On optimal Padé-type cuts. *

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One looks for rules of determination of denominators of rational approximations to analytic functions (Padé-type approximations). Functions with branch points are analytic outside systems of cuts which have an influence on the rate of convergence. The quest for the best system of cuts is discussed.

1 Convergence regions of transformations of Taylor series.

Let f be an analytic function in a known domain (i.e., a connected open subset of the extended complex plane) D, known by its Taylor coefficients about a point $z_0 \in D$:

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k. \tag{1}$$

Of course, if we only know that f is analytic in D, we can only ensure that this series converges in the part of D which is the largest disk $\{z : |z - z_0| < r_0\}$ contained in D. Moreover, if z is in this disk, the partial sums of increasing degrees of (1) converge to f with a guaranteed rate of convergence $|(z - z_0)/r_0|$. Some functions will exhibit a better rate of convergence, but there are functions analytic in D which will have this rate of convergence.

The problem is to be able to approximate f in the whole of D, and to achieve the best rate of convergence, which means: if R_n is the approximation using n coefficients of (1), we want

$$\limsup_{n\to\infty} |f(z) - R_n(z)|^{1/n}$$

to be as small as possible. Actually, we consider the worst case with respect to functions analytic in D, so to try to minimize

$$\sup_{f\in A(D)} \limsup_{n\to\infty} |f(z) - R_n(z)|^{1/n},$$

*Dedicated to T. Rivlin on the occcasion of his 70^{th} anniversary and to J. Meinguet on the occcasion of his 65^{th} anniversary on the way of constructing R_n, i.e., on the mapping $f \longrightarrow R_n$, which shows that we are close to be dealing with a problem of *optimal recovery* [25, 32], or of determination of *n-width* [49], but the present study will not try to formalize this approach, it will only give progressive hints. Moreover, what is usually considered to be the most stable base of these problems, the domain D itself, will be questionned (in § 3).

An interesting example of convergence enhancement of series is the Euler transformation [22, 35, 42, 46] with a parameter. For a power series of the form (1), the Euler transformation amounts to

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k = \sum_{k=0}^{\infty} d_k \left(\frac{z - z_0}{z - p} \right)^k , \tag{2}$$

with p outside D. Each d_k is a linear combination of $c_0, \ldots, c_k$. Remark also that the partial sum of the $n+1$ terms of the last series of (2) is a *rational approximation* of degree n to f, with denominator $Q_n(z) = (z-p)^n$, showing a first example of Padé-type approximation (see [12, 14, 20] for more on the connections between series transformations and rational approximation). The new series has a rate of convergence

$$|\rho(z)| = \text{constant } \times |(z - z_0)/(z - p)|$$

increasing from $z = z_0$ up to a circle touching the boundary of D. Figure 1 shows a "typical" (but is it so typical? see § 3) elliptic-like domain D, examples of convergence domains of (1) and (2) for two values of p, together with level lines of rate of convergence $= 1/4, 1/2$ and $3/4$. These level lines are Apollonius circles related to z_0 and p [46, p. PA-32]. A discussion of what should be the best choice of p is made in [45] for Stieltjes functions.

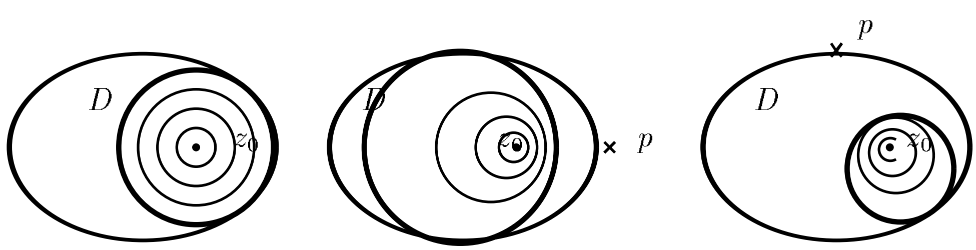

Figure 1: Examples of convergence domains of Euler transformations of Taylor series.

Much better results can be reached if a new parameter is introduced in each term:

$$f(z) = \sum_{k=0}^{\infty} e_k \frac{(z - z_0)^k}{(z - p_1)(z - p_2) \ldots (z - p_k)} . \tag{3}$$

For a first analysis, let us work with a finite number of r distinct p's, i.e., $p_k = p_{k-r}, k > r$, outside D. The series (3) is then basically a combination of Taylor series

$$\sum_{m=0}^{\infty} e_{rm+s} \left(\frac{(z-z_0)^r}{(z-p_1)\dots(z-p_r)} \right)^m ,$$

$s = 0, \dots, r-1$, of the variable $(z-z_0)^r/((z-p_1)\dots(z-p_r))$, therefore convergent in the domain $D_r = \{z : |\rho(z)| < 1\}$, with

$$\rho(z) = \frac{(z-z_0)/[(z-p_1)\dots(z-p_r)]^{1/r}}{\sup_{w\in D} |w-z_0|/|(w-p_1)\dots(w-p_r)|^{1/r}}. \tag{4}$$

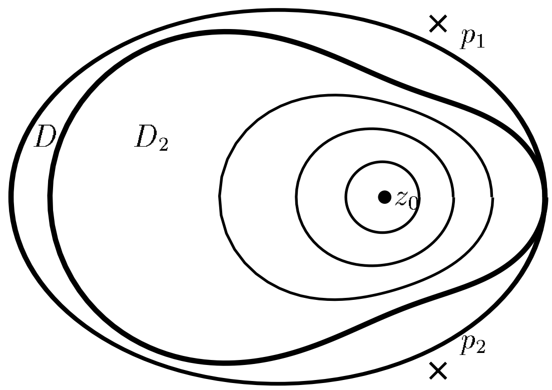

Figure 2: Convergence domain of (3) with 2 different points p_1 and p_2.

To the largest domain of convergence corresponds the best rate of convergence. This is a special case of a deep result of complex approximation theory, see for instance [48, § 5.1].

For a proof using (4), suppose $D_s \subseteq D_r (\subseteq D)$, then $|\rho_s| = 1$ on the boundary ∂D_s of D_s (by definition of D_s), $|\rho_r| \leqslant 1$ on ∂D_s (as $|\rho_r| \leqslant 1$ in the whole of D_r containing D_s), and ρ_r/ρ_s is regular[1] in D_s (no singularity at z_0), has a modulus $\leqslant 1$ on the boundary of D_s, therefore $|\rho_r(z)/\rho_s(z)| \leqslant 1$ for any $z \in D_s$, by the maximum principle.

Remark that such discussions usually are found about approximations by polynomials on compact sets [23, 48]. However, they also apply to our rational approximations constructions, thanks to *Walsh duality* [48, § 8.4]: we are in the conditions of [48, § 8.4, Example IIa, IIb] (with $z_0 = \infty$, but, with Walsh's $F(t) = f(z_0 + t^{-1})$ approximated by the rational function $S_n : F(t) - S_n(t) = O(|t|^{-n-1}), n \to \infty$, it is

[1]of course, one chooses a single determination for the root in (4).

clear that $R_n(z) = S_n((z-z_0)^{-1})$ is still a rational function of degree n satisfying now $f(z) - R_n(z) = O(|z-z_0|^{n+1})$ when $z \to z_0$).

One can fill arbitrarily closely the whole domain D by a domain D_r if r is large enough: D_r is the interior of a *lemniscate*, which can indeed approximate the boundary ∂D under mild conditions [48, § 4.2]. Indeed, let us write $(D-z_0)^{-1}$ the (unbounded) domain $\{t : z_0 + t^{-1} \in D\}$. Then, if the boundary of $(D-z_0)^{-1}$ is made of a finite number of mutually exterior Jordan curves and arcs, one can construct a polynomial $\Pi_r(t) = (t-\tau_1)\dots(t-\tau_r)$ such that $(D-z_0)^{-1}$ is close to a set $\{t : |\Pi_r(t)| > \text{ constant }\}$. Moreover, $|\Pi_r(t)|^{1/r}$ is then close to constant $\times \exp(G_0(t))$ in $(D-z_0)^{-1}$, where G_0 is the harmonic function (Green function) in $(D-z_0)^{-1}$ such that $G_0(t) - \log|t|$ remains bounded in a neighbourhood of $t = \infty$, and $G_0(t) \to 0$ when $t \to$ the boundary of $(D-z_0)^{-1}$. Acceptable polynomials may be constructed through conformal mapping data, Fekete points, Fejér points, Leja points, Faber polynomials and points [13, 21, 23, 30, 48] associated to the complement of $(D-z_0)^{-1}$.

Returning to D and the $z = z_0 + t^{-1}$ variable, we find that D can indeed be approximated by a set $D_r = \{z : |\prod_{j=1}^{r}((z-z_0)^{-1} - \tau_j)| > \text{ constant}\}$, i.e., a set defined by $|\rho(z)| <$ constant with the $\rho(z)$ of (4) and $p_j = z_0 + 1/\tau_j, j = 1, \dots, r$. The rate of convergence in D is arbitrarily close to

$$\rho_{\text{opt}}(z) = \exp(-G(z; z_0)), \qquad z \in D, \tag{5}$$

where $G(z; z_0)$ is the Green function of D with singularity at z_0.

We arrived at (5) as the ultimate performance of series (3) still constructible with the data of (1). The partial sum of degree n of (3) is a rational function N_n/P_n of degree n with preassigned denominator $P_n(z) = (z-p_1)\dots(z-p_n)$ and a numerator N_n constructed such that $f(z) - N_n(z)/P_n(z) = O(|z-z_0|^{n+1})$. We just essentially discussed the construction of optimal *Padé-type rational approximations.*

2 Padé-type rational approximation.

Whenever we are able to represent the coefficients of (1) as results of applying a linear functional $\mathscr{L}$ to successive powers:

$$c_k = \mathscr{L}(t^k), \qquad k = 0, 1, \dots, \tag{6}$$

so that $\mathscr{L}(p)$ is known for any polynomial p, we may approximate $f(z) = \mathscr{L}\left(\dfrac{1}{1-t(z-z_0)}\right)$ by applying $\mathscr{L}$ to a polynomial interpolant of $(1-t(z-z_0))^{-1}$:

$$R_n(z) = \mathscr{L}\left(\text{ interpolant of } \frac{1}{1-t(z-z_0)} \text{ at } t = t_0, t_1, \dots, t_n.\right). \tag{7}$$

We so get an "integration formula" at preassigned points, which is called Padé-type approximation, emphasising the similarity with the representation of Padé approximation as formal Gaussian quadrature ([3, 9] [10, p.34]). The result is a combination of $(1 - t_0(z-z_0))^{-1}, \dots, (1-t_n(z-z_0))^{-1}$, i.e., a rational function of z.

Let $t_0 = 0$, then, $R_n(z) = N_n(z)/P_n(z)$ with N_n and $P_n(z) =$ constant $\times (1 - t_1(z -$

$z_0))\dots(1-t_n(z-z_0))$ of degrees $\leqslant n$. The approximation error is

$$\begin{aligned} f(z)-R_n(z) &= \mathscr{L}\left(\frac{1}{1-t(z-z_0)}\frac{t(t-t_1)\dots(t-t_n)}{\dfrac{1}{z-z_0}\left(\dfrac{1}{z-z_0}-t_1\right)\dots\left(\dfrac{1}{z-z_0}-t_n\right)}\right) \\ &= (z-z_0)^{n+1}\mathscr{L}\left(\frac{t(t-t_1)\dots(t-t_n)}{(1-t(z-z_0))(1-t_1(z-z_0))\dots(1-t_n(z-z_0))}\right) \\ &= O(|z-z_0|^{n+1}), \end{aligned} \tag{8}$$

showing that the Padé-type approximation of degree n to f with preassigned denominator P_n is N_n/P_n with N_n of degree $\leqslant n$ such that $P_n(z)f(z)-N_n(z)=O(|z-z_0|^{n+1})$.

We now see that the partial sums R_n of $n+1$ terms of the various series transformations discussed in section 1 are indeed Padé-type approximations of degree n to f: these partial sums are indeed rational functions (of denominator $P_n(z)\equiv 1$ in (1); $P_n(z)=(z-p)^n$ in (2); $P_n(z)=(z-p_1)\dots(z-p_n)$ in (3)), and satisfy $f(z)-R_n(z)=O(|z-z_0|^{n+1})$ when $z\to z_0$.

The determination of a good denominator P_n in terms of D and z_0 is achieved through a contour integral representation of (6):

$$\mathscr{L}(P)=\frac{1}{2\pi i}\int_{C_0}P(t)\,f(z_0+t^{-1})\,t^{-1}dt \quad \text{for any polynomial } P \tag{9}$$

where C_0 is a contour $\subset \{t : z_0+t^{-1}\in D\}$, let us write this as $C_0\subset(D-z_0)^{-1}$. With $P_n(z)=$ constant $\times\prod_{k=1}^{n}(1-t_k(z-z_0))$, (8) becomes

$$f(z)-R_n(z)=\frac{1}{2\pi i}\frac{(z-z_0)^{n+1}}{P_n(z)}\int_{C_0}\frac{t^{n+1}P_n(z_0+t^{-1})}{1-t(z-z_0)}f(z_0+t^{-1})t^{-1}dt. \tag{10}$$

We now see how to choose good denominators P_n: when D, z_0, and z are given, we take C_0 close or equal to the boundary of $(D-z_0)^{-1}$, and P_n such that $\max_{t\in C_0}|t^nP_n(z_0+t^{-1})|$ is as small as possible with respect to $|P_n(z)/(z-z_0)^n|$.

This is basically achieved by Chebyshev polynomials (least norm polynomials) on C_0: if Π_n is a monic polynomial of least norm, let P_n be such that $t^nP_n(z_0+t^{-1})=\Pi_n(t)$, i.e., $P_n(z)=(z-z_0)^n\Pi_n((z-z_0)^{-1})$. We know ([23, 26, 48] etc.) that $|\Pi_n(t)|^{1/n}\to$ constant $\exp G_0(t)$ when $t\in(D-z_0)^{-1}$.

Returning to the initial geometry, we get

$$\lim_{n\to\infty}\frac{|z-z_0|}{|P_n(z)|^{1/n}}=\text{constant }\times\exp(-G_0((z-z_0)^{-1})=\text{constant }\times\exp(-G(z;z_0)),$$

and we recover exactly (5)!

From (8),

$$\limsup_{n\to\infty}|f(z)-R_n(z)|^{1/n}\leqslant\exp(-G(z;z_0)), \tag{11}$$

uniformly on z in compact sets of D.

Example 1
For instance, with $f(z) = z^{-1}\,Log\,(1+z) = \sum_0^\infty (-1)^k z^k/(k+1)$, $z_0 = 0$, $D = \mathbb{C} \setminus (-\infty, -1]$, $D^{-1} = \mathbb{C} \setminus [-1, 0]$, *we may immediately choose* $C_0 = [-1, 0]$, *as we have the obvious representation* $c_k = (-1)^k/(k+1) = \mathscr{L}(t^k) = \int_{-1}^0 t^k\,dt$. *Least norm polynomials on* C_0 *are of course the historical Chebyshev polynomials [18, 37, 39]* $\Pi_n(t) =$ *constant* $T_n(2t+1)$, *so,* $P_n(z) =$ *constant* $z^n T_n(2/z+1)$. *The preassigned poles of the approximant of degree n are therefore* $2/(\cos((k+1/2)\pi/n)-1)$, $k = 0, \ldots, n-1$. *Successful numerical tests have been reported by C. Brezinski [8].*

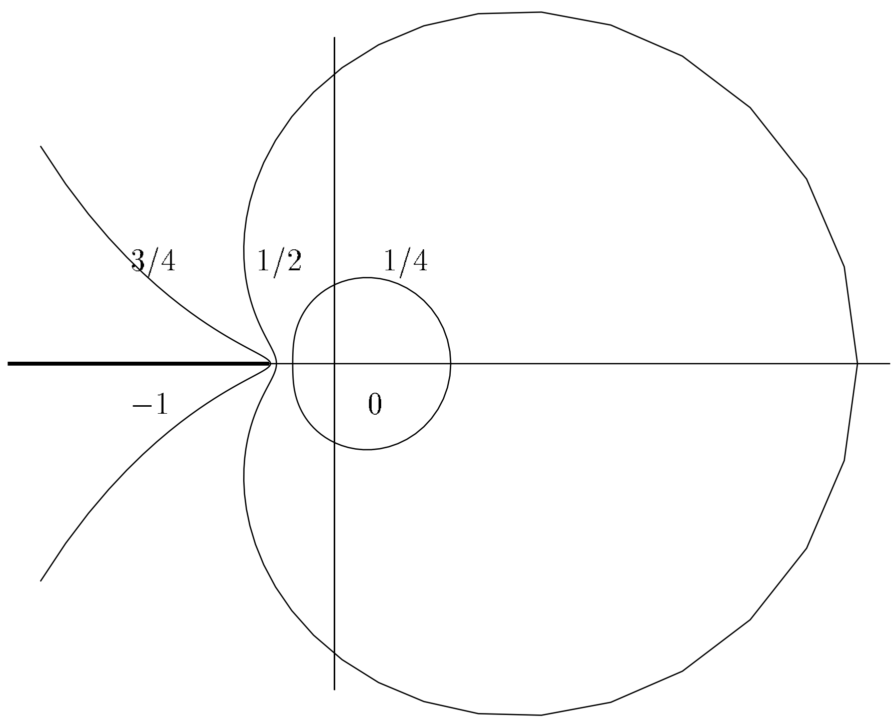

Figure 3: Convergence rates of approximants to z^{-1} Log $(1+z)$, $z_0 = 0$, using Chebyshev denominators.

The corresponding rate of convergence is

$$|\rho(z)| = \lim_{n\to\infty} \frac{|z|}{|P_n(z)|^{1/n}} = \lim_{n\to\infty} \frac{1}{|T_n(2/z+1)|^{1/n}} = \frac{|z|}{|z+2+2(z+1)^{1/2}|},$$

where the square root is such that $|\rho(z)| < 1$ *in* $D = \mathbb{C} \setminus (-\infty, 0]$. *One may also check that the Green function is* $G(z;0) = \log(|z+2+2(z+1)^{1/2}|/|z|)$. *Also, as* D *is simply connected,* $G(z;0)$ *is the real part of the logarithm of* $\Phi(z;0) = (z+2+2(z+1)^{1/2})/z$ *which maps conformally* D *on the exterior of the unit disk, with* $z_0 = 0$ *mapped on* ∞.

The level lines $|\rho| = 1/4, 1/2$, *and* $3/4$ *are indicated in Fig. 3. These lines are the boundaries of larger and larger domains tending to fill* $D = \mathbb{C} \setminus (-\infty, -1]$. *These lines meet the real axis at* $-16/25$ *and 16/9 when* $|\rho| = 1/4$*; at* $-8/9$ *and 8 when* $|\rho| = 1/2$*; and at* $-48/49$ *and 48 when* $|\rho| = 3/4$*!*

3 From Padé-type to Padé.

The preceding sections showed how to construct rational approximants N_n/P_n to a function f whose Taylor series (1) is known, taking into account a known domain D where f is presumed to be analytic (the consequences of a wrong assumption about D are not discussed here, let me just say that convergence still holds in a subset of D defined by a level line of $G(z; z_0)$).

The approximant is determined by its preassigned denominator P_n and the property $f(z) - N_n(z)/P_n(z) = O(|z - z_0|^{n+1})$ when $z \to z_0$, i.e., N_n is the truncated Taylor expansion of degree n of the product fP_n.

Such approximants are called Padé-type approximants by C. Brezinski [9] (and Padé-like approximants by other authors [5]).

Interpolatory Padé-type approximation, in particular two-point Padé-type approximation has also been investigated [15, 19], as well as approximations based on other than Taylor expansions [12, 31].

On the other hand, Padé approximation is characterized by the removal of any reference to any possible domain of analyticity D, the denominator P_n as well as the numerator N_n are constructed with the coefficients of (1) in such a way to achieve an error of the highest possible order near z_0, usually $O(|z - z_0|^{2n+1})$. This does not mean that the Padé approximant is better than other ones, nor even that it is good! It simply happens that the Padé construction summarizes parts of valuable works on continued fractions and approximations (by Gauss, Hermite, Stieltjes, Markov, etc., see [11]), but does not contain itself any hint of convergence.

However, as Padé approximation calculations do not require any knowledge on the domain D, they soon became used for exploring functions only known from the coefficients of (1). In particular, sequences of Padé approximants to functions with branchpoints have zeros and poles (often) concentrating on a beautiful net of lines [6, 36, 43], [24, pp. 283–291] joining these branchpoints in some way (the usual principal value cuts [27] for most elementary functions), and convergence occurs (in a weak sense) in the domain $D_{\text{Padé}}$ bounded by these lines (*Padé cuts*). The domain $D_{\text{Padé}}$ associated to f is such that $(D_{\text{Padé}} - z_0)^{-1}$ has a boundary of minimal capacity (see [36, 43, 44] for a survey and main results; least capacity property is already discussed in [4, p.192] with credit to J.

Nuttall). In fortunate cases, the rate of convergence is

$$|\rho_{\text{Padé}}(z)| = \exp(-2G_{\text{Padé}}(z; z_0)), \qquad (12)$$

but has only been proved in general in a weak sense, in capacity: if $\varepsilon(z)$ is the error of the n^{th} degree approximant, the subset of $D_{\text{Padé}}$ where $|\varepsilon(z)|^{1/n} > |\rho_{\text{Padé}}(z)|$ has capacity $\to 0$ when $n \to \infty$.

Recently, a similar discrepancy between proved knowledge and expected performance has been partially solved [38]: best rational approximation error norms ε_n on a compact $K \subset D$ were known to satisfy $\limsup_{n\to\infty} \varepsilon_n^{1/n} \leqslant \rho^{1/2}$ (with a ρ similar to the right-hand side of (12), but with $G(z; z_0)$ replaced by an appropriate harmonic measure). The result can not be improved, as, given D and K, there are functions for wich the lim sup is actually this $\rho^{1/2}$. However, most of of the ε_n's are expected to be much smaller than $\rho^{n/2}$, and to behave essentially like ρ^n. Stahl [44, eq. (7.7) p. 630] gives classes of functions for which the limit of $\varepsilon_n^{1/n}$ is indeed ρ, but Prokhorov [38] succeeded to prove that one has always $\limsup_{n\to\infty}(\varepsilon_1 \dots \varepsilon_n)^{2/n^2} \leqslant \rho$, a deep and clever new way to look at error behaviour. One may wonder if a similar enhancement will be achieved in Padé convergence theory ...

4 Optimal Padé-type cuts for functions with branch points.

Variation of Padé-type cuts would never have been imagined if experiments would not have been performed with Padé approximants of functions with branch points. Padé-type approximation would not even bear this name, it would still be called "rational approximation with preassigned poles", as Walsh [48] calls it.

The need to move cuts appeared when one encountered Padé approximants with lines of zeros and poles dangerously near the region of interest, or even exactly upon this region [16, 28]!

One may then perform a nonlinear change of variable in the series (1) [28, 29], or even choose a new z_0 if we can afford it [16], [24, p.291].

Padé-type approximation allows to *create* a system of cuts bounding a domain D where convergence can be controlled [5, 6, 29]. In order to have the best possible performance at a fixed point z, we see from (11) that we have to arrange the cuts so that $G(z; z_0)$ is maximized. Only necessary conditions will be examined here:

Proposition .1
*Let the domain D be bounded by one or several smooth arcs (*cuts*) C, and z and z_0 be two fixed points in D. If the cuts are such that the value of the Green function $G(z; z_0)$ is extremal, then the product of the exterior normal derivatives takes equal values on the two sides of C:*

$$\frac{\partial G(\eta; z)}{\partial n_1}\frac{\partial G(\eta; z_0)}{\partial n_1} = \frac{\partial G(\eta; z)}{\partial n_2}\frac{\partial G(\eta; z_0)}{\partial n_2}, \qquad \forall \eta \in C. \qquad (13)$$

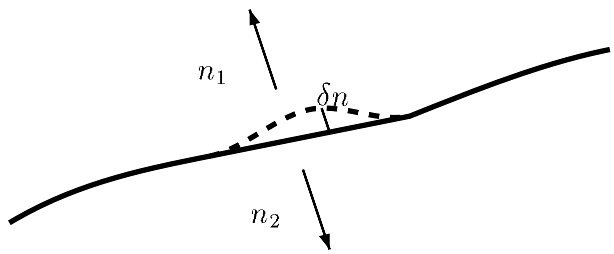

Figure 4: Variation of cut.

Proof
Indeed, from Hadamard's variation formula [7, p.126], [34, Chap. 1, § 11], [40], the variation of $G(z;z_0)$ is

$$\delta G(z;z_0) = \frac{1}{2\pi}\int_\Gamma \frac{\partial G(\eta;z)}{\partial n}\frac{\partial G(\eta;z_0)}{\partial n}\delta n(s)ds + o(\|\delta n\|),$$

where Γ is the limit of a contour including C, and where δn is the variation of C along the exterior normal direction. This means that an integral on Γ is an integral on C performed twice, once on each side of C. As δn takes opposite values on the two sides of C (Fig. 4), (13) follows. □

Remark that if $z \to z_0$, (13) is compatible with Stahl's *symmetry characterization* [43, 44] of Padé cuts. This is not surprising, as Padé approximation is a limit of best rational approximation on smaller and smaller neighbourhoods of z_0 [47].

Example 2
Let C be the arc of circle of endpoints $\pm i$ and midpoint $-\tan\alpha$, with $-\pi/2 < \alpha < \pi/2$ (Fig. 5). Let us show that this C is optimal for a problem with z and z_0 real.

First, we find $G(\eta;\xi)$ for real ξ through conformal mapping: $G(\eta;\xi)$ is the real part of $\log\Phi(\eta;\xi)$ where $\Phi(\eta;\xi)$ maps the exterior of C on the exterior of the unit disk, with a pole at $\eta=\xi$. As $\exp(-2i\alpha)(\eta+i)/(\eta-i)$ maps C on the exterior of the negative real axis,

$$\gamma\frac{A+\exp(-i\alpha)\sqrt{(\eta+i)/(\eta-i)}}{\overline{A}-\exp(-i\alpha)\sqrt{(\eta+i)/(\eta-i)}}$$

maps the exterior of C to the exterior of the unit disk, provided $|\gamma|=1$ and Re $A>0$. In order to have a pole at $\eta=\xi$, we find A from $\overline{A}=\exp(-i\alpha)\sqrt{(\xi+i)/(\xi-i)}$, i.e., $A=\exp(i\alpha)\sqrt{(\xi-i)/(\xi+i)}$, as ξ is real. More calculations yield

$$\Phi(\eta;\xi) = \frac{(\eta\cos\alpha+\sin\alpha)\sqrt{\xi^2+1}+(\xi\cos\alpha+\sin\alpha)\sqrt{\eta^2+1}}{\eta-\xi}. \quad (14)$$

The values on the two sides of the cut C correspond to opposite choices of the square root of η^2+1 in (14). A convenient change of variable is $\eta = -\tan\zeta$ with ζ in the infinite vertical strip $\alpha <$ Re $\zeta < \alpha+\pi$. Indeed, as we know that $\exp(-2i\alpha)(i+\eta)/(i-\eta)$ maps the exterior of C to the exterior of the positive real axis, this gives with $\eta = -\tan\zeta$, $\exp(2i(\zeta-\alpha))$ not real positive, meaning $0 <$ Re $(\zeta-\alpha) < \pi$, with the two sides of C mapped on Re $\zeta = \alpha$ and Re $\zeta = \alpha+\pi$. We now have

$$\Phi(\eta;\xi) = -\frac{\sin(\alpha+(\delta-\zeta)/2)}{\sin((\delta+\zeta)/2)},$$

where δ is defined by $\xi = \tan\delta$.

We now come to the normal derivatives of $G(\eta;\xi)$ along the two sides of C. As $G=0$ on C, $\partial G/\partial n$ is the norm of the gradient of $G =$ Re $\log\Phi$, so $\partial G/\partial n = |\Phi'/\Phi| = |\Phi'|$ on C. We find $\Phi' = (d\Phi/d\zeta)(d\zeta/d\eta) = -\sin(\alpha+\delta)/[2(1+\eta^2)\sin^2((\delta+\zeta)/2)]$. Let $z=\tan\beta$, $z_0 = \tan\beta_0$ (β and β_0 are the angles between the imaginary axis and the lines joining i to z and z_0 in Fig. 5). In order to check the equality of $|\Phi'(\eta;z)\Phi'(\eta;z_0)|$ on the two sides of C, we only have to look at $|\sin((\beta+\zeta)/2)\sin((\beta_0+\zeta/2)| = |\cos((\beta-\beta_0)/2)-\cos(\zeta+(\beta+\beta_0)/2)|/2$ on Re $\zeta=\alpha$ and Re $\zeta = \alpha+\pi$. This amounts to $\cos(\zeta+(\beta+\beta_0)/2)$ being pure imaginary on Re $\zeta=\alpha$, finally to $\alpha+(\beta+\beta_0)/2 = \pi/2$.

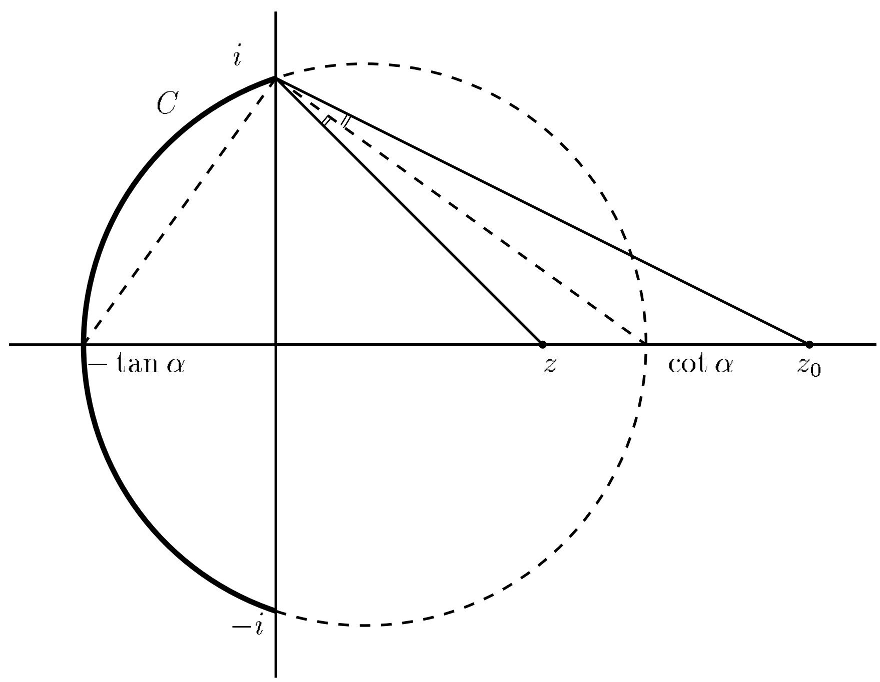

Figure 5: Circular optimal cut.

The cut is therefore a part of the circle with diameter $[-\tan\alpha, \cot\alpha]$*, the line joining* i *to* $\cot\alpha$ *bisecting the angle made by the lines joining* i *to* z *and* z_0 *(Fig. 5).*

In particular, if $z = z_0$*,* z *is on the circle containing the Padé cut [4, p. 123]. If* $z_0 = \infty$*,* z *is the center of the circle.*

We now come to an alternate description of optimal cuts, through *Schiffer's (interior) variation formula*, adapting [2, chap. 7], [7, chap.8, § 3], [40, pp.298–305] , [41]:

Let D be bounded by a system of cuts C joining points $b_1, \ldots, b_m$ (the branch points) in some way. We shall consider a family of domains $\widetilde{D}$ close to D and examine $\widetilde{G}(z; z_0) - G(z; z_0)$ for each $\widetilde{D}$.

For each interior point $\xi \in D$, let us consider

$$\Psi(z;\xi) = z + \varepsilon e^{i\alpha} \frac{B(z)}{(z-\xi)^m}, \tag{15}$$

where $B(z) = (z-b_1)\ldots(z-b_m)$. Let $D_\eta(\xi)$ be the disk of center ξ and radius $\eta > 0$. We suppose that η is small enough so that $D_\eta(\xi)$ does not intersect C. Therefore η depends on ξ (and ε will depend on ξ too). Let us look at the image of $D \setminus D_\eta(\xi)$ under (15): let $K_1 = \max_{|z-\xi|=\eta} |B(z)|$, as $B(z)/(z-\xi)^m$ is a polynomial in $(z-\xi)^{-1}$, $|B(z)/(z-\xi)^m|$ is bounded by K_1/η^m when $|z-\xi| > \eta$ (maximum principle), so that $\Psi(z;\xi)$ is close to z when $|z-\xi| > \eta$ provided $\varepsilon > 0$ is small with respect to $\eta^m : \varepsilon = o(\eta^m)$. We *define*

$$\widetilde{D} = \{\Psi(z) : z \in D \setminus D_\eta(\xi)\} \bigcup D_{K_1\varepsilon/\eta^m}(\xi).$$

Remark that $\widetilde{D}$ is still bounded by a system of cuts $\widetilde{C}$ joining the same points $b_1, \ldots, b_m$. Later on, it will be useful to be sure of a one-one correspondence between z and $\Psi(z)$ when $z \in D \setminus D_\eta(\xi)$: the equation $\Psi(z') = \Psi(z)$ has then only one solution $z' = z$ in $z' \in D \setminus D_\eta(\xi)$: should $z' \neq z$ be another solution, one should have $|[(z'-\xi)^{-m}B(z') - (z-\xi)^{-m}B(z)]/(z'-z)| = 1/\varepsilon$, impossible if $|d[(z-\xi)^{-m}B(z)]/dz| < 1/\varepsilon$ for all $|z-\xi| > \eta$, requiring simply a stronger condition of the form $\varepsilon < K_2\eta^{m+1}$ for ε.

We now look at $\widetilde{G}(z; z_0) - G(z; z_0) = \widetilde{G}(\Psi(z;\xi); \Psi(z_0;\xi)) - G(z; z_0) + \widetilde{G}(z; z_0) - \widetilde{G}(\Psi(z;\xi); \Psi(z_0;\xi))$.

First, $\widetilde{G}(\Psi(z;\xi); \Psi(z_0;\xi)) - G(z; z_0)$ is harmonic in $D \setminus D_\eta(\xi)$ (without singularity at z_0, remember that $\Psi(z;\xi) - \Psi(z_0;\xi)$ vanishes only at $z = z_0$) and vanishes on C. From Green function identities ([2, p.100])

$$\begin{aligned}
&\widetilde{G}(\Psi(z;\xi); \Psi(z_0;\xi)) - G(z; z_0) \\
&= -\frac{1}{2\pi}\int_{\partial D_\eta(\xi)} \left[\widetilde{G}(\Psi(t;\xi); \Psi(z_0;\xi))\frac{\partial G(t;z)}{\partial n_t} - G(t;z)\frac{\partial \widetilde{G}(\Psi(t;\xi); \Psi(z_0;\xi))}{\partial n_t}\right] |dt| \\
&= \frac{1}{\pi}\,\mathrm{Im}\int_{\partial D_\eta(\xi)} \left[\widetilde{G}(\Psi(t;\xi); \Psi(z_0;\xi))\Gamma(t;z) - G(t;z)\widetilde{\Gamma}(\Psi(t;\xi); \Psi(z_0;\xi))\frac{d\Psi(t;\xi)}{dt}\right] dt,
\end{aligned}$$

where $\Gamma(t;z) = \partial G(t;z)/\partial t$ is the analytic function $\Phi'(t;z)/(2\Phi(t;z))$, using $G(t;z) =$

Re $\log\Phi(t;z)$. Expanding to first order in ε, one finds

$$\widetilde{G}(\Psi(z;\xi);\Psi(z_0;\xi)) - G(z;z_0) =$$
$$2 \text{ Re residue at } t=\xi \text{ of } \left[\varepsilon e^{i\alpha}\Gamma(t;z_0)\Gamma(t;z)\frac{B(t)}{(t-\xi)^m}\right] + o(\varepsilon).$$

Next, a direct first order expansion yields

$$\widetilde{G}(z;z_0) - \widetilde{G}(\Psi(z;\xi);\Psi(z_0;\xi)) =$$
$$= -\varepsilon \text{ Re } e^{i\alpha}\left[\Gamma(z;z_0)\frac{B(z)}{(z-\xi)^m} + \Gamma(z_0;z)\frac{B(z_0)}{(z_0-\xi)^m}\right] + o(\varepsilon).$$

The Schiffer variation formula needed here is therefore

$$\widetilde{G}(z;z_0) - G(z;z_0) =$$
$$=\varepsilon \text{ Re } e^{i\alpha}\left[\frac{2}{(m-1)!}\frac{d^{m-1}}{d\xi^{m-1}}\left[\Gamma(\xi;z_0)\Gamma(\xi;z)B(\xi)\right] - \Gamma(z;z_0)\frac{B(z)}{(z-\xi)^m} - \Gamma(z_0;z)\frac{B(z_0)}{(z_0-\xi)^m}\right] + o(\varepsilon).$$

A necessary condition for the Green function at z to be stationary with respect to variation of the boundary cuts is therefore

$$\Gamma(\xi;z)\Gamma(\xi;z_0)B(\xi) = Q_1(\xi) - \frac{\Gamma(z;z_0)B(z)}{2(z-\xi)} - \frac{\Gamma(z_0;z)B(z_0)}{2(z_0-\xi)}, \tag{16}$$

where Q_1 is a polynomial of degree $\leqslant m-2$. The final characterization is

Proposition .2
For a domain D bounded by a system C of cuts joining fixed points $b_1,\dots,b_m$ to be optimal with respect to Padé-type approximation at a fixed point $z \in D$, it is necessary that the function $\Phi(\xi;z_0)$ mapping D on the exterior of the unit disk (or on several copies of the exterior of the unit disk if D is multiply connected [17, p.183-187] [26, p.277]) satisfies a quadratic differential equation

$$\frac{(d\Phi(\xi;z_0))^2}{\Phi(\xi;z_0)[1-\overline{\Phi(z;z_0)}\Phi(\xi;z_0)][\Phi(\xi;z_0)-\Phi(z;z_0)]} = \frac{Q(\xi)(d\xi)^2}{(z-\xi)(z_0-\xi)B(\xi)}, \tag{17}$$

where $B(\xi) = (\xi-b_1)\cdots(\xi-b_m)$, Q is a polynomial of degree $\leqslant m-2$.

Proof
Indeed, from (16),

$$\Gamma(\xi;z)\Gamma(\xi;z_0)B(\xi) = \frac{Q_2(\xi)}{(z-\xi)(z_0-\xi)},$$

where Q_2 is a polynomial of degree $\leqslant m$. We write everything in terms of $\Phi(\xi;z_0)$: $\Gamma(\xi;z_0) = \Phi'(\xi;z_0)/(2\Phi(\xi;z_0))$, $\Phi(\xi;z)$ maps D on $|\Phi|>1$ with a pole at $\xi=z$: $\Phi(\xi;z) = (1-\overline{\Phi(z;z_0)}\Phi(\xi;z_0))/(\Phi(\xi;z_0)-\Phi(z;z_0)$ ([2, p.103] [40, p.301] [41]). So, $2\Gamma(\xi;z) = [-\overline{\Phi(z;z_0)}/(1-\overline{\Phi(z;z_0)}\Phi(\xi;z_0)) - 1/(\Phi(\xi;z_0)-\Phi(z;z_0))]\Phi'(\xi;z_0)$, and (16) follows.

Actually, as ∞ is normally an ordinary interior point of D, any determination of Φ behaves like $A + B/\xi + \cdots$, when ξ is large, with $A \neq 0$, so that Γ behaves there like constant $/\xi^2$, and the degree of Q in (16) is not larger than $m - 2$. □

Quadratic differentials are indeed met in Padé cuts descriptions [36, 43]. The discussion of the solutions of (17) promises to show rich structures [33] (elliptic functions of hyperelliptic integrals). The polynomial Q must probably be such that $\log \Phi$ has only pure imaginary periods (as in [36, p.186, eq. 3]).

It is not yet clear if (17) actually gives a maximum value for $G(z; z_0) = \text{ Re } \log \Phi(z; z_0)$, nor if there are several local maxima.

When $|\Phi| = 1$, the left-hand side of (17) is a negative real number, so that the cut C correspond to

$$\frac{Q(\xi)(d\xi)^2}{(z-\xi)(z_0-\xi)B(\xi)} < 0.$$

For instance, as in the example of section 2, let $m = 2$, $z_0 = 0$ and $B(\xi) = \xi + 1$ (the branchpoints of the function f are -1 and ∞). One has

$$[(z-\xi)\xi(\xi+1)]^{-1/2} d\xi = e^{-i\alpha/2} dt$$

on C, where t is real, and where α is the still unknown phase of the constant Q. A parametric representation of the cut C is therefore $\xi(t) = \mathscr{E}(e^{-i\alpha/2}t + \text{ constant })$, where $\mathscr{E}$ is some (Weierstrass-like [1]) elliptic function. As C must join the known branch points -1 and ∞, the line $e^{-i\alpha/2}t + \text{ constant }$, t real, must join the solutions of $\mathscr{E}(\tau_1) = -1$ and $\mathscr{E}(\tau_2) = \infty$. As $\mathscr{E}$ is doubly periodic, τ_1 and τ_2 are determined up to integer combinations of the periods. This gives quite a lot (a countable infinity) of possible cuts, which are believed to be optimal with respect to investigation of f in various Riemann sheets (see [24, p.256, p.291]). The subject obviously deserves more developments, which will perhaps be achieved in the future.

Conclusion.

It has first been shown that familiar transformations of Taylor series lead to special rational approximants called Padé-type approximants. Slightly more general transformations are discussed through the usual convergence radius theory and already allow to grasp the principles of construction of best Padé-type approximation to classes of analytic functions, in section 1.

An example of function with branchpoints is then given, with a denominator of best approximant related to Chebyshev polynomials (section 2).

Comparison with Padé approximation results leads to inquiry about the possibility to change domain boundaries (cuts) (section 3).

Elementary steps towards the description of "best cuts" are presented (section 4).

It is hoped that more research will lead to more solid theoretical knowledge, and that credible implementation will be tested on some benchmark.

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