CFGT determination of Varga's constant '1/9'.

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Summary. '1/9' $= \exp(-\pi K'/K)$, where K is the complete elliptic integral of the first kind such that $K = 2E$.

Statement. Gutknecht and Trefethen (GT) have shown how to adapt the Carathéodory-Fejér (CF) approximation scheme to the study of polynomial and rational approximation. In doing so, they have expressed a paradigm. A demonstration follows.

Introductory material. Consider the best L_∞ rational n/n approximation to $F(x) = \sum_0^{\infty}{}' c_k T_k(x)$ on $[-1,1]$. The error norm E_n is often very close to σ_n, the nth singular value $(\sigma_1 \geq \sigma_2 \geq \dots)$ of the Hankel matrix $H = [c_{k+m-1}]$ $k,m = 1,2,\dots$ (see L.N. TREFETHEN and M.H. GUTKNECHT, The Carathéodory-Fejér method for real rational approximation, *SIAM J. Numer. Anal.* **20** (1983) 420–436).

Problem: appreciate if σ_n must be expected to show a q^n behaviour and give q.

If F is real, $\sigma_n = |\lambda_n|$, nth eigenvalue of H. $P(\mu) = \prod_{n=1}^{\infty}(1-\lambda_n\mu) = \det(I-\mu H) = \sum_{j=0}^{\infty} d_j \mu^j$. (Determinant exists if $\sum_k \sum_n |h_{k,n}| = \sum_1^{\infty} n|c_n| < \infty$).

Problem becomes: appreciate $|d_n| \sim q^{n^2/2}$ and give q.

d_n is a series of determinants of order n: $d_0 = 1$, $d_1 = -\sum_0^{\infty} c_{2k+1}$, $d_n = (-1)^n \sum_{k_1=0}^{\infty} \sum_{k_2>k_1} \cdots \det[c_{k_m+k_j+1}]$

Now, use $c_k = \dfrac{2}{\pi}\displaystyle\int_{-1}^{1} F(x)T_k(x)(1-x^2)^{-1/2}dx = \dfrac{1}{\pi}\int_{-\pi}^{\pi} F(\cos\theta)\exp(ik\theta)\,d\theta = \dfrac{1}{\pi i}\int_{\mathrm{C}} F((u+u^{-1})/2)u^{k-1}\,du$ (C = unit circle):

$$d_n = (-\pi i)^{-n} \sum_{k_1=0}^{\infty} \sum_{k_2>k_1} \cdots \sum_{k_n>k_{n-1}} \int_{\mathrm{C}} F\left(\frac{u_1+u_1^{-1}}{2}\right) du_1 \cdots \int_{\mathrm{C}} F\left(\frac{u_n+u_n^{-1}}{2}\right) du_n \det\left[u_m^{k_m+k_j}\right]_{m,j=1}^{n}$$

$$= \frac{1}{(-\pi i)^n\, n!}\int_{\mathrm{C}} \cdots \int_{\mathrm{C}} D(u_1,u_2,\dots,u_n) F\left(\frac{u_1+u_1^{-1}}{2}\right) \cdots F\left(\frac{u_n+u_n^{-1}}{2}\right) du_1 \cdots du_n \text{ , after } (1/n!)$$

sum of permutations on $u_1,\dots,u_n$, in order to have a symmetric $D(u_1,\dots,u_n)$. This function $D(u_1,\dots,u_n)$ does not depend on F, it can be found through special finite rank Hankel matrices.

The result is $D(u_1,\dots,u_n) = \displaystyle\prod_{1\leqslant m<j\leqslant n} \left(\frac{u_m-u_j}{1-u_m u_j}\right)^2 \prod_{m=1}^{n}(1-u_m^2)^{-1}$, checked by André Hautot.

Applications.

[1] If $F((u+u^{-1})/2)$ is analytic between a contour C_1 inside C and its inverse $(C_1)^{-1}$, the integrals may be performed on C_1 instead of C. The integral for d_n is then dominated by configurations $(u_1,\dots,u_n)$ maximizing $|D(u_1,\dots,u_n)|$. This gives $q = \exp(-2/\kappa)$, where κ is the capacity of the condenser (C_1, C_1^{-1}), $u_1,\dots,u_n$ are the positions of charges repelling each other on C_1, attracted by

[1] Date added on the September 2000 retyping. The text has been sent to several people in October 1985, and been given to all the participants of the Padé approximation meeting organized by C. Brezinski at Marseille-Luminy, 14–18 Oct. 1985. A.P.M.

C_1^{-1}, according to the logarithmic potential. Such behaviours have indeed been reported (Gončar, *MATH USSR Sb.* **23** (1974) 254-270). If there are still degrees of freedom on C_1, minimize κ on the admissible C_1's.

$\boxed{2}$ If $F(x) = G((x-1)/(x+1))$ and G is ***entire***, ($\leftarrow$ approximation to $G(z)$ on $z \in (-\infty, 0]$), C_1 and C_1^{-1} touch at $x = -1 (\Rightarrow \kappa = \infty)$, but there is no other constraint on C_1, which can decrease in size when $n \to \infty$.

Then, with $v = \dfrac{1-u}{1+u}, d_n = \dfrac{1}{(2\pi i)^n n!} \displaystyle\int_{\Gamma_1} \cdots \int_{\Gamma_1} \prod_{1 \leqslant m < j \leqslant n} \left(\frac{v_m - v_j}{v_m + v_j} \right)^2 G(v_1^2) \cdots G(v_n^2)\, dv_1 \cdots dv_n / (v_1 \cdots v_n)$.

$\Gamma_1 = \dfrac{1 - C_1}{1 + C_1} \to \infty$ when $n \to \infty$, estimated by values at saddle-points (Nuttall's technique[2]):

$4 \displaystyle\sum_{\substack{j=1 \\ j \neq m}}^{n} \frac{v_j}{v_m^2 - v_j^2} = -2 v_m \frac{G'(v_m^2)}{G(v_m^2)} + \frac{1}{v_m}, m = 1, 2, \ldots, n.$ (Any relation with Opitz-Scherer saddle-points? [*Constr. Approx.* **1** (1985) 195-216])

$\boxed{3}$ Solution when $G(z) = \exp(z)$.

Assume the v_m's distributed on a curve $\Gamma_1 = n^{1/2} \Gamma_0$, where Γ_0 joins two fixed points X and Y in the right half-plane: $\displaystyle\sum_{m=1}^{n} f(n^{-1/2} v_m) \sim n \int_X^Y f(w) \varphi(w)\, dw$. This yields the equation for φ: $\displaystyle⨍_X^Y \frac{w}{x^2 - w^2} \varphi(w)\, dw = -\frac{x}{2}$, $x \in \Gamma_0$, solved (assuming further φ to be analytic up to branch points) through $\chi(x) = \displaystyle\int_X^Y \frac{w}{x^2 - w^2} \varphi(w)\, dw$ (definition of χ) $= -\dfrac{x}{2} \pm \dfrac{\pi i}{2} \varphi(x)$ (from the equation and the properties of such integrals, $x \notin \Gamma_0$)
$= \dfrac{2}{\pi i} [(x^2 - X^2)(x^2 - Y^2)]^{1/2} \displaystyle\int_X^Y [(w^2 - X^2)(w^2 - Y^2)]^{-1/2} \frac{w}{x^2 - w^2} \frac{w}{2}\, dw$ (inversion trick). The remaining conditions $\displaystyle\int_X^Y [(w^2 - X^2)(w^2 - Y^2)]^{-1/2} w \frac{w}{2}\, dw = 0$ ($\chi(x)$ must be $o(1)$ when $x \to \infty$) and $\displaystyle\int_X^Y \varphi(w)\, dw = 1$ determine X and Y. Practically, with X and $Y = R \exp(\pm i \theta_0)$, the conditions turn into a complete elliptic integrals equation $\boldsymbol{K(\sin\theta_0) = 2E(\sin\theta_0)}$ giving

$$
\begin{aligned}
k = \sin\theta_0 &= 0.908908557548541478236118908744793504901013969340 41 \\
k' = \cos\theta_0 &= 0.41699548440604205639041957807087776692610248051382 \\
K = K(\sin\theta_0) &= 2.32104973253042114734283739983633918849213061106173 \\
E = K/2 &= 1.16052486626521057367141869991816959424606530553086 \\
K' = K(\cos\theta_0) &= 1.64669144431946837372958069030713103423036178930922 \\
E' = E(\cos\theta_0) &= 1.50010688965199892576311071782207995131063998866470 \\
X, Y = \pi(k' \pm ik)/K &= 0.564412701731271 \pm 1.230228033100522\, i \;,
\end{aligned}
$$

$$
q = \exp\left[2 \int_X^Y \int_X^Y \log\left| \frac{x-y}{x+y} \right| \varphi(x)\varphi(y)\, dxdy \right] = \ldots = \exp(-\pi K'/K);
$$

$$
\boxed{q = 1/9.28902549192081891875544943595174506103169486775012}
$$

[2]The reference is J. Nuttall, Location of poles of Padé approximants to entire functions, pp. 354-363 *in Rational Approximation and Interpolation, Proceedings, Tampa, Florida, 1983* (P.R. Graves-Morris, E.B. Saff and R.S. Varga, editors), *Springer Lecture Notes Math.* **1105**, Springer, Berlin, 1984. [added on September 2000 retyping, A.P.M.]