let

$$a_0 = 1\,, \quad b_0 = k' = \sqrt{1-k^2}\,, \quad c_0 = k,$$

then,

$$a_{n+1} = \frac{a_n + b_n}{2}\,, \quad b_{n+1} = \sqrt{a_n b_n}\,, \quad c_{n+1} = \frac{c_n^2}{4a_{n+1}}$$

produces

$$\mathbf{K}(k) = \frac{\pi}{2\lim_{n\to\infty} a_n}\,, \quad \frac{\mathbf{E}(k)}{\mathbf{K}(k)} = 1 - \sum_{n=0}^{\infty} 2^{n-1} c_n^2.$$

One finds, according to next section,

$\overset{\bullet}{k} = \cos\varphi/2 = 0.908908557548541478236118 9\,,$
$\overset{\bullet}{k}{}' = \sin\varphi/2 = 0.416995484406042056390419 5.$

5. Appendix. The GP-PARI program.

See here the program used to produce the numerical values. The Jacobi data $\overset{\bullet}{k}$ and $\overset{\bullet}{q}$ are first found thanks to the very efficient AGM algorithm. The Weierstrass form follows in the determination of the periods $2\omega, 2\omega'$, and a formula for the complex potential $\mathcal{V}$.
The program can be fetched at `http://www.math.ucl.ac.be/~magnus/nine.gp`

```
{
/* nine.gp:  launch gp and type   \r nine


Elliptic functions & integrals related to '1/9'


    gp-pari program: see
          ftp://megrez.math.u-bordeaux.fr/pub/pari/
*/
default(realprecision,40);      default(format,"g7.21");
flone=1.0;pr=precision(flone);ep=10^(-pr);
/*
   AGM calculation of E and K such that K=2E
  from J. & P. Borwein, Pi and the AGM, Wiley 1987, algorithm 1.2
*/
k=0.9;ek=1; cnt=1;
```

```
while((abs(2*ek-1)> 1000*ep)&&(cnt<20), cnt=cnt+1;
     a=1;b=sqrt(1-k^2);c=k;ek=1;cnt2=1;p2=1;
      while( (a-b > 1000*ep)&&(cnt2<20), cnt2=cnt2+1;
       ek=ek-p2*c^2/2;a1=(a+b)/2;b=sqrt(a*b);a=a1;c1=c^2/(4*a);
       cc=c1/c;c=c1;p2=p2*2;
           );  \\ end while a-b
   dek= (2*ek-1 -ek^2/(1-k^2) )/k;   \\ derivative of E/K wrt k
   k = k -(2*ek-1)/(2*dek) ;   \\ Newton step
   print(k," ",2*ek-1);
   );  \\ end while 2ek-1
 /*  q = lim (c_{n+1}/c_n)^{1/2^n} :  */
   dotq=cc;for(iq=1,cnt2-3,dotq=sqrt(dotq));
   print(" rho = '1/9' = 1/",1/dotq);  \\   rho=dotq
   q=sqrt(-dotq);   \\  dotq = -q^2
   tau=log(q)/(Pi*I);
    print("tau = ",tau);

/* Weierstrass functions with periods  2 and 2 tau :  */

   perio=2*[1,tau];
   print(" check eta+e omega=0: ",ellzeta(perio,1)+ellwp(perio,1));
   e21=ellwp(perio,tau)-ellwp(perio,1);
/* actual periods */
   om1=abs(e21)/Pi;om2=tau*om1;om=[om1,om2];perio=2*om;
print("half periods= ",om);
e1=ellwp(perio,om1);     eta1=ellzeta(perio,om1);
print("e= ",e1); print("eta= ",eta1);
}

/* z as a function of u : */
{
 zu(u)=    ellwp(perio,u)-e1
}
{
  a=zu(om2);    absa=abs(a);absa2=absa^2;  print("a= ",a);
}

/* V as a function of u : */
{
```

```
 vu(u)=
  locz=zu(u);
  (1*I/absa)*(locz*(ellzeta(perio,om1-u)-eta1-e1*u)-absa2*u);
}
{
 vu2(u)=
  locz=zu(u);
  vu(u)+locz/2;
}
{
default(format,"g12.7");
print("You may now use functions zu, vu, and vu2 for z,V, and V+z/2");
print("as functions of the complex number u");
print(zu(om2)," ",vu2(om2));
}
```

Precision may be set (almost) at will! Here is a run with a little more than 1000 digits. Successive errors in the Newton-Raphson iteration needed to solve $2\dot{\mathbf{E}} = \dot{\mathbf{K}}$ are given first:

```
Script V1.1 session started Fri Jul 02 11:17:05 1999
C:\calc\pari>gp
...
/* ninem.gp:  launch gp and type   \r ninem
default(realprecision,1100);
...
? \r ninem

0.909106389042564496377  0.0275569449838391609734
0.908908659948146424888 -0.000626191643655082797822
0.908908557548568882889 -0.000000323955610037733639977
0.908908557548541478236 -8.66984654923741374392 E-14
0.908908557548541478236 -6.20959400962069829582 E-27
0.908908557548541478236 -3.18542069815699074789 E-53
0.908908557548541478236 -8.38250806957750567338 E-106
0.908908557548541478236 -5.80481449065487332117 E-211
0.908908557548541478236 -2.78366568096331142362 E-421
0.908908557548541478236 -6.40139365695332962324 E-842
```

```
0.90890855754854147823 6 -1.79438244558797800674 E-1108
 rho = '1/9' = 1/9.
2890254919 2081891875 5449435951 7450610316 9486775012
4408239700 6142172937 5247286507 0705241587 0614247144
3912681967 7128554634 7619722718 8544788518 0420483981
0589915280 9484682552 1982511074 8658254081 5458401564
8136915011 7186189524 6761040890 8308993768 4760280340
5868937594 6010024587 8577308304 3751302874 4779961340
7190579322 3955637594 9783816768 3632589307 7334560097
3350182664 9343035931 6218132137 3666611613 8225969837
0767080119 5884844115 4302660175 5483089394 7104785320
1025861840 7185058908 4662450431 6462436998 7930626775
1488801274 1726769451 4490394090 7097402173 3227620975
0075560031 6605883906 6451215528 0694284302 7393563193
4928943995 4835453174 0869105909 8687547545 5565122660
3233226675 4811472225 9063831342 4685872428 1552421119
0218737835 4467747118 4720787880 1401612334 4268072847
0863803318 7195631203 9582366412 0392486822 9780060331
8692746560 2489555817 7940630266 7701570912 4711699116
7847538280 1366128253 7836690422 1897763449 9799342464
5700490122 1605086501 6281257252 7002899916 4390980736
8153525888 9110283973 7792295987 2981897188 7234445326
? quit      Good bye!
Script completed Fri Jul 02 11:17:20 1999
```

References

[1] N.I. Akhiezer, *Elements of the Theory of Elliptic Functions*, 2nd edition, "Nauka", Moscow, 1970 = Translations Math. Monographs **79**, AMS, Providence, 1990.

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