# SHARP CONSTANTS FOR RATIONAL APPROXIMATIONS OF ANALYTIC FUNCTIONS

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12.03.2001

**Abstract.** Theorems are proved describing sharp constants for the approximation of a general class of analytic functions by rational functions. The Magnus hypothesis on the sharp constant for the approximation of $e^{-z}$ on $[0, \infty]$ is proved as a consequence. For the proof of the theorems new formulae of the strong asymptotics of polynomials orthogonal with respect to a varying complex weight are obtained.

Bibliography: 68 titles.

## Introduction

**0.1. Statement of problem.** Let

$$\widehat{\rho}_n(z) := \int_F \frac{\rho_n(t)}{t-z}\,\frac{dt}{2\pi i}\,, \qquad F \Subset \mathbb{C}, \quad n \in \mathbb{N}, \tag{0.1}$$

be a sequence of holomorphic (in $\overline{\mathbb{C}} \setminus F$) functions defined by integrals of Cauchy type over a rectifiable Jordan arc $F$ and let

$$d_n := d_n(\widehat{\rho}_n, E) := \inf_{r \in \mathfrak{R}_n} \big\{ \max_E |\widehat{\rho}_n - r| \big\}, \qquad E \cap F = \varnothing, \tag{0.2}$$

be the distance in the Chebyshev metric on a compact set $E$ from $\widehat{\rho}_n$ to the set $\mathfrak{R}_n$ of rational functions $r$ of order $n$.

In this work we study the rate of rational approximation of the functions $\widehat{\rho}_n$, that is, the asymptotic behaviour of $d_n$ as $n \to \infty$. The modern theory of this collection of classical problems was developed by A. A. Gonchar (see [1]–[12]; this list of key works is far from being complete). Under the most general assumptions on $F$, $E$, and $\rho_n$ the result describing the exponent of the rate of rational approximation

$$d := \lim_{n \to \infty} d_n^{1/(2n)} \tag{0.3}$$

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*AMS* 2000 *Mathematics Subject Classification.* Primary 30E10, 41A20

This work was carried out with the financial support from the Russian foundation for Fundamental Research (grants no. 99-01-01149 and no. 00-15-96132) and from INTAS (grant no. 00-272).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was obtained in the work of Gonchar and Rakhmanov [11].

The purpose of this work is to refine the rate of approximation (0.3) for real-symmetric functions (0.1) being approximated on finite or infinite intervals of the real axis. The question is the existence and the value of the limit

$$\lim_{n\to\infty} \frac{d_n(\widehat{\rho}_n, E)}{\alpha_n d^{2n}}, \quad \text{where} \quad \alpha_n^{1/n} = 1 + o(1), \tag{0.4}$$

which is called the *sharp constant for the approximation* of an analytic function by rational functions. (In connection with sharp constants of rational approximations of *continuous* functions $|x|$, $x^\alpha$ on $[-1,1]$ we mention recent papers of Stahl [13]–[15].) In § 1 we discuss the formulations of the general theorem of Gonchar and Rakhmanov for functions of the form (0.1) and the refinements of it (Theorems 1 and 1′) that are proved in this work.

**0.2. The rate $d_n(e^{-x}, [0,\infty])$.** One of the remarkable applications of the general theorem from [11] was the proof of the existence and the determination of the limit

$$d := d(e^{-x}, [0,\infty]) = \lim_{n\to\infty} d_n^{1/(2n)}(e^{-x}, [0,\infty]). \tag{0.5}$$

This problem (well-known sometime ago) was posed by Varga in connection with numerical methods for the solution of the heat equation [16], [17] and has been followed by dozens of works (see the reviews in [18], [19]). It was solved in 1986. Gonchar and Rakhmanov (see [1], [11]) have shown that the limit (0.5) exists and is equal to

$$d = \sqrt{v}\,, \tag{0.6}$$

where $v$ is the unique root of the equation

$$\sum_{l=1}^{\infty} a_l v^l = \frac{1}{8}, \qquad a_l := \Bigl|\sum_{b|l} (-1)^b b\Bigr|. \tag{0.7}$$

We note that in the same year of 1986, when the results of [11] were announced in Gonchar's talk at the International Congress of Mathematicians in Berkeley (see [1]), the exact value of the constant

$$v = \frac{1}{9.2890254919208189\ldots} \tag{0.8}$$

has independently been found by Magnus (without rigorous proof of the existence of the limit (0.5)). Magnus (see [20]–[22]) described $v$ in a form equivalent to (0.7):

$$v = \exp\left[-\pi \frac{K'}{K}\right], \qquad K(k) = 2E(k), \tag{0.9}$$

where $K$ and $K'$ are the complete elliptic integrals of the first kind for moduli $k$ and $k' = \sqrt{1-k^2}$ and $E(k)$ is the complete elliptic integral of the second kind.

A remarkable fact (also discovered by Magnus): it turned out that the exact value of the constant $v$ (0.8) was found in 1886 (exactly one hundred years earlier!) in Halphen's treatise [23]) on elliptic functions, where $v$ appeared as the root of the equation (equivalent to (0.7))

$$\sum \frac{nv^n}{1-(-v)^n} = \frac{1}{8}.$$

Analyzing the results of numerical experiments in [19], [25] Magnus later conjectured (see [24]) that

$$\frac{d_n(e^{-x},[0,\infty])}{d^{2n}} \to 2d \quad \text{as} \quad n \to \infty, \tag{0.10}$$

that is, the constant refining the exponent of the rate of rational approximations (the degree of rational approximations) of $e^{-x}$ is again connected with Halphen's constant!

The validity of the hypothesis of Magnus follows from the general theorem 1 on asymptotics (0.4) proved in this paper. The implication "Theorem 1 $\Rightarrow$ (0.10)" is given in § 1.

**0.3. Asymptotic behaviour of polynomials orthogonal with a varying complex weight.** An important tool for the study of rational approximations is the theory of *polynomials* $\{P_n(z) = z^n + \cdots\}$ *orthogonal* on contours and arcs $F$ in the complex plane:

$$\int_F P_n(z) z^\nu h(z)\, dz = 0, \qquad \nu = 0, 1, \dots, n-1, \tag{0.11}$$

with respect to a complex-valued *weight* $h(z)$.

Perhaps precisely in the above cited works on rational approximations Gonchar for the first time considered polynomials (0.11) orthogonal with respect to a *varying* (depending on $n$) *weight* $h$ of the form

$$h(z) := h_n(z) := e^{-2nQ(z)}. \tag{0.12}$$

Their so-called *weak asymptotics*

$$|P_n(z)|^{1/n} \underset{n\to\infty}{\longrightarrow} \Phi(z) := \exp\{-V^\lambda(z)\} \tag{0.13}$$

was connected with a *logarithmic potential*

$$V^\lambda(z) := \int \ln\frac{1}{|t-z|}\, d\lambda(t) \tag{0.14}$$

of the *extremal measure* $\lambda$ providing *equilibrium of the potential in the external field* $\varphi := \operatorname{Re} Q$:

$$V^\lambda + \varphi = \begin{cases} \text{const} =: \gamma & \text{on } \operatorname{supp}\lambda, \\ \geqslant \gamma & \text{on } F \setminus \operatorname{supp}\lambda. \end{cases} \tag{0.15}$$

We point out that the asymptotic theory of polynomials orthogonal with a varying weight (0.12) is nowadays rapidly developing, being not only a powerful tool in analysis and mathematical physics, but also having applications from number theory to random matrix ensembles and asymptotic combinatorics (see, for instance, Deift's lecture at the International Congress of Mathematicians in Berlin [26]).

Returning to the problems considered in this work we note that weak asymptotics (0.13) of orthogonal polynomials make it possible to establish the existence and to find the value of the limits (0.3) for the exponents of the best rational approximations, while for the study of more refined limits of the type (0.4) (which is our goal) the so-called formulae of *strong asymptotics*

$$\frac{P_n}{\Phi^n} \to ? \tag{0.16}$$

are required. The success in finding best constants (0.4) became a reality after a recent substantial progress in the development of complex methods in the study of strong asymptotics. A brief review of this scope of problems and the formulation of the theorems (proved in this work) on the strong asymptotics of polynomials orthogonal with respect to a varying complex weight (Theorems 2 and 3) are given in § 2 below.

**0.4. Structure of the paper.** As mentioned, the next two sections are introductory. They contain the necessary definitions and main results related to this work and the formulations of the theorems proved in this work. § 1 is devoted to the results on the rate and sharp constants of rational approximations, § 2 is devoted to the results on the asymptotic behaviour of polynomials orthogonal with a varying complex weight. Then we give the proofs of the formulated theorems. In § 3, assuming the theorems on the strong asymptotics (Theorems 2 and 3) formulated in § 2, we prove the results of § 1 (Theorems 1 and 1′). Then in § § 4–7 we prove the results formulated in § 2.

The author would like to express deep gratitude to A. A. Gonchar and A. Magnus for their attention to this work and helpful discussions of its results. The author is also grateful to his colleagues and friends A. Kujlaars, S. P. Suetin, E. M. Chirka, W. Van Assche, V. S. Buyarov, V. N. Sorokin, and V. A. Kalyagin for the discussions and advices during the work on the paper.

## § 1. Formulations of theorems on the rate of rational approximation

In this section we give necessary definitions and formulate theorems on the rate of best rational approximations (the Gonchar–Rakhmanov theorem) and also the refinements of it proved in this work. Then we discuss applications of the presented theorems, one of which is the proof of the hypothesis of Magnus (0.10).

**1.1. Equilibrium of $S$-condenser in an external field. Gonchar–Rakhmanov theorem.** We recall the basic concepts of the theory of logarithmic potential in an external field used in [11] for the description of the input data and the solution of the problem on the exponent of the geometric progression in the rate of approximations (0.3) (for more details see [11], [27], [28]).

Let $K$ be a compact set in the extended complex plane $\overline{\mathbb{C}}$ and let $M(K)$ be the set of all unit positive Borel measures $\nu$ on $K$ (that is, $S(\nu) := \operatorname{supp}\nu \subseteq K$) such that

$$\int_{|t|\geqslant 1} \log|t|\, d\nu(t) < +\infty.$$

A pair of disjoint compacts with positive capacities $E$ and $F$ (plates of a condenser),

$$\operatorname{cap} E, \operatorname{cap} F > 0, \quad E \cap F = \varnothing, \quad E \Subset \overline{\mathbb{C}}, \quad F \Subset \mathbb{C}, \tag{1.1$_1$}$$

and a function $\varphi$ harmonic in a neighbourhood $\Omega$ of the plate $F$,

$$\varphi \in \operatorname{Harm}\Omega, \qquad F \Subset \Omega, \tag{1.1$_2$}$$

denote a *condenser*

$$(E; F, \varphi) \tag{1.1$_3$}$$

in the *field* $\varphi$ acting on the plate $F$.

Let $M(E,F)$ be the set of all charges $\mu = \mu_F - \mu_E$, $\mu_E \in M_E$, $\mu_F \in M_F$. There exists a unique charge (see [11])

$$\lambda(t) := \lambda(E; F, \varphi) \in M(E,F) \tag{1.2$_1$}$$

which minimize the energy functional $I_\varphi(\mu) := \displaystyle\int (V^\mu + 2\varphi)\, d\mu$, $\mu \in M(E,F)$, where $\varphi \equiv 0$ on $E$ and $V^\mu$ denotes the logarithmic potential (0.14). In addition, the extremal charge has the equilibrium property:

$$\begin{aligned} V^\lambda + \varphi &= \begin{cases} \gamma_\Delta & \text{on } \Delta := S(\lambda)\cap F,\\ \geqslant \gamma_\Delta & \text{on } F,\end{cases}\\ V^\lambda &= \gamma_E \quad \text{on} \quad E. \end{aligned} \tag{1.2$_2$}$$

The equilibrium relations (1.2$_2$) uniquely define the *extremal* (*equilibrium*) *charge* $\lambda$ *and the equilibrium constants* $\gamma_\Delta$, $\gamma_E$. We introduce the notation that will be important in what follows

$$\gamma := \gamma(E; F, \varphi) = \gamma_\Delta - \gamma_E. \tag{1.3}$$

The key role in problems of rational approximations is played by the following symmetry property introduced by Stahl in [29], [30]. In the case when $\Delta \subseteq F$ is a union of piecewise analytic arcs we say that a condenser has the $S$-*property* (or the symmetry property) for the plate $F$:

$$(E; F, \varphi) \in S, \quad \text{if} \quad \frac{\partial(V^\lambda + \varphi)}{\partial n_+} = \frac{\partial(V^\lambda + \varphi)}{\partial n_-} \quad \text{everywhere on} \quad \Delta \tag{1.4}$$

with possible exception of finitely many points. As usual the subscripts '+' or '−' denote (non-tangential) limiting values as one approaches the oriented arc $\Delta$ from the left and from the right, respectively.

We now can give one of the formulations of the main theorem of Gonchar and Rakhmanov (see [11]).

**Theorem** (Gonchar–Rakhmanov). *Given a sequence of analytic functions* (0.1),

$$\widehat{\rho}_n(z) := \int_F \frac{\rho_n(t)}{t-z}\,\frac{dt}{2\pi i},$$

*consider the deviations $d_n$ of its best rational approximations* (0.2) *on the set $E$. Suppose that*

1) *$E$ is the union of finitely many continua*[1] *in $\overline{\mathbb{C}}$;*
2) *$F$ is the union of finitely many rectifiable arcs or Jordan curves;*
3) *$\rho_n \in H(\Omega)$, $F \Subset \Omega$, and* $\displaystyle \frac{1}{2n}\log\frac{1}{|\rho_n|} \underset{\Omega}{\rightrightarrows} \varphi$ *as $n \to \infty$;* (1.5)
4) *$(E; F, \varphi) \in S$.*

*Then*

$$\lim_{n\to\infty} d_n(\widehat{\rho}_n, E)^{1/(2n)} = d = e^{-\gamma}, \tag{1.6}$$

*where $\gamma$ is defined in* (1.3).

**1.2. Boundary-value problem for a complex potential.** To formulate the main theorem of this work we have to extend the concept of equilibrium of an $S$-condenser.

We consider a condenser

$$(E, \varphi_E; F, \varphi_F), \tag{1.7$_1$}$$

on both plates of which acts the field ($\varphi_E$ on $E$ and $\varphi_F$ on $F$, respectively). Suppose that the supports of the equilibrium charge $\lambda = \lambda_F - \lambda_E$ on both plates are the *connected arcs* $\Delta_E$ and $\Delta_F$:

$$V^\lambda + \varphi_\alpha = \begin{cases} \gamma_{\Delta_\alpha} & \text{on } \Delta_\alpha := S(\lambda_\alpha), \\ \geqslant \gamma_{\Delta_\alpha} & \text{on } \alpha, \end{cases} \qquad \alpha = E, F, \tag{1.7$_2$}$$

and suppose that the symmetry property in the field (1.4) is satisfied on both plates

$$(E, \varphi_E; F, \varphi_F) \in S \iff \frac{\partial(V^\lambda + \varphi_\alpha)}{\partial n_+} = \frac{\partial(V^\lambda + \varphi_\alpha)}{\partial n_-} \quad \text{on } \Delta_\alpha, \quad \alpha = E, F. \tag{1.8}$$

Then for the complex potential of the charge $\lambda$

$$\mathcal{V}^\lambda(z) := V^\lambda(z) + i\widetilde{V}^\lambda(z) = \int \log\frac{1}{t-z}\,d\lambda,$$

defined up to an imaginary constant by the Cauchy–Riemann relations, the $S$-property (1.8) can equivalently be written as

$$\frac{d}{dz}(\mathcal{V}^\lambda_+ + Q_\alpha) = -\frac{d}{dz}(\mathcal{V}^\lambda_- + Q_\alpha) \quad \Delta_\alpha, \quad \alpha = E, F,$$
$$Q_\alpha = \varphi_\alpha + i\widetilde{\varphi}_\alpha, \quad Q_\alpha \in H(\Omega_\alpha), \quad \alpha = E, F,$$

[1] Continuum is a connected compact set containing more than one point.

which, in turn, implies the following boundary-value problem for the complex potential:

$$\mathcal{V}_+^\lambda + \mathcal{V}_-^\lambda = \begin{cases} -2Q_F + 2\Gamma_F & \Delta_F, \\ -2Q_E + 2\Gamma_E & \Delta_E, \end{cases} \tag{1.9}$$

where the uniquely defined real parts of the constants $\Gamma_\alpha$

$$\gamma^*_{\Delta_\alpha} := \operatorname{Re}\Gamma_\alpha, \qquad \alpha = E, F, \tag{1.10}$$

are equal, in view of the equilibrium relations ($1.7_2$), to the equilibrium constants

$$\gamma^*_{\Delta_\alpha} = \gamma_{\Delta_\alpha}, \qquad \alpha = E, F. \tag{1.11}$$

We note that the equivalent reformulation of the $S$-property (1.9) means that not only the potential of the extremal charge is in equilibrium with the field on the support, but also that the conjugate potential is in equilibrium with the conjugate field.

Thus, the holomorphic function

$$f = \exp[-\mathcal{V}^\lambda] \tag{1.12}$$

is the solution of the following boundary-value problem:

$$\begin{gathered} f \in H^\infty(\overline{\mathbb{C}} \setminus (\Delta_F \cup \Delta_E)), \quad f \neq 0 \ \text{ in } \overline{\mathbb{C}} \setminus (\Delta_F \cup \Delta_E), \\ f(\infty) = 1, \quad \underset{F}{\Delta} \arg f = 2\pi, \quad \underset{E}{\Delta} \arg f = -2\pi, \\ f_+ f_- = \exp\{2Q_\alpha - 2\Gamma_\alpha\} \ \text{ on } \Delta_\alpha, \quad \alpha = E, F. \end{gathered} \tag{1.13}$$

We note that for any input data:

$$\begin{aligned} &\text{a)} \quad \text{two smooth disjoint arcs } \Delta_F \text{ and } \Delta_E, \\ &\text{b)} \quad \text{Hölder continuous complex-valued functions } Q_F \text{ and } Q_E \end{aligned} \tag{$1.13_1$}$$

the boundary-value problem (1.13) has a unique solution:

$$\text{the function } f \text{ and the constants } \Gamma_F, \Gamma_E. \tag{$1.13_2$}$$

(This statement reduces to the standard facts of the theory of boundary-value problems, see [31]. The proof of it will be given in § 3.)

However, not for all input data the solution of (1.13) can be represented in the form (1.12), where $\lambda$ is the equilibrium charge of the condenser ($1.7_1$). As follows from the uniqueness of the solution of the problem (1.13) and the above argument, this can only take place for the input data

$$(\Delta_E, \operatorname{Re} Q_E; \Delta_F, \operatorname{Re} Q_F) \in S,$$

and then, preserving for ($1.13_2$) notation (1.10), we shall have (1.11). In the remaining cases $\mathcal{V}^\lambda$ in (1.12) can be interpreted as a complex potential of the charge that is equilibrium in the sense of relation (1.9).

By analogy with (1.3) we set

$$\gamma^* := \gamma^*(\Delta_E, Q_E; \Delta_F, Q_F) := \gamma^*_{\Delta_F} - \gamma^*_{\Delta_E}. \tag{1.14}$$

Explicit formulae for the solution ($1.13_2$) of the problem (1.13), and, consequently, for the quantity $\gamma^*$ (1.14) will be given in § 3 (see (3.4), (3.5)).

**1.3. Formulation of the main theorem. Hypothesis of Magnus.** In this work we prove the following theorem.

**Theorem 1.** *Given a sequence of analytic functions* (0.1)

$$\widehat{\rho}_n(z) := \int_F \frac{\rho_n(t)}{t-z} \frac{dt}{2\pi i},$$

*we consider the deviations of the best rational approximations* (0.2) *to it on the set* $E$

$$d_n := d_n(\widehat{\rho}_n, E).$$

*Suppose that*

1) $E$ *is a finite or semi-infinite interval on* $\mathbb{R}$;
2) $F$ *is an* $\mathbb{R}$*-symmetric rectifiable Jordan arc* ($F$ *is connected*, $\overline{F} = F$);
3) $\rho_n$ *is a real-symmetric analytic (in a neighbourhood* $\Omega$ *of the arc* $F$*) function such that* *(1.15)*

$$\rho_n := \exp\{-2(nQ + Q_1)\}, \qquad Q, Q_1 \in H(\Omega);$$

4) $(E; F, \operatorname{Re} Q) \in S$, *the equilibrium set on* $F$

$$\Delta := \{z : V^{\lambda(E;F,\operatorname{Re} Q)}(z) + \operatorname{Re} Q(z) = \gamma_\Delta\}$$

*is connected and* $\lambda' > 0$ *at the interior points of the arc* $\Delta$.

*Then*

$$d_n = 2d^* d^{2n}(1 + o(1)), \tag{1.16$_1$}$$

*where*

$$\begin{aligned} d &= \exp[-\gamma(E; F, \operatorname{Re} Q)] = \exp[-\gamma^*(E; \Delta, Q)], \\ d^* &= \exp[-\gamma^*(E; \Delta, 2Q_1)], \end{aligned} \tag{1.16$_2$}$$

*and* $\gamma^*$ (1.14) *is defined by the parameters of the solution of the boundary-value problem* (1.13).

*Remark.* The requirement that the equilibrium set be connected (see condition 4) of the theorem) is slightly more restrictive than just connectedness of the support $S(\lambda)$ of the equilibrium measure $\lambda_F$. A sufficient condition for 4) in (1.15) to hold is the convexity of $\operatorname{Re} Q$ on $F$ (see, for instance, [28]).

The hypotheses of Theorem 1 are clearly much more restrictive than those of the Gonchar–Rakhmanov theorem. Part of these additional restrictions are technical (for example, the $\mathbb{R}$-symmetry of the problem or strict positiveness of $\lambda'$ inside $\Delta$) and could have been removed. However, it seems that the main restriction — the connectedness of the support of the equilibrium measure — is adequate for the existence of the limit (0.4).

The theorem implies the following fact:

$$\begin{gathered} \rho_n := \exp\left\{-2\left(n+\frac{1}{2}\right)Q\right\} \\ \Downarrow \\ d_n = 2dd^{2n}(1+o(1)), \qquad d = \exp[-\gamma(E;F,\operatorname{Re}Q)]. \end{gathered} \tag{1.17}$$

The hypothesis of Magnus can easily be deduced from this. In fact, following [11] we have

$$d_n(e^{-x},[0,\infty]) = d_n(e^{-(n+1/2)x},[0,\infty]). \tag{1.18}$$

We write $e^{-2(n+1/2)x}$ as the Cauchy integral over the contour $\widetilde{F}$ in $\overline{\mathbb{C}}$ containing $[0,\infty]$ and observe that the integral over the ends of $\widetilde{F}$ in a neighbourhood of $\infty$ make an inessential contribution to (1.18). Therefore

$$d_n(e^{-x},[0,\infty]) = d_n\left(\int_F \frac{e^{-2(n+1/2)t/2}}{t-z}\,dt\right)(1+o(1)),$$

where $F$ is a finite Jordan arc, which, as shown in [11], can be deformed so that condition 4) will be satisfied. Thus, by the consequence of the theorem (see (1.17)) the hypothesis of Magnus is valid.

The similar application of Theorem 1 to the rational approximations to the function $x^k e^{-x}$, $k\in\mathbb{N}$, on $[0,\infty]$ gives in (0.4) the sequence $\alpha_n := n^k$, which refines the rate of approximation and, in addition,

$$\frac{d_n(x^k e^{-x},[0,\infty])}{n^k d^{2n}} \xrightarrow[n\to\infty]{} 2e^{-\gamma^*}, \qquad \gamma^* := \gamma^*\left([0,\infty];F,\frac{k}{2}\log z\right),$$

where $d^2$ is Halphen's constant (0.8) and for each $k=0,1,2,\dots$, $F$ is a Gonchar–Rakhmanov arc such that $\left([0,\infty];F,\frac{1}{2}\operatorname{Re} z\right)\in S$.

**1.4. Sharp constants for the approximation of the square root and logarithm.** We consider the situation, when the density $\rho_n$ in the integrals of Cauchy type (0.1) does not have an analytic continuation to the domain $\Omega \Supset F$ and can only by continued from the arc $F$ on both sides of it to a lens-like domain $\widetilde{\Omega}$ containing on the boundary $\partial\widetilde{\Omega}$ the end-points of the arc $F$. This is the case, for example, when $\rho_n$ has branching points at the end-points of $F$. Of interest in this situation is the case when

$$F = \Delta := S(\lambda_F), \tag{1.19}$$

since otherwise in practice Theorem 1 is applicable, as a rule. We note that (1.19) holds if in a neighbourhood $\widetilde{\Omega}_j$ of the end-point $c_j$ of the arc $\Delta$ we have

$$\lambda'(z) = \frac{m_j(z)}{\sqrt{z-c_j}}, \qquad m_j\in H(\widetilde{\Omega}_j), \quad m_j(c_j)\neq 0, \quad j=1,2 \tag{1.20}$$

(we note that (1.20) is a situation of general position for (1.19)).

We leave the general treatment of this situation beyond the scope of this work and only give an example, when the density in (0.1) has quadratic branching points at the end-points of $F$. We set

$$w_F(z) := \sqrt{(z-c_1)(z-c_2)}, \qquad \frac{w_F(z)}{z}\to 1, \quad z\to\infty, \tag{1.21}$$

where $c_1$, $c_2$ are the end-points of the arc $F$. The following theorem holds.

**Theorem 1′.** *Given a sequence of analytic functions*

$$\widehat{\rho}_n(z) := \int_F \frac{\rho_n(t)}{t-z}\,\frac{dt}{2\pi w_{F+}(t)}\,, \tag{1.22}$$

*we consider the deviations $d_n$ of the best rational approximations* (0.2) *to them on the set $E$. Suppose that conditions* 1)–4) *of Theorem* 1 *hold (see* (1.15)*). Suppose further that in addition to condition* 4) *in* (1.15) *conditions* (1.19), (1.20) *are satisfied. Then*

$$d_n = 2d^* d^{2n}(1+O(\delta^n)), \qquad 0<\delta<1,$$

*where*

$$\begin{aligned} d &:= \exp[-\gamma(E;F,\operatorname{Re}Q)] = \exp[-\gamma^*(E;F,Q)],\\ d^* &:= \exp\left[-\gamma^*\left(E,\log\frac{\Phi_{0,F}}{w_F}\,;F,2Q_1\right)\right], \end{aligned} \tag{1.23}$$

*here*

$$\Phi_{0,F}(z) := \frac{z-(c_1+c_2)/2+\sqrt{z^2-z(c_1+c_2)+c_1c_2}}{(c_1-c_2)/2}\,. \tag{1.24}$$

We now give applications of Theorems 1 and 1′ in the absence of the external field on the plate $F$, that is, $Q=0$ and we also set $Q_1=0$. We are dealing with the approximation on the interval $E$ of the functions

$$\frac{1}{2\pi}\log\frac{z-c_2}{z-c_1} \quad \text{and} \quad \frac{1}{2\sqrt{(z-c_1)(z-c_2)}}\,,$$

where $\bar c_1 = c_2$ or $c_1, c_2 \in \mathbb{R}$. We assume for simplicity that

$$\begin{gathered} \operatorname{Re} c_1, \operatorname{Re} c_2 \leqslant 0,\\ E \subseteq [0,+\infty]. \end{gathered}$$

In this case there exists Stahl's arc joining the points $c_1$ and $c_2$ so that

$$(F,E)\in S, \qquad F\cap E=\varnothing.$$

Then Theorems 1 and 1′ give the existence of the limits as $n\to\infty$

$$\begin{gathered} \frac{d_n\left(\frac{1}{2\pi}\log\frac{z-c_2}{z-c_1},E\right)}{d^{2n}} \to 2d,\\ \frac{d_n\left(\frac{1}{2\sqrt{(z-c_1)(z-c_2)}},E\right)}{d^{2n}} \to 2d_{\mathrm{sq}}, \end{gathered}$$

and also the expressions for the constants $d$ and $d_{\mathrm{sq}}$

$$d=\exp[-\gamma(E;F)] \quad \text{and} \quad d_{\mathrm{sq}} = \exp\left[-\gamma^*\left(E,\log\frac{\Phi_{0,F}}{w_F};F\right)\right.$$

Furthermore, in the case when $c_1,c_2\in\mathbb{R}$, we have $F=[c_1,c_2]$ and

$$d_{\mathrm{sq}} = \exp\left[-\gamma\left(E,\log\left|\frac{\Phi_{0,F}}{w_F}\right|;F\right)\right].$$

## § 2. Formulations of the theorems on the strong asymptotics of polynomials orthogonal with varying complex weight

We now briefly discuss main approaches to the study of the asymptotic behaviour of orthogonal polynomials. Our interest in this work is connected with the methods based on the Riemann-Hilbert (scalar and matrix) boundary-value problem, which are adapted to the proof of the required theorems on the strong asymptotics (Theorems 2 and 3). We give here the necessary definitions and formulate Theorems 2 and 3.

**2.1. Brief review.** From a variety of results of the asymptotic theory of polynomials defined by the orthogonality relations (0.11):

$$\int_F P_n(z) z^\nu h(z)\, dz = 0, \qquad P_n(z) = z^n + \cdots, \quad \nu = 0, \dots, n-1, \tag{2.1}$$

we mention only those which have a direct relation to the theorems on the strong asymptotics formulated below.

The polynomials defined by (2.1) (the so-called complex-orthogonal or non-Hermitian-orthogonal polynomials, $F \subset \mathbb{C}$, $h\big|_F \subset \mathbb{C}$), are essentially different in their properties from the Hermitian-orthogonal polynomials:

$$\int_F P_n(z) \overline{z}^\nu |h(z)|\, |dz| = 0, \qquad \nu = 0, \dots, n-1. \tag{2.2}$$

For example, the problem of the existence of polynomials (2.1) for a given complex weight $h$ requires a special attention, while the existence of polynomials (2.2) is guaranteed by their definition. Polynomials (2.2) minimize the $L^2_{|h(\xi)|\,|d\xi|}(F)$-norm among all the polynomials with the leading coefficient equal to one. This external property is one of the key tools for the analysis of their asymptotics. Accordingly, in the case of polynomials (2.1) connected with rational approximations such a tool no longer exists. However, in the real-valued case, when

$$F \subset \mathbb{R}, \qquad h \geqslant 0, \tag{2.3}$$

both constructions (2.1) and (2.2) coincide and the asymptotic theory of polynomials orthogonal on the axis (which goes back to the classical works of Bernstein [32] and Szegö [33]) is applicable to the study of rational approximations.

The situation when it is necessary to consider a varying orthogonality weight (0.12)

$$h(z) := h_n(z), \qquad \frac{1}{2n} \log |h_n(z)| \to \varphi(z), \quad n \to \infty,$$

arises in the study of

a) orthogonal polynomials on unbounded sets $F$,
b) multipoint Padé approximations, (2.4)
c) Hermite–Padé approximations.

The weak asymptotics (0.13) of polynomials (2.1), (2.3), (0.12) was studied in this connection in [7], [9], [10], [12], [34]–[37]. The most compactly formulated and at the same time the most general result on the weak asymptotics of polynomials orthogonal with respect to a varying real weight belongs to Gonchar and Rakhmanov [8]. A detailed treatment of this scope of problems can be found in the monographs [28], [38], [39].

The beginning of the study of the purely complex orthogonality case has been set forth by Nuttall (see the review in [40]). An essential breakthrough in the justification of the ideas and proving the hypotheses of Nuttall was made by Stahl. For compacts of 'minimal capacity' of Nuttall he established the local symmetry property (1.4) (see [29]) and proved the weak asymptotics (0.13) for polynomials (2.1) orthogonal on such compacts [30]. The results of Stahl were generalized by Gonchar and Rakhmanov in [1] to the case of a varying weight (2.4), which was an important step in the proof of the above theorem on the rate of rational approximation of analytic functions.

We now turn to the results on the strong asymptotics (0.16). Szegö's theory for Hermitian-orthogonal polynomials (2.2) in the case when $F$ is the union of finitely many Jordan curves and arcs was most completely developed in Widom's paper [41] (in [42] Widom's formulae of strong asymptotics were modified by means of the Riemann $\theta$-functions).

The first general results on the strong asymptotics for polynomials (2.1), (2.3) orthogonal with respect to a varying real weight were motivated by the same applications (2.4) (see [43]–[51]). The most general theorem in this direction belongs to Totik (see [52], [53]) and is proved by the methods connected with the real (Hermitian) orthogonality (that is, with the extremal properties of orthogonal polynomials).

The complex methods for the study of strong asymptotics of orthogonal polynomials are based on the boundary-value problems for analytic functions (Riemann–Hilbert problems). These problems are based on the boundary condition

$$R_{n+} + R_{n-} = P_n h \quad \text{on} \quad F, \tag{2.5}$$

which follows by the Sokhotskiĭ–Plemel' formulae from the definition of the functions of the second kind $R_n$ for polynomials (2.1):

$$R_n(z) := \frac{1}{2\pi i}\int_F \frac{P_n(t)h(t)}{t-z}\,dt, \qquad R_n(z) = O\left(\frac{1}{z^{n+1}}\right) \quad \text{as} \quad n\to\infty. \tag{2.6}$$

(We recall that the orthogonal polynomials (2.1) are the denominators of the Padé approximations of the function $\displaystyle\int_F \frac{h(t)}{t-z}\frac{dt}{2\pi i}$, and function (2.6) is called the remainder function of the Padé approximations.)

This approach appeared in the works of Nuttall in connection with the study of the strong asymptotics of the Hermite–Padé polynomials (see the review [40] and also [54]). In [55] the boundary condition (2.5) was considered as a singular integral equation and on this basis the formulae of the strong asymptotics of the polynomials (2.1) orthogonal on the interval $F \subset \mathbb{R}$ with respect to the complex weight

$$h := \frac{\widetilde{h}}{w_{F+}} \quad \text{on} \quad F, \qquad \widetilde{h} \subset H(\Omega), \quad F \Subset \Omega, \quad \widetilde{h} \neq 0 \ \text{in} \ \Omega \tag{2.7}$$

were obtained. Here $w_F$ is defined in (1.21) (and $\widetilde{h}$ can also be taken from a Hölder class). Let us mention in this direction the work [56]. In Suetin's paper [57] the approach of Nuttall was developed for the polynomials (2.1) orthogonal with respect to weight (2.7) on the union of finitely many $S$-symmetric arcs in $\mathbb{C}$, where $w_F$, as in (1.21), is the square root of the polynomial with zeroes at the end-points $\{c_j\}$ of the arcs making up $F$. Suetin reduced relation (2.5) to a certain Riemann boundary-value problem on a two-sheeted (hyperelliptic) Riemann surface with quadratic branch points at $\{c_j\}$.

In this work we adapt the approach of Nuttall and Suetin for the polynomials (2.1) orthogonal with respect to a *varying* complex weight (0.12), that is, we consider in (2.7)

$$\widetilde{h} := h_n := e^{-2nQ}\widetilde{h}_n, \qquad \|\widetilde{h}_n - \widetilde{h}_\infty\|_{H(\Omega)} = o(1), \quad n \to \infty, \tag{2.8}$$

and $F$ is an arc with $S$-property in the field $\operatorname{Re} Q$ (see the formulation of Theorem 3 below in section 2.3).

The most impressive development of the methods for the study of the strong asymptotics of orthogonal polynomials based on the Riemann-Hilbert problem has been made in a cycle of works by Deift and coauthors [26], [58]–[62]. As was noticed by Its and coauthors in [63], relations (2.5) can be reformulated in terms of a matrix Riemann-Hilbert problem. The solution of it as $n \to \infty$ is the essence of the approach of Deift and coauthors. Although the results in the above works concern polynomials orthogonal on the real axis with respect to varying real weights, nevertheless the method is essentially 'complex', as we show in this paper adapting it for the proof of the strong asymptotics of the polynomials (2.1) orthogonal with respect to a varying *complex* weight

$$h := \widetilde{h} := h_n.$$

Here $h_n$ is defined in (2.8), and $F$ is an $S$-symmetric arc in the field $\operatorname{Re} Q$ (see the formulation of Theorem 2 below in section 2.3).

**2.2. Auxiliary boundary-value problem. Szegö function and main term of asymptotics.** We now introduce certain functions in terms of which the formulae of the strong asymptotics will be written below.

Let us be given a sequence of holomorphic functions $h_n$ non-vanishing in the domain $\Omega$ and defined by (2.8):

$$h_n := e^{-2nQ}\widetilde{h}_n, \qquad \|\widetilde{h}_n - \widetilde{h}_\infty\|_{H(\Omega)} = o(1), \quad n \to \infty. \tag{2.8$'$}$$

Let $\Delta$ be an arbitrary smooth Jordan arc in $\Omega$. We assume for convenience that $\Delta$ joins the points $-1$ and $0$ (that is, $-1$ and $0$ are the end-points of the arc $\Delta$ oriented from $-1$ to $0$).

It is known (see [31], section 42.2) that the boundary-value problem

$$\begin{gathered} f \in H^\infty(\overline{\mathbb{C}} \setminus \Delta), \quad f \neq 0 \ \text{ in } \overline{\mathbb{C}} \setminus \Delta, \\ f_+ f_- \widetilde{h}_n = 1 \ \text{ on } \Delta \end{gathered} \tag{2.9}$$

has a unique (up to ± sign) solution in the class of bounded functions given by the formula

$$(2.10_1)\qquad f(z)=\exp\left\{w(z)\int_\Delta \frac{\log\widetilde{h}_n(t)}{z-t}\,\frac{dt}{2\pi i w_+(t)}\right\},$$

where

$$w(z):=(z(z+1))^{1/2},\qquad w>0\quad\text{for}\quad z>0,$$

and

$$(2.10_2)\qquad f(\infty)=\exp\left\{\int_\Delta \frac{\log\widetilde{h}_n(t)\,dt}{2\pi i w_+(t)}\right\}.$$

By analogy with the 'real-valued' case we shall call the solution of the boundary-value problem (2.9) (the function $(2.10_1)$) the *Szegö function* for the weight $\widetilde{h}_n i/w_+$ on $\Delta$ and denote by

$$(2.10_3)\qquad f\left(\frac{\widetilde{h}_n i}{w_+};z\right).$$

To introduce the analogue of the Szegö function for the complex-valued weight $\widetilde{h}_n$ we consider the function

$$(2.11_1)\qquad \varkappa(z):=\left(\frac{\Phi_0(z)}{w(z)}\right)^{1/2},$$

where for the function $(\,\cdot\,)^{1/2}$ we take the main branch (positive on $\mathbb{R}^+$) and the function $\Phi_0$ (see (1.24)),

$$(2.11_2)\qquad \Phi_0(z):=4\left(z+\frac{1}{2}+\sqrt{z^2+z}\right),$$

is a rational function on a two-sheeted Riemann surface with branch points at $-1$ and 0 having on one sheet a simple pole at the point $\infty$ and a simple zero at $\infty$ on the other sheet. Denoting by superscripts $(+)$ and $(-)$ the values (of the branches) of the functions on the sheets of this surface we obtain

$$(2.11_3)\qquad \Phi_0^{(+)}\Phi_0^{(-)}=1\quad\text{and}\quad w^{(+)}=-w^{(-)}\ \text{everywhere in}\ \overline{\mathbb{C}}.$$

Therefore for each arc $\Delta$ joining the branch points $-1$ and 0 we have

$$(2.11_4)\qquad \varkappa_+\varkappa_-=\left(\frac{\Phi_{0+}\Phi_{0-}}{w_+w_-}\right)^{1/2}=\left(\frac{-1}{w_+^2}\right)^{1/2}=\frac{i}{w_+}.$$

By analogy with the real-valued case we shall again call the function

$$(2.12)\qquad f(\widetilde{h}_n;z):=\varkappa f\left(\frac{\widetilde{h}_n i}{w_+};z\right)$$

(see ($2.11_1$) and ($2.10_3$)) the Szegö function for the complex-valued weight $\widetilde{h}_n$. In view of (2.9) and ($2.11_4$) the function $f$ (2.12) is the solution (in the sense of continuous boundary values on the open arc $\mathring{\Delta} := \Delta \setminus (\{-1\} \cup \{0\})$) of the following boundary-value problem:

$$f \in H^2(\overline{\mathbb{C}} \setminus \Delta), \quad f \neq 0 \text{ in } \overline{\mathbb{C}} \setminus \Delta, \qquad (2.13)$$
$$f_+ f_- \widetilde{h}_n = \frac{i}{w_+} \text{ on } \mathring{\Delta}.$$

We now turn to the definition of the function describing the main term of the asymptotics. Suppose that in $\Omega$ there is a Jordan arc $F \Subset \Omega$ with $S$-property in the field

$$q := \operatorname{Re} Q,$$

which means that there exists a measure $\lambda \in M(F)$ such that

$$(2.14) \quad V^\lambda + q = \begin{cases} \gamma_\Delta & \text{on } S(\lambda), \\ \geqslant \gamma_\Delta & \text{on } F \end{cases} \quad \text{and} \quad \frac{\partial(V^\lambda + q)}{\partial n_+} = \frac{\partial(V^\lambda + q)}{\partial n_-} \quad \text{on} \quad S(\lambda).$$

The fulfillment of conditions (2.14) will be denoted as follows:

$$(2.15_1) \qquad (F, q) \in S.$$

In what follows we assume that the equilibrium set

$$(2.15_2) \qquad \Delta := \{z : V^\lambda(z) + q(z) = \gamma_\Delta\}$$

is a connected analytic arc, whose end-points (as agreed) are $-1$ and 0. For the description of the main term of the asymptotics we consider the function

$$(2.16) \qquad \Phi_q := \exp\{\Gamma - \mathcal{V}^\lambda\}, \qquad \operatorname{Re}\Gamma = \gamma.$$

Repeating the argument in section 1.2 (see the passage from (1.8) to (1.9)) we see that, thanks to the $S$-property, $\Phi_q$ satisfies the boundary condition

$$(2.17) \qquad \Phi_{q+}\Phi_{q-}e^{-2Q} = 1 \quad \text{on} \quad \Delta.$$

We observe that taking into account the boundary conditions for $\Phi_q$ and $\Phi_0$ the function $\Phi_q/\Phi_0$ is the solution of the boundary-value problem (2.9) with $\widetilde{h}_n := e^{-2Q}$, which gives by (2.10) the explicit expressions for $\Phi_q$ and $\Gamma$.

**2.3. Formulations of Theorems 2 and 3.** In this work we prove the following theorem.

**Theorem 2.** *Suppose that the sequence of weight functions $h_n \in H(\Omega)$, $h_n \neq 0$ in $\Omega$, has the form* (2.8), *that is,*

1) $h_n := e^{-2nQ}\widetilde{h}_n$, $\|\widetilde{h}_n - \widetilde{h}_\infty\|_{H(\Omega)} = o(1)$,

*and let the smooth Jordan arc $F$ satisfy conditions* $(2.15_1)$, $(2.15_2)$, *that is,*

2) $(F, \operatorname{Re} Q) \in S$, *the equilibrium set* $\Delta$ $(2.15_2)$ *is connected and* $\lambda' > 0$ *inside* $\Delta$.

*Then*

A) *for all sufficiently large* $n$ *there exist polynomials orthogonal on* $F$ *with weight* $h_n$ $P_n(z) = z^n + \cdots$:

$$\int_F P_n(z) z^\nu h_n(z)\, dz = 0, \qquad \nu = 0, \dots, n-1;$$

B) *for the polynomials* $P_n$ *and functions of the second kind* $R_n = \dfrac{1}{2\pi i} \displaystyle\int_F \frac{P_n(t) h_n(t)\, dt}{t - z}$ *the following asymptotic formulae hold as* $n \to \infty$:

B1) $$\begin{cases} P_n = C\varphi(1 + o(1)), \\ R_n = \dfrac{i}{w} \dfrac{C}{\varphi}(1 + o(1)) \end{cases} \quad \textit{uniformly on } K \Subset \overline{\mathbb{C}} \setminus \Delta, \tag{2.18$_1$}$$

B2) $$\begin{cases} P_n = C(\varphi_+ + \varphi_-)(1 + o(1)), \\ R_{n\pm} = \dfrac{Ci}{(w\varphi)_\pm}(1 + o(1)) \end{cases} \quad \textit{uniformly on } K \Subset \Delta, \tag{2.18$_2$}$$

*where* $\varphi$ *is defined by* (2.12) *and* (2.16) *as follows:*

$$\varphi(z) := \Phi_q^n(z) f(\widetilde{h}_\infty; z), \tag{2.19$_1$}$$

*and the constant* $C$ *is equal to*

$$C := (e^{n\Gamma} f(\widetilde{h}_\infty; \infty))^{-1}. \tag{2.19$_2$}$$

The weight functions in Theorem 2 are analytic perturbations of the Legendre weight ($Q \equiv 0$, $\widetilde{h}_n \equiv 1$). Therefore it is not surprising that asymptotic formulae B2) (see $(2.18_2)$) hold only 'inside' $\overset{\circ}{\Delta}$. In fact, in the course of the proof of Theorem 2 we obtain the asymptotics of $P_n$ and $R_n$ in neighbourhoods of the end-points of $\Delta$. However, since this local asymptotics is not used in the proof of the main Theorem 1, it is not included in the formulation of Theorem 2 (to avoid complications) and the corresponding formulae will be given below (see section 6.1.5, formulae (6.28), (6.27)). As in connection with Theorem 1 we point out that a 'rough' sufficient condition for the connectedness of $\Delta$ in condition 2) of Theorem 2 is the convexity of the external field $\operatorname{Re} Q$ on $F$.

In the course of the proof of Theorem 1 we shall require polynomials with a single uniform oscillating asymptotic behaviour on the whole arc $\Delta$ (including the end-points). This can be achieved by considering the weights which are analytic perturbations of the Chebyshev weight. The following theorem holds.

**Theorem 3.** *Let conditions* 1) *and* 2) *of Theorem* 2 *hold and let, in addition to* 2), *the following condition be satisfied*

3) $F = \Delta$ *and in a neighbourhood* $\Omega_j$ *of the end-point* $c_j$, $j = 1, 2$ *of the arc* $\Delta$ *the following equality holds:*

$$\lambda'(F, \operatorname{Re} Q; z) = \frac{m_j(z)}{\sqrt{z - c_j}}, \quad m_j \in H(\Omega_j), \ \ m_j(c_j) \neq 0, \ \ j = 1, 2. \tag{2.20}$$

*Then*

A) *for sufficiently large $n$ there exist polynomials $P_n(z) = z^n + \cdots$ orthogonal on $F$ with weight $h_n i/w_+$;*

B) *for the polynomials $P_n$ and the functions of the second kind $R_n$ (see* (2.6)*) the following asymptotic formulae hold as $n \to \infty$:*

$$\text{B1)} \begin{cases} P_n = C\varphi(1 + O(\delta^n)), \\ R_n = \dfrac{i}{w}\dfrac{C}{\varphi}(1 + O(\delta^n)), \end{cases} \qquad \delta := \delta(K) \in (0,1), \quad K \Subset \overline{\mathbb{C}} \setminus \Delta,$$

$$\text{B2)} \begin{cases} P_n = C(\varphi_+ + \varphi_-)(1 + O(\delta^n)), \\ R_{n\pm} = \dfrac{iC}{(w\varphi)_\pm}(1 + O(\delta^n)) \end{cases} \qquad \textit{uniformly on } \Delta,\ 0 < \delta < 1, \tag{2.21}$$

*where $\varphi$ is defined by* (2.10) *and* (2.16) *as follows*

$$\varphi(z) := \Phi_q^n(z) f\left(\frac{\widetilde{h}_n i}{w_+}; z\right), \tag{2.22}$$

*and the constant $C$ is equal to*

$$C := \left(e^{n\Gamma} f\left(\frac{\widetilde{h}_n i}{w_+}; \infty\right)\right)^{-1}.$$

We note that in the case when $Q = 0$, $F \in \mathbb{R}$, and $\widetilde{h}_n \equiv \widetilde{h}_\infty$ Theorem 3 goes over into the theorem of Nuttall mentioned above (see [55]).

## § 3. Proof of the theorem on the rate and sharp constant of the rational approximation of analytic functions

In this section we give the solution to the boundary-value problem (1.13). Then assuming theorems on the strong asymptotics (see Theorems 2 and 3) we prove the main results — Theorems 1 and 1′.

**3.1. Solution of the auxiliary boundary-value problem.** Let $F$ and $E$ be two arbitrary smooth arcs with end-points $c_1$, $c_2$ and $c_3$, $c_4$, respectively. Let $Q_F$ and $Q_E$ be analytic functions in neighbourhoods of the arcs $F$ and $E$, respectively, and let $\Gamma_F$ and $\Gamma_E$ be some complex constants. We recall the notation (1.24):

$$\Phi_0(z; c_1, c_2) := \frac{z - (c_1 + c_2)/2 + \sqrt{z^2 - z(c_1 + c_2) + c_1 c_2}}{(c_1 - c_2)/2}. \tag{3.1$_1$}$$

Accordingly, we set

$$\begin{aligned} \widetilde{\Phi}_{0,F}(z) &:= C_{0,F}\Phi_0(z; c_1, c_2), \qquad C_{0,F} := \frac{c_1 - c_2}{4}, \\ \widetilde{\Phi}_{0,E}(z) &:= C_{0,E}\Phi_0(z; c_3, c_4), \qquad C_{0,E} := \frac{c_3 - c_4}{4}. \end{aligned} \tag{3.1$_2$}$$

We define on $F$ and $E$ the functions

$$
(3.2_1)\qquad
\begin{aligned}
H_F &:= Q_F + \log\frac{C_{0,F}}{\widetilde{\Phi}_{0,E}} \quad \text{on} \quad F,\\
H_E &:= Q_E - \log\frac{C_{0,E}}{\widetilde{\Phi}_{0,F}} \quad \text{on} \quad E,
\end{aligned}
$$

also on $E \cup F$ we set

$$
(3.2_2)\qquad H(t;\Gamma_F,\Gamma_E) := 2(H_\alpha(t) - \Gamma_\alpha), \qquad t \in \alpha, \quad \alpha = F, E.
$$

Finally, we set

$$
(3.3)\qquad w_{E,F} := \sqrt{\prod_{j=1}^{4}(z-c_j)}\,; \qquad \frac{w_{E,F}(z)}{z^2} \to 1 \quad \text{as} \quad z \to \infty.
$$

The following statement holds.

**Statement 1.** *There exists a unique*[2] *collection*

$$
(f(z), \Gamma_F, \Gamma_E),
$$

*satisfying the boundary-value problem* (1.13). *More precisely,*

$$
(3.4)\qquad
\begin{gathered}
f \in H^\infty(\overline{\mathbb{C}} \setminus (F \cup E)), \quad f \neq 0 \ \ \textit{in} \ \ \overline{\mathbb{C}} \setminus (F \cup E),\\
f(\infty) = 1, \quad \mathop{\Delta}_{F} \arg f = 2\pi, \quad \mathop{\Delta}_{E} \arg f = -2\pi,\\
f_+ f_- = \exp\{2(Q_\alpha - \Gamma_\alpha)\} \ \ \textit{on} \ \ \alpha, \quad \alpha = E, F.
\end{gathered}
$$

*Moreover, the constants $\Gamma_E$ and $\Gamma_F$ here are*

$$
(3.5_1)\qquad \Gamma_E = \frac{D_E}{D}, \qquad \Gamma_F = \frac{D_F}{D},
$$

*where*

$$
(3.5_2)\qquad
\begin{aligned}
D_E &:= \begin{vmatrix} \displaystyle\int_F \frac{dt}{w_{EF+}(t)} & \displaystyle\int_E \frac{H_E(t)\,dt}{w_{EF+}(t)} \\ \displaystyle\int_F \frac{t\,dt}{w_{EF+}(t)} & \displaystyle\int_E \frac{H_E(t)t\,dt}{w_{EF+}(t)} \end{vmatrix},\\
D_F &:= \begin{vmatrix} \displaystyle\int_F \frac{H_F(t)\,dt}{w_{EF+}(t)} & \displaystyle\int_E \frac{dt}{w_{EF+}(t)} \\ \displaystyle\int_F \frac{H_F(t)t\,dt}{w_{EF+}(t)} & \displaystyle\int_E \frac{t\,dt}{w_{EF+}(t)} \end{vmatrix},\\
D &:= \begin{vmatrix} \displaystyle\int_F \frac{dt}{w_{EF+}(t)} & \displaystyle\int_E \frac{dt}{w_{EF+}(t)} \\ \displaystyle\int_F \frac{t\,dt}{w_{EF+}(t)} & \displaystyle\int_E \frac{t\,dt}{w_{EF+}(t)} \end{vmatrix},
\end{aligned}
$$

[2] The uniqueness of $\Gamma_F$ and $\Gamma_E$ is understood up to $\pi k$, $k \in \mathbb{Z}$.

*the function $f(z)$ is*

$$f = \frac{\widetilde{\Phi}_{0,F}}{\widetilde{\Phi}_{0,E}} e^W, \tag{3.6$_1$}$$

*where*

$$W(z) := w_{E,F}(z) \int_{E\cup F} \frac{H(t;\Gamma_F,\Gamma_E)}{w_{E,F}(t)} \frac{dt}{2\pi i(t-z)}, \tag{3.6$_2$}$$

*and the functions on the right-hand sides of* (3.5) *and* (3.6) *are defined above (see* (3.1)–(3.3)*).*

*Proof.* We consider the function

$$\mathcal{F} := f \frac{\widetilde{\Phi}_{0,E}}{\widetilde{\Phi}_{0,F}}.$$

Since

$$\widetilde{\Phi}_{0,\alpha+}\widetilde{\Phi}_{0,\alpha-} = C_{0,\alpha}^2 \quad \text{on} \quad \alpha = E, F$$

and taking into account (3.4), we see that the function $\mathcal{F}$ must satisfy the following boundary-value problem:

$$\begin{gathered} \mathcal{F} \in H^\infty(\overline{\mathbb{C}} \setminus (F \cup E)), \quad \mathcal{F} \neq 0 \ \text{ in } \overline{\mathbb{C}} \setminus (F \cup E), \\ \mathcal{F}(\infty) = 1, \quad \underset{E}{\Delta} \arg \mathcal{F} = 0, \quad \underset{F}{\Delta} \arg \mathcal{F} = 0, \\ \mathcal{F}_+\mathcal{F}_- = e^H \ \text{ on } E \cup F. \end{gathered}$$

Since now the increment of the argument of $\mathcal{F}$ along any closed path in $\overline{\mathbb{C}}\setminus(E\cup F)$ is zero and $\mathcal{F} \neq 0$ in this domain, it follows that we can select a holomorphic branch:

$$W := \log \mathcal{F}.$$

The function $W$ must, in turn, satisfy, the boundary-value problem

$$\begin{gathered} W \in H^\infty(\overline{\mathbb{C}} \setminus (F \cup E)), \\ W(\infty) = 0, \\ W_+ + W_- = H \ \text{ on } E \cup F. \end{gathered} \tag{3.7}$$

The boundary-value problem (3.7) is well studied (even for an arbitrary number of arcs). The condition for solubility of it (see [31], section 42.2) in the class of bounded functions is the relation

$$\int_{F\cup E} \frac{H(t)t^\nu \, dt}{w_{EF+}(t)} = 0, \qquad \nu = 0, 1,$$

which gives a linear system for finding $\Gamma_F$ and $\Gamma_E$; the solution is written down in (3.5). Once the constants $\Gamma_F$ and $\Gamma_E$ are found we can express the unique solution (3.7) in the form (3.6$_2$). The statement is proved.

**3.2. Proof of Theorem 1.** We recall that we have to find the asymptotics on an interval $E$ of the real axis $\mathbb{R}$ of the deviation of the rational approximations from the function

$$\widehat{\rho}_n(z) := \int_F \frac{\exp\{-2(nQ(t)+Q_1(t))\}}{t-z}\,\frac{dt}{2\pi i}, \tag{3.8}$$

where $Q, Q_1 \in H(\Omega)$ are real-symmetric functions and the rectifiable Jordan arc $F \Subset \Omega$ and the interval $E$ make up a condenser such that the plate $F$ has the $S$-property in the field $\operatorname{Re} Q$. In addition, the equilibrium set $\Delta$ is connected (and, consequently, $\Delta = S(\lambda_F)$, where $\lambda_F$ is the component of the equilibrium charge $\lambda$ on $F$). Since $Q$ is analytic, the arc $\Delta$ joining the points $\widetilde{c}_1$ and $\widetilde{c}_2$ is also an analytic arc.

We observe that since the input data for our problem, that is, the integrals of Cauchy type (3.8) and their best rational approximations go over under a linear change of variables into integrals of Cauchy type (with other densities) and, accordingly, a rational function goes over into a rational function and, finally, since the hypotheses of Theorem 1 (see (1.15)) and the asymptotic formulae in (1.16) are invariant with respect a conformal change of variables, it follows that we can assume that $E$ is a finite interval on $\mathbb{R}$:

$$E := [c_3, c_4].$$

We also assume that the equilibrium problem for the condenser $(E; F, \operatorname{Re} Q)$ is solved and we are given the analytic arc $\Delta$, the components of the equilibrium charge (that is, the measures $\lambda_\Delta := \lambda_F$ and $\lambda_E$), and also the equilibrium constants $\gamma_\Delta$ and $\gamma_E$.

We set

$$\widetilde{\Phi}_\alpha := \exp\{-\mathcal{V}^{\lambda_\alpha}\}, \qquad \alpha = \Delta, E.$$

By the $S$-property of the plate $F$ we have (see (1.9))

$$\left(\frac{\widetilde{\Phi}_\Delta}{\widetilde{\Phi}_E}\right)_+ \left(\frac{\widetilde{\Phi}_\Delta}{\widetilde{\Phi}_E}\right)_- = \exp\{2(Q - \Gamma_\Delta)\} \quad \text{on} \quad \Delta.$$

Since our problem is real-symmetric, we have, in addition,

$$\Gamma_\alpha = \overline{\Gamma}_\alpha = \gamma_\alpha, \qquad \alpha = \Delta, E.$$

We set

$$\Phi_\alpha := \exp\{-\mathcal{V}^{\lambda_\alpha} + \gamma_\alpha\}, \qquad \alpha = \Delta, E. \tag{3.9}$$

The boundary conditions on $\Delta$, the equilibrium condition on $E$, and the $\mathbb{R}$-symmetry give that

$$\left(\frac{\widetilde{\Phi}_E}{\Phi_\Delta}\right)_+ \left(\frac{\widetilde{\Phi}_E}{\Phi_\Delta}\right)_- = \begin{cases} e^{-2Q} & \text{on } \Delta, \\ e^{-2\gamma} & \text{on } E, \end{cases} \tag{3.10}$$

where $\gamma$ is given by (1.3), that is, $\gamma := \gamma_\Delta - \gamma_E$.

Finally, we assume that the boundary-value problem (1.13) is solved with input data $(E; \Delta, 2Q_1)$ and we have the analytic (in $\overline{\mathbb{C}} \setminus (E \cup \Delta)$) function $f$ normalized at the point $\infty$ so that $f(\infty) = 1$ and the real (in view of the $\mathbb{R}$-symmetry) constants $\Gamma^*_\Delta = \overline{\Gamma}^*_\Delta = \gamma^*_\Delta$ and $\Gamma^*_E = \overline{\Gamma}^*_E = \gamma^*_E$ such that the following boundary conditions hold:

$$f_+ f_- = \begin{cases} \exp\{4Q_1 - 2\gamma^*_\Delta\} & \text{on } \Delta, \\ \exp\{-2\gamma^*_E\} & \text{on } E \end{cases} \tag{3.11}$$

and the following index conditions are satisfied

$$\underset{F}{\Delta} \arg f = 2\pi, \qquad \underset{E}{\Delta} \arg f = -2\pi. \tag{3.12}$$

We define the function

$$\widetilde{f} := \frac{\widetilde{\Phi}_E}{w_\Delta} f,$$

where

$$w_\Delta(z) := \sqrt{(z - \widetilde{c}_1)(z - \widetilde{c}_2)}\,, \qquad \frac{w_\Delta(z)}{z} \to 1, \quad z \to \infty.$$

Since $\widetilde{f}$ in $\overline{\mathbb{C}} \setminus (\Delta \cup E)$ has index zero, that is (3.12) gives that

$$\underset{F}{\Delta} \arg \widetilde{f} = \underset{E}{\Delta} \arg \widetilde{f} = 0, \tag{3.12$'$}$$

it follows that $\widetilde{f}$ admits a unique decomposition

$$\widetilde{f} := \widetilde{f}^2_\Delta \widetilde{f}^{-1}_E$$

such that

$$\widetilde{f}_\alpha \in H(\overline{\mathbb{C}} \setminus \alpha), \quad \widetilde{f}_\alpha \neq 0 \quad \text{in} \quad \overline{\mathbb{C}} \setminus \alpha, \quad \widetilde{f}_\alpha(\infty) = 1, \quad \alpha = \Delta, E.$$

In fact, since we can take the single-valued logarithm of $\widetilde{f}$ (see (3.12$'$)), we have by the Cauchy integral formula

$$\widetilde{f}^{k_\alpha}_\alpha(z) = \exp\left\{ \oint_\alpha \frac{\log \widetilde{f}(t)}{t - z} \frac{dt}{2\pi i} \right\}, \qquad \alpha = \Delta, E, \quad k_\Delta = 2, \quad k_E = -1, \tag{3.13}$$

where $\oint_\alpha$ denotes the integral over an arbitrary rectifiable Jordan contour enclosing the arc $\alpha = \Delta, E$ and the point $z$ is taken outside the contour.

By analogy with (3.9) we set

$$f_\Delta := \widetilde{f}_\Delta e^{\gamma^*_\Delta}, \tag{3.13$'$}$$

and obtain that the functions so defined satisfy, in view of (3.11), the boundary conditions

$$(3.14)\qquad \left(\frac{\widetilde{\Phi}_E\widetilde{f}_E}{f_\Delta^2 w_\Delta}\right)_+\left(\frac{\widetilde{\Phi}_E\widetilde{f}_E}{f_\Delta^2 w_\Delta}\right)_- = \begin{cases} e^{-4Q_1} & \text{on } \Delta, \\ e^{-2\gamma^*} & \text{on } E, \end{cases}$$

where $\gamma^*$ is given by (1.14), that is, $\gamma^* = \gamma_\Delta^* - \gamma_E^*$.

We observe that if in the boundary condition (3.14) considered on $\Delta$ we are given the functions $\widetilde{\Phi}_E$, $\widetilde{f}_E$, $Q_1$, then they up to a sign define the function $f_\Delta^2$:

$$f_\Delta \in H(\overline{\mathbb{C}} \setminus \Delta), \quad f_\Delta \neq 0 \quad \text{in} \quad \overline{\mathbb{C}} \setminus \Delta.$$

A similar statement also holds with respect the boundary condition on $E$ and with respect to the boundary conditions (3.10).

Having constructed and described the useful properties of the functions $\Phi_\Delta$,$f_\Delta$, $\widetilde{\Phi}_E$, and $\widetilde{f}_E$ we conclude the preliminary stage and turn to the proof of the theorem.

1) We first construct a sequence of polynomials

$$\omega_{2n+1}(z) = \prod_{j=1}^{2n+1}(z - x_{j,2n+1}), \qquad x_{j,2n+1} \in E, \quad j = 1, \dots, 2n+1,$$

such that

$$(3.15_1)\qquad \frac{\omega_{2n+1}}{\widetilde{\Phi}_E^{2n+1}\widetilde{f}_E} \underset{n\to\infty}{\rightrightarrows} 1 \quad \text{uniformly on} \quad K \Subset \overline{\mathbb{C}} \setminus E$$

and

$$(3.15_2)\qquad \frac{\omega_{2n+1}}{|\widetilde{\Phi}_E|^{2n+1}} = |\widetilde{f}_E| 2\cos\big((2n+1)\arg\widetilde{\Phi}_E + \arg\widetilde{f}_E\big) + o(1)$$

uniformly on the whole interval $E$.

For the construction of such sequence we can use, for example, Theorem 3 (formulated in § 2 and proved below in § 4). We set

$$N := 2n+1.$$

As $\omega_N$ we take the polynomials orthogonal with respect the weight $h_N i/w_{E+}$ on the interval $E$:

$$(3.16)\qquad \int_E \omega_N(x)x^\nu \frac{h_N i}{w_{E+}}\,dx = 0, \qquad \nu = 0, \dots, N-1,$$

where

$$w_E(z) := \sqrt{(z-c_3)(z-c_4)},$$

and we chose

$$(3.17)\qquad h_N := \widetilde{\Phi}_\Delta^{-2N}\widetilde{h}_N, \qquad \widetilde{h}_N := \text{const}\cdot\frac{\Phi_\Delta^2}{f_\Delta^4 w_\Delta^2}$$

as $h_N$. In what follows we set for convenience

$$\text{const} := e^{2(\gamma^*-\gamma)}.$$

By Theorem 3 the asymptotic behaviour of polynomials (3.16) is described by the function (see (2.22))

$$\varphi(z) := \Phi_q^N(z) f\left(\frac{\widetilde{h}_N i}{w_{E+}}\,; z\right). \tag{3.18}$$

The main term of the asymptotics is characterized by the boundary condition (2.17)

$$\Phi_{q+}\Phi_{q-}\exp\{-2\log\widetilde{\Phi}_\Delta\} = 1 \quad \text{on} \quad E.$$

Renormalizing the left-hand side of the last equality and taking into account the $\mathbb{R}$-symmetry we obtain

$$\left(\frac{\widetilde{\Phi}_q}{\Phi_\Delta}\right)_+\left(\frac{\widetilde{\Phi}_q}{\Phi_\Delta}\right)_- = e^{-2\gamma} \quad \text{on} \quad E. \tag{3.19}$$

Comparing the equality (3.19) so obtained with (3.10) and taking into account the remark after (3.14), we obtain

$$\widetilde{\Phi}_q = \widetilde{\Phi}_E. \tag{3.20}$$

We denote for brevity the Szegö function in (3.18) by

$$\widehat{f} := f\left(\frac{\widetilde{h}_N i}{w_{E+}}\,; z\right).$$

From the definition of the Szegö function (2.9) we deduce

$$\widehat{f}_+\widehat{f}_-\widetilde{h}_N = 1 \quad \text{on} \quad E.$$

Substituting into the left-hand side the value of $\widetilde{h}_N$ from (3.17) and taking into account (3.19) and (3.20), we have

$$\widehat{f}_+\widehat{f}_-\frac{1}{e^{2(\gamma-\gamma^*)}}\frac{\Phi_\Delta^2}{f_\Delta^4 w_\Delta^2} = e^{2\gamma^*}\left(\frac{\widehat{f}\widetilde{\widetilde{\Phi}}_E}{f_\Delta^2 w_\Delta}\right)_+\left(\frac{\widehat{f}\widetilde{\widetilde{\Phi}}_E}{f_\Delta^2 w_\Delta}\right)_- = 1 \quad \text{on} \quad E.$$

Hence, comparing this with (3.14) we obtain

$$\widehat{f} = \widetilde{f}_E, \qquad \widehat{f}(\infty) = 1. \tag{3.21}$$

Theorems 3 gives for the polynomials $\omega_N$ the relations (see (2.21))

$$\begin{aligned}
\frac{\omega_N}{C\varphi} &= 1 + O(\delta^n) \quad \text{on} \quad K \Subset \overline{\mathbb{C}} \setminus E, \quad 0 < \delta(K) < 1, \\
\frac{\omega_N}{C} &= (\varphi_+ + \varphi_-)(1 + O(\delta^n)) \quad \text{on} \quad E, \quad 0 < \delta < 1.
\end{aligned}$$

Substituting (3.18), (3.20), and (3.21) into these formulae we immediately deduce ($3.15_1$) from the first one. As far as the second one is concerned, we have (since $\Phi_{E+} = \overline{\Phi}_{E-} \Rightarrow |\Phi_{E+}| = |\Phi_{E-}| =: |\Phi_E|$) the relation

$$(3.22) \qquad \frac{\omega_N}{|\widetilde{\Phi}_E|^N} = \left\{ |f_E| \frac{(\Phi_E^N \widetilde{f}_E)_+ + (\Phi_E^N \widetilde{f}_E)_-}{|\Phi_E^N \widetilde{f}_E|} \right\} (1 + O(\delta^n)) \quad \text{on} \quad E.$$

Since the expression in braces is bounded uniformly in $n$, (3.22) gives ($3.15_2$). Thus, we have constructed the sequence $\{\omega_{2n+1}\}$ for which asymptotic formulae (3.15) hold.

2) We now consider a rational function $r_n$ of order not higher than $n$ which interpolates the given function $\widehat{\rho}_n(z)$ (see (3.8)) at $2n+1$ points of the interval $E$, for which we chose the zeros of our polynomial $\omega_{2n+1}$. This rational function is called the *multi-point Padé approximation.* It is well known (and easy to show) that its denominator $P_n$ is orthogonal on $F$ to powers of $z$ with respect to the varying weight $h_n$:

$$(3.23) \qquad \int_F P_n(z) z^\nu h_n(z)\, dz = 0, \qquad \nu = 0, 1, \ldots, n-1,$$

where

$$(3.24) \qquad h_n := \frac{\exp\{-2(nQ + Q_1)\}}{\omega_{2n+1}} i^\varepsilon, \qquad \varepsilon := \begin{cases} 1, & \Delta \subset \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

By the Hermite interpolation formula we have for the deviation of $r_n$ from $\widehat{\rho}_n$[3]

$$(3.25) \qquad i^\varepsilon \widehat{\rho}_n - r_n = \frac{\omega_{2n+1}}{P_n} R_n,$$

where

$$R_n(z) = \frac{1}{P_n(z)} \int_F \frac{P_n^2(t) h_n(t)}{t - z} \frac{dt}{2\pi i} = \int_F \frac{P_n(t) h_n(t)}{t - z} \frac{dt}{2\pi i}.$$

Substituting in (3.24) the asymptotic formula ($3.15_1$) we obtain

$$h_n = \exp\{-2n(Q + \log \widetilde{\Phi}_E)\} \widetilde{h}_n,$$

where

$$\widetilde{h}_n = \widetilde{h}_\infty + o(1), \qquad \widetilde{h}_\infty = \frac{e^{-2Q_1} i^\varepsilon}{\widetilde{\Phi}_E \widetilde{f}_E} \quad \text{on} \quad \Omega,$$

and see that the weight function (3.24) satisfies the hypotheses of Theorem 2 formulated in § 3 and to be proved below in § § 5–7. (We note that $Q$ in the formulation of Theorem 2 corresponds here to $Q + \log \widetilde{\Phi}_E$.) Therefore for sufficiently

[3] The factor $i^\varepsilon$ is introduced in order that (3.25) be real on $E$.

large $n$ the polynomials $P_n(z) = z^n + \cdots$ exist and satisfy the asymptotic formulae (2.18), (2.19). We require only the formulae of the outer asymptotics ($2.18_1$). We recall that they are of the form

$$\begin{aligned} P_n &= (\Phi_q e^{-\gamma})^n \left( \frac{\widehat{f}}{\widehat{f}(\infty)} + o(1) \right), \\ R_n &= \frac{i}{w_\Delta} \cdot \frac{e^{-n\gamma}}{\widehat{f}(\infty)} \cdot \frac{1+o(1)}{\Phi_q^n \widehat{f}} \end{aligned} \qquad \text{on} \quad K \Subset \overline{\mathbb{C}} \setminus F, \tag{3.26}$$

where the main term of the asymptotics $\Phi_q$ satisfies the boundary condition (2.17)

$$\Phi_{q+} \Phi_{q-} e^{-2(Q + \log \widetilde{\Phi}_E)} = 1 \quad \text{on} \quad \Delta, \tag{3.27}$$

and the Szegö function denoted here by

$$\widehat{f}(z) := f(\widetilde{h}_\infty; z),$$

is defined by the boundary-value problem (2.13) with boundary condition

$$i^\varepsilon \widehat{f}_+ \widehat{f}_- \frac{e^{-Q_1}}{\widetilde{\Phi}_E \widetilde{f}_E} = \frac{i}{w_{\Delta+}} \quad \text{in} \quad \mathring{\Delta}. \tag{3.28}$$

Comparing (3.27) with the boundary condition on $\Delta$ in (3.10) we conclude that

$$\Phi_q = \Phi_\Delta.$$

Taking the square of (3.28) and taking into account that $w_{\Delta+} = -w_{\Delta-}$ and the fact that for $\varepsilon = 0$ (see ($3.1_1$)) the normalization at infinity of the Szegö function (2.12), ($2.11_1$) turns out to be complex, we find from (3.14) that

$$\widehat{f}^2 = f_\Delta^2 i.$$

Thus, the asymptotic formulae (3.26) for $P_n$ and $R_n$ are written in terms of $\Phi_\Delta$ and $f_\Delta$ (defined above in (3.9) and (3.13), respectively) as follows

$$\begin{aligned} P_n &= (\Phi_\Delta e^{-\gamma})^n \left( \frac{f_\Delta}{f_\Delta(\infty)} + o(1) \right), \\ R_n &= \frac{1}{w_\Delta} \cdot \frac{e^{-n\gamma}}{f_\Delta(\infty)} \cdot \frac{1+o(1)}{\Phi_\Delta^n f_\Delta} \end{aligned} \qquad \text{on} \quad K \Subset \overline{\mathbb{C}} \setminus F. \tag{3.29}$$

3) To complete the proof of Theorem 1 it remains to substitute the asymptotic formulae (3.29) and (3.15) on $E$ into the expression for deviation (3.25):

$$\begin{aligned} i^\varepsilon \widehat{\rho}_n - r_n &= \frac{\omega_{2n+1}}{P_n} R_n = \frac{\omega_{2n+1}}{|\widetilde{\Phi}_E|^{2n+1}} \cdot \frac{|\widetilde{\Phi}_E|^{2n+1}}{(\Phi_\Delta e^{-\gamma})^n \widetilde{f}_\Delta} \cdot \frac{1}{w_\Delta} \cdot \frac{e^{-\gamma n}(1+o(1))}{f_\Delta(\infty) \Phi_\Delta^n f_\Delta} \\ &= 2 \left| \frac{\widetilde{\Phi}_E}{\Phi_\Delta} \right|^{2n} \frac{|\widetilde{\Phi}_E \widetilde{f}_E|}{f_\Delta^2 w_\Delta} \big( \cos((2n+1) \arg \widetilde{\Phi}_E + \arg \widetilde{f}_E) + o(1) \big). \end{aligned} \tag{3.30}$$

Taking into account the boundary conditions (3.10) and (3.14) on $E$ and the $\mathbb{R}$-symmetry we obtain the relation

$$i^\varepsilon \widehat{\rho}_n - r_n = 2 e^{-2\gamma n} e^{-\gamma^*} \cos\big((2n+1) \arg \widetilde{\Phi}_E + \arg \widetilde{f}_E\big)(1 + o(1))$$

holding uniformly on $E$. Thus, our rational approximations provide an asymptotic (uniformly on the whole $E$) alternance of the approximation. Hence, Theorem 1 follows from the Vallée-Poussin theorem (see [64], Chapter II, section 32).

**3.3. Proof of Theorem 1′.** The proof of Theorem 1′ goes along the same scheme as the proof of Theorem 2. Moreover, the proof of Theorem 1′ requires fewer tools than the proof of Theorem 1, since both the asymptotic behaviour of the polynomials $\omega_{2n+1}$ with zeros at optimal points of interpolations and the asymptotic behaviour of the polynomials $P_n$ (the denominators of the multi-point Padé approximations) are described by the same Theorem 3, whose proof is much easier than the proof of Theorem 2 (as we shall see in § § 4–7 below).

We recall that we have to find the asymptotics of the deviation on $E$ of the best rational approximations from the functions (1.22)

$$\widehat{\rho}_n(z) := \frac{1}{2\pi i}\int_F e^{-2(nQ+Q_1)}\frac{i\,dt}{w_{\Delta+}(t-z)} .$$

Let $\widetilde{\Phi}_E$ and $\widetilde{f}_E$ be some functions such that

$$\text{(3.31)}\qquad \widetilde{\Phi}_E \in H(\mathbb{C}\setminus E), \qquad \exists\, \widetilde{\Phi}_{E\pm} \in L^\infty(E), \qquad \frac{\widetilde{\Phi}_E}{z} \to 1, \quad z\to\infty,$$
$$\widetilde{f}_E \in H^\infty(\mathbb{C}\setminus E), \qquad \widetilde{f}_E(\infty) = 1;$$

the particular choice of the functions satisfying (3.31) will be specified below.

Let $\omega_N$ be a polynomial

$$\omega_N(z) := \prod_{j=1}^{N}(z-z_{j,N}), \qquad \{z_{j,N}\}\in E,$$

satisfying the asymptotic formulae (3.15):

$$\text{(3.32)}\qquad \begin{gathered} \frac{\omega_N}{\widetilde{\Phi}_E^{2n+1}\widetilde{f}_E} \rightrightarrows 1 \quad \text{on} \quad K \Subset \overline{\mathbb{C}}\setminus E, \\ \left[\frac{\omega_N}{|\widetilde{\Phi}_E|^{2n+1}} - |\widetilde{f}_E|2\cos(N\arg\widetilde{\Phi}_E + \arg\widetilde{f}_E)\right] \rightrightarrows 0 \quad \text{on} \quad E, \end{gathered}$$

converging at the rate of a geometric progression as $n\to\infty$. These polynomials will be constructed below after the specification of their choice of the functions $\widetilde{\Phi}_E$ and $\widetilde{f}_E$.

We consider the multi-point Padé approximation $r_n$ of order not higher than $n$ interpolating $\widehat{\rho}_n$ at the zeros of $\omega_{2n+1}$. The denominators of the rational function $r_n$ are the polynomials $P_n(z) = z^n + \cdots$ orthogonal to powers of $z$ on $\Delta = F$ with respect to the varying weight (see (3.24))

$$h := \frac{h_n i}{w_{\Delta+}}, \qquad h_n := \frac{\exp\{-2(nQ+Q_1)\}}{\omega_{2n+1}},$$

where, in view of (3.32),

$$h_n = e^{-2n(Q+\log\widetilde{\Phi}_E)}\left(\frac{e^{-Q_1}}{\widetilde{\Phi}_E\widetilde{f}_E} + o(1)\right), \qquad n\to\infty.$$

By Theorem 3 the polynomials $P_n$ and the corresponding functions of the second kind $R_n$ (2.6) satisfy the asymptotic formulae (2.21):

$$\begin{aligned} &P_n = (\Phi_\Delta C_\Delta)^n\Big(\frac{f_\Delta}{f_\Delta(\infty)} + O(\delta^n)\Big), \\ &R_n = \frac{i}{w_\Delta}\,\frac{C_\Delta^n}{f_\Delta(\infty)}\,\frac{1+O(\delta^n)}{\Phi_\Delta^n f_\Delta^2}, \end{aligned} \qquad 0<\delta<1, \quad \text{on} \quad K\Subset\overline{\mathbb{C}}\setminus\Delta, \tag{3.33}$$

where the functions $\Phi_\Delta$ and $f_\Delta$,

$$\Phi_\Delta \in H(\mathbb{C}\setminus\Delta), \qquad \exists\,\Phi_{\Delta\pm}\in L^\infty(E), \qquad \frac{\Phi_\Delta(z)}{z}\to C_\Delta>0, \quad z\to\infty,$$
$$f_\Delta \in H^\infty(\overline{\mathbb{C}}\setminus\Delta),$$

are defined by $\widetilde{\Phi}_E$ and $\widetilde{f}_E$ and the boundary conditions on $\Delta$

$$\begin{cases} \Phi_{\Delta+}\Phi_{\Delta-}e^{-2(Q+\log\widetilde{\Phi}_E)} = 1, \\ f_{\Delta+}f_{\Delta-}\dfrac{e^{-2Q_1}}{\widetilde{\Phi}_E\widetilde{f}_E} = 1, \end{cases}$$

which can written in the form

$$\begin{cases} \left(\dfrac{\widetilde{\Phi}_E}{\Phi_\Delta}\right)_+\left(\dfrac{\widetilde{\Phi}_E}{\Phi_\Delta}\right)_- = e^{-2Q}, \\ \left(\dfrac{\widetilde{\Phi}_E\widetilde{f}_E}{f_\Delta^2}\right)_+\left(\dfrac{\widetilde{\Phi}_E\widetilde{f}_E}{f_\Delta^2}\right)_- = e^{-4Q_1} \end{cases} \qquad \text{on} \quad \Delta. \tag{3.34}$$

Substituting the asymptotic formulae (3.33) and (3.32) into the expression (3.25) for the deviation of the rational function $r_n$ from $\widehat{\rho}_n$, we, as before (see (3.30)), obtain the asymptotic relation

$$\widehat{\rho}_n - r_n = \frac{\omega_{2n+1}}{P_n}R_n = 2\left|\frac{\widetilde{\Phi}_E}{\Phi_\Delta}\right|^{2n}\frac{|\widetilde{\Phi}_E\widetilde{f}_E|i}{f_\Delta^2 w_\Delta}\big(\cos\big((2n+1)\arg\widetilde{\Phi}_E + \arg\widetilde{f}_E\big) + o(1)\big)$$

holding uniformly on $E$. To obtain the asymptotic alternance (uniformly on $E$) we set the amplitude in the last formula equal to a constant. As a result we obtain:

$$\begin{cases} \left|\dfrac{\widetilde{\Phi}_E}{\Phi_\Delta}\right| =: d =: e^{-\gamma}, \\ \left|\dfrac{\widetilde{\Phi}_E\widetilde{f}_E}{f_\Delta^2 w_\Delta}\right| =: d_1 =: e^{-\gamma^*} \end{cases} \qquad \text{on} \quad E.$$

Hence, taking into account the $\mathbb{R}$-symmetry we obtain the boundary condition:

$$\begin{cases} \left(\dfrac{\widetilde{\Phi}_E}{\Phi_\Delta}\right)_+\left(\dfrac{\widetilde{\Phi}_E}{\Phi_\Delta}\right)_- = e^{-2\gamma}, \\ \left(\dfrac{\widetilde{\Phi}_E\widetilde{f}_E}{f_\Delta^2}\right)_+\left(\dfrac{\widetilde{\Phi}_E\widetilde{f}_E}{f_\Delta^2}\right)_- = e^{-2\gamma^*}w_\Delta^2 \end{cases} \qquad \text{on} \quad E. \tag{3.35}$$

Comparing the first relations in (3.34) and (3.35) with (3.10) we see that for finding $\widetilde{\Phi}_E/\Phi_\Delta$ and $\gamma$ we have the same boundary-value problem as in the proof of Theorem 1. Hence,

$$\gamma = \gamma(E; F, \operatorname{Re} Q) = \gamma^*(E; F, Q)$$

and the solution of this problem provides us with concrete $\widetilde{\Phi}_E$ and $\Phi_\Delta$.

However, the second relations in (3.34) and (3.35) do not yet make up a boundary-value problem of the form (1.13). Therefore we introduce the function

$$\frac{\widetilde{\Phi}_E \widetilde{f}_E}{\Phi_{0,F} f_\Delta^2}, \tag{3.36}$$

where the function $\Phi_{0,F}$ defined in (1.24) satisfies on $\Delta = F$ the boundary condition

$$\Phi_{0,F+}\Phi_{0,F-} = 1.$$

We obtain from this the following boundary conditions for the function (3.36):

$$\left(\frac{\widetilde{\Phi}_E \widetilde{f}_E}{\Phi_{0,F} f_\Delta^2}\right)_+ \left(\frac{\widetilde{\Phi}_E \widetilde{f}_E}{\Phi_{0,F} f_\Delta^2}\right)_- = \begin{cases} e^{-4Q_1} & \text{on } \Delta, \\ e^{-2\gamma^*} \dfrac{w_\Delta^2}{\Phi_{0,F}^2} & \text{on } E, \end{cases} \tag{3.37}$$

which already make up the boundary-value problem (similar to (3.14)), which by renormalization at $\infty$ reduces to (1.13). Since the right-hand sides in (3.37) satisfy the Hölder condition, we obtain that

$$\gamma^* = \gamma^*\left(E, \log \frac{\Phi_{0,F}}{w_F}; F, 2Q_1\right)$$

and factorizing the solution of the boundary-value problem (1.13) with boundary conditions (3.37) (that is, the function (3.36)) we fix the choice of the concrete functions $\widetilde{f}_E$ and $f_\Delta$.

To complete the proof it remains to present the polynomials $\omega_N(z)$ with asymptotic formulae (3.32). As before, we take as $\omega_N$ the polynomial orthogonal on $E$ with respect to the weight $\widetilde{\Phi}_\Delta^{-2N} \dfrac{\Phi_\Delta^2}{f_\Delta^4 w_\Delta^2} \dfrac{i}{w_E}$.

Repeating the corresponding argument from the proof of Theorem 1 again based on Theorem 3 we obtain that the required polynomials are constructed. Theorem 1′ is proved.

## § 4. Proof of Theorem 3. The method of scalar Riemann problem

In this section we prove Theorem 3 on the strong asymptotics of polynomialas orthogonal with respect to a complex varying analytic weight with branch points (of quadratic order) at the end-points of the arc (the support of the orthogonality measure). The proof is based on an analysis of a certain scalar Riemann problem on a two-sheeted Riemann surface.

**4.1. Boundary-value problem on Riemann surface.** We recall that we have to obtain asymptotic formulae (2.21) for the polynomials $P_n(z) = z^n + \cdots$,

$$\int_F P_n(z) z^\nu \frac{h_n(z)i}{w_+(z)}\,dz = 0, \qquad \nu = 0, 1, \ldots, n-1, \tag{4.1}$$

and the functions of the second kind (2.6)

$$R_n(z) := \frac{1}{2\pi i}\int_F \frac{P_n(t)h_n(t)i}{(t-z)w_+(t)}\,dt, \tag{4.2}$$

for which, in view of the orthogonality conditions, the following relation holds:

$$R_n(z) = O\left(\frac{1}{z^{n+1}}\right) \quad \text{as} \quad z \to \infty. \tag{4.2$'$}$$

The varying component of the orthogonality weight in (4.1) is holomorphic in the neighbourhood $\Omega$ of the arc $F$, does not vanish in $\Omega$, and is of the form

$$h_n := e^{-2nQ}\widetilde{h}_n, \qquad \|\widetilde{h}_n - \widetilde{h}_\infty\|_{H(\Omega)} = o(1), \tag{4.3}$$

where $Q$ is such that the equilibrium measure $\lambda$ in the field $\operatorname{Re} Q$ on $F$ has as the support the arc $\Delta$, which coincides with $F$:

$$\Delta := S(\lambda), \qquad \Delta \equiv F,$$

and in the neighbourhood $\omega_j$ of the end-point $\widetilde{c}_j$, $j = 1, 2$, of the arc $\Delta$ the derivative of $\lambda$ has the form

$$\lambda'(z) = \frac{m_j(z)}{w(z)}, \qquad m_j \in H(\omega_j), \quad m_j(\widetilde{c}_j) \neq 0, \quad j = 1, 2, \tag{4.4}$$

where

$$w(z) = ((z-\widetilde{c}_1)(z-\widetilde{c}_2))^{1/2}, \qquad \frac{w(z)}{z} \to 1 \quad \text{as} \quad z \to \infty.$$

We also recall that the asymptotic formulae (2.21), which we have to prove, are written in terms of the function $\varphi$ (see (2.22)). This function is the solution of the following boundary-value problem:

$$\begin{gathered} \varphi \in H(\mathbb{C}\setminus\Delta), \qquad \varphi \neq 0 \quad \text{in} \quad \mathbb{C}\setminus\Delta, \quad \exists\, \varphi_\pm \in L^\infty(\Delta), \\ \varphi(z) = \frac{z^n}{C_n} + \cdots, \qquad z \to \infty, \quad C_n > 0, \\ \varphi_+\varphi_- = \frac{1}{h_n} \quad \text{on} \quad \Delta. \end{gathered} \tag{4.5}$$

The starting point of various methods connected with the Riemann boundary-value problem is the application of the Sokhotskiĭ–Plemel' formulae to the expression (4.2) for the functions of the second kind $R_n$

$$R_{n+} - R_{n-} = P_n \frac{h_n i}{w_+} \quad \text{at the interior points of} \quad \Delta.$$

For a function $R_n w$ with boundary values bounded on $\Delta$ this gives

$$(R_n w)_+ + (R_n w)_- = h_n P_n i \quad \text{on} \quad \Delta. \tag{4.6}$$

Identical transformations of the last equality by means of the decomposition (4.5) of the weight function $h_n$ on $\Delta$ lead to the equalities

$$\begin{aligned} (R_n w\varphi)_+ + \left(\frac{R_n w}{\varphi h_n}\right)_- &= \left(\frac{iP_n}{\varphi}\right)_-, \\ \left(\frac{R_n w}{\varphi h_n}\right)_+ + (R_n w\varphi)_- &= \left(\frac{iP_n}{\varphi}\right)_+. \end{aligned}$$

That is, for functions with boundary values bounded on $\Delta$ we have the relation

$$\left(\frac{wR_n\varphi}{i}\right)_\pm - \left(\frac{P_n}{\varphi}\right)_\mp = -\left(\frac{wR_n}{i\varphi h_n}\right)_\mp \quad \text{on} \quad \Delta. \tag{4.6$'$}$$

We consider the two-sheeted Riemann surface $\mathfrak{R}$ of the function $w$. This surface is made up of the two copies of the extended complex plane cut along the arc $\Delta$ (denoted by $\mathfrak{R}^+$ and $\mathfrak{R}^-$), which are glued in the standard way so that the left 'bank' $\Delta_+ \subset \mathfrak{R}^+$ is identified with the right 'bank' $\Delta_- \subset \mathfrak{R}^-$ and, respectively, $\Delta_- \subset \mathfrak{R}^+$ is identified with $\Delta_+ \subset \mathfrak{R}^-$. We denote by $\pi$ the canonical projection of $\mathfrak{R}$ onto $\mathbb{C}$:

$$\pi(\mathfrak{R}) = \overline{\mathbb{C}}.$$

Finally, we denote by $\widetilde{\Delta}$ the closed (on $\mathfrak{R}$) contour dividing $\mathfrak{R}$ into two sheets $\mathfrak{R}^+$ and $\mathfrak{R}^-$. The projection of $\widetilde{\Delta}$ coincides with 'bank' (+) and 'bank' (−) of the arc $\Delta$:

$$\mathfrak{R} := \mathfrak{R}^+ \cup \widetilde{\Delta} \cup \mathfrak{R}^-. \tag{4.7}$$

We assume that the contour $\widetilde{\Delta}$ is oriented so that in going along it the domain $\mathfrak{R}^+$ is on the left and $\mathfrak{R}^-$ is on the right.

We define on $\mathfrak{R}$ the piecewise analytic function

$$\mathcal{F}(z) := \begin{cases} \left(\dfrac{wR_n\varphi}{i}\right)(\pi(z)), & z \in \mathfrak{R}^+, \\ \left(\dfrac{P_n}{\varphi}\right)(\pi(z)), & z \in \mathfrak{R}^-, \end{cases} \tag{4.8}$$

and also the function

$$j(z) := -\left(\frac{wR_n}{\varphi h_n}\right)(\pi(z)), \qquad z \in \Omega^- \subset \mathfrak{R}^-, \quad \pi(\Omega^-) = \Omega. \tag{4.9}$$

Suppose that for a fixed $n$ the orthogonal polynomial $P_n(z) = z^n + \cdots$ exists and that (see (4.2$'$)) $R_n$ has at the point $\infty$ the zero of order $n+1$. Such indices $n$ are called *normal indices*. We shall see below that our assumption holds for large $n$. The normality of the index $n$, the definition (4.5) of the function $\varphi$, and the boundary conditions (4.6) imply that the function $\mathcal{F}$ is the unique solution of the following boundary-value problem on $\mathfrak{R}$:

$$
\begin{aligned}
&\mathcal{F} \in H^\infty(\mathfrak{R} \setminus \widetilde{\Delta}),\\
&\mathcal{F}_+ - \mathcal{F}_- = j_- \quad \text{on} \quad \widetilde{\Delta},\\
&\mathcal{F}(\infty^{(-)}) = C_n + O\left(\frac{1}{z}\right).
\end{aligned}
\tag{4.10}
$$

The solution of the boundary-value problem (4.10) can be written by means of the Cauchy integral formula on the Riemann surface

$$
\mathcal{F}(z) = \frac{1}{2\pi i} \int_{\widetilde{\Delta}} j_-(\xi)\, d\Omega(\xi; z, \infty^{(-)}) + C_n, \qquad z \in \mathfrak{R} \setminus \widetilde{\Delta},
\tag{4.11}
$$

where $d\Omega$ is the meromorphic differential on $\mathfrak{R}$ with the simple pole at the point $z$ with residue $+1$ and the simple pole at the point $\infty^{(-)}$ with residue $-1$. The explicit expression for it is known

$$
d\Omega(\xi; z, \infty^{(-)}) := \left(\frac{1}{2}\,\frac{w(\xi) + w(z)}{(\xi - z)w(\xi)} - \frac{1}{w(\xi)}\right) d\xi.
$$

**4.2. Deformation of the contour in the integral equation.** For $z \in \overline{\mathfrak{R}^+}$ formula (4.11) is an integral equation for the functions of the second kind $R_n$. We note that under the integral sign in (4.11) we have the limiting value of the holomorphic (in $\Omega^- \subset \mathfrak{R}^-$) function $j(\xi)$ as $\xi$ tends to $\widetilde{\Delta}$ from the side of the sheet $\mathfrak{R}^-$. Therefore we can deform the contour $\widetilde{\Delta}$ to a contour $\widetilde{\Delta}' \subset \Omega^-$ so that for points $z$ outside the annulus-like domain $\mathcal{A} \subset \Omega^-$ bounded by $\widetilde{\Delta}$ and $\widetilde{\Delta}'$ relation (4.11) still holds:

$$
\mathcal{F}(z) = \frac{1}{2\pi i} \int_{\widetilde{\Delta}'} j(\xi)\, d\Omega(\xi; z, \infty^{(-)}) + C_n, \qquad z \in \mathfrak{R} \setminus \overline{\mathcal{A}}.
\tag{4.12}
$$

We write the jump function $j$ in the form

$$
j = -wR_n\varphi J,
$$

where (see (4.9))

$$
J := \frac{1}{\varphi^2 h_n}\,.
$$

The following result holds.

**Lemma 1.** *There exists a domain $\widetilde{d} \subset \Omega^-$ such that on each contour $\widetilde{\Delta}' \subset \widetilde{d} \setminus \Delta$ the following relation holds:*

$$
\|J\|_{C(\widetilde{\Delta}')} = O(\delta^n), \qquad 0 < \delta < 1,
\tag{4.13}
$$

*where the constant $\delta$ depends on $\widetilde{\Delta}'$.*

*Proof.* We project the sheet $\mathfrak{R}^-$ onto the complex plane $\mathbb{C}$. By the representation (2.22) for the function $\varphi$ we have

$$J = \left(\frac{1}{f^2_{\widetilde{h}_n}\,\widetilde{h}_n}\right)\left(\frac{e^{2nQ}}{\Phi_q^{2n}}\right), \tag{4.14}$$

where the function $f_{\widetilde{h}_n} := f\left(\frac{\widetilde{h}_n i}{w_+}\,; z\right)$ is defined in (2.10) and $\Phi_q$ is defined in (2.16). In view of the uniform boundedness (from above and from below) of the family $\{\widetilde{h}_n\}$ in $\Omega$ (see (4.3)) we have

$$(f^2_{\widetilde{h}_n}\,\widetilde{h}_n)^{-1} \leqslant C \quad \text{on} \quad K \Subset \Omega \tag{4.15}$$

uniformly in $n$. We consider the second factor in (4.14). Taking into account the $S$-symmetry (see (2.17) and (2.16)) we have on the arc $\Delta$

$$\frac{e^{2nQ}}{\Phi^{2n}_{q\pm}} = \left(\frac{\Phi_{q\mp}}{\Phi_{q\pm}}\right)^n = \exp\{-n(\mathcal{V}^\lambda_\mp - \mathcal{V}^\lambda_\pm)\} \quad \text{on} \quad \Delta.$$

Since for the boundary values of the complex potential

$$\mathcal{V}^\lambda(z) = -\int_\Delta \ln(z-t)\,d\lambda(t) = V^\lambda(z) - i\int_\Delta \arg(z-t)\,d\lambda(t)$$

we have on $\Delta$ the equality

$$\mathcal{V}^\lambda_\pm(x) = V^\lambda(x) - i\int_{-1}^x \arg(x-t)\,d\lambda(t) + i\int_0^x [\arg(t-x)\pm\pi]\,d\lambda(t), \qquad x\in\Delta$$

(recall that we assumed that the end-points of $\Delta$ are $-1$ and $0$), it follows that

$$\frac{e^{2nQ(x)}}{\Phi^{2n}_{q\pm}(x)} = \exp\left\{\mp 2\pi i n\int_x^0 d\lambda(t)\right\}, \qquad x\in\Delta.$$

We set

$$l_\pm(x) := \mp\int_x^0 d\lambda(t), \qquad x\in\Delta. \tag{4.16}$$

By the positiveness of the equilibrium measure $\lambda$ we have

$$\operatorname{Im} l_\pm = 0 \quad \text{on} \quad \Delta. \tag{4.17}$$

The fact that the external field $Q$ is holomorphic in $\Omega$ gives that the equilibrium measure is absolutely continuous and its density $\lambda'$ is holomorphic at the interior points of the arc $\Delta$. Hence it is holomorphic in a lens-like domain $d$: $\Delta \subset \overline{d} \subset \Omega$. Thus, the function

$$l(z) := \begin{cases} -\displaystyle\int_z^0 \lambda'(t)\,dt, & z \in d^+, \\ +\displaystyle\int_z^0 \lambda'(t)\,dt, & z \in d^-, \end{cases} \qquad \overline{d^+ \cup \Delta \cup d^-} = \overline{d}, \tag{4.18}$$

as the primitive of a holomorphic function realizes a holomorphic continuation of the functions $l_+$ and $l_-$ from the 'banks' of the arc $\Delta$ into the simply connected domains $d^+$ and $d^-$.

Using the local representation (2.20) of the density $\lambda'$ in the neighbourhoods $O_1$ and $O_0$ of the end-points $-1$ and $0$ of the arc $\Delta$ we obtain that

$$l(z) = \begin{cases} \widetilde{m}_1(z)\sqrt{z+1}\,, & \widetilde{m}_1 \in H(O_1),\ z \in O_1, \\ \widetilde{m}_0(z)\sqrt{z}\,, & \widetilde{m}_0 \in H(O_0),\ z \in O_0. \end{cases} \tag{4.19}$$

The right-hand sides of (4.19) at points corresponding to different banks of $\Delta$ differ only in sign after going around the end-points. Therefore (4.18) realizes a holomorphic continuation of (4.16) into a wider domain $\widetilde{d} \setminus \Delta$ containing the end-points:

$$l \in H(\widetilde{d} \setminus \Delta), \qquad \widetilde{d} := d^+ \cup d^- \cup O_1 \cup O_0.$$

We now make sure that

$$\operatorname{Im} l > 0 \quad \text{in} \quad \widetilde{d} \setminus \Delta. \tag{4.20}$$

In domains $d^+$ and $d^-$ inequality (4.20) holds in view of the Cauchy–Riemann relations:

$$\left.\frac{\partial \operatorname{Im} l}{\partial n}\right|_x = \left.\frac{\partial \operatorname{Re} l}{\partial \tau}\right|_x = \lambda'(x) > 0, \qquad x \in \Delta,$$

which gives (with (4.17) taken into account) that

$$\operatorname{Im} l > 0 \quad \text{in} \quad d^+ \cup d^-. \tag{4.20$'$}$$

In neighbourhoods of the end-points of the arc the local representation (4.19) of $l$ implies the following: starting from real values of $l$ on $\Delta$ (see (4.17)) the argument of the function $l$ increases by $\pi$ after going around the end-point. With (4.20$'$) taken into account this leads to the inequality

$$0 < \arg l(z) < \pi, \qquad z \in O_1 \cup O_0,$$

that is,

$$\operatorname{Im} l > 0 \quad \text{in} \quad O_1 \cup O_0.$$

Thus, the function

$$J = (f^2_{\widetilde{h}_n} \widetilde{h}_n)^{-1} \exp\{2\pi i n l\}$$

realizes a holomorphic continuation of the function (4.14) from both 'banks' of the arc $\Delta$ into the domain $\widetilde{d} \setminus \Delta$, and, in view of (4.15) and (4.20) the estimate (4.13) holds on any contour $\Delta' \subset \widetilde{d} \setminus \Delta$ and the number $\delta$ in (4.13) depends on the distance between $\Delta'$ and $\Delta$. The lemma is proved.

**4.3. Asymptotic formulae for $R_n$ and $Q_n$.** We first estimate the boundary values of the functions of the second kind $R_n$ on $\Delta$. We set

$$M_n := \|wR_n\varphi\|_{C(\Delta)} =: (wR_n\varphi)(z_0), \qquad z_0 \in \Delta.$$

We consider the equation (4.12) for $z \in \mathfrak{R}^+$

$$\left(\frac{wR_n\varphi}{i}\right)(\pi(z)) = -\frac{1}{2\pi i}\int_{\widetilde{\Delta}'} (wR_n\varphi J)(\pi(\xi))\, d\Omega(\xi; z, \infty^{(-)}) + C_n. \tag{4.21}$$

We observe that the integral is no longer singular as $z \to \widetilde{\Delta}$ and since the function $wR_n\varphi$ has continuous limiting values on $\Delta$, we consider (4.21) at the point $z_0$ and obtain that

$$M_n \leqslant M_n O(\delta^n) + C_n, \tag{4.22}$$

that is,

$$M_n \leqslant \frac{C_n}{1 - O(\delta^n)}\,. \tag{4.22$'$}$$

Returning to (4.21) for $z \in \widetilde{\Delta}_+$ we have

$$\left|\left(\frac{wR_n\varphi}{i}\right)(\pi(z)) - C_n\right| \leqslant M_n O(\delta^n),$$

which, in view of (4.22′), gives

$$\left(\frac{wR_n\varphi}{iC_n}\right)_\pm = 1 + O(\delta^n) \quad \text{on} \quad \Delta.$$

This is the required asymptotic formula for (2.21)-B2) for $R_n$ on $\Delta$. From this formula we see, in the first place, that by the maximum modulus principle the asymptotic formula (2.21)-B2) holds for $R_n$ on compacts in $\overline{\mathbb{C}} \setminus \Delta$. Secondly, since (taking into account (4.5))

$$\frac{(wR_n)_\pm}{h_n} = \frac{C_n}{\varphi_\pm h_n}(1 + O(\delta^n)) = C_n\varphi_\mp(1 + O(\delta^n)),$$

the boundary condition (4.6) gives that

$$P_n = \left(\frac{wR_n}{h_n}\right)_+ + \left(\frac{wR_n}{h_n}\right)_- = C_n(\varphi_+ + \varphi_-)(1 + O(\delta^n)), \tag{4.23}$$

that is, we have obtained the asymptotic formula (2.21)-B2) for the polynomial $P_n$ on $\Delta$. On compact sets $K \Subset \overline{\mathbb{C}} \setminus \Delta$ the asymptotic formula for the polynomials $P_n$ (see (2.21)-B1)) is obtained as follows. We fix $K$ and choose $\widetilde{\Delta}'$ so that

$$K \Subset \overline{\mathbb{C}} \setminus \pi(\mathcal{A}).$$

We consider the equation (4.12) for $z \in \mathfrak{R}^- \setminus \mathcal{A}$ and see that

$$\left(\frac{P_n}{\varphi}\right)(\pi(z)) - C_n = -\frac{1}{2\pi i}\int_{\widetilde{\Delta}'} (wR_n\varphi J)(\pi(\xi))\, d\Omega(\xi; z, \infty^{(-)}).$$

Estimating the right-hand side by (4.22′) and (4.13) we obtain the required formula (2.21)-B1).

To complete the proof Theorem 3 it remains to make sure that our assumption on the normality of indices as they tend to infinity is satisfied. If we assume the contrary, that is, there exists an infinite subsequence $\Lambda$ of indices $n$, which are not normal, then for these indices there exist polynomials $P_n(z)$ orthogonal to the $n$-powers of $z$ such that

$$\deg P_n < n \quad \text{and} \quad R_n = O\left(\frac{1}{z^{n+1}}\right), \qquad z \to \infty, \qquad n \in \Lambda.$$

We can write for these $P_n$ and $R_n$ the boundary-value problem (4.10), for which the normalization condition goes over into the relation $\mathcal{F}(\infty^{(-)}) = O(1/z)$. Repeating the above analysis of the solution of the boundary-value problem so obtained we come to the estimate (4.22) in the form

$$M_n \leqslant M_n O(\delta^n), \qquad 0 < \delta < 1,$$

which leads to a contradiction for sufficiently large $n$. Theorem 1 is proved.

## § 5. Proof of theorem 2. The method of matrix Riemann-Hilbert problem

In this and the following sections we prove Theorem 2 on the strong asymptotics of polynomials orthogonal with respect to varying complex weight holomorphic in a neighbourhood of the support, where, in addition, the support of the equilibrium measure in the field of the varying weight is assumed to be connected. For the proof we use the method based on an analysis of a certain matrix Riemann-Hilbert problem. This method was proposed and developed by Deift and coauthors (see [26], [58]–[61]). The new point in the present paper (in comparison with the above cited works) is the fact that the varying orthogonality weight is *complex* and the orthogonality is considered on an arc in the *complex* plane. However, to be fair we point out that the general scheme of our proof and many details of it are often a repetition (with certain methodological innovations) of the corresponding argument in [59], [61].

In this section we give a conditional proof of Theorem 2 assuming the existence of the solution of certain matrix boundary-value problems in *neighbourhoods of the end-points* of the support of the equilibrium measure.

**5.1. Matrix Riemann–Hilbert problem for orthogonal polynomials.** We recall that we have to prove the asymptotic formulae (2.18), (2.19) for the orthogonal polynomials $P_n(z) = z^n + \cdots$,

$$\int_F P_n(z) z^\nu h_n(z)\, dz = 0, \qquad \nu = 0, 1, \dots, n-1, \tag{5.1}$$

and the functions of the second kind

$$(5.1') \qquad R_n(z) := \frac{1}{2\pi i}\int_F \frac{P_n(t)h_n(t)}{t-z}\,dt, \qquad R_n(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z\to\infty.$$

The weight function $h_n$ is holomorphic in a neighbourhood $\Omega$ of the Jordan arc $F$, does not vanish in $\Omega$, and is of the form

$$(5.2) \qquad h_n := e^{-2nQ}\widetilde{h}_n, \qquad \|\widetilde{h}_n - \widetilde{h}_\infty\|_{H(\Omega)} = o(1),$$

where $Q$ is such that the support of the equilibrium measure $\lambda$ in the field $\operatorname{Re} Q$ is $S$-symmetric and connected (see (2.14)).

Along with the pair $P_n$, $R_n$ we consider the pair $\widetilde{P}_{n-1}$, $\widetilde{R}_{n-1}$ such that

$$\int_F \widetilde{P}_{n-1}(z)z^\nu h_n(z)\,dz = 0, \qquad \nu = 0,\dots,n-2, \quad \widetilde{P}_{n-1}(z) = z^{n-1}+\cdots,$$
$$\widetilde{R}_{n-1}(z) := \frac{1}{2\pi i}\int_F \frac{\widetilde{P}_{n-1}(t)h_n(t)\,dt}{t-z}, \qquad \widetilde{R}_{n-1}(z) = \frac{1}{mz^n}+\cdots, \quad z\to\infty,$$

where $m$ is a constant independent of $n$.

As in the proof of Theorem 3 we assume statement A) of Theorem 2, that is, for a sufficiently large $n$ the pairs $(P_n, R_n)$ and $(\widetilde{P}_{n-1}, \widetilde{R}_{n-1})$ exist. This result will be proved at the end of this section (see subsection 5.8).

Following [63] we consider the matrix-valued functions

$$Y := \begin{pmatrix} P_n & R_n \\ m\widetilde{P}_{n-1} & m\widetilde{R}_{n-1}\end{pmatrix}, \qquad W := \begin{pmatrix} 1 & h_n \\ 0 & 1\end{pmatrix}.$$

The function $Y \in H_{2\times2}(\mathbb{C}\setminus F)$ has continuous (non-tangential) boundary values $Y_+$ and $Y_-$ 'inside' $F$ (that is, except the end-points of $F$) and these boundary values are square-integrable on the whole $F$. Moreover, in view of the Sokhotskiĭ relations

$$P_n h_n = R_{n+} - R_{n-}$$

we have

$$Y_- W = \begin{pmatrix} P_n & P_n h_n + R_{n-} \\ m\widetilde{P}_{n-1} & m(\widetilde{P}_{n-1}h_n + \widetilde{R}_{(n-1)-})\end{pmatrix} = Y_+.$$

Therefore the function $Y$ is the unique solution of following matrix Riemann-Hilbert problem:

$$(5.3) \qquad \begin{gathered} Y \in H_{2\times2}(\mathbb{C}\setminus F), \quad \exists\, Y_\pm \in L^2_{2\times2}(F), \\ Y_+ = Y_- W \quad \text{'inside'} \quad F, \\ Y(z) = \left(I + O\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}, \qquad z\to\infty, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}. \end{gathered}$$

The boundary-value problem (5.3) in an identical reformulation of the definition of orthogonal polynomials (5.1).

**5.2. Normalization of the matrix boundary-value problem.** We recall that the asymptotic formulae (2.18) to be established are written in terms of the function $\varphi$ (see (2.19)), which is the solution of the following boundary-value problem:

$$\begin{gathered}\varphi \in H(\mathbb{C}\setminus\Delta), \quad \varphi \neq 0 \qquad \mathbb{C}\setminus\Delta, \quad \exists\,\varphi_\pm \in L^2(\Delta),\\ \varphi(z) = \frac{z^n}{C_n} + \cdots, \qquad z\to\infty,\\ \varphi_+\varphi_- h_n \omega_{\Delta+} = 1 \qquad \Delta.\end{gathered} \tag{5.4}$$

Here we use the notation

$$\omega_\Delta := \frac{w_\Delta}{i}\,. \tag{5.5}$$

We also recall that for notational convenience we assume that the arc $\Delta$ (the support of the equilibrium measure) starts at the point $-1$ and ends at the point $0$. Therefore

$$w_\Delta = \sqrt{z(z+1)}\,, \qquad \omega_\Delta = \sqrt{z(-z-1)}\,.$$

Our nearest goal is to transform the boundary-value problem (5.3) so that the solution of the new problem (unlike $Y$) is a function holomorphic in $\overline{\mathbb{C}}\setminus F$, that is, with no singularity at the point $\infty$. For this purpose by means of the diagonal matrices

$$S := \operatorname{diag}\{\varphi^{-1},\varphi\}; \qquad C := \operatorname{diag}\{C_n^{-1}, C_n\}$$

we define the matrix function:

$$Z := CYS = \begin{pmatrix} \dfrac{Q_n}{C_n\varphi} & \dfrac{\varphi}{C_n}R_n \\ \dfrac{C_n m\widetilde{Q}_{n-1}}{\varphi} & C_n m\varphi\widetilde{R}_{n-1}\end{pmatrix}. \tag{5.6}$$

This function is the unique solution of the boundary-value problem

$$\begin{gathered} Z \in H^2_{2\times2}(\overline{\mathbb{C}}\setminus F),\\ Z_+ = Z_- J \quad \text{on} \quad F,\\ Z(z) = I + O\Big(\frac{1}{z}\Big), \qquad z\to\infty,\end{gathered} \tag{5.7}$$

where the matrix-valued jump function $J$ equals

$$J := S_-^{-1} W S_+ = \begin{pmatrix} \varphi_-/\varphi_+ & \varphi_+\varphi_- h_n \\ 0 & \varphi_+/\varphi_-\end{pmatrix} \quad \text{on} \quad F. \tag{5.8}$$

**5.3. Remark on the scheme of the proof.** Before we carry out the subsequent steps of the proof of the asymptotic formulae (2.18) we briefly clarify the essence of our method. We have to find the limit as $n\to\infty$ of the solution of the boundary-value problem (5.3). If the problem (5.7) (equivalent to (5.3)) had been a scalar problem, then without difficulty we could have written down the solution for $\log Z$

(as in the proof of Theorem 3) by means of the Cauchy integral and would thereby have not only asymptotic, but exact formulae (for finite $n$) for the orthogonal polynomials by expressing them in terms of the known jump function $J$. However, since (5.7) is a matrix problem, we cannot just take the logarithm of it thereby reducing it to a problem with a known solution. Therefore we have to continue the chain of transformations (5.3)$\to$(5.7)$\to \cdots$ until we get a problem (equivalent to the initial problem (5.3)) for which we can either write down the solution or its asymptotics. We say at once that such will be a problem in which the jump matrix on the system of contours $\Sigma$ tends as $n \to \infty$ to the identity matrix uniformly on $\Sigma$:

$$\begin{gathered} \mathcal{I} \in H^2_{2\times 2}(\overline{\mathbb{C}} \setminus \Sigma), \\ \mathcal{I}_+ = \mathcal{I}_-(I + o(1)) \quad \text{uniformly on } \Sigma \text{ as } n \to \infty, \\ \mathcal{I}(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty. \end{gathered} \tag{5.9}$$

The solution of this problem is known to satisfy the relation

$$\mathcal{I} = I + o(1), \qquad n \to \infty, \tag{5.10}$$

uniformly on compact sets in $\overline{\mathbb{C}}$. Therefore reversing the chain of identical transformations (5.9)$\to \cdots \to$(5.7)$\to$(5.3), we get from (5.10) the required asymptotic formulae.

**5.4. Factorization of the jump matrix $J$.** We observe that the jump matrix (5.8) of the problem (5.7) on the support $\Delta$ of the equilibrium measure $\lambda$ goes over into (see (5.4))

$$J = \begin{pmatrix} \varphi_-/\varphi_+ & 1/\omega_+ \\ 0 & \varphi_+/\varphi_- \end{pmatrix} \quad \text{on} \quad \Delta.$$

This matrix has a convenient factorization

$$J = \begin{pmatrix} 1 & 0 \\ w_+\varphi_+/\varphi_- & 1 \end{pmatrix} \begin{pmatrix} 0 & \omega_+^{-1} \\ -\omega_+ & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ w_+\varphi_-/\varphi_+ & 1 \end{pmatrix} \quad \text{on} \quad \Delta.$$

Denoting the central matrix (independent of $n$) by

$$\mathcal{J} := \begin{pmatrix} 0 & \omega_+^{-1} \\ -\omega_+ & 0 \end{pmatrix} \tag{5.11}$$

and introducing in $\Omega \setminus F$ the matrix function

$$D := \begin{pmatrix} 1 & 0 \\ 1/(h_n\varphi^2) & 1 \end{pmatrix}, \tag{5.12}$$

we see by the boundary condition of the problem (5.4) that the factorization of the matrix $J$ on $\Delta$ takes the form

$$J = D_-\mathcal{J}D_+ \quad \text{on} \quad \Delta. \tag{5.13}$$

We note at once that since the solution of the boundary-value problem (5.4) is of the form (see (2.19))

$$\varphi = \Phi_q^n f_{\widetilde{h}_n}, \tag{5.14}$$

where $\Phi_q$ is defined in (2.16) and $f_{\widetilde{h}_n}(z) = f(\widetilde{h}_n; z)$ in (2.12), it follows that the boundary values on $\Delta$ of the entry in the lower left corner of the matrix $D$ (with (2.17) and (5.2) taken into account) can be represented in the form

$$\frac{1}{h_n \varphi_{\mp}^2} = \frac{1}{\widetilde{h}_n f_{\widetilde{h}_n}} \left( \frac{\Phi_{q\pm}}{\Phi_{q\mp}} \right) \quad \text{on} \quad \Delta.$$

The first factor here has, in view of (5.2), a uniformly (in $n$) bounded analytic continuation into $\Omega \setminus \Delta$. The second factor (as in Lemma 1 of the previous section) can be transformed as follows:

$$\frac{\Phi_{q\pm}}{\Phi_{q\mp}} = \exp\{-n(\mathcal{V}_{\mp}^{\lambda} - \mathcal{V}_{\pm}^{\lambda})\} = \exp\{2\pi i n l_{\pm}\} \quad \text{on} \quad \Delta,$$

where (see (4.16)) $l$ denotes

$$l_{\pm}(x) := \mp \int_x^0 d\lambda(t) \quad \text{on} \quad \Delta. \tag{5.15}$$

In addition, as mentioned in the proof of Lemma 1 (see (4.20′)), an important consequence of the $S$-property of the arc $\Delta$ is the fact that $l$ can be continued from both 'banks' of $\Delta$ into a lens-like domains $d^+$ and $d^-$ so that

$$\operatorname{Im} l > 0 \qquad d^+ \cup d^-. \tag{5.16}$$

Thus, the jump matrix $J$ on $\Delta$ can be factorized in the form (5.13), where the central matrix $\mathcal{J}$ is independent of $n$ and the analytic continuations of the right and left matrices $D_+$ and $D_-$ into the domains $d^+$ and $d^-$, respectively, have the property that

$$D_\alpha \rightrightarrows I \quad \text{as} \quad n \to \infty \quad \text{on} \quad K \Subset d^\alpha, \qquad \alpha = +, -, \tag{5.17}$$

at the rate of a geometric progression.

**5.5. 'Opening of the lens' – deformation of the boundary $\Delta$.** The convenient factorization of the jump on $\Delta$ makes it possible to go over in equivalent way from the boundary value (5.7), (5.8) to the problem more suiting the goals stated in subsection 5.3.

Let $\Delta^+$ be an arc in $d^+$ joining the end-point of the arc $\Delta$ (that is, the points $-1$ and 0). Accordingly, the arc $\Delta^-$ in $d^-$ joins the end-points of $\Delta$. The arcs $\Delta^+$ and $\Delta^-$ are oriented as $\Delta$: they are directed from the point $-1$ to the point 0. We denote by $\widetilde{d}^+$ the domain bounded by the arcs $\Delta^+$ and $\Delta$; the domain $\widetilde{d}^-$ is bounded by $\Delta^-$ and $\Delta$, respectively:

$$\widetilde{d}^\alpha \subset d^\alpha, \quad \Gamma(\widetilde{d}^\alpha) = \Delta^\alpha \cup \Delta, \qquad \alpha = +, -.$$

Finally, we denote by $F_1$ the part of $F$ joining the initial point of the arc $F$ with the initial point of $\Delta$ (that is, with $-1$) and by $F_0$ we denote the part of $F$ joining the end-point of $\Delta$ (the point 0) with the end-point of the arc $F$:

$$F = F_1 \cup \Delta \cup F_0.$$

We note that depending on the varying field $Q$ the arcs $F_1$ and/or $F_0$ can be empty.

We now go over from the problem (5.7), (5.8) with boundary conditions on the arc $F$ to the problem with boundary conditions on the contour $\widetilde{\Sigma}$ (see Fig. 1):

$$\widetilde{\Sigma} := F \cup \Delta^+ \cup \Delta^- = F_1 \cup \Delta \cup \Delta^+ \cup \Delta^- \cup F_0. \tag{5.18}$$

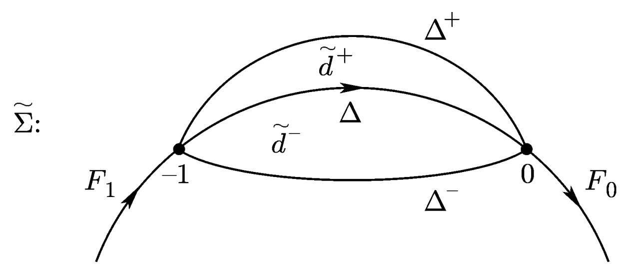


FIG. 1

For this purpose we define by means of (5.3) the piecewise-analytic matrix-valued function $\widetilde{Z}$:

$$\widetilde{Z} := \begin{cases} Z & \text{in } \overline{\mathbb{C}} \setminus \overline{(\widetilde{d}^+ \cup \widetilde{d}^-)}, \\ ZD^{-1} & \text{in } \widetilde{d}^+, \\ ZD & \text{in } \widetilde{d}^-. \end{cases} \tag{5.19}$$

For the boundary values of $\widetilde{Z}$ on $\Delta^+ \cup \Delta \cup \Delta^-$ we have by definition (5.19) the equality

$$\text{on } \ \Delta^+\colon \ \widetilde{Z}_+ = Z = ZD^{-1}D = \widetilde{Z}_- D,$$
$$\text{on } \ \Delta^-\colon \ \widetilde{Z}_+ = ZD = \widetilde{Z}_- D$$

and with (5.13) taken into account this gives

$$\text{on } \ \Delta\colon \ \widetilde{Z}_+ = Z_- J D_+^{-1} = \widetilde{Z}_- (D_-^{-1} J D_+^{-1}) = \widetilde{Z} \mathcal{J}.$$

Thus, the function $\widetilde{Z}$ is the unique solution of the following boundary-value problem:

$$\begin{gathered} \widetilde{Z} \in H^2_{2\times 2}(\overline{\mathbb{C}} \setminus \widetilde{\Sigma}), \\ \widetilde{Z}_+ = \widetilde{Z}_- \cdot \begin{cases} D & \text{on } \Delta^+ \cup \Delta^-, \\ \mathcal{J} & \text{on } \Delta, \\ J & \text{on } F_1 \cup F_0, \end{cases} \\ \widetilde{Z}(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty, \end{gathered} \tag{5.20}$$

where the jump functions are defined in (5.12), (5.11) and (5.8).

The boundary-value problem (5.20) has the following advantages in comparison with the equivalent problem (5.7):

a) on the part $\Delta$ of the boundary contour $\widetilde{\Sigma}$ the jump matrix $J$ is independent of $n$,

b) at the 'interior' points of the arcs $\Delta^+$ and $\Delta^-$, $F_0$, and $F_1$ the jump matrix tends to the identity matrix $I$.

Let us clarify item b). We fix the neighbourhoods of the end-points of $\Delta$:

$$O_\alpha, \qquad \alpha = 1, 0$$

($\alpha = 1$ corresponds to the initial point $-1$ of the arc $\Delta$, $\alpha = 0$ corresponds to the other end, that is, the point 0). We denote the end-points of the arcs of the contour $\widetilde{\Sigma}$ adjoining the end-points of $\Delta$ by

$$\begin{aligned} \Delta_\alpha^\pm &:= \Delta^\pm \cap O_\alpha, \qquad \alpha = 1, 0, \\ F_\alpha^\varepsilon &:= F_\alpha \cap O_\alpha, \qquad \alpha = 1, 0. \end{aligned}$$

Then we have by (5.17)

$$D \rightrightarrows I \quad \text{as} \quad n \to \infty \quad \text{on} \quad \Delta^\alpha \setminus (\Delta_1^\alpha \cup \Delta_0^\alpha), \qquad \alpha = +, -, \tag{5.21}$$

and noting that outside $\Delta$ on $F$ the jump matrix $J$ (see (5.8)) goes over (with (2.19) taken into account) into

$$J = \begin{pmatrix} 1 & \varphi^2 h_n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \widetilde{h}_n f_{\widetilde{h}_n}^2 \exp\{2n(\gamma - \mathcal{V}^\lambda - Q)\} \\ 0 & 1 \end{pmatrix},$$

we have by the hypotheses of the theorem (see (2.14), ($2.15_2$) and (5.2)) that

$$J \rightrightarrows I \quad \text{as} \quad n \to \infty \quad \text{on} \quad F_\alpha \setminus F_\alpha^\varepsilon, \qquad \alpha = 1, 0. \tag{5.22}$$

Moreover, the uniform convergence in (5.21) and (5.22) has the rate of a geometric progression.

**5.6. Solution of the auxiliary problem outside $\Delta$.** For the transformation of the problem (5.20) to the form (5.9) we consider the following boundary-value problem:

$$\begin{gathered} X \in H^2_{2\times 2}(\overline{\mathbb{C}} \setminus \Delta), \\ X_+ = X_- \mathcal{J} \quad \text{on} \quad \Delta, \\ X(z) = I + O\left(\frac{1}{z}\right), \qquad z \to \infty. \end{gathered} \tag{5.23}$$

The problem (5.23) differs from (5.20) by the absence of the jumps (depending on $n$) on the arcs $\Delta^\alpha$, $\alpha = +, -$, and $F_\alpha$, $\alpha = 1, 0$. The solution of (5.23) can be found in the explicit form.

We now write the problem (5.23) in the element-wise form for

$$X := \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

We obtain two scalar boundary-value problems for the functions $x_{ij} \in H^2(\mathbb{C} \setminus \Delta)$, $i, j = 1, 2$:
(5.24₁)

$$\begin{cases} x_{11+} = -\omega_+ x_{12-}, \\ x_{12+} = \dfrac{x_{11-}}{\omega_+} \end{cases} \quad \text{on} \quad \Delta, \qquad \begin{cases} x_{11}(z) = 1 + O\Big(\dfrac{1}{z}\Big), \\ x_{12}(z) = O\Big(\dfrac{1}{z}\Big) \end{cases} \quad \text{as} \quad z \to \infty$$

and
(5.24₂)

$$\begin{cases} x_{21+} = -\omega_+ x_{22-}, \\ x_{22+} = \dfrac{x_{21-}}{\omega_+} \end{cases} \quad \text{on} \quad \Delta, \qquad \begin{cases} x_{21}(z) = O\Big(\dfrac{1}{z}\Big), \\ x_{22}(z) = 1 + O\Big(\dfrac{1}{z}\Big) \end{cases} \quad \text{as} \quad z \to \infty.$$

Taking into account the definition (5.5) of the function $\omega$ the boundary conditions here can be written in the form

$$x_{11\pm} = (\omega x_{12})_{\mp} \quad \text{and} \quad x_{21\pm} = (\omega x_{22})_{\mp} \quad \text{on} \quad \Delta.$$

Therefore $\omega x_{12}$ (respectively, $\omega x_{22}$) is an analytic continuation of the function $x_{11}$ (respectively, $x_{21}$) on the other sheet of the Riemann surface $\mathfrak{R}$ of the function $w$ (see (4.7)). If in accordance with the normalization at $\infty$ we choose

$$x_{11} :\equiv 1 \quad \text{in} \quad \mathbb{C} \setminus \Delta,$$

then the continuation of $x_{11}$ on the second sheet $\mathfrak{R}$ will again be identically equal to one, which gives

$$x_{12} = \omega^{-1} = \frac{i}{w}.$$

Checking the normalization at the point $\infty$ we see that $x_{11}$ and $x_{12}$ satisfy the first problem (5.24₁).

For the solution of the second problem (5.24₂) we have to find a function $x_{21}$ vanishing at the point $\infty^{(+)}$ and having a pole at the point $\infty^{(-)}$ (when it is continued on the second sheet of $\mathfrak{R}$). These conditions will be satisfied if we choose $x_{21}$ in the form

$$x_{21} := \frac{c}{i}\,\Phi_0^{(-)},$$

where the algebraic function $\Phi_0$ (inverse to the Zhukovskiĭ function) is defined in (2.11₂). Then

$$x_{22} = \frac{c\Phi_0^{(+)}}{w}, \qquad \Phi_0^{(+)}(z)\big|_{\infty} = \frac{z}{c} + \cdots,$$

and checking the normalization of $x_{22}$ at the point $\infty$:

$$x_{22}(z)\big|_{\infty} = 1 + O\left(\frac{1}{z}\right),$$

we see that we have found the solution for the second problem ($5.24_2$). Finally, the following function is the solution of the boundary-value problem (5.23):

$$X = \begin{pmatrix} 1 & i/w \\ c/(i\Phi_0) & c\Phi_0/w \end{pmatrix}. \tag{5.25}$$

**5.7. Statement of the auxiliary boundary-value problem in a neighbourhood of the end-point of the support $\Delta$.** We recall that our problem (5.20) differs from (5.9) (which is our goal) by the behaviour of the jump matrices in neighbourhoods of the end-points of the support $\Delta$ of the equilibrium measure $\lambda$. More precisely, in the neighbourhoods of the points $-1$ and $0$ the jump matrices of the problem (5.20) do not tend as $n \to \infty$ to the identity matrices (as we would like them to). In order to go over in equivalent way from the problem (5.20) to the problem (5.9) we separately consider the following boundary-value problem in the domain $O_0$ (see Fig. 2):

$$\begin{gathered} U_0 \in H^2_{2\times 2}(O_0 \setminus (\Delta_0 \cup \Delta_0^+ \cup \Delta_0^- \cup F_0^\varepsilon)), \\ U_{0+} = U_{0-} \cdot \begin{cases} D & \text{on } \Delta_0^+ \cup \Delta_0^-, \\ \mathcal{J} & \text{on } \Delta_0, \\ J & \text{on } F_0^\varepsilon, \end{cases} \\ U_0 = (I + o(1))X \quad \partial O_0 \text{ as } n \to \infty. \end{gathered} \tag{5.26}$$

The same problem (with subscript 0 replaced by 1) is posed for finding the function $U_1$ in the neighbourhood $O_1$ of the other end-point $-1$.

$O_0$:

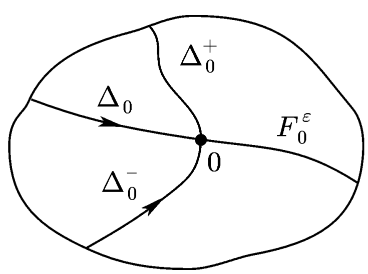


FIG. 2

The main technical aspect of the method that we follow here consists in finding the solution (or in the proof of the existence of the solution) of the problem (5.26). The subsequent sections of this paper are devoted to the study of the boundary-value problem (5.26).

*In this section we suppose* that there exists a martix function $U_0$ solving the problem (5.26) and also a matrix $U_1$ solving the similar problem in the neighbourhood of the other end-point $-1$.

We now carry out the proof of Theorem 2 under these assumptions.

**5.8. Final transformation to a boundary-value problem with jumps tending to $I$.** We define the function

$$\mathcal{I} := \begin{cases} \widetilde{Z}X^{-1} & \text{in } \overline{\mathbb{C}} \setminus (O_0 \cup O_1), \\ \widetilde{Z}U_0^{-1} & \text{in } O_0, \\ \widetilde{Z}U_1^{-1} & \text{in } O_1, \end{cases} \tag{5.27}$$

where $\widetilde{Z}$ is the solution of (5.20), and $X$, $U_0$, and $U_1$ are the solutions of the auxiliary problems (5.23), (5.26). By construction, the function $\mathcal{I}$ is holomorphic in the domain $\overline{\mathbb{C}} \setminus \Sigma$, where $\Sigma$ is the following contour (see Fig. 3):

$$\Sigma := \partial O_0 \cup \partial O_1 \cup (\Delta^+ \setminus (\Delta_0^+ \cup \Delta_1^+)) \cup (\Delta^- \setminus (\Delta_0^- \cup \Delta_1^-)) \cup (F_0 \setminus F_0^\varepsilon) \cup (F_1 \setminus F_1^\varepsilon).$$

In addition, we have on $\Sigma$

$$\mathcal{I}_+ = \mathcal{I}_- \widetilde{I}$$

(5.28)

$$:= \begin{cases} \widetilde{Z}DX^{-1} = \mathcal{I}_-(XDX^{-1}) & \text{on } \Delta^\alpha \setminus (\Delta_0^\alpha \cup \Delta_1^\alpha), \quad \alpha = +, -, \\ \widetilde{Z}X^{-1} = \widetilde{Z}U_\alpha^{-1}U_\alpha X^{-1} = \mathcal{I}_-(U_\alpha X^{-1}) & \text{on } O_\alpha, \quad \alpha = 1, 0, \\ \widetilde{Z}_- JX^{-1} = \mathcal{I}_-(XJX^{-1}) & \text{on } F_\alpha \setminus F_\alpha^\varepsilon, \quad \alpha = 1, 0. \end{cases}$$

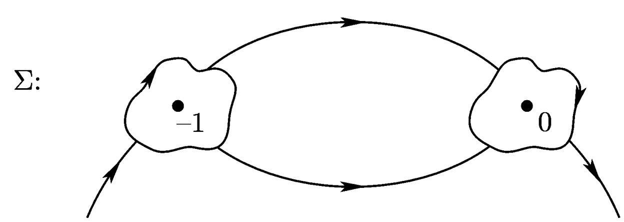


FIG. 3

Thus, the matrix function (5.27) is the solution of the problem:

$$\begin{gathered} \mathcal{I} \in H^2_{2\times 2}(\overline{\mathbb{C}} \setminus \Sigma), \\ \mathcal{I}_+ = \mathcal{I}_- \widetilde{I} \quad \text{on} \quad \Sigma, \\ \mathcal{I}(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty, \end{gathered} \tag{5.29}$$

where the jump matrix $\widetilde{I}$ is defined in (5.28) and in, view of (5.21), (5.22), and (5.26), satisfies the relation

$$\widetilde{I} \rightrightarrows I, \qquad n \to \infty, \tag{5.30}$$

uniformly on $\Sigma$.

As mentioned in section 5.3 (see (5.10)), for sufficiently large $n$ the solution of the problem (5.29), (5.30) exists and has the property that

$$\mathcal{I} \rightrightarrows I \quad \text{as} \quad n \to \infty \tag{5.31}$$

uniformly on compact sets in $\overline{\mathbb{C}}$. The proof of this result can be found, for instance, in [59], § 7.5, Corollary 7.108. For the sake of completeness we prove in the next section the implication (5.29), (5.30) $\Rightarrow$ (5.31) assuming that the jump matrix $\widetilde{I}$ is analytic in a neighbourhood of the contour $\Sigma$ (which is, in fact, true in our case) and that the contour $\Sigma$ is a rectifiable Jordan curve. Under these essentially more restrictive (than in [59]) conditions on the input data we can give an elementary proof, which does not use harmonic analysis (as in [59]).

*Remark on asymptotic normality*. We note that since the solution of the problem (5.28), (5.29) exists for large $n$, it follows that our earlier assumption (see section 5.1) on the normality of the index $n$ (that is, on the existence of the solution $Y$ of the problem (5.3), which is equivalent to (5.28), (5.29)) is satisfied. In fact, the analytic (in $\mathbb{C} \setminus F$) functions $P_n$ and $R_n$, making up the unique solution of the problem (5.3), satisfy the relation

$$P_n(z) = z^n + \cdots, \quad R_n(z) = O\left(\frac{1}{z^{n+1}}\right), \qquad z \to \infty,$$

where it follows from the boundary conditions (5.3) that $P_{n+} = P_{n-}$ on $F$ and, consequently, $P_n$ is a polynomial. The relation (5.1′) between this polynomial and the function of the second kind again follows from the boundary conditions (5.3). This, in turn, leads to the orthogonality of this polynomial with respect to the weight $h_n$ on $F$.

**5.9. Asymptotics of the solution of the matrix boundary-value problem with a jump tending to $I$.** The following result holds.

**Lemma 2.** *Let $\Sigma$ be a simple rectifiable Jordan contour and let the matrix function $\widetilde{I}$ be analytic in the domain $\mathcal{A}$ containing $\Sigma$,*

$$\widetilde{I} \in H(\mathcal{A}), \qquad \Sigma \subset \mathcal{A}, \tag{5.32}$$

*where*

$$\widetilde{I} = I + \varepsilon_n, \quad \varepsilon_n \rightrightarrows 0 \ \text{ as } \ n \to \infty \ \text{ on } \ K \Subset \mathcal{A}. \tag{5.33}$$

*Then the following relation holds for the solution of the boundary-value problem* (5.29)*:*

$$\mathcal{I} = I + O(\varepsilon_n). \tag{5.34}$$

*Proof.* Substituting (5.33) into the boundary condition (5.29) we obtain the following boundary-value problem:

$$\begin{gathered} \mathcal{I} \in H^2_{2\times 2}(\overline{\mathbb{C}} \setminus \Sigma), \\ \mathcal{I}_+ = \mathcal{I}_- + \varepsilon_n \mathcal{J}_- \quad \text{on} \quad \Sigma, \\ \mathcal{I}(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty. \end{gathered}$$

Applying the Cauchy integral formula to each entry of the matrix $\mathcal{I}$ we obtain

$$\mathcal{I}(z) = \frac{1}{2\pi i}\int_\Sigma (\varepsilon_n \mathcal{I}_-)(\xi)\,\frac{d\xi}{\xi - z} + I, \qquad z \in \overline{\mathbb{C}} \setminus \Sigma. \tag{5.35}$$

In view of (5.32) we then can deform $\Sigma$ into the contour $\Sigma'$ lying in the domain $\mathcal{A} \cap \operatorname{Int}\Sigma$:

$$\mathcal{I}(z) = \frac{1}{2\pi i}\int_{\Sigma'} (\varepsilon_n \mathcal{I}_-)(\xi)\,\frac{d\xi}{\xi - z} + I, \qquad z \in \overline{\mathbb{C}} \setminus \widetilde{\mathcal{A}}, \qquad \Gamma(\widetilde{\mathcal{A}}) = \Sigma \cup \Sigma'. \tag{5.36}$$

The integral is no longer singular for $z \in \Sigma$ and both sides of (5.36) have a limit as $z \to \Sigma_+$. Hence,

$$\mathcal{I}_-(z)\widetilde{I}(z) = \mathcal{I}_+(z) = \frac{1}{2\pi i}\int_{\Sigma'} (\varepsilon_n \mathcal{I}_-)(\xi)\,\frac{d\xi}{\xi - z} + I, \qquad z \in \Sigma. \tag{5.37}$$

Multiplying (5.37) from the left by $\widetilde{I}^{-1}$ and choosing $z_0 \in \Sigma$ so that

$$\|\mathcal{I}_-(z_0)\| = \max_\Sigma \|\mathcal{I}_-(z)\| =: M_n,$$

we find from (5.37) that

$$M_n \leqslant M_n \varepsilon_n \operatorname{const}_{\Sigma,\Sigma'} + 1 + O(\varepsilon_n)$$

and, consequently,

$$M_n \leqslant \frac{1 + O(\varepsilon_n)}{1 - O(\varepsilon_n)}\,.$$

Substituting this estimate into (5.35) we come to (5.34). The lemma is proved.

**5.10. Obtaining asymptotic formulae.** Carrying out in (5.31) the transformations inverse to those, which have led us from the matrix function (5.6) to the matrix function (5.27), we obtain the required asymptotic formulae for the polynomials $Q_n$ and functions of the second kind $R_n$. In fact, we have (see (5.19), (5.12), (5.6))

$$\widetilde{Z} = \begin{cases} \begin{pmatrix} \dfrac{Q_n}{C_n\varphi} & \dfrac{\varphi}{C_n} R_n \\ \dfrac{C_n m \widetilde{Q}_{n-1}}{\varphi} & C_n m \varphi \widetilde{R}_{n-1} \end{pmatrix} & \text{on } K \Subset \overline{\mathbb{C}} \setminus (d^+ \cup d^-), \\[3ex] \begin{pmatrix} \dfrac{Q_n}{C_n\varphi} \mp \dfrac{R_n}{C_n\varphi h_n} & \dfrac{R_n\varphi}{C_n} \\ \dfrac{C_n m \widetilde{Q}_{n-1}}{\varphi} \mp \dfrac{C_n m \widetilde{R}_{n-1}}{\varphi h_n} & m C_n \varphi \widetilde{R}_{n-1} \end{pmatrix} & \text{on } K \Subset d^\pm. \end{cases} \tag{5.38}$$

It follows from (5.31) that

$$\widetilde{Z} = (I + o(1))X \tag{5.39}$$

uniformly on $K \Subset \overline{\mathbb{C}} \setminus \{O_1 \cup O_0\}$, where $X$ is explicitly given by matrix (5.25). Comparing the first row in the upper matrix in (5.38) with the first row in the matrix (5.25) we obtain by (5.39) the formulae for the outer asymptotics ($2.18_1$). We consider the limiting values on $\Delta$ of the second (lower) matrix in (5.38) and the matrix $X$ (see (5.25)) and compare the entries in the upper right corner of the matrix relation (5.39). As a result we obtain the asymptotic formula ($2.18_2$) for the functions of the second kind on the compact sets of the arc $\Delta$. The corresponding formula in ($2.18_2$) for the polynomials $P_n$ is obtained from the above formula for the functions of the second kind by means of the Sokhotskiĭ–Plemel' formula (2.5) in the same way as in the proof of Theorem 3 (see (4.23)).

Thus, the asymptotic formulae ($2.18_1$) and ($2.18_2$) in Theorem 2 are obtained under the assumption that the solution of the boundary-value problem (5.26) exists.

## § 6. Proof of Theorem 2. Explicit solutions of regular boundary-value problems in a neighbourhood of the end-point of $\Delta$

To complete the proof of Theorem 2 it remains to make sure that there exists a solution of the boundary-value problem (5.26) defined in a neighbourhood of the end-point of the support $\Delta$ of the equilibrium measure $\lambda$ in the field $\operatorname{Re} Q$. The analysis of this boundary-value problem depends on the two circumstances. First, does the end-point of $\Delta$ (recall that we put it at the point 0) coincide with the end-point of the arc $F$ (the support of the orthogonality measure) or the point 0 is an interior point of $F$. In other words, is the arc $F_0^\varepsilon$ (see Fig. 1) empty or not.

$$(A) := \{F_0^\varepsilon \neq \varnothing\}, \qquad (B) := \{F_0^\varepsilon = \varnothing\}. \tag{6.1}$$

Secondly, the analysis of the problem (5.26) depends on the behaviour of the derivative $\lambda'$ of the equilibrium measure in a neighbourhood of the end-point of $\Delta$. It is known (for case $(A)$ see, for instance, [62]; case $(B)$ is treated in the similar way) that if the external field $Q$ is analytic in the neighbourhood $\Omega$ of the arc $F$, then the equilibrium measure $\lambda$ is absolutely continuous and the derivative of it in a neighbourhood of the end-point of $\Delta$ (the point 0) has the form:

$$\lambda'(z) = z^{n-1/2} m(z), \qquad m(z) \in H(O_0), \quad m(0) \neq 0, \tag{6.2}$$

where $n$ in (6.2) (depending on case $(A)$ or $(B)$) takes the values:

$$\begin{aligned} &(A) \Rightarrow n = 2k+1, \qquad k = 0,1,2,\dots, \\ &(B) \Rightarrow n = 0,1,2,\dots . \end{aligned} \tag{6.3}$$

In this section we explicitly construct the solutions of the problem (5.26) for the main (regular) cases:

$$(A), \quad n = 1, \tag{6.4$_1$}$$

$$(B), \quad n = 0. \tag{6.4$_2$}$$

The next section § 7 is devoted to the proof of the existence of the solution of the problem (5.26) in the remaining (singular) cases.

Since the solution of the boundary-value problem (5.26) describes the asymptotic behaviour of the orthogonal polynomials near the extreme zeros of them, the examples of the solution of this problem go back to the asymptotic theory of *classical* orthogonal polynomials. For example, the solution of the problem for the special case $(6.4_1)$ can be obtained from the Plancherel–Rotach formulae (see [33]) for the Hermite and Laguerre polynomials, where the asymptotics is written in terms of the Airy functions. For the solution of a certain special case of the problem (5.26) for $(6.4_2)$ the Heine–Mehler and Hilb formulae for the asymptotics of the Jacobi polynomials near the end-points of the orthogonality interval are also given in terms of the Bessel functions. In connection with the representation of the asymptotics of *general* orthogonal polynomials in a neighbourhood of the end-point of the orthogonality interval in terms of the Bessel functions we also mention the works [65] and [66].

The solution of the general case of the problem (5.26) for $(A)$, $n=1$, in terms of the Airy functions was constructed by Deift and coauthors (see, for example, [59]). On the whole, we follow this approach adapting it to the goals of this work, in particular, adding to the jump matrices in (5.26) the Szegö functions for the weakly varying (as $n\to\infty$) component of the varying weight $\widetilde{h}_n$, $\widetilde{h}_n\to\widetilde{h}_\infty$. The main steps of the solution of the problem (5.26) by means of the Bessel functions for $(B)$, $n=0$, are borrowed from [67].

**6.1. Solution of boundary-value problem in a neighbourhood of the end-point in terms of the Airy functions.** We construct in this section the solution of the boundary-value problem (5.26):

$$\begin{gathered} U_0\in H^2_{2\times 2}(O_0\setminus\Sigma_0),\\ U_{0+}=U_{0-}\cdot\begin{cases} D & \text{on } \Delta_0^+\cup\Delta_0^-,\\ \mathcal{J} & \text{on } \Delta_0,\\ J & \text{on } F_0^\varepsilon\neq\varnothing\quad(\text{case } (A)),\end{cases}\\ U_0=(I+o(1))X \ \text{ on } \partial O_0 \ \text{ as } n\to\infty, \end{gathered}\tag{6.5}$$

where (see Fig. 2)

$$\Sigma_0:=\Delta_0\cup\Delta_0^+\cup\Delta_0^-\cup F_0^\varepsilon,$$

the jump matrices in (6.5) are equal to

$$\begin{aligned} D&:=\begin{pmatrix}1&0\\ 1/(h_n\varphi^2)&1\end{pmatrix}\quad\text{in}\quad\Omega\setminus F,\\ \mathcal{J}&:=\begin{pmatrix}0&\omega^{-1}\\ -\omega&0\end{pmatrix}\quad\text{on}\quad\Delta,\\ J&:=\begin{pmatrix}1&h_n\varphi^2\\ 0&1\end{pmatrix}\quad\text{on}\quad F_0^\varepsilon,\end{aligned}\tag{6.6}$$

and the matrix $X$ (independent of $n$) is defined in (5.25). In addition, we assume that the equilibrium measure is such that condition (6.2) holds with $n=1$.

**6.1.1. Transformation of the problem (6.5) to a boundary-value problem with jump independent of $n$.** We denote by $A$ the diagonal matrix

$$A := \operatorname{diag}\{\varphi h_n^{1/2}, (\varphi h_n^{1/2})^{-1}\}, \tag{6.7}$$

where $\varphi$ is the solution of the boundary-value problem (5.4). We observe that the jump matrices (6.6) admits the following decomposition:

$$\begin{aligned} D &= A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} A^{-1} \quad \text{in} \quad \Omega \setminus F, \\ J &= A \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A^{-1} \quad \text{on} \quad F_0^\varepsilon, \\ \mathcal{J} &= A_- \begin{pmatrix} 0 & (\varphi_+\varphi_- h_n\omega)^{-1} \\ -\varphi_+\varphi_- h_n\omega & 0 \end{pmatrix} A_+^{-1} \\ &= A_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A_+^{-1} \quad \text{on} \quad \Delta_0. \end{aligned} \tag{6.8}$$

We define the matrix function

$$V_0 := U_0 A. \tag{6.9}$$

Then the boundary-value problem (6.5) can be transformed by means of (6.8) to the following boundary-value problem for $V_0$:

$$\begin{gathered} V_0 \in H^\infty_{2\times 2}(O_0 \setminus \Sigma_0), \\ V_{0+} = V_{0-} \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{on } \Delta_0^+ \cup \Delta_0^-, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } \Delta_0, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{on } F_0^\varepsilon, \end{cases} \\ V_0 = (I + o(1))XA \quad \text{on} \quad \partial O_0. \end{gathered} \tag{6.10}$$

**6.1.2. Auxiliary boundary-value problem for the Airy functions.** We denote by

$$\gamma_0 := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$$

the union of the four rays (see Fig. 4)

$$\gamma_1 := \mathbb{R}^-, \qquad \gamma_2 := e^{2\pi i/3} \cdot \mathbb{R}^+, \qquad \gamma_3 := \mathbb{R}^-, \qquad \gamma_4 := e^{-2\pi i/3} \cdot \mathbb{R}^+.$$

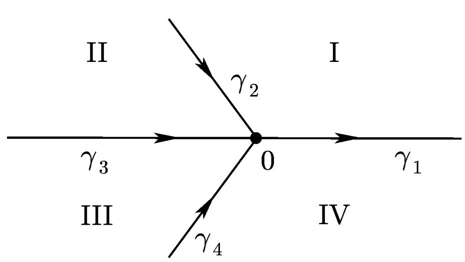


FIG. 4

For the construction of the solution of the problem (6.10) we consider the auxiliary problem of finding a matrix function $\Psi$ which is analytic in $\mathbb{C}\setminus\gamma_0$ and has on $\gamma_0$ the same jumps as $V_0$ on $\Sigma_0$ in the problem (6.10):

$$\Psi \in H(\mathbb{C}\setminus\gamma_0), \quad \exists\,\Psi_\pm \in \mathbb{C}(\gamma_0),$$

$$\Psi_+ = \Psi_- \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{on } \gamma_2\cup\gamma_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } \gamma_3, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{on } \gamma_1. \end{cases} \tag{6.11}$$

It is known (see [59]) that the solution of (6.11) is represented in terms of the special entire functions, the Airy functions. We recall that the Airy functions are the solutions of the differential equation (see [68]):

$$\operatorname{Ai}''(\zeta) = \zeta\operatorname{Ai}(\zeta), \qquad \operatorname{Ai}\in H(\mathbb{C}).$$

They have the following asymptotics as $z\to\infty$ in the angle $|\arg\zeta|<\pi$:

$$\begin{aligned} \operatorname{Ai}(\zeta) &= \frac{1}{2\sqrt{\pi}}\zeta^{-1/4}\exp\Big\{-\frac{2}{3}\zeta^{3/2}\Big\}(1+O(\zeta^{-3/2})), \\ \operatorname{Ai}'(\zeta) &= -\frac{1}{2\sqrt{\pi}}\zeta^{1/4}\exp\Big\{-\frac{2}{3}\zeta^{3/2}\Big\}(1+O(\zeta^{-3/2})). \end{aligned} \tag{6.12}$$

We set

$$A^u(\zeta) := \begin{pmatrix} \operatorname{Ai}(\zeta) & \operatorname{Ai}(\varepsilon_3^2\zeta) \\ \operatorname{Ai}'(\zeta) & \varepsilon_3^2\operatorname{Ai}'(\varepsilon_3^2\zeta) \end{pmatrix}; \qquad A^l(\zeta) := \begin{pmatrix} \operatorname{Ai}(\zeta) & -\varepsilon_3^2\operatorname{Ai}(\varepsilon_3\zeta) \\ \operatorname{Ai}'(\zeta) & -\operatorname{Ai}'(\varepsilon_3\zeta) \end{pmatrix};$$

$$\varepsilon_3 = e^{2\pi i/3},$$

and also

$$\widetilde{\sigma} := \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix}.$$

Using the known relations for the Airy functions (see [68]):

$$\begin{aligned} \operatorname{Ai}(\zeta) + \varepsilon_3\operatorname{Ai}(\zeta\varepsilon_3) + \varepsilon_3^2\operatorname{Ai}(\zeta\varepsilon_3^2) &= 0, \\ \operatorname{Ai}'(\zeta) + \varepsilon_3^2\operatorname{Ai}'(\zeta\varepsilon_3) + \varepsilon_3\operatorname{Ai}(\zeta\varepsilon_3^2) &= 0 \end{aligned}$$

one can verify (see [59; p. 226]) that the jumps on $\gamma_0$ of the piecewise analytic function

$$\Psi := \begin{cases} A^u\widetilde{\sigma} & \text{in I}, \\ A^u\widetilde{\sigma}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{in II}, \\ A^l\widetilde{\sigma}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{in III}, \\ A^l\widetilde{\sigma} & \text{in IV} \end{cases} \tag{6.13}$$

satisfy the boundary conditions of the problem (6.11). Therefore the function $\Psi$ defined in (6.13) solves the problem (6.11).

**6.1.3. Choice of the arcs $\Delta_0^+$, $\Delta_0^-$, $F_0^\varepsilon$ and the boundary $\partial O_0$. Solution of the jump problem on $\Sigma_0$.** Our aim is to construct a function that solves the problem (6.10). It will be sought in the form

$$V_0(z) = E(z)\Psi(\widetilde{c}_n\zeta(z)), \tag{6.14}$$

where $E$ is a holomorphic (in $O_0$) matrix function and $\zeta(z)$ is a conformal map, whose task is to transform the contour $\Sigma_0$ in the plane $z$ into the contour $\gamma_0$ in the plane $\zeta$ (see Fig. 5).

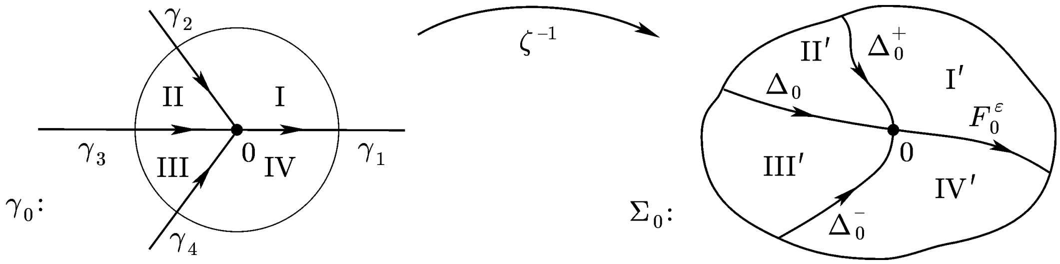


FIG. 5

Therefore, in view of (6.11), the function (6.14) will satisfy the jump relations of the problem (6.10). In this subsection we construct this map. We note that only the arc $\Delta_0$ (part of the support of the equilibrium measure, the $S$-property of which has already been used) of $\Sigma_0$ is fixed, while the choice of the disposition of the arcs $\Delta_0^+$ and $\Delta_0^-$ and the boundary $\partial O_0$ is at our disposal. As well as the position of the arc $F_0^\varepsilon$ can vary since by the analyticity of the weight function the contour of integration in the orthogonality relations can be deformed. Thus, we shall seek the conformal map $\zeta(z)$, which transforms $\Delta_0$ into $\gamma_3$. At the same time, the preimages of $\gamma_1$, $\gamma_2$, and $\gamma_4$ for this map will fix for us $\Delta_0^+$, $\Delta_0^-$, and $F_0^\varepsilon$.

We recall that the solution of the boundary-value problem (5.4) and the weight function (5.2) are factorized as follows (see (5.14)):

$$h_n\varphi^2 = \widetilde{h}_n f^2_{\widetilde{h}_n}\Phi_q^{2n}e^{-2nQ} =: \widetilde{h}_n f^2_{\widetilde{h}_n}\exp\{-2n\widetilde{l}\}, \tag{6.15}$$

where the function on the right-hand side denoted by $\widetilde{l}$ has the following property (see (5.15))

$$\widetilde{l} \in H(O_0 \setminus \Delta_0), \qquad \widetilde{l}_\pm = \mp i\pi\int_x^0 d\lambda \quad \text{on} \quad \Delta_0. \tag{6.16}$$

Since in our case the equilibrium measure has the property (see (6.2), (6.4$_1$)):

$$\lambda'(z) = z^{1/2}m(z), \qquad z \in O_0, \quad m(0) \neq 0,$$

we carry out the integration in (6.16) and obtain that

$$\widetilde{l}(z) = \frac{2}{3}z^{3/2}M(z),$$

where

$$M(z) \in H(O_0), \qquad M(0) \neq 0.$$

Therefore the function

$$\zeta(z) := \left(\frac{3}{2}\widetilde{l}(z)\right)^{2/3} = zM^{2/3}(z) \tag{6.17}$$

realizes a conformal map of the neighbourhood $O_0$ into a neighbourhood of the point 0. Let us verify that the conformal map $\zeta(z)$ defined in (6.17) has the required properties, that is, transforms the arc $\Delta_0$ into $\gamma_3$. It follows from (6.16) that

$$\operatorname{Re}\widetilde{l} = 0,$$
$$\operatorname{Im}\widetilde{l} = \begin{cases} < 0 & \text{on } \Delta_{0+}, \\ > 0 & \text{on } \Delta_{0-}, \end{cases}$$

moreover,

$$\frac{\partial \operatorname{Re}\widetilde{l}(x)}{\partial n} = \frac{\partial}{\partial n}\left(\pi \operatorname{Im}\int_x^0 d\lambda\right) = \pi\frac{\partial}{\partial \tau}\operatorname{Re}\int_x^0 d\lambda < 0.$$

Therefore $\widetilde{l}(z)$ transforms the simply connected domain

$$d \cap O_0 := (d^+ \cup \Delta \cup d^-) \cap O_0$$

into the left half-plane so that the image of $d^+ \cap O_0$ is the lower left quadrant, the image of $d^- \cap O_0$ goes over into the upper left quadrant and the subsequent map by means of $(\,\cdot\,)^{2/3}$ 'glues' the two 'halfs' of these images giving, as a result, a simply connected domain in the left half-plane (see Fig. 6).

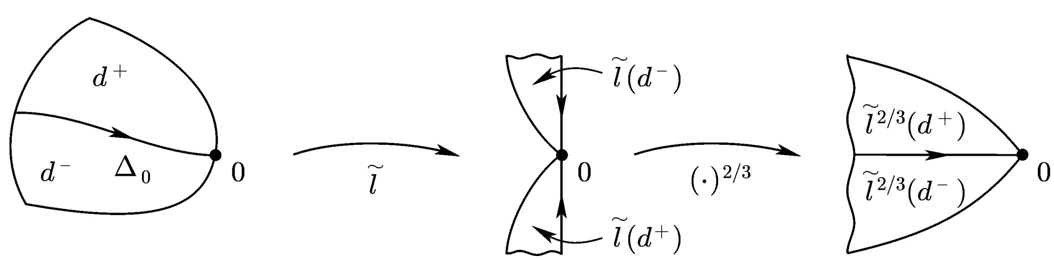


FIG. 6

We have here

$$\begin{cases} \operatorname{Im}\zeta = 0, \\ \operatorname{Re}\zeta \leqslant 0 \end{cases} \quad \text{on} \quad \Delta_0,$$

therefore

$$\zeta(\Delta_0) \subset \gamma_3.$$

We fix in the plane $\zeta$ the disc $D_\varepsilon$ with centres at $\zeta = 0$ and radius $\varepsilon$. The neighbourhood $O_0$ of the end-point 0 of the arc $\Delta$ is defined as follows

$$O_0 := \zeta^{-1}(D_\varepsilon). \tag{6.18}$$

Then we have by construction

$$\Delta_0 = \zeta^{-1}(\gamma_3 \cap D_\varepsilon),$$

and the remaining parts of the contour $\Sigma_0$ are fixed as follows

$$\begin{aligned} \Delta_0^+ &:= \zeta^{-1}(\gamma_2 \cap D_\varepsilon), \\ \Delta_0^- &:= \zeta^{-1}(\gamma_4 \cap D_\varepsilon), \\ F_0^\varepsilon &:= \zeta^{-1}(\gamma_1 \cap D_\varepsilon). \end{aligned} \tag{6.19}$$

Therefore for each constant $\widetilde{c}_n$ the function

$$\Psi(\widetilde{c}_n \zeta(z))$$

satisfies the boundary condition of the problem (6.10) on the contour $\Sigma_0$ (which is fixed by (6.19)).

**6.1.4. 'Matching' of boundary conditions on $\partial O_0$.** We recall that the solution of the problem (6.10) is sought in the form (6.14)

$$V_0(z) = E\Psi(\widetilde{c}_n \zeta(z)).$$

To satisfy the boundary condition on $\partial O_0$ as $n \to \infty$ we need the asymptotics of $\Psi(\widetilde{c}_n\zeta)$ as $\widetilde{c}_n \to \infty$, when $n \to \infty$. We first consider $z \in I'$ $(\zeta \in I)$, see Fig. 5. By definition (6.13) we have

$$\Psi(\widetilde{c}_n\zeta) = \begin{pmatrix} \mathrm{Ai}(\widetilde{c}_n\zeta) & \mathrm{Ai}(\varepsilon_3^2\widetilde{c}_n\zeta) \\ \mathrm{Ai}'(\widetilde{c}_n\zeta) & \varepsilon_3^2\,\mathrm{Ai}'(\varepsilon_3^2\widetilde{c}_n\zeta) \end{pmatrix} \widetilde{\sigma};$$

since both $|\arg\zeta|$ and $|\arg\varepsilon_3^2\zeta|$ are less than $\pi$, we choose

$$\widetilde{c}_n := n^{2/3},$$

and obtain for $z \in \partial O_0$ $(\zeta \in \partial D_\varepsilon)$ that

$$|n^{2/3}\zeta| = |\varepsilon_3^2 n^{2/3}\zeta| = n^{2/3}\varepsilon \to \infty, \qquad n \to \infty.$$

We now use the asymptotic formulae (6.12):

$$\Psi(n^{2/3}\zeta) = \begin{pmatrix} \dfrac{1}{2\sqrt{\pi}}(n^{2/3}\zeta)^{-1/4}e^{-2/3(n^{2/3}\zeta)^{3/2}}(1+o(1)) & \dfrac{1}{2\sqrt{\pi}}(\varepsilon_3^2 n^{2/3}\zeta)^{-1/4}e^{-2/3(\varepsilon_3^2 n^{2/3}\zeta)^{3/2}}(1+o(1)) \\ -\dfrac{1}{2\sqrt{\pi}}(n^{2/3}\zeta)^{1/4}e^{-2/3(n^{2/3}\zeta)^{3/2}}(1+o(1)) & -\dfrac{\varepsilon_3^2}{2\sqrt{\pi}}(\varepsilon_3^2 n^{2/3}\zeta)^{1/4}e^{-2/3(\varepsilon_3^2 n^{2/3}\zeta)^{3/2}}(1+o(1)) \end{pmatrix}\widetilde{\sigma}.$$

Since by definition

$$\zeta^{3/2} = \frac{3}{2}\widetilde{l}$$

and

$$(\varepsilon_3^2\zeta)^{3/2} = |\zeta|^{3/2}\exp\left\{i\left(\arg\zeta - \frac{2\pi}{3}\right)^{3/2}\right\} = -(\zeta)^{3/2} = -\frac{3}{2}\widetilde{l},$$

we can write

$$\Psi(n^{2/3}\zeta) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} (n^{2/3}\zeta)^{-1/4} & (n^{2/3}\zeta\varepsilon_3^2)^{-1/4} \\ -(n^{2/3}\zeta)^{1/4} & -\varepsilon_3^2(n^{2/3}\zeta\varepsilon_3^2)^{1/4} \end{pmatrix} \times \left(I + O\left(\frac{1}{n}\right)\right)\begin{pmatrix} e^{-n\widetilde{l}} & 0 \\ 0 & e^{n\widetilde{l}} \end{pmatrix}\widetilde{\sigma}.$$

Then substituting the values for $\varepsilon_3^2 = e^{-\pi i/6}$ and $\widetilde{\sigma} = \operatorname{diag}\{e^{-i\pi/6}, e^{i\pi/6}\}$ we have

$$\Psi(n^{2/3}\zeta) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} (n^{2/3}\zeta)^{-1/4} & 0 \\ 0 & (n^{2/3}\zeta)^{1/4} \end{pmatrix}\begin{pmatrix} e^{-i\pi/6} & e^{i\pi/3} \\ -e^{-i\pi/6} & -e^{i4\pi/3} \end{pmatrix} \times \left(I + O\left(\frac{1}{n}\right)\right)\begin{pmatrix} e^{-n\widetilde{l}} & 0 \\ 0 & e^{n\widetilde{l}} \end{pmatrix}.$$

Singling out the common factor $e^{-i\pi/6}$ in the central matrix and factorizing it we finally obtain

$$\Psi(n^{2/3}\zeta(z)) = \frac{e^{-i\pi/6}}{2\sqrt{\pi}}\begin{pmatrix} (n^{2/3}\zeta(z))^{-1/4} & 0 \\ 0 & (n^{2/3}\zeta(z))^{1/4} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \times \left(I + O\left(\frac{1}{n}\right)\right)\begin{pmatrix} e^{-n\widetilde{l}(z)} & 0 \\ 0 & e^{n\widetilde{l}(z)} \end{pmatrix}, \qquad n \to \infty. \tag{6.20}$$

The asymptotic formula (6.20) was derived for $z \in I' \cap O_0$ $(\zeta \in I \cap D_\varepsilon)$. Repeating the same argument for other parts of the boundary $\partial O_0$ we obtain the same formula. Thus, (6.20) holds uniformly on $\partial O_0$.

The boundary condition on $\partial O_0$ in (6.10) (which we have to satisfy) is of the form

$$E\Psi(n^{2/3}\zeta(z)) = (I + o(1))XA \quad \text{uniformly for} \quad z \in \partial O_0 \quad \text{as} \quad n \to \infty, \tag{6.21}$$

where the diagonal matrix $A$ (see (6.7)) (with (6.15) taken into account) has the form

$$A := \operatorname{diag}\{\varphi h_n^{1/2}, (\varphi h_n^{1/2})^{-1}\} = \operatorname{diag}\{e^{-n\widetilde{l}}, e^{n\widetilde{l}}\}\operatorname{diag}\{s, s^{-1}\}. \tag{6.22}$$

Here the following notation was used:

$$s := f_{\widetilde{h}_n}\widetilde{h}_n^{1/2}. \tag{6.23}$$

We define the matrix function $E$ as follows

(6.24)
$$E := 2\sqrt{\pi}\, e^{i\pi/6} X \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} n^{1/6}\zeta^{1/4} & 0 \\ 0 & n^{-1/6}\zeta^{-1/4} \end{pmatrix}.$$

Then by the asymptotic formula (6.20) the function

$$V_0 = E\Psi(n^{2/3}\zeta) \tag{6.25}$$

satisfies the boundary condition (6.21) of the problem (6.10).

Finally, to see that (6.25) is the solution of the problem (6.10) it remains to verify that the function $E$ defined in (6.24) is holomorphic in $O_0$ or, in other words, that $E$ does not 'spoil' the jumps of $\Psi(n^{2/3}\zeta(z))$ on $\Sigma_0$, which already satisfy the problem (6.10). We substitute into (6.24) the expression (5.25) for $X$:

$$X = \begin{pmatrix} 1 & \dfrac{i}{w} \\ \dfrac{c}{i\Phi_0} & \dfrac{c\Phi_0}{w} \end{pmatrix}.$$

Multiplying the matrices we obtain

$$\begin{aligned} E = \sqrt{\pi}\, e^{i\pi/6} \begin{pmatrix} 1 & 0 \\ 0 & ic \end{pmatrix} &\begin{pmatrix} -\left(s + \dfrac{1}{ws}\right)\zeta^{1/4} & \left(s - \dfrac{1}{ws}\right)\zeta^{-1/4} \\ \left(\dfrac{s}{\Phi_0} + \dfrac{\Phi_0}{ws}\right)\zeta^{1/4} & \left(-\dfrac{s}{\Phi_0} + \dfrac{\Phi_0}{ws}\right)\zeta^{-1/4} \end{pmatrix} \\ &\times \begin{pmatrix} n^{1/6} & 0 \\ 0 & n^{-1/6} \end{pmatrix}. \end{aligned} \tag{6.26}$$

Obviously, the function $E(z)$ is analytic in the punctured neighbourhood $O_0 \setminus \{0\}$. First we show that it is holomorphic in $O_0 \setminus \{0\}$. To see this we make sure that the entries of the central matrix in (6.26) taken on $\Delta_{0+}$ do not change on $\Delta_{0-}$ after going around the point 0. We see from (6.17) that

$$\zeta(z) \simeq z \quad \text{in} \quad O_0.$$

The boundary conditions (2.13)

$$f_{\widetilde{h}_n+} f_{\widetilde{h}_n-} \widetilde{h}_n = \frac{i}{w_+} \quad \text{on} \quad \Delta$$

imply the following relation for $s$

$$s_\pm = \frac{i}{w_+ s_\mp} \quad \text{on} \quad \Delta.$$

Besides that, recalling (2.11$_3$) we have

$$\Phi_{0+} = \frac{1}{\Phi_{0-}} \quad \text{on} \quad \Delta.$$

Therefore, for example, for the entry in the upper right corner of the central matrix in (6.26) we check going around the point 0 on $\Delta_0$ :

$$\left(s_+ + \frac{1}{w_+s_+}\right)z_+^{1/4} \to \left(s_- + \frac{1}{w_-s_-}\right)z_-^{1/4}$$
$$= \left(\frac{i}{w_+s_+} - \frac{s_+}{i}\right)\frac{z_+^{1/4}}{i} = \left(\frac{1}{w_+s_+} + s_+\right)z_+^{1/4}.$$

The remaining entries in (6.26) are treated similarly. Therefore

$$E \in H(O_0 \setminus \{0\}).$$

Finally, it follows from the boundary conditions on $\Delta$ that

$$s \simeq \frac{1}{ws} \simeq z^{-1/4} \qquad O_0,$$

therefore the entries of the matrix in (6.26) cannot have at the point 0 a singularity of order higher than $O(1/z^{1/2})$. But since we have shown that 0 is not a branch point for $E$, it follows that $E$ is bounded in $O_0$ and hence

$$E \in H(O_0).$$

**6.1.5. Asymptotics of orthogonal polynomials in an neighbourhood of the end-point of $\Delta$.** Summarizing, we see that for the case $(6.4_1)$ the function $U_0$ (the solution of the boundary-value problem (6.5)) is equal to

$$U_0 = E\Psi(n^{2/3}\zeta)A^{-1} \qquad O_0, \tag{6.27}$$

where $E$ is defined in (6.24) (see also (6.26)), $\Psi$ is defined in (6.13), $\zeta$ in (6.17), and $A$ in (6.7).

Thus, in case $(6.4_1)$ under consideration we not only completed the proof of Theorem 2, but for the matrix $Z$ (see (5.6))

$$Z = \begin{pmatrix} \dfrac{Q_n}{C_n\varphi} & \dfrac{\varphi}{C_n}R_n \\ \dfrac{C_nm\widetilde{Q}_{n-1}}{\varphi} & C_nm\varphi\widetilde{R}_{n-1} \end{pmatrix}$$

we obtained from (5.27) and (5.31) the following asymptotic formula (see (5.19)):

$$Z = (I + o(1))U_0 \cdot \begin{cases} I & \text{in I}' \cup \text{IV}', \\ D & \text{in II}', \\ D^{-1} & \text{in III}', \end{cases} \qquad n \to \infty, \tag{6.28}$$

uniformly in $O_0$, where $D$ is defined in (6.6) and $U_0$ is defined in (6.27).

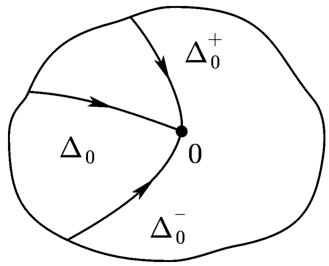


FIG. 7

**6.2. Solution of the boundary-value problem in a neighbourhood of the end-point of $\Delta$ by means of the Bessel functions.** In this section we are dealing with the contour $\Sigma_0$ corresponding to case $(B)$ in (6.1) (see Fig. 7):

$$\Sigma_0 := \Delta_0 \cup \Delta_0^+ \cup \Delta_0^- .$$

For the external field $Q$ corresponding to the case $n = 0$ in (6.2), that is,

$$\lambda'(z) = z^{-1/2} m(z), \qquad m \in H(O_0), \quad m(0) \neq 0, \tag{6.29}$$

we explicitly construct the solution of the boundary-value problem (5.26):

$$\begin{gathered} U_0 \in H^2_{2\times 2}(O_0 \setminus \Sigma_0), \\ U_{0+} = U_{0-} \cdot \begin{cases} D & \text{on } \Delta_0^+ \cup \Delta_0^-, \\ \mathcal{J} & \text{on } \Delta_0, \end{cases} \\ U_0 = (I + o(1))X \ \text{ on } \partial O_0 \ \text{ as } n \to \infty, \end{gathered} \tag{6.30}$$

where $D$ and $\mathcal{J}$ are the same as in (6.6) and $X$ is (5.25). The construction of the solution will go along the same scheme as in the previous section.

**6.2.1. Boundary-value problem with jump independent of $n$.** We go over from (6.30) to the problem for the function (6.9):

$$V_0 = U_0 A,$$

where $A$ is (6.7), as before. We have (see (6.10))

$$\begin{gathered} V_0 \in H^\infty_{2\times 2}(O_0 \setminus \Sigma_0), \\ V_{0+} = V_{0-} \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{on } \Delta_0^+ \cup \Delta_0^-, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } \Delta_0, \end{cases} \\ V_0 = (I + o(1))XA \quad \text{on} \quad \partial O_0. \end{gathered} \tag{6.31}$$

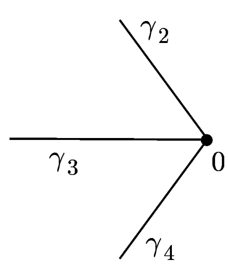


FIG. 8

**6.2.2. Boundary-value problem for the Bessel functions.** We now have $\gamma_0 = \gamma_2 \cup \gamma_3 \cup \gamma_4$ (see Fig. 8):

$$\begin{aligned}\gamma_2 &:= \left\{\zeta : \arg\zeta = \frac{2\pi}{3}\right\},\\ \gamma_3 &:= \{\zeta : \arg\zeta = -\pi\},\\ \gamma_4 &:= \left\{\zeta : \arg\zeta = -\frac{2\pi}{3}\right\}.\end{aligned}$$

The problem for $\Psi(\zeta)$ (see (6.11))

$$\Psi \in H^\infty_{2\times 2}(\mathbb{C}\setminus\gamma_0),$$

$$\Psi_+ = \Psi_- \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{on } \gamma_2 \cup \gamma_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } \gamma_3 \end{cases} \tag{6.32}$$

was solved in [67]. The solution of it is written in terms of the modified Bessel functions $I$, $K$ and the Hankel functions $H^{(1)}$ and $H^{(2)}$ (for their definitions and properties see [68]):

(6.33)

$$\Psi := \begin{cases} \begin{pmatrix} I(2\zeta^{1/2}) & \frac{i}{\pi}K(2\zeta^{1/2}) \\ 2\pi i\zeta^{1/2}I'(2\zeta^{1/2}) & -2\zeta^{1/2}K'(2\zeta^{1/2}) \end{pmatrix}, & |\arg\zeta| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2}H^{(1)}(2(-\zeta)^{1/2}) & \frac{1}{2}H^{(2)}(2(-\zeta)^{1/2}) \\ \pi\zeta^{1/2}(H^{(1)})'(2(-\zeta)^{1/2}) & \pi\zeta^{1/2}(H^{(2)})'(2(-\zeta)^{1/2}) \end{pmatrix}, & \frac{2\pi}{3} < \arg\zeta < \pi, \\ \begin{pmatrix} \frac{1}{2}H^{(2)}(2(-\zeta)^{1/2}) & -\frac{1}{2}H^{(1)}(2(-\zeta)^{1/2}) \\ -\pi\zeta^{1/2}(H^{(2)})'(2(-\zeta)^{1/2}) & \pi\zeta^{1/2}(H^{(1)})'(2(-\zeta)^{1/2}) \end{pmatrix}, & -\pi < \arg\zeta < -\frac{2\pi}{3}. \end{cases}$$

**6.2.3. Conformal correspondence of $(O_0, \Sigma_0)$ and $(D_\varepsilon, \gamma_0 \cap D_\varepsilon)$.** As before (see (6.16)) the required conformal map is constructed by means of the function $\widetilde{l}$:

$$\begin{cases} \widetilde{l} \in H(O_0 \setminus \Delta_0), \\ \widetilde{l}_\pm = \mp i\pi \displaystyle\int_x^0 d\lambda(t) \quad \text{on} \quad \Delta_0, \end{cases}$$

which, in view of (6.29), now takes the form

$$\widetilde{l}(z) = 2z^{1/2}M(z), \qquad M \in H(O_0), \quad M(0) \neq 0.$$

The conformal map

$$\zeta(z) := \left(\frac{1}{2}\widetilde{l}(z)\right)^2 = zM^2(z) \tag{6.34}$$

defines $O_0$ as the image of the disc $D_\varepsilon$. It maps $D_\varepsilon \cap \gamma_3$ into $\Delta_0$,

$$\zeta(D_\varepsilon \cap \gamma_3) = \Delta_0,$$

and defines $\Delta_0^+$ $\Delta_0^-$ as

$$\zeta(D_\varepsilon \cap \gamma_2) =: \Delta_0^+,$$
$$\zeta(D_\varepsilon \cap \gamma_3) =: \Delta_0^-.$$

Therefore, in view of (6.32), the function

$$\Psi(c_n \zeta(z))$$

satisfies now for each constant $c_n$ the boundary conditions of the problem (6.31) on $\Sigma_0$.

**6.2.4. Boundary conditions on $\partial O_0$.** We seek the solution of (6.31) in the form

$$V_0(z) = E(z)\Psi(c_n\zeta(z)), \qquad E \in H(O_0). \tag{6.35}$$

The asymptotic behaviour of the functions (6.33) as $\zeta \to \infty$ in the sector $|\arg\zeta| < \pi$ is of the form (see [67], [68]):

$$\Psi(\zeta) := \begin{pmatrix} \frac{1}{2\sqrt{\pi}}\zeta^{-1/4}e^{2\zeta^{1/2}}(1+O(\zeta^{-1/2})) & \frac{i}{2\sqrt{\pi}}\zeta^{-1/4}e^{-2\zeta^{1/2}}(1+O(\zeta^{-1/2})) \\ \sqrt{\pi}i\zeta^{1/4}e^{2\zeta^{1/2}}(1+O(\zeta^{-1/2})) & \sqrt{\pi}\zeta^{1/4}e^{-2\zeta^{1/2}}(1+O(\zeta^{-1/2})) \end{pmatrix},$$
$$\zeta \to \infty.$$

Taking

$$c_n = n^2,$$

in view of (6.34), we obtain

$$\begin{aligned} \Psi(n^2\zeta(z)) = \frac{1}{\sqrt{2}} &\begin{pmatrix} (2\pi n)^{-1/2} & 0 \\ 0 & (2\pi n)^{1/2} \end{pmatrix} \begin{pmatrix} \zeta^{-1/4}(z) & 0 \\ 0 & \zeta^{1/4}(z) \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ &\times \left(I + O\left(\frac{1}{n}\right)\right) \begin{pmatrix} e^{-n\tilde{l}(z)} & 0 \\ 0 & e^{n\tilde{l}(z)} \end{pmatrix} \end{aligned} \tag{6.36}$$

as $n \to \infty$ uniformly for $z \in \partial O_0 \cap \zeta^{-1}(\{\zeta : |\arg\zeta| < \pi\})$. Since the entries of the matrices on the right-hand side of (6.36) are uniformly bounded, it follows that this relation holds uniformly on $\partial O_0$. Therefore to satisfy the boundary conditions (6.31)

$$V_0 = (I + o(1))XA \qquad \partial O_0,$$

we (taking into account (6.36)) choose the matrix $E$ in (6.35) in the form

$$E := \sqrt{2}\,XA \begin{pmatrix} e^{n\tilde{l}} & 0 \\ 0 & e^{-n\tilde{l}} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta^{1/4} & 0 \\ 0 & \zeta^{-1/4} \end{pmatrix} \begin{pmatrix} (2\pi n)^{1/2} & 0 \\ 0 & (2\pi n)^{-1/2} \end{pmatrix}.$$

Substituting the expression (5.25) for $X$ and the expression (6.7) for $A$ we obtain

$$E = \sqrt{2}\begin{pmatrix} 1 & \dfrac{i}{w} \\ \dfrac{c}{i\Phi_0} & \dfrac{c\Phi_0}{w} \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta^{1/4} & 0 \\ 0 & \zeta^{-1/4} \end{pmatrix} \times \begin{pmatrix} (2\pi n)^{1/2} & 0 \\ 0 & (2\pi n)^{-1/2} \end{pmatrix}.$$

Multiplying the matrices we obtain

$$E = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} \left(s + \dfrac{1}{ws}\right)\zeta^{1/4} & i\left(\dfrac{1}{ws} - s\right)\zeta^{-1/4} \\ \dfrac{1}{i}\left(\dfrac{s}{\Phi_0} + \dfrac{\Phi_0}{ws}\right)\zeta^{1/4} & \left(\dfrac{\Phi_0}{ws} - \dfrac{s}{\Phi_0}\right)\zeta^{1/4} \end{pmatrix} \times \begin{pmatrix} (2\pi n)^{1/2} & 0 \\ 0 & (2\pi n)^{-1/2} \end{pmatrix}. \tag{6.37}$$

We have already shown in subsection 6.1.4 that the entries on the right-hand side of (6.37) are holomorphic in $O_0$. Therefore the function

$$V_0(z) = E(z)\Psi(n^2\zeta(z)), \qquad z \in O_0,$$

where $E$ is defined in (6.37), $\Psi$ in (6.33), and $\zeta(z)$ in (6.34), solves the boundary-value problem (6.31). Accordingly, the function

$$U_0 = E\Psi(n^2\zeta)A^{-1} \quad \text{in} \quad O_0,$$

where $A$ is (6.7), gives the solution of our problem (6.31), which completes the proof of Theorem 2 for case ($6.4_2$).

## § 7. Completion of the proof of Theorem 2. Existence of the solutions of the singular boundary-value problems in a neighbourhood of the end-point of $\Delta$

It remains to consider the singular cases of the boundary-value problem (5.26), which have been classified by the order of zero of the derivative of the equilibrium measure at the end-points of its support (see (6.2))

$$\lambda'(z) = z^{n-1/2}m(z), \qquad m \in H(O_0), \quad m(0) \neq 0. \tag{7.1}$$

We recall that we are dealing with two cases (see (6.1)). Case $(A)$ corresponds to the case, when the end-point of the support $\Delta$ of the equilibrium measure $\lambda$ is an interior point of the arc $F$ (the support of the orthogonality weight $h_n$). In this case the boundary-value problem (5.26) is singular for the following values of the exponent $n$ in (7.1):

$$(A), \quad n = 2k+1, \quad k = 1, 2, \dots. \tag{7.2}$$

Case $(B)$ corresponds to the case, when the end-point of $\Delta$ coincide with the end-point of $F$, and then the problem (5.26) is singular for

$$(B), \quad n = 1, 2, \dots . \tag{7.3}$$

The existence of the solution of the boundary-value problem (5.26) in case (7.2) for $\widetilde{h}_n \equiv 1$ is proved in [61; p. 1395–1409]. The modification of the proof in [61] for the case of weakly varying component $\widetilde{h}_n$ of the varying weight (see (5.2)) is not very difficult (see below). Therefore we do not turn our attention to the analysis of case (7.2). In this section we adapt the method from [61] to the proof of the existence of the solution of the problem (5.26) in case (7.3).

**7.1. Reduction to a boundary-value problem with globally continuous jump function.** We recall that our boundary-value problem (5.26) is of the form (see Fig. 7)

$$\begin{gathered} U_0 \in H^2_{2\times 2}(O_0 \setminus \Sigma_0), \\ U_{0+} = U_{0-} \cdot \begin{cases} D & \text{on } \Delta_0^+ \cup \Delta_0^-, \\ \mathcal{J} & \text{on } \Delta_0, \end{cases} \\ U_0 = (I + o(1))X \ \text{ on } \partial O_0 \ \text{ as } n \to \infty, \end{gathered} \tag{7.4}$$

where $X$ is defined in (5.25), $\mathcal{J}$ in (6.6), and $D$ (in view of (6.6) and (6.15)) is written in the form

$$D = \begin{pmatrix} 1 & 0 \\ e^{2n\widetilde{l}(z)}/(\widetilde{h}_n f^2_{\widetilde{h}_n}) & 1 \end{pmatrix}.$$

Here

$$\begin{cases} \widetilde{l} \in H(O_0 \setminus \Delta), \\ \widetilde{l}(x)\big|_{\Delta_\pm} = \pm i\pi \displaystyle\int_0^x d\lambda \end{cases}$$

and taking into account (7.3) we have in our case

$$\widetilde{l}(z) = z^{\frac{3+2\nu}{2}} \widetilde{M}(z), \qquad \widetilde{M} \in H(O_0), \quad \widetilde{M}(0) \neq 0, \quad \nu = 0, 1, 2, \dots . \tag{7.5}$$

By the change of variables

$$\widetilde{V}_0 := U_0 \widetilde{A}, \qquad \widetilde{A} := \begin{pmatrix} f\widetilde{h}_n^{1/2} & 0 \\ 0 & (f\widetilde{h}_n^{1/2})^{-1} \end{pmatrix},$$

we go over from the problem (7.4) with integrable boundary conditions to an equivalent problem, which we consider in the class of functions with continuous boundary values:

$$\begin{gathered} \widetilde{V}_0 \in H(O_0 \setminus \Sigma_0), \\ \widetilde{V}_{0+} = \widetilde{V}_{0-} \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2n\widetilde{l}} & 1 \end{pmatrix} & \text{on } \Delta_0^+ \cup \Delta_0^-, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } \Delta_0, \end{cases} \\ \widetilde{V}_0 = (I + o(1))X\widetilde{A} \ \text{ on } \partial O_0 \ \text{ as } n \to \infty. \end{gathered} \tag{7.6}$$

Carrying out the conformal change of variables corresponding to (7.5)

$$\zeta(z) := (2n\widetilde{l}(z))^{\frac{2}{3+2\nu}} = z(2n\widetilde{M}(z))^{\frac{2}{3+2\nu}}, \tag{7.7}$$

we obtain for the function $\widetilde{\mathcal{V}}_0(\zeta) := \widetilde{V}(z(\zeta))$ the following problem (see Fig. 8):

$$\widetilde{\mathcal{V}}_0 \in H\Big(\mathbb{C} \setminus \bigcup_2^4 \gamma_j\Big),$$

$$\widetilde{\mathcal{V}}_{0+} = \widetilde{\mathcal{V}}_{0-} \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ \exp\{\zeta^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix} & \text{on } \gamma_2 \cup \gamma_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } \gamma_3, \end{cases} \tag{7.8}$$

$$\widetilde{\mathcal{V}}_0(\zeta) = \Big[I + O\Big(\frac{1}{\zeta}\Big)\Big] \begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix} \cdot \mathrm{Const}, \qquad \zeta \to \infty, \quad \zeta \in \mathbb{C} \setminus \bigcup_2^4 \gamma_j,$$

where Const is a matrix of absolute constants. The branch of the function $\zeta^\alpha$ is fixed by the cut along $(-\infty, 0)$, and the angles $\gamma_2$ and $\gamma_4$ of the slope of the straight lines are chosen so that the function $\exp\{\zeta^{\frac{3+2\nu}{2}}\}$ decays exponentially as $\zeta \to \infty$ along $\gamma_2 \cup \gamma_4$. For the proof of the existence of the solution of the problem (7.8) we also consider the problem for the function $\tilde{\widetilde{\mathcal{V}}}_0(\zeta)$, which differs from (7.8) by the jump condition on $\gamma_2 \cup \gamma_4$:

$$\tilde{\widetilde{\mathcal{V}}}_{0+}(\zeta) = \tilde{\widetilde{\mathcal{V}}}_{0-}(\zeta) \cdot \begin{pmatrix} 1 & 0 \\ \exp\{\zeta^{1/2}\} & 1 \end{pmatrix} \quad \text{on} \quad \gamma_2 \cup \gamma_4.$$

Carrying out in the problem for $\tilde{\widetilde{\mathcal{V}}}_0$ the change of variables inverse to (7.7) and setting $2\widetilde{n} := (2n)^{1/3}$, we come to the problem (7.6) with a modified jump matrix on $\Delta_0^+ \cup \Delta_0^-$:

$$\tilde{\widetilde{V}}_{0+} = \tilde{\widetilde{V}}_{0-} \begin{pmatrix} 1 & 0 \\ e^{2\widetilde{n}\tilde{\widetilde{l}}} & 1 \end{pmatrix} \quad \text{on} \quad \Delta_0^+ \cup \Delta_0^-, \tag{7.9}$$

where

$$\tilde{\widetilde{l}} := \widetilde{l}^{\frac{1}{3+2\nu}} = z^{1/2}\tilde{\widetilde{M}}, \qquad \tilde{\widetilde{M}} \in H(O_0), \quad \tilde{\widetilde{M}}(0) \neq 0.$$

The solution of the last problem (that is, (7.6) with the modified jump (7.9)) exists and was constructed by means of the Bessel functions in the previous section (see section. 6.2).

Therefore to achieve our goal it suffices to prove the existence of the function

$$\widehat{\mathcal{V}}_0 := \widetilde{\mathcal{V}}_0 \tilde{\widetilde{\mathcal{V}}}_0^{-1},$$

satisfying (in the sense of continuous boundary values) the problem

$$\begin{gathered} \widehat{\mathcal{V}}_0 \in H(\mathbb{C} \setminus \widehat{\Gamma}), \\ \widehat{\mathcal{V}}_{0+} = \widehat{\mathcal{V}}_{0-} T \quad \text{on} \quad \widehat{\Gamma}, \\ \widehat{\mathcal{V}}_0(\zeta) = \Big(I + O\Big(\frac{1}{\zeta}\Big)\Big), \quad \zeta \to \infty, \quad \zeta \in \mathbb{C} \setminus \widehat{\Gamma}, \end{gathered} \tag{7.10}$$

where $\widehat{\Gamma} := \gamma_2 \cup \gamma_4^-$, and the jump matrix on $\widehat{\Gamma}$ is defined as follows:

$$(7.10') \qquad T := \begin{cases} \tilde{\tilde{\mathcal{V}}}_{0-} \begin{pmatrix} 1 & 0 \\ \exp\{\zeta^{\frac{3+2\nu}{2}}\} - \exp\{\zeta^{\frac{1}{2}}\} & 1 \end{pmatrix} \tilde{\tilde{\mathcal{V}}}_{0-}^{-1} & \text{on } \gamma_2, \\ \tilde{\tilde{\mathcal{V}}}_{0-} \begin{pmatrix} 1 & 0 \\ \exp\{\zeta^{\frac{1}{2}}\} - \exp\{\zeta^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix} \tilde{\tilde{\mathcal{V}}}_{0-}^{-1} & \text{on } \gamma_4^-. \end{cases}$$

We note that we have changed the orientation of $\gamma_4$ (see Fig. 9) so that the oriented contour $\gamma_2 \cup \gamma_4^-$ splits $\mathbb{C}$ into the two domains $\widetilde{\Omega}^{(-)}$ and $\widetilde{\Omega}^{(+)}$.

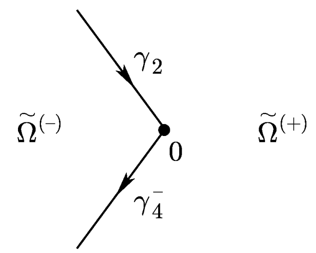


FIG. 9

The problem (7.10) has the advantage that the jump function (7.10′) is now continuous on the whole $\widehat{\Gamma}$.

**7.2. Resolvent of the Cauchy operator and solution of the matrix Riemann-Hilbert problem.** The general method of the proof of the existence of the solution of the problem (7.10) is as follows. Let $C_\pm$ be the Cauchy operator:

$$(C_\pm f)(\zeta) = \lim_{\zeta' \to \zeta_\pm} \int_{\widehat{\Gamma}} \frac{f(s)}{s - \zeta'} \frac{ds}{2\pi i}, \qquad \zeta \in \widehat{\Gamma},$$

where $\zeta' \to \zeta_\pm$ denotes the non-tangential limit to $\zeta$ from side $(+)$ (respectively, side $(-)$) of the oriented contour $\widehat{\Gamma}$. By the theorem of M. Riesz, under general conditions on $\widehat{\Gamma}$ we have for $f \in L^p(\widehat{\Gamma}, |d\zeta|)$

$$\|C_\pm f\|_{L^p(\widehat{\Gamma})} \leqslant c_p \|f\|_{L^p(\widehat{\Gamma})},$$

and it follows from the Sokhotskiĭ–Plemel' formula that

$$C_+ - C_- = E, \qquad Ef := f.$$

Let

$$u := I - T^{-1}.$$

We set

$$(7.11) \qquad C_T(f) := C_+(fu).$$

If $u \in C(\widehat{\Gamma})$, then $C_T$ clearly is a bounded operator $L^2(\widehat{\Gamma}) \to L^2(\widehat{\Gamma})$. Suppose that the equation

$$(7.12) \qquad (E - C_T)\mu = I \qquad \widehat{\Gamma}$$

has a solution $\mu - I \in L^2(\widehat{\Gamma})$, that is,

$$(E - C_T)(\mu - I) = C_T I = C_+ u$$

in the sense of $L^2(\widehat{\Gamma})$-boundary values. Then the solution of the matrix boundary-value problem (7.10) is written as follows

$$\widehat{\mathcal{V}}_0(\zeta) = I + \int_{\widehat{\Gamma}} \frac{\mu(s)u(s)}{s - \zeta} \frac{ds}{2\pi i}, \qquad \zeta \notin \widehat{\Gamma}. \tag{7.13}$$

In fact,

$$\widehat{\mathcal{V}}_{0+} = I + C_+(\mu u) = I + C_T(\mu) = \mu,$$
$$\widehat{\mathcal{V}}_{0-} = \mu(I - u)^{-1} = \mu T^{-1}.$$

Therefore to prove the existence of the solution of the problem (7.10) in the $L^2$-sense for boundary values (it becomes clear in what follows that in view of continuity of the jump this will also guarantee the existence of the solution in the sense of continuous boundary values) it suffices to make sure (see (7.12)) that the operator

$$E - C_T,$$

is invertible. For this purpose it suffices to show that

A) $E - C_T$ is a Fredholm operator
B) the index of $E - C_T$ is 0, (7.14)
C) $\operatorname{Ker}(E - C_T) = \{0\}$.

The validity of A) and B) is a fairly general fact following from the continuity of $T$ on $\widehat{\Gamma}$ (the proof of A) and B) below is basically a repetition of the corresponding argument from [61], and we present it here for the sake of completeness only).

**7.3. Fredholm property of $E - C_T$.** We now prove statement A) from (7.14). Along with the operator (7.11) we consider the operator

$$C_{T^{-1}} f := C_+(f\widehat{u}), \qquad \widehat{u} := I - T.$$

The operator $E - C_{T^{-1}}$ is pseudoinverse to $E - C_T$. In fact, we have for $f \in L^2(\widehat{\Gamma})$

$$\begin{aligned}
(E - C_T)&(E - C_{T^{-1}})f \\
&= f - C_T f - C_{T^{-1}} f + C_+[\{f(I - T) + C_-(f(I - T))\}(I - T^{-1})] \\
&= f + C_+[C_-(f\widehat{u})u].
\end{aligned}$$

Therefore for the proof of A) we have to show that

$$K\colon f \to C_+[C_-(f\widehat{u})u] \tag{7.15}$$

is a compact operator in the sense of $L^2(\widehat{\Gamma}) \to L^2(\widehat{\Gamma})$.

Let $u_j$ be a rational approximation

$$u_j(\xi) := \sum_{\nu=1}^{N} \frac{\alpha_\nu}{\xi - \xi_\nu}, \qquad \xi_\nu \notin \widehat{\Gamma},$$

of a function $u$ continuous on $\widehat{\Gamma}$:

$$\|u_j - u\|_{C(\widehat{\Gamma})} \leqslant \frac{1}{j},$$

and let $K_j$ be the operator (7.15) corresponding to $u_j$. Since

$$\|K_j - K\| \leqslant \frac{\text{const}}{j},$$

it suffices to verify that $K_j$ is compact for each $j$. Let

$$f_n \in L^2(\widehat{\Gamma}), \qquad f_n \xrightarrow{*} 0. \tag{7.16}$$

We have

$$\begin{aligned}(K_j f_n)(\xi) &= \lim_{\substack{\zeta' \to \zeta \\ \zeta' \in \Omega^{(+)}}} \sum_{\nu=1}^{N} \alpha_\nu \int_{\widehat{\Gamma}} \frac{C_-(f_n \widehat{u})(\xi)\, d\xi}{(\xi - \xi_\nu)(\xi - \zeta')2\pi i} \\ &= -\sum_{\nu : \xi_\nu \in \Omega^{(-)}} \frac{\alpha_\nu}{\xi_\nu - \zeta} \int_{\widehat{\Gamma}} \frac{f_n(t)\widehat{u}(t)}{t - \xi_\nu} \frac{dt}{2\pi i},\end{aligned}$$

which, in view of (7.16), implies that

$$\|K_j f_n\|_{L^2(\widehat{\Gamma})} \to 0 \quad \text{as} \quad n \to \infty.$$

Statement A) from (7.14) is proved.

**7.4. The index of $E - C_T$.** We consider the operator $E - C_{T_t}$ constructed by the jump matrices $T_t$ depending on the parameter $t \in [0,1]$:

$$T_t(\zeta) := \begin{cases} \tilde{\tilde{\mathcal{V}}}_{0-} \begin{pmatrix} 1 & 0 \\ t(\exp\{\zeta^{\frac{3+2\nu}{2}}\} - \exp\{\zeta^{\frac{1}{2}}\}) & 1 \end{pmatrix} \tilde{\tilde{\mathcal{V}}}_{0-}^{-1} & \text{on } \gamma_2, \\ \tilde{\tilde{\mathcal{V}}}_{0-} \begin{pmatrix} 1 & 0 \\ t(\exp\{\zeta^{\frac{1}{2}}\} - \exp\{\zeta^{\frac{3+2\nu}{2}}\}) & 1 \end{pmatrix} \tilde{\tilde{\mathcal{V}}}_{0-}^{-1} & \text{on } \gamma_4^-. \end{cases}$$

As shown above, $E - C_{T_t}$ is a Fredholm operator for each $t$ and for $t = 0$ (that is, $E = E - C_{T_0}$) it has index zero. By the continuous dependence of the index on the parameter $t$ we have

$$\operatorname{Ind}(E - C_T) = \operatorname{Ind}(E - C_{T_t}) = 0, \qquad t \in [0,1].$$

**7.5. The kernel of $E - C_T$.** It remains to prove statement ) from (7.14). Assume the contrary, that is, let there exist a function $\mu_0 \in L^2(\Gamma_0)$ such that

$$(E - C_T)\mu_0 = 0. \tag{7.17}$$

Then, by repeating the derivation of (7.13) from (7.12) we obtain that the function

$$\widehat{\mathcal{V}}_0^{\#}(\zeta) = \int_{\widehat{\Gamma}} \frac{\mu_0(s)u(s)}{s-\zeta}\,\frac{ds}{2\pi i}, \qquad \zeta \notin \widehat{\Gamma},$$

is the solution (in the $L^2$-sense) of the following boundary-value problem:

$$\begin{gathered} \widehat{\mathcal{V}}_0^{\#} \in H(\mathbb{C} \setminus \widehat{\Gamma}), \\ \widehat{\mathcal{V}}_{0+}^{\#} = \widehat{\mathcal{V}}_{0-}^{\#} T \ \text{ on } \widehat{\Gamma}, \\ \widehat{\mathcal{V}}_0^{\#}(\zeta) = O\Big(\frac{1}{\zeta}\Big), \quad \zeta \to \infty, \ \zeta \in \widehat{\Gamma}. \end{gathered} \tag{7.18}$$

Moreover, by the piecewise analyticity and continuity of $\mathcal{V}$ one can show (see [61; Proposition 5.7]) that $\mathcal{V}_0^{\#}$ is uniformly bounded and solves (7.18) in the sense of continuous boundary values.

Now, in connection with (7.18), we reverse the argument that has led us from (7.8) to (7.10) and obtain that (7.17) implies the existence (in the sense of continuous boundary values) of the solution of the problem (7.8) with modified normalization conditions at infinity. More precisely, there exists $\widetilde{\mathcal{V}}_0^{\#}$:

$$\begin{gathered} \widetilde{\mathcal{V}}_0^{\#} \in H\Big(\mathbb{C} \setminus \bigcup_2^4 \gamma_j\Big), \\ \widetilde{\mathcal{V}}_{0+}^{\#} = \widetilde{\mathcal{V}}_{0-}^{\#} \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ \exp\{\zeta^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix} & \text{on } \gamma_2 \cup \gamma_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } \gamma_3, \end{cases} \\ \widetilde{\mathcal{V}}_0^{\#}(\zeta) = O\Big(\frac{1}{\zeta}\Big)\begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix}, \qquad \zeta \to \infty, \quad \zeta \in \mathbb{C} \setminus \bigcup_2^4 \gamma_j. \end{gathered} \tag{7.19}$$

We now show that (7.19) can have, in fact, only the trivial solution. This will imply statement ) in (7.14).

We go over to the auxiliary function $B(\zeta)$:

$$B(\zeta) := \begin{cases} \widetilde{\mathcal{V}}_0^{\#}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \zeta \in \mathrm{I}, \\ \widetilde{\mathcal{V}}_0^{\#}\begin{pmatrix} 1 & 0 \\ \exp\{\zeta^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \zeta \in \mathrm{II}, \\ \widetilde{\mathcal{V}}_0^{\#}\begin{pmatrix} 1 & 0 \\ \exp\{\zeta^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix}^{-1}, & \zeta \in \mathrm{III}, \\ \widetilde{\mathcal{V}}_0^{\#}, & \zeta \in \mathrm{IV}, \end{cases}$$

where, as we recall, the domains I, II, III, and IV are bounded by the rays $\gamma_j$, $j = 1, \dots, 4$ (see Fig. 4). Let us check the jumps of $B$ on $\bigcup_{j=1}^4 \gamma_j$. The jumps on $\gamma_2 \cup \gamma_4$ no longer exist, the jump $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ appeared on $\gamma_1$, and on $\gamma_3$ we have

$$
\begin{aligned}
B_+\big|_{\gamma_3} &= \widetilde{\mathcal{V}}^{\#}_{0+} \begin{pmatrix} 1 & 0 \\ \exp\{\zeta_+^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \widetilde{\mathcal{V}}^{\#}_{0-} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \exp\{\zeta_+^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= B_-\big|_{\gamma_3} \begin{pmatrix} 1 & 0 \\ \exp\{\zeta_-^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \exp\{\zeta_+^{\frac{3+2\nu}{2}}\} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= B_-\big|_{\gamma_3} \begin{pmatrix} 1 & -\exp\{\zeta_+^{\frac{3+2\nu}{2}}\} \\ \exp\{\zeta_-^{\frac{3+2\nu}{2}}\} & 0 \end{pmatrix}.
\end{aligned}
$$

Therefore the function $B(\zeta)$ solves the following boundary-value problem:

$$
B \in H(\mathbb{C} \setminus \mathbb{R}), \quad \exists\, B_\pm \subset C(\mathbb{R}),
$$

$$
B_+(\zeta) = B_-(\zeta) \cdot \begin{cases} \begin{pmatrix} 1 & -\exp\{\zeta_+^{\frac{3+2\nu}{2}}\} \\ \exp\{\zeta_-^{\frac{3+2\nu}{2}}\} & 0 \end{pmatrix}, & \zeta \in \mathbb{R}^{(-)} := (-\infty, 0), \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \zeta \in \mathbb{R}^{(+)} := (0, +\infty), \end{cases}
$$

$$
B(\zeta) = O\left(\frac{1}{\zeta^{3/4}}\right), \quad \zeta \to \infty, \quad \text{uniformly in} \quad \mathbb{C} \setminus \mathbb{R}. \tag{7.20}
$$

We define the function

$$
B^{\#}(\zeta) := B(\zeta) B^*(\overline{\zeta}). \tag{7.21}
$$

Here $^*$ denotes adjoint matrices and $^-$ denotes the complex conjugation.

The function $B^{\#}(\zeta)$ is analytic in $\mathbb{C}^{(+)} := \{\operatorname{Im} \zeta > 0\}$, continuous up to $\mathbb{R}$, and decays as $\zeta^{-3/2}$ when $\zeta \to \infty$. Hence, by the Cauchy theorem we have

$$
\int_{\mathbb{R}} B_+^{\#}(s)\, ds = 0.
$$

Substituting into the last relation the definition of $B^{\#}$ (see (7.21)) and taking into account the jump matrices of the function $B$ (see (7.20)), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{(-)}} B_-(s) \begin{pmatrix} 1 & -\exp\{s_+^{\frac{3+2\nu}{2}}\} \\ \exp\{-s_+^{\frac{3+2\nu}{2}}\} & 0 \end{pmatrix} B_-^*(s)\, ds \\
&\quad + \int_{\mathbb{R}^{(+)}} B_-(s) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B_-^*(s)\, ds = 0.
\end{aligned}
$$

Next, adding to the relation so obtained the conjugate relation we have

$$\int_{\mathbb{R}^{(-)}} B_-(s)\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix} B_-^*(s)\,ds = 0.$$

Since the diagonal elements of the product of the matrices under the integral sign are the positive numbers $2B_{11}\overline{B}_{11}$ and $2B_{21}\overline{B}_{21}$, it follows that

$$\begin{pmatrix}B_{11}\\ B_{21}\end{pmatrix}\Bigg|_{\mathbb{R}^{(-)}} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

and by the Privalov theorem we obtain

$$\begin{pmatrix}B_{11}\\ B_{21}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{in} \quad \overline{\mathbb{C}}^{(-)},$$

and in view of the relations on the jumps in (7.20) this gives

$$\begin{pmatrix}B_{12}\\ B_{22}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{in} \quad \overline{\mathbb{C}}^{(+)}.$$

Thus, it remains to consider the vector function

$$b := \begin{cases} \begin{pmatrix}B_{11}\\ B_{21}\end{pmatrix} & \text{in } \mathbb{C}^{(+)},\\[2ex] \begin{pmatrix}B_{12}\\ B_{22}\end{pmatrix} & \text{in } \mathbb{C}^{(-)},\end{cases}$$

which (as follows from (7.20)) satisfies the following boundary-value problem:

$$\begin{gathered} b \in H(\mathbb{C}\setminus\mathbb{R}^{(-)}), \quad \exists\, b_\pm \in C(\mathbb{R}^{(-)}),\\ b_+(\zeta) = b_-(\zeta)\exp\{-\zeta_+^{\frac{3+2\nu}{2}}\}, \quad \zeta\in\mathbb{R}^{(-)},\\ b(\zeta) = O(\zeta^{-3/4}) \quad \text{uniformly as} \quad \zeta\to\infty, \quad \zeta\in\mathbb{C}\setminus\mathbb{R}^{(-)}. \end{gathered} \tag{7.22}$$

Essentially, we can assume that $b(\zeta)$ in (7.22) is a scalar function. We now show that only the function identically equal to zero can be a solution of the problem (7.22). For this purpose we cut the complex plane along $\mathbb{R}^{(-)}$ and glue to both 'banks' $\mathbb{R}^{(-)}_\pm$ the sectors of angle $\frac{\pi}{2}\eta$ with some small positive $\eta$. We continue on the Riemann surface so obtained the function $b$ by projecting it onto the glued sectors and multiplying the results by $\exp\{\zeta^{\frac{3+2\nu}{2}}\}$. The function so obtained is analytic on the Riemann surface. By means of the map

$$\zeta(w) = w^{2+\eta}$$

it goes over into the function

$$\widetilde{b}(w) = b(\zeta(w)),$$

which is

$$\begin{aligned} &1)\ \text{analytic in } \{w : \operatorname{Re} w > 0\},\\ &2)\ \text{bounded in } \{w : \operatorname{Re} w \geqslant 0\},\\ &3)\ |\widetilde{b}(ix)| \leqslant C_1\exp\{-C_2|x|^{\frac{(3+2\nu)(2+\eta)}{2}}\} \leqslant C_1' e^{-C_2'|x|},\ \ x\in\mathbb{R}. \end{aligned} \tag{7.23}$$

It is more convenient to deal with the function $a(w) \in H(\{w : \operatorname{Im} w \geqslant 0\})$:

$$a(w) := \widetilde{b}(iw).$$

This function has all the power moments equal to zero:

$$\int_{-\infty}^{\infty} w^n a(w)\, dw = 0, \qquad n = 0, 1, 2, \dots. \tag{7.24}$$

In fact, taking into account 1) and 2) from (7.23) we have by the Cauchy theorem

$$\int_{-\infty}^{\infty} \frac{w^n a(w)\, dw}{(i\varepsilon w - 1)^{n+2}} = 0, \qquad \varepsilon > 0, \quad n = 0, 1, 2, \dots.$$

Passing to the limit as $\varepsilon \to 0$ and taking into account inequality 3) from (7.23) we obtain (7.24) by the Lebesgue theorem. Finally, let us see that

$$a(w) \equiv 0, \qquad \{w : \operatorname{Re} w \geqslant 0\}. \tag{7.25}$$

In fact, by inequality 3) from (7.23) the Fourier transform $\widehat{a}(k)$ of the function $a(w)$ is analytic in the strip

$$\widehat{a}(k) \in H(\{k : |\operatorname{Im} k| < C_2'\}). \tag{7.26}$$

The derivatives of $\widehat{a}(k)$ at $k = 0$ are proportional to the moments of $a(w)$, and by (7.24), (7.26) we obtain that

$$\widehat{a}(k) \equiv 0, \qquad k \in \mathbb{R}.$$

Hence, $a(w) \equiv 0$ on $\mathbb{R}$ and by the uniqueness of the solution of the Dirichlet problem in the half-plane in the class of *bounded* functions we obtain (7.25).

Thus, the problem (7.19) has only the trivial solution, that is, the kernel of the operator $E - C_T$ is trivial, which completes the proof of the invertibility of it.

Hence the existence of the solution of the boundary-value problem (5.26) in the final singular case (7.3) is proved. The proof of Theorem 2 is complete.

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Translated by A.A. ILYIN

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