Appeared in *Journal of Computational and Applied Mathematics* **57** (1–2) (1995) 215-237. Received 23 October 1992 (Évian proceedings).

With correction of a misprint p.227 (an α_n^2 should have been a a_n^2).

Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials.

Alphonse P. Magnus
Institut Mathématique, Université Catholique de Louvain
Chemin du Cyclotron 2
B-1348 Louvain-la-Neuve
Belgium
E-mail: magnus@anma.ucl.ac.be

Abstract.

Recurrence coefficients of semi-classical orthogonal polynomials (orthogonal polynomials related to a weight function w such that w'/w is a rational function) are shown to be solutions of non linear differential equations with respect to a well-chosen parameter, according to principles established by D.& G. Chudnovsky. Examples are given. For instance, the recurrence coefficients in $a_{n+1}p_{n+1}(x) = xp_n(x) - a_n p_{n-1}(x)$ of the orthogonal polynomials related to the weight $\exp(-x^4/4 - tx^2)$ on $\mathbb{R}$ satisfy $4a_n^3\ddot{a}_n = (3a_n^4 + 2ta_n^2 - n)(a_n^4 + 2ta_n^2 + n)$, and a_n^2 satisfies a Painlevé $\mathrm{P_{IV}}$ equation.

1. Introduction: measures and recurrence coefficients of orthogonal polynomials.

Let $\{p_n\}_0^\infty$ be the set of orthonormal polynomials related to some measure $d\mu$ on its support S:

$$\int_S p_n(x)p_m(x)\,d\mu(x) = \delta_{m,n}. \tag{1}$$

The most remarkable property of the p_n's is the recurrence relation joining them:

$$a_{n+1}p_{n+1}(x) = (x - b_n)p_n(x) - a_n p_{n-1}(x). \tag{2}$$

An often encountered problem in applied and numerical mathematics as well as in physics is then to relate the coefficients a_n and b_n of (2) to properties of the measure $d\mu$.

For instance, interesting solid-state Hamiltonian operators submitted to the so-called "recursion method" (or Lanczos method) show a tri-diagonal matrix representation. Investigation of spectral properties of the operator is then equivalent to investigating the measure of (1) from the recurrence coefficients of (2) [GaCL] [Hay] [HayN] [LaG] [LiMu] [OW].

The study of special partition functions in statistical physics and quantum physics leads to relations which can be translated as properties of particular recurrence coefficients. Much important work is currently done on this subject [Bes] [BIZ] [Fok1] [Fok2] [Fra1] [Fra2] [GrM1] [GrM2] [HH] [KvM] [LW] [Mo] [Y] [Zu] [91-93].

Numerical implementation of spectral methods and quadrature formulas needs accurate determination of recurrence coefficients for various measures. This appears in the survey [Gau] and in some recent papers as [Chin] and [ClS] (see also the references in [BeR]).

Quite a number of theoretical studies have appeared on this problem of relating properties of the orthogonality measure to the recurrence coefficients, especially to their asymptotic behaviour. See at least the books [Chi], [Fr1], [VA] and the survey papers [Lub] and [GFOPCF].

To give just a taste of the matter, the asymptotic behaviour of the recurrence coefficients associated to $d\mu(x) = |x|^\rho \exp(-x^4)dx$ on $x \in \mathbb{R}$ appears in an amazing number of fields:

1. This extension of Hermite polynomials is studied by Shohat [Sho], using methods of Laguerre [Lag]. Later on, Freud [Fr2] rediscovered Shohat's formulas (see (31) with $t = 0$) and proved that $a_n \sim (n/12)^{1/4}$ when $n \to \infty$. Much more has been done since then [Lub] [Mag2] [Mag3] [Nev] [GFOPCF] [Nev2], it has been shown that a behaviour $d\mu(x) \sim \exp(-|x|^\alpha)dx$ for $x \to \pm\infty$ implies a_n and $b_n \sim$ constant $n^{1/\alpha}$ for large n.

2. Similar exponential weights were encountered in solid-state and statistical physics, where the same asymptotic connections have been used (sometimes after clever guesswork) [LiMu] [OW].

3. These extensions of Hermite and Laguerre polynomials also appear in numerical quadrature methods intended to solve Boltzmann and Fokker-Planck equations [ClS], where they are called "speed", "bimode" and "Druyvesteyn" polynomials. The recurrence coefficients can be computed safely from a suitable algorithmic use of the Shohat-Freud equations [LeQ], or from asymptotic expansions [ClS] (see [Nev2] pp.462-463).

4. The same orthogonal polynomials reappear in special solutions of important differential equations of mathematical physics ([Bes] [KvM]; Shohat-Freud's equations are called "discrete Painlevé equations" in [Fok2] [93]), as well as in continued fraction expansions of special irrational numbers [Chu2].

Any advance in one of these fields is liable to benefit to the other ones, although the dialog is not always obvious: established theorems may sometimes have poor constructive contents and be unable to inspire valuable algorithms; explicit formulas (using for instance exotic special functions or high-order determinants) may be delightful solutions for some people and useless nightmares for other ones; successful numerical tricks or self-consistent "*Ansatze*" may be out of reach of contemporary methods of proof, etc.

Let us return now to the problem considered here: to deduce properties of the recurrence coefficients a_n and b_n from the measure $d\mu(x)$. The Chebyshev orthogonal polynomials are related to measures involving the square root of a polynomial of degree 2 and yield constant recurrence coefficients (the simplest case!). The classical orthogonal polynomials (Jacobi polynomials and their limit cases) have a known measure and known recurrence coefficients (a_n^2 and b_n are special rational functions of n). We may define a more general class by allowing a_n^2 and b_n to be general rational functions of n (Pollaczek class [Chi]) but then the orthogonality measure becomes difficult to control. Natural extensions of Chebyshev polynomials are related to measures involving the square root of a polynomial of degree > 2. One finds then an oscillatory behaviour of the recurrence coefficients ([GV1] [GV2] [Gr] [I] [VA]), whose description may even need elliptic or hyperelliptic functions [Ak] [Apt] [GaN]. We will deal here with a further extension, the so-called semi-classical class (to be defined in the next section) which represents still a reasonable trade-off between measure description (easy and containing useful cases) and the possibility of description of recurrence coefficients (interesting nonlinear relations). Much of the work was already done in the end of the nineteenth century by Laguerre [Lag] who recognized (in 1885) that special cases (worked by Jacobi and Borchardt) would indeed involve elliptic functions. As he could not establish the general recurrence coefficients behaviour, we could suspect that *special functions still unknown in the nineteenth century* would be needed... Painlevé transcendents will indeed appear, and they were investigated in the early twentieth century (see the foreword of [Pain]).

2. Formal semi-classical orthogonal polynomials.

Orthogonal polynomials p_n are usually defined through a measure, so to satisfy (1). The construction of these polynomials only needs the sequence of moments $\mu_k = \int_S x^k d\mu(x), k = 0, 1, \ldots$ *Formal* orthogonal polynomials are only related to a numerical (real or complex) sequence of numbers $\mu_k, k = 0, 1, \ldots$, ignoring whether these numbers are actual moments of some weight or distribution on some support or not. The polynomial $p_n(z) = \gamma_n z^n + \gamma_{n,1} z^{n-1} + \cdots + p_n(0)$ is then obtained from the equations $\gamma_n \mu_{n+k} + \gamma_{n,1}\mu_{n+k-1} + \cdots + p_n(0)\mu_k = 0, k = 0, 1, \ldots, n-1$ and $\gamma_n[\gamma_n\mu_{2n} + \gamma_{n,1}\mu_{2n-1} +$

$\cdots + p_n(0)\mu_n] = 1$. These equations can be solved for any $n = 0, 1, \ldots$ if the Hankel determinants built with $\mu_0, \ldots, \mu_{2n}$ do not vanish ([Bre], [dBvR] § 7 , see the definition of *regular* formal orthogonal polynomials on p.47 of [Dra] § 1.1-1.3).

If we define a linear form $\mathcal{L}$ on the space of polynomials by $\mathcal{L}(x^n) = \mu_n, n = 0, 1, \ldots$, the polynomials p_n satisfy $\mathcal{L}(p_n p_m) = \delta_{n,m}, n, m = 0, 1, \ldots$ ([Mar], [Mar2], where $\mathcal{L}$ is written $\mathcal{L}_0$).

Regular formal orthogonal polynomials always satisfy the *recurrence relation* (2), with $p_0 = \gamma_0 = 1/\sqrt{\mu_0}$, $a_1 p_1(z) = (z - b_0)p_0(z)$, and where $b_0 = -\gamma_{1,1}/\gamma_1$, $a_n = \gamma_{n-1}/\gamma_n$, $b_n = \gamma_{n,1}/\gamma_n - \gamma_{n+1,1}/\gamma_{n+1}, n = 1, 2, \ldots$ ([Bre] , [Dra] § 1.4).

By introducing the formal series

$$f(z) = \sum_0^\infty \mu_k z^{-k-1}, \tag{3}$$

the equations for p_n are summarized as

$$f(z)p_n(z) = p_{n-1}^{(1)}(z) + \varepsilon_n(z), \qquad \varepsilon_n(z) = \gamma_n^{-1} z^{-n-1} + O(z^{-n-2}), \tag{4}$$

where $p_{n-1}^{(1)}$ is a polynomial of degree $n-1$, (associated polynomial to p_n). These polynomials, as well as the ε_n's, satisfy the same recurrence relations (2), but with $p_{-1}^{(1)} = 0$, $p_0^{(1)} = \mu_0\gamma_1 = 1/(a_1\gamma_0)$. The following relation

$$p_n p_{n-2}^{(1)} - p_{n-1}p_{n-1}^{(1)} = p_{n-1}\varepsilon_n - p_n\varepsilon_{n-1} = -1/a_n \tag{5}$$

is well known ([Chi] , [Fr1] , etc.) From the recurrence relations (2), we have the main terms in the expansions of p_n and ε_n, which will be useful later:

$$p_n(z) = \gamma_n \left[z^n - \left(\sum_0^{n-1} b_i \right) z^{n-1} + \left(\sum_{i<j<n} b_i b_j - \sum_1^{n-1} a_i^2 \right) z^{n-2} + \cdots \right] \tag{6}$$

$$\varepsilon_n(z) = \gamma_n^{-1} \left[z^{-n-1} + \left(\sum_0^{n} b_i \right) z^{-n-2} + \left(\sum_{i\le j\le n} b_i b_j + \sum_1^{n+1} a_i^2 \right) z^{-n-3} + \cdots \right] \tag{7}$$

(for the latter one, use $\gamma_n\varepsilon_n(z) = (z-b_n)^{-1}\gamma_{n-1}\varepsilon_{n-1}(z) + (z-b_n)^{-1}a_{n+1}^2\gamma_{n+1}\varepsilon_{n+1}(z)$).

Of course, if we happen to know a true function of the complex variable z having the asymptotic expansion (3) when $z \to \infty$ in some way, and if this function is analytic outside a set S made of contours and arcs, we may use a Cauchy-like integral representation

$$f(z) = \int_S w(x)(z-x)^{-1}dx\,, \qquad z \notin S \tag{8}$$

allowing to recover the convenient description in terms of a "weight function" w, but the description is not unique and w may be complex. We then have an integral representation of the form $\mathcal{L}$: $\mathcal{L}\varphi = \int_S \varphi(x)w(x)dx$. For instance the Bessel orthogonal polynomials are defined by $\mu_n = 1/n!, n = 0, 1, \dots$ and can be considered as orthogonal with respect to the complex weight $(2\pi i)^{-1} \exp x^{-1}$ on any contour containing the origin in its interior. Remark that orthogonality of two complex function φ and ψ always involves here the product $\varphi\psi$ and *not* the product $\varphi\overline{\psi}$ (as in [StT]).

For an example showing how formal orthogonal polynomials can be investigated through their generating function of formal moments (3), consider $f(z) = [A(z) - B(z)^{1/2}]/C(z)$, where A, B and C are given polynomials (*formes du second degré* in [Mar2] p.122, Def. 7.4). Such a function can be represented as (8) outside a systems of cuts S joining the zeros of B in some way. Here $w(x)$ will have the form $w(x) = \pm(\pi i)^{-1}B(x)^{1/2}/C(x)$ ([N]§1.2 & 4.3.1). If B has only real zeros, this is a way to introduce special orthogonal polynomials on several intervals (the intervals where $B(x) \leq 0$). Now, (4) gives here $-B^{1/2}p_n = q_n + C\varepsilon_n$, with $q_n = -Ap_n + Cp_{n-1}^{(1)}$. Squaring yields $Bp_n^2 - q_n^2 = L_n$, where L_n *must* be a polynomial of degree bounded by a constant, as the left-hand side is a polynomial, and as the right-hand side is $2q_nC\varepsilon_n + C^2\varepsilon_n^2$. So, p_n is such that the square of this polynomial times a given polynomial B equals the square of another polynomial plus a polynomial of bounded degree. This is enough for experts to describe p_n in terms of (hyper)elliptic function and integrals, theta functions, etc. (see [Ak] §53, [Apt], [Brez] pp. 296-298, [N]§4.3 , [Peh]), and to discuss periodic features in the sequence of the recurrence coefficients ([GV1], [GV2], [Gr], [I], [Peh1]). For arithmetic continued fractions connected to Pell's equation, see [Brez] pp. 39.43.

A similar technique will now be applied to a more general class of functions f.

Many special families of orthogonal polynomials have been studied. In most cases, the knowledge of a special family is considered satisfactory when an explicit formula for the recurrence coefficients a_n and b_n as functions of n is associated to a definite formula for the weight w, or measure of orthogonality, see for instance the final tables of Chihara's book [Chi], whereas the starting point of the study may be generating functions, Rodrigues formulas, special functions identities, differential equations, etc.

The simplest way to start the study of the class of *semi-classical* orthogonal polynomials is to define them through a differential equation of their function f:

Definition: The sequence $\{p_n(z) = \gamma_n z^n + \cdots\}_{n=0}^{\infty}$ is a set of formal semi-classical orthogonal polynomials if (3) holds with a function f satisfying the first order linear differential equation

$$Wf' = 2Vf + U \tag{9}$$

where W, V and U are polynomials ($W \not\equiv 0$).

This is equivalent to the existence of a linear recurrence relation of the form $\sum_{k=0}^{d}(n\xi_k+\eta_k)\mu_{n+k}=0$ for the formal moments μ_n [BeR].

Moreover, only regular semi-classical orthogonal polynomials will be considered here, so that $\gamma_n \neq 0, n=0,1,\ldots$

Of course, (4) must be possible with an expansion of the form (3), so that degree $U \leq \max($ degree $W-2$, degree $V-1)$. All the classical families are recovered when degrees of W and $V \leq 2$ and 1.

We will consider especially

Definition: Generic semi-classical orthogonal polynomials are semi-classical orthogonal polynomials where $m=$ degree $W \geq 2$, degree $V<m$, the zeros $x_1, x_2, \ldots x_m$ of W are distinct, and the residues $\alpha_k = 2V(x_k)/W'(x_k)$ are not integers, $k=1,2,\ldots,m$. The Jacobi polynomials correspond to $m=2$.

We have then:

Proposition: *Generic semi-classical orthogonal polynomials are orthogonal with respect to a (possibly complex) generalized Jacobi weight function* $w(x)=A_j\Pi_1^m(x-x_k)^{\alpha_k}$ *on arcs* $S_j, j=1,2,\ldots m$ *of the complex plane.*

Indeed, (9) has exactly one holomorphic solution $f_j(z)=c_{j,0}+c_{j,1}(z-x_j)+\cdots$ in a neighbourhood of the singular point x_j, as the equations for the $c_{j,i}$'s are $2V(x_j)c_{j,0}+U(x_j)=0$ and $W'(x_j)ic_{j,i}+\cdots=2V(x_j)c_{j,i}+\cdots, i=1,2,\ldots$ have exactly one solution, as $V(x_j)\neq 0$ and $W'(x_j)i-2V(x_j)=W'(x_j)(i-\alpha_j)$ cannot vanish (this can also be seen as a most elementary application of L.Fuchs theory of linear differential equations). As $\Pi_1^m(x-x_k)^{\alpha_k}$ is a solution of the homogeneous equation (9), one has $f(z)=f_j(z)+B_j\Pi_1^m(z-x_k)^{\alpha_k}$ near x_j, on one side of the cut. A Cauchy integral expression of $f(z)$ will, after a distorsion of the integration contour (as in [N] §1.2), involve the difference of the limit functions f_+ and f_- which is a multiple of $\Pi_1^m(x-x_k)^{\alpha_k}$ on a cut. This gives w on S. Let $w(z)$ be a continuation of w on some side of the cut, then we have

$$f(z)=f_j(z)+C_jw(z) \tag{10}$$

near x_j.

Non generic semi-classical orthogonal polynomials can be considered as limit cases, for instance, a weight $\exp P(x)$, where P is a polynomial, is the limit of $(1+P(x)/N)^N$ when $N\to\infty\ldots$ See [Al] and [Bel] for other proofs and examples.

Anyhow, as f_+ and f_- along the two sides of a system of cuts are solutions of the same equation (9), their difference must be a solution of the homogeneous equation: *semi-classical orthogonal polynomials are orthogonal with respect to a (possibly complex) weight function w satisfying*

$$Ww' = 2Vw \tag{11}$$

on a system of cuts, masspoints may also be present if f has poles. Examples have been given in [BoN], [HvR1], [HvR2] and [Sho]; the whole class of true positive semi-classical measures on real sets is given in [BLN].

Conversely, Shohat [Sho] develops the theory starting from a weight function satisfying (11) on an interval. Let us generalize this to a given set of arcs S, and show that (9) is recovered: if needed, we multiply W and V by common factors in order to have $\lim W(x)w(x) = 0$ when x tends to any endpoint (eq. (6) of [Sho]). Then, from (8), $W(z)f(z) = \int_S W(x)w(x)(z-x)^{-1}\,dx$ plus a polynomial ($\int_S[(W(z)-W(x))/(z-x)]w(x)\,dx$ is a polynomial in z). The derivative gives

$$(W(z)f(z))' = -\int_S W(x)w(x)(z-x)^{-2}dx + \text{ pol. } = \int_S (W(x)w(x))'(z-x)^{-1}dx + \text{ pol. },$$

by integration by parts, using $Ww \to 0$ at the endpoints of S. As $(Ww)' = (W' + 2V)w$, and $\int_S(W'(x) + 2V(x))(z-x)^{-1}dx = (W'(z) + 2V(z))\int_S(z-x)^{-1}dx+$ a polynomial, we find indeed $Wf' = 2Vf+$ a polynomial, i.e, (9).

3. Differential relations and equations for formal semi-classical orthogonal polynomials.

Now, we go further, following Laguerre ([Lag] sec. 2, see also [HvR1], [Per] § 76): from (4) and (9),

$$\begin{aligned}
0 &= W\left[\frac{p_{n-1}^{(1)}}{p_n} + \frac{\varepsilon_n}{p_n}\right]' - 2V\left[\frac{p_{n-1}^{(1)}}{p_n} + \frac{\varepsilon_n}{p_n}\right] - U \\
&= \frac{W[p_{n-1}^{(1)}{}'p_n - p_n'p_{n-1}^{(1)}] - 2Vp_{n-1}^{(1)}p_n - Up_n^2}{p_n^2} + W\left[\frac{\varepsilon_n}{p_n}\right]' - 2V\frac{\varepsilon_n}{p_n}
\end{aligned}$$

so,

$$\Theta_n = W[p_{n-1}^{(1)}{}'p_n - p_n'p_{n-1}^{(1)}] - 2Vp_{n-1}^{(1)}p_n - Up_n^2 \tag{12}$$

is a polynomial of degree bounded by a constant, as

$$\Theta_n = -p_n^2 W\left[\frac{\varepsilon_n}{p_n}\right]' + 2V\varepsilon_n p_n = W(\varepsilon_n p_n' - \varepsilon_n' p_n) + 2V\varepsilon_n p_n \tag{13}$$

is bounded by a power $\le \max($ degree $W\ -2$, degree $V\ -1)$ for large argument. For given W and V, (13) with (6) and (7) allow to give Θ_n in terms of n and the recurrence coefficients

a's and b's. Moreover, expanding (13) up to *negative* powers of z yields equations for these coefficients. This is a first hint towards identities (**Laguerre-Freud's equations**) for the recurrence coefficients of semi-classical orthogonal polynomials. See [BeR] for a technique involving Turán determinants.

Identities like (12) involving orthogonal polynomials of arbitrary high degree on one side and polynomials of bounded degree with respect to n on the other side occur whenever one has a functional equation $P(f) = 0$ for f, provided the elimination of f in $P(p_{n-1}^{(1)}/p_n + O(z^{-2n-1})) = 0$ is simple enough. This happens if P applied to a rational function φ/ψ produces another rational function with denominator ξ of degree not much larger that *twice* the degree of ψ. Then, multiplication by this denominator ξ will produce polynomials and, roughly speaking, products of ξ and the error term $O(z^{-2n-1})$ which will keep a small rate of growth at ∞. Exemples of valid functionals P are quadratic polynomials (discussed in the preceding section: $f = (A - B^{1/2})/C \Rightarrow (Cf - A)^2 - B = 0$), linear differential operators of first order discussed here, both giving $\xi = p_n^2$, and Riccati differential operators (theory of Laguerre-Hahn orthogonal polynomials [Mag1]). Difference operators may also be considered, they can leave things like $\xi(z) = p_n(z)p_n(z+h)$, $\xi(z) = p_n(z)p_n(qz)$, etc. [Mag4]

In the generic case, let $W(z) = \Pi_1^m(z - x_k) = z^m - (\sum_1^m x_k)z^{m-1} + \cdots$, then $2V(z) = W(z)\sum_1^m(\alpha_k/(z-x_k)) = (\sum_1^k \alpha_k)z^{m-1} + [\sum_1^m(\alpha_k x_k) - (\sum_1^m x_k)(\sum_1^m \alpha_k)]z^{m-2} + \cdots$, using (13), (6) and (7):

$$\Theta_n(z) = \left(2n + 1 + \sum_1^m \alpha_k\right) z^{m-2} +$$
$$+ \left[\left(2n + 1 + \sum_1^m \alpha_k\right)\left(b_n - \sum_1^m x_k\right) + 2\sum_0^{n-1} b_i + b_n + \sum_1^m(\alpha_k x_k)\right] z^{m-3} + \cdots \qquad (14)$$

From (5), replace Θ_n in (12) by $(p_{n-1}p_{n-1}^{(1)} - p_n p_{n-2}^{(1)})a_n\Theta_n$:
$p_{n-1}^{(1)}[Wp_n{}' + Vp_n + a_n\Theta_n p_{n-1}] = p_n[Wp_{n-1}^{(1)}{}' - Vp_{n-1}^{(1)} + a_n\Theta_n p_{n-2}^{(1)} - Up_n]$, which must therefore have the form $\Omega_n p_n p_{n-1}^{(1)}$, where Ω_n is a new auxiliary polynomial of bounded degree. Using again (5), one has

$$\Omega_n = a_n W[p_{n-1}^{(1)}{}' p_{n-1} - p_n' p_{n-2}^{(1)}] - a_n V[p_{n-1}^{(1)} p_{n-1} + p_n p_{n-2}^{(1)}] - a_n U p_n p_{n-1}$$

And, with (4):

$$\Omega_n = a_n W(\varepsilon_{n-1}p_n' - \varepsilon_n' p_{n-1}) + a_n V(\varepsilon_{n-1}p_n + \varepsilon_n p_{n-1}). \qquad (15)$$

This yields the two *differential relations*:

$$Wp_n' = (\Omega_n - V)p_n - a_n\Theta_n p_{n-1}$$
$$Wp_{n-1}^{(1)}{}' = (\Omega_n + V)p_{n-1}^{(1)} - a_n\Theta_n p_{n-2}^{(1)} + Up_n$$

We get rid of the Up_n term of the second equation by forming an equation for fp_n, using (9), and subtracting the second equation: $W\varepsilon_n' = (\Omega_n + V)\varepsilon_n - a_n\Theta_n\varepsilon_{n-1}$. We recover the form of the first equation by using (11): $W(\varepsilon_n/w)' = (\Omega_n - V)\varepsilon_n/w - a_n\Theta_n\varepsilon_{n-1}/w$. In order to have a differential system, we have to give y_{n-1}' ($y = p$ or ε/w) in terms of y_n and y_{n-1}. As y_n satisfies the recurrence relations (2), $Wy_{n-1}' = (\Omega_{n-1} - V)y_{n-1} - a_{n-1}\Theta_{n-1}y_{n-2}$ turns easily as $Wy_{n-1}' = a_n\Theta_{n-1}y_n + (\Omega_{n-1} - V - (z - b_{n-1})\Theta_{n-1})y_{n-1}$. From (15), (13) and (2),

$$\Omega_{n+1}(z) = (z - b_n)\Theta_n(z) - \Omega_n(z), \tag{16}$$

so we finally have the *differential system*:

$$Y' = AY : \quad \begin{bmatrix} p_n & \varepsilon_n/w \\ p_{n-1} & \varepsilon_{n-1}/w \end{bmatrix}' = \frac{1}{W}\begin{bmatrix} \Omega_n - V & -a_n\,\Theta_n \\ a_n\Theta_{n-1} & -\Omega_n - V \end{bmatrix}\begin{bmatrix} p_n & \varepsilon_n/w \\ p_{n-1} & \varepsilon_{n-1}/w \end{bmatrix} \quad . \tag{17}$$

This differential system gives the whole differential history of the semi-classical orthogonal polynomials. Laguerre [Lag] and many other people ([AtE] [Ha1] [Ha2] [Nev] [Sho] etc.) have preferred the scalar second order form obtained from eliminating y_{n-1} in $Wy_n' = (\Omega_n - V)y_n - a_n\Theta_n y_{n-1}$ and $Wy_{n-1}' = a_n\Theta_{n-1}y_n - (\Omega_n + V)y_{n-1}$:

$$W\Theta_n y_n'' = (W\Theta_n' - W'\Theta_n - 2V\Theta_n)y_n' + K_n y_n, \tag{18}$$

with $K_n = (\Omega_n - V)'\Theta_n - (\Omega_n - V)\Theta_n' + \Theta_n(\Omega_n^2 - V^2 - a_n^2\Theta_n\Theta_{n-1})/W$, which is a polynomial, as putting $a_{n+1}y_{n+1}' = (z - b_n)y_n' + y_n - a_n y_{n-1}'$ (derivative of (2)) in $a_{n+1}Wy_{n+1}' = a_{n+1}(\Omega_{n+1} - V)y_{n+1} - a_{n+1}^2\Theta_{n+1}y_n$, using again (2), and the differential equation (17) for Wy_{n-1}' gives an expression of the form $Ay_n = By_{n-1}$, with $B = 0$ from (16), whence $A = 0$, which is

$$(z - b_n)(\Omega_{n+1} - \Omega_n) = W + a_{n+1}^2\Theta_{n+1} - a_n^2\Theta_{n-1}. \tag{19}$$

Multiplying by (16) and summing on n, one finds

$$\Omega_n^2 - a_n^2\Theta_n\Theta_{n-1} = V^2 + W\sum_0^{n-1}\Theta_i, \tag{20}$$

knowing that $\Omega_0 = V$.

With $z_n = (Ww/\Theta_n)^{1/2}y_n$, we have a form without first derivative

$$z_n'' = \left\{\frac{3}{4}\left(\frac{\Theta_n'}{\Theta_n}\right)^2 - \frac{1}{2}\frac{\Theta_n''}{\Theta_n} - \frac{1}{2}\frac{\Theta_n'}{\Theta_n}\frac{W' + 2\Omega_n}{W} + \frac{4V^2 - W'^2}{4W^2} + \frac{W'' + 2\Omega_n'}{2W} + \frac{\sum_0^{n-1}\Theta_i}{W}\right\} z_n, \tag{21}$$

used by R. Fuchs [RFu] in the case $m =$ degree $W = 3$.

Laguerre ([Lag], see also [GaN]) finds equations for the recurrence coefficients and the coefficients of Θ_n and Ω_n by using (16) and (19), keeping the degrees of Θ_n and Ω_n bounded when n increases. We may express everything in terms of the recurrence coefficients alone, then the expansion of Ω_n, constructed on the same lines as (14), will be useful:

$$\Omega_n(z) = \left[n + (\sum_1^m \alpha_k)/2\right] z^{m-1} +$$
$$+ \left[\sum_0^{n-1} b_i - n\sum_1^m x_k + \left(\sum_1^m (\alpha_k x_k) - (\sum_1^m x_k)(\sum_1^m \alpha_k)\right)/2\right] z^{m-2} +$$
$$+ \left[\sum_0^{n-1} b_i^2 + 2\sum_1^{n-1} a_i^2 - (\sum_1^m x_k)\left(\sum_0^{n-1} b_i + \sum_1^m (\alpha_k x_k)/2\right) + \right.$$
$$\left. +(n + \sum_1^m \alpha_k/2)\left(\sum_{k<\ell\le m} x_k x_\ell\right) + \sum_1^m \alpha_k x_k^2/2 + \left(2n+1+\sum_1^m \alpha_k\right) a_n^2\right] z^{m-3} + \ldots \quad (22)$$

Consider for instance the case $m = 3$ (simplest generalized Jacobi polynomials): from (14), Θ_n is a polynomial of degree 1 with a known coefficient of z and a constant coefficient depending on the b_i's up to b_n; from (22), Ω_n is a polynomial of degree 2 with a known coefficient of z^2 and two other coefficients depending on the b_i's and the a_i's up to the index $n-1$ (see example 1 in section 5) . The constant coefficients of (16) and (20) give nonlinear relations for a_n and b_n. The meaning of the solutions of these recurrence relations for the recurrence coefficients of (2) is not obvious. Even the simplest relations found in nongeneric cases (as (30) or (31)) are baffling.

The explanation in terms of Painlevé transcendents and similar functions, i.e., solutions of remarkable high-order nonlinear differential equations in terms of a well-chosen parameter, will be given now. The derivation is based on the isomonodromy properties of (18). Later on, examples will show that a more elementary derivation is possible.

4. Monodromy matrices and isomonodromy identities.

D. & G. Chudnovsky remarked ([Chu2], see also (5.1.18) in [N]) how (18) has a form already investigated in the period 1890-1910 by authors working on isomonodromy deformations ([RFu], [Pain]).

Let $Y(z)$ be a fundamental matrix of solutions of the differential system $Y'(z) = A(z)Y(z)$, defined outside a system of cuts joining the singular points (poles of A) of the equation. When z follows a contour about a singular point x_j, let us solve $Z'(z) = A(z)Z(z)$ with the initial value $Z(z_0) = Y(z_0)$ at a starting point on the contour. As long as no cut is crossed, $Z(z) = Y(z)$. This is no more true when one or several cuts are crossed but, when we come back in a neighbourhood of z_0, the columns of the matrix of solutions $Z(z)$ must be fixed combinations of the columns of the initial fundamental matrix of solutions: $Z(z) = Y(z)M_j$. This matrix M_j is called the **monodromy matrix** of $Y' = AY$ at the singular point x_j (only *regular* singularities are considered here).

Theorem 1. *Generic formal semi-classical orthogonal polynomials satisfy differential systems (17) with monodromy matrices*

$$M_j = \begin{bmatrix} 1 & C_j[1-\exp(-2\pi i\alpha_j)] \\ 0 & \exp(-2\pi i\alpha_j) \end{bmatrix}$$

at the singular points $x_j, j = 1, 2, \dots m$.

Indeed, p_n and p_{n-1} are not modified after a circle about x_j, but ε_n, ε_{n-1} and w have a branchpoint there. According to the discussion made in the proof of (10), $f(z) = f_j(z) + B_j\Pi_1^m(z-x_k)^{\alpha_k}$ with some determination of the powers near x_j, near a side of a cut. By following a contour about x_j, f_j returns to its previous value, but $\Pi_1^m(z-x_k)^{\alpha_k}$ has been multiplied by $\exp(2\pi i\alpha_j)$. The same happens with w. Therefore, from (10), $\varepsilon_n/w = (fp_n - p_{n-1}^{(1)})/w = (f_jp_n - p_{n-1}^{(1)})/w + C_jp_n$ becomes $\exp(-2\pi i\alpha_j)(f_jp_n - p_{n-1}^{(1)})/w + C_jp_n = \exp(-2\pi i\alpha_j)\varepsilon_n/w + [1-\exp(-2\pi i\alpha_j)]C_jp_n$.

This shows that the monodromy matrices of (17) at the singular points remain unchanged if the exponents α_k *remain unchanged and if the weight* w *on* S *is adapted so that the multipliers* C_k *remain the same.* However, one may vary the positions of the singular points x_k. The quantities f, p_n, a_n, b_n, Θ_n etc. will then be subject to extrememely interesting *isomonodromy deformations*. Here is a sketch ([LD] III, from p.128 onwards), applied to the specific equation (17):

Let the x_k depend on a single parameter t, and let us define the matrix

$$H = \frac{\partial Y}{\partial t}Y^{-1},$$

as $\partial M_j/\partial t = 0$, H does not change when z achieves a contour about x_j. So, H has no branchpoints at the x_j's. To get a better view of what happens at the singular points, we expand H (using $\det Y = 1/(a_nw)$, from (5)):

$$H = a_n \begin{bmatrix} \dot{p}_n\varepsilon_{n-1} - p_{n-1}\dot{\varepsilon}_n + p_{n-1}\varepsilon_n\dot{w}/w & -\dot{p}_n\varepsilon_n + p_n\dot{\varepsilon}_n - p_n\varepsilon_n\dot{w}/w \\ \dot{p}_{n-1}\varepsilon_{n-1} - p_{n-1}\dot{\varepsilon}_{n-1} + p_{n-1}\varepsilon_{n-1}\dot{w}/w & -\dot{p}_{n-1}\varepsilon_n + p_n\dot{\varepsilon}_{n-1} - p_n\varepsilon_{n-1}\dot{w}/w \end{bmatrix}, \tag{23}$$

where the dot derivative is $\partial/\partial t$. From (4) and (10), one has $\varepsilon_n = \varepsilon_{n,j} + C_jwp_n$ near x_j, where $\varepsilon_{n,j}$ is regular near x_j. The singular terms cancel nicely in the combinations of (23) (remember that $\dot{C}_j = 0$!); the ratio $\dot{w}/w$ has a simple pole at x_j with residue $-\alpha_j\dot{x}_j$ (as $\dot{\alpha}_j = 0$). We are left with

$$H = H_\infty + \sum_{j=1}^{m} H_j(z - x_j)^{-1},$$

with

$$H_j = -\alpha_j\dot{x}_j a_n \begin{bmatrix} p_{n-1}\varepsilon_{n,j} & -p_n\varepsilon_{n,j} \\ p_{n-1}\varepsilon_{n-1,j} & -p_n\varepsilon_{n-1,j} \end{bmatrix} \qquad j = 1, \ldots, m,$$

where the $p_r\varepsilon_{s,j}$'s are the values at $z = x_j$. As $W(x_j) = 0$, (13) tells that $\Theta_n = 2V\varepsilon_{n,j}p_n$ at $z = x_j$, and (15) with (5) gives $\Omega_n = V + 2a_nV\varepsilon_{n,j}p_{n-1} = -V + 2a_nV\varepsilon_{n-1,j}p_n$ at x_j. With $\alpha_j = 2V/W'$ at x_j, one finds from (17):

$$A = \sum_{j=1}^{m}(z - x_j)^{-1}A_j \qquad \Rightarrow \qquad H = H_\infty - \sum_{j=1}^{m}(z - x_j)^{-1}\dot{x}_jA_j,$$

A direct inspection of (23) when $z \to \infty$ gives, using (6) and (7),

$$H_\infty = \begin{bmatrix} \dot{\gamma}_n/\gamma_n & 0 \\ 0 & -\dot{\gamma}_{n-1}/\gamma_{n-1} \end{bmatrix}$$

in the generic case, as $\dot{w}/w = -\sum_1^m \alpha_k\dot{x}_k/(z - x_k) \to 0$ when $z \to \infty$.

Finally, the *differential equations in t* appear by working

$$\begin{aligned} &\partial^2Y/\partial z\partial t = \partial\dot{Y}/\partial z = (HY)' = H'Y + HY' = H'Y + HAY = \\ &= \partial^2Y/\partial t\partial z = \partial Y'/\partial t = (A\dot{Y}) = \dot{A}Y + A\dot{Y} = \dot{A}Y + AHY, \end{aligned}$$

whence

$$\dot{A} = H' + HA - AH. \qquad (24)$$

C'étaient les cieux ouverts
Stendhal

This equation (24) has an incredibly inspiring form, explaining how this theory is related to integrable Hamiltonians, Bäcklund transformations, Lax pairs, Toda lattices, solitons, etc. whereas the connection with orthogonal polynomials, special functions, continued fractions, Diophantine approximations has been worked with great virtuosity by G. & D. Chudnovsky [Chua] [Chub] [Chu0] [Chu1] [Chu2],...

In the generic case, we have for the residue matrices

$$\dot{A}_j = H_\infty A_j - A_jH_\infty + \sum_{\substack{k=1 \\ k\neq j}}^{k=m} \frac{\dot{x}_j - \dot{x}_k}{x_j - x_k}(A_kA_j - A_jA_k), \qquad j = 1, \ldots, m$$

called the *Schlesinger equations* (see [Chua]).

Now, we show how these equations lead to differential equations for the recurrence coefficients:

Theorem 2. *Let a_n and b_n be the recurrence coefficients of (2) for generalized Jacobi orthogonal polynomials related to a (possibly complex) weight of the form $\Pi_1^m(x-x_j)^{\alpha_j}$, on a set of arcs joining the x_j's, where at least one of the x_j's depend on a parameter t. Then, we have the Toda equations*

$$\frac{\dot{a}_n}{a_n} = \frac{1}{2}\sum_{k=1}^{m} \frac{(\Theta_n(x_k)-\Theta_{n-1}(x_k))\dot{x}_k}{W'(x_k)}, \tag{25}$$

$$\dot{b}_n = \sum_{k=1}^{m} \frac{(\Omega_{n+1}(x_k)-\Omega_n(x_k))\dot{x}_k}{W'(x_k)}, \tag{26}$$

where $W(x)=\Pi_1^m(x-x_k)$, and Θ_n and Ω_n are polynomials introduced in (12) – (15).

Indeed, from (17), the residue matrix A_j is

$$A_j = \frac{1}{W'(x_j)}\begin{bmatrix} \Omega_n(x_j)-V(x_j) & -a_n\Theta_n(x_j) \\ a_n\Theta_{n-1}(x_j) & -\Omega_n(x_j)-V(x_j) \end{bmatrix}$$

we have

$$H_\infty A_j - A_j H_\infty = -\frac{\dot{\gamma}_n/\gamma_n + \dot{\gamma}_{n-1}/\gamma_{n-1}}{W'(x_j)}\, a_n \begin{bmatrix} 0 & \Theta_n(x_j) \\ \Theta_{n-1}(x_j) & 0 \end{bmatrix},$$

$$A_kA_j - A_jA_k = \frac{a_n}{W'(x_j)W'(x_k)} \times$$
$$\times \begin{bmatrix} a_n(\Theta_n(x_j)\Theta_{n-1}(x_k)-\Theta_n(x_k)\Theta_{n-1}(x_j)) & 2(\Theta_n(x_k)\Omega_n(x_j)-\Theta_n(x_j)\Omega_n(x_k)) \\ 2(\Theta_{n-1}(x_k)\Omega_n(x_j)-\Theta_{n-1}(x_j)\Omega_n(x_k)) & a_n(\Theta_n(x_k)\Theta_{n-1}(x_j)-\Theta_n(x_j)\Theta_{n-1}(x_k)) \end{bmatrix}.$$

The Schlesinger equations for the off-diagonal elements of A_j are

$$-\frac{\partial}{\partial t}\frac{\Theta_n(x_j)}{W'(x_j)} = -2\frac{\dot{\gamma}_n}{\gamma_n}\frac{\Theta_n(x_j)}{W'(x_j)} + 2\sum_{k\neq j}\frac{\dot{x}_j-\dot{x}_k}{x_j-x_k}\frac{\Theta_n(x_k)\Omega_n(x_j)-\Theta_n(x_j)\Omega_n(x_k)}{W'(x_j)W'(x_k)}, \tag{27}$$

$$\frac{\partial}{\partial t}\frac{\Theta_{n-1}(x_j)}{W'(x_j)} = -2\frac{\dot{\gamma}_{n-1}}{\gamma_{n-1}}\frac{\Theta_{n-1}(x_j)}{W'(x_j)} + 2\sum_{k\neq j}\frac{\dot{x}_j-\dot{x}_k}{x_j-x_k}\frac{\Theta_{n-1}(x_k)\Omega_n(x_j)-\Theta_{n-1}(x_j)\Omega_n(x_k)}{W'(x_j)W'(x_k)},$$

where $a_n\gamma_n=\gamma_{n-1} \Rightarrow \dot{a}_n/a_n + \dot{\gamma}_n/\gamma_n = \dot{\gamma}_{n-1}/\gamma_{n-1}$ has been used. Increasing n by 1 in the second equation and adding to the first one,

$$0 = -4\frac{\dot{\gamma}_n}{\gamma_n}\frac{\Theta_n(x_j)}{W'(x_j)} + 2\sum_{k\neq j}(\dot{x}_j-\dot{x}_k)\frac{\Theta_n(x_j)\Theta_n(x_k)}{W'(x_j)W'(x_k)},$$

where (16) has been used. At this point, we don't have to avoid the term $k = j$ anymore in the sum. Moreover, as any polynomial $P(z) = \pi_0 z^{m-1} + \cdots$ satisfies $\pi_0 = \sum_1^m P(x_k)/W'(x_k)$, (coefficient of z^{-1} in $P(z)/W(z) = \sum_1^m P(x_k)/((W'(x_k)(z-x_k)))$, and as the degree of Θ_n is $m-2$ ((14)), $\dot{x}_j$ disappears from the sum:

$$\frac{\dot{\gamma}_n}{\gamma_n} = -\frac{1}{2}\sum_{k=1}^{m} \frac{\Theta_n(x_k)\,\dot{x}_k}{W'(x_k)}$$

and (25) follows from $a_n\gamma_n = \gamma_{n-1}$.
Now, we come to the first diagonal element of the Schlesinger's equations:

$$\frac{\partial}{\partial t}\frac{\Omega_n(x_j) - V(x_j)}{W'(x_j)} = a_n^2 \sum_{k\neq j} \frac{\dot{x}_j - \dot{x}_k}{x_j - x_k}\frac{\Theta_n(x_j)\Theta_{n-1}(x_k) - \Theta_n(x_k)\Theta_{n-1}(x_j)}{W'(x_j)W'(x_k)}, \tag{28}$$

for $j = 1, \ldots, m$. As $\dfrac{\Theta_n(x)\Theta_{n-1}(y) - \Theta_n(y)\Theta_{n-1}(x)}{x-y}$ is some polynomial, say $\sum_{p,q} \tau_{p,q} x^p y^q$, of degree $\max(p,q) < m-2$ in x and y, the term $k = j$ may be included in the sum as before. Still using $\sum_1^m P(x_k)/W'(x_k) = 0$ for polynomials P of degree less than $m-1$, $\dot{x}_j$ may also be removed, and

$$\begin{aligned}
\sum_{j=1}^{m} \frac{1}{z-x_j}\frac{\partial}{\partial t}\frac{\Omega_n(x_j) - V(x_j)}{W'(x_j)} &= -a_n^2 \sum_{k=1}^{m} \frac{\dot{x}_k}{W'(x_k)} \sum_{p,q} \tau_{p,q} x_k^q \sum_{j=1}^{m} \frac{x_j^p}{(z-x_j)W'(x_j)} \\
&= -a_n^2 \sum_{k=1}^{m} \frac{\dot{x}_k}{W'(x_k)} \sum_{p,q} \tau_{p,q} x_k^q \frac{z^p}{W(z)} \\
&= -a_n^2 \sum_{k=1}^{m} \frac{\dot{x}_k}{W'(x_k)} \frac{\Theta_n(z)\Theta_{n-1}(x_k) - \Theta_n(x_k)\Theta_{n-1}(z)}{W(z)\,(z-x_k)}.
\end{aligned}$$

The left-hand side is

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\sum_{j=1}^{m} \frac{1}{z-x_j}\frac{\Omega_n(x_j) - V(x_j)}{W'(x_j)}\right) - \sum_{j=1}^{m} \frac{\partial}{\partial t}\left(\frac{1}{z-x_j}\right)\frac{\Omega_n(x_j) - V(x_j)}{W'(x_j} &= \\
= \frac{\partial}{\partial t}\frac{\Omega_n(z) - V(z)}{W(z)} - \sum_{k=1}^{m} \frac{\dot{x}_k}{(z-x_k)^2}\frac{\Omega_n(x_k) - V(x_k)}{W'(x_k}&,
\end{aligned}$$

and we take the z^{-2} term in the expansion about ∞, the right-hand side vanishes as degree $\Theta_n < m-1$, and using (22) in the left-hand side:

$$\sum_0^{n-1} \dot{b}_i - \sum_1^m \dot{x}_k \frac{\Omega_n(x_k) - V(x_k)}{W'(x_k)} = 0$$

yields (26).

In concrete situations, (25) and (26) will be used, together with other non differential identities (Freud Laguerre equations for the recurrence coefficients), but we may prefer to return to (24), or even use *ad hoc* differential relations. In the generic case, (27) and (28) for $j = 1, 2, \ldots, m$ give a differential system for $2m$ unknowns $\Theta_n(x_j)$ and $\Omega_n(x_j)$, $j = 1, 2, \ldots, m$, when the quantities $a_n^2\Theta_{n-1}(x_j)$ are eliminated with the help of (20) at x_j (recall that $W(x_j) = 0$). However, considering from (14) and (22) that there are only $2m-3$ unknown coefficients in Θ_n and Ω_n, further eliminations are possible. We start with an example of generic semi-classical weight with $m = 3$.

5. Example 1. Generalized Jacobi weight with three factors $(1-x)^\alpha x^\beta (t-x)^\gamma$.

So, $W(z) = z(z-1)(z-t)$, $V(z) = (\alpha z(z-t) + \beta(z-1)(z-t) + \gamma z(z-1))/2$, the support S joins $0, 1$, and t in some way, or is an arc joining only two of these points. (14) and (22) yield readily

$$\Theta_n(z) = \nu_n z + \vartheta_n, \qquad \Omega_n(z) = \frac{\nu_n - 1}{2} z^2 + \kappa_n z + \omega_n,$$

with $\nu_n = 2n + 1 + \alpha + \beta + \gamma$, $\vartheta_n = \nu_n(b_n - 1 - t) + 2\sum_0^{n-1} b_i + b_n + \alpha + \gamma t$, $\kappa_n = \sum_0^{n-1} b_i - (\nu_n - 1)(1+t)/2 + (\alpha + \gamma t)/2$, and $\omega_n = \sum_0^{n-1}(b_i^2 - (t+1)b_i + 2a_i^2) - (t+1)(\alpha + \gamma t)/2 + (\nu_n - 1)t/2 + (\alpha + \gamma t^2)/2 + \nu_n a_n^2$.

(25) and (26) are here, with $\dot{x}_k = \delta_{k,3}$, $W'(x_3) = W'(t) = t(t-1)$,

$$\frac{\dot{a}_n}{a_n} = \frac{-2 + (\nu_n + 1)b_n - (\nu_n - 3)b_{n-1}}{2t(t-1)}, \quad \dot{b}_n = \frac{b_n(b_n - 1) + (\nu_n + 2)a_{n+1}^2 - (\nu_n - 2)a_n^2}{t(t-1)}.$$

One would have a true differential system if b_{n-1} and a_{n+1}^2 were simple functions of b_n and a_n, but this does not seem to be the case here. So, we try with the unknowns ϑ_n, κ_n and ω_n instead. In (27), using $[\Theta_n(x)\Omega_n(y) - \Theta_n(y)\Omega_n(x)]/(y-x) = (\nu_n - 1)\nu_n xy/2 + \vartheta_n(\nu_n - 1)(x+y)/2 + \zeta_n$, with $\zeta_n = \vartheta_n\kappa_n - \nu_n\omega_n$, with $x_j = 0, 1, t$, one finds three equations which are all equivalent to

$$\dot{\vartheta}_n = \frac{-\vartheta_n - \vartheta_n^2 + 2\zeta_n}{t(t-1)}. \tag{29}$$

In (28), using $[\Theta_n(x)\Theta_{n-1}(y) - \Theta_n(y)\Theta_{n-1}(x)]/(x-y) = \nu_n\vartheta_{n-1} - \vartheta_n\nu_{n-1}$, one finds two independent equations

$$\dot{\omega}_n = \frac{\omega_n}{t} - \frac{a_n^2(\nu_n\vartheta_{n-1} - \vartheta_n\nu_{n-1})}{t(t-1)},$$

$$\dot{\kappa}_n = \frac{\nu_n - 1}{2(t-1)} + \frac{\kappa_n}{t-1} + \frac{\omega_n}{t(t-1)}.$$

Now, the three non differential equations (20) at $x = 0, 1, t$ allow to eliminate $a_n^2\vartheta_{n-1}$, $a_n^2\nu_{n-1}$:

$$a_n^2\vartheta_{n-1} = \frac{\omega_n^2 - \beta^2 t^2/4}{\vartheta_n}, a_n^2\nu_{n-1} = \frac{((\nu_n-1)/2+\kappa_n+\omega_n)^2 - \alpha^2(t-1)^2/4}{\nu_n+\vartheta_n} - a_n^2\vartheta_{n-1},$$

and a third equation allowing to eliminate either κ_n or ω_n, actually it is simpler to give everything in function of ζ_n: from $a_{n-1}^2[\vartheta_{n-1}/t - (\vartheta_{n-1}+\nu_{n-1})/(t-1) + (\vartheta_{n-1} + \nu_{n-1}t)/(t(t-1))] = 0$,

$$\omega_n = -\frac{\alpha^2\vartheta_n(t-1)/4}{(\nu_n-1)(\nu_n+\vartheta_n)} + \frac{\beta^2 t/4}{\nu_n-1} + \frac{\gamma^2\vartheta_n t(t-1)/4}{(\nu_n-1)(\nu_n t+\vartheta_n)} - \\ - \frac{(\nu_n-1)\vartheta_n[\nu_n t(t+1)+\vartheta_n(t^2+t+1)]/4 + [\nu_n t+\vartheta_n(t+1)]\zeta_n + \zeta_n^2/(\nu_n-1)}{(\nu_n+\vartheta_n)(\nu_n t+\vartheta_n)}.$$

This allows to give $\dot\zeta_n$ as a function of ϑ_n and ζ_n, so to complete (29):

$$\dot\zeta_n = \dot\vartheta_n\kappa_n + \vartheta_n\dot\kappa_n - \nu_n\dot\omega_n \\ = \frac{-\vartheta_n - \vartheta_n^2 + 2\zeta_n}{t(t-1)}\kappa_n + \frac{(\nu_n-1)\vartheta_n}{2(t-1)} + \frac{\kappa_n\vartheta_n}{t-1} + \frac{\omega_n\vartheta_n}{t(t-1)} - \frac{\nu_n\omega_n}{t} + \frac{a_n^2\nu_n(\nu_n\vartheta_{n-1} - \vartheta_n\nu_{n-1})}{t(t-1)}.$$

Using the preceding calculations, $a_n^2\vartheta_{n-1}$ and $a_n^2\nu_{n-1}$ are replaced in terms of κ_n and ω_n, then $\kappa_n = (\zeta_n + \nu_n\omega_n)/\vartheta_n$ is used, and ω_n is finally replaced as a function of ζ_n, and what comes out is

$$\dot\zeta_n = \frac{1}{t(t-1)}\left\{\frac{\alpha^2(t-1)\vartheta_n(\nu_n t+\vartheta_n)}{4(\nu_n+\vartheta_n)} - \frac{\beta^2 t(\nu_n+\vartheta_n)(\nu_n t+\vartheta_n)}{4\vartheta_n} + \right. \\ + \frac{(1-\gamma^2)t(t-1)\vartheta_n(\nu_n+\vartheta_n)}{4(\nu_n t+\vartheta_n)} + \left(\frac{1}{\nu_n+\vartheta_n} + \frac{1}{\vartheta_n} + \frac{1}{\nu_n t+\vartheta_n}\right)(\zeta_n - \vartheta_n(\vartheta_n+1)/2)^2 + \\ \left. + \left(2\vartheta_n + 1 + \frac{\nu_n t(t-1)}{\nu_n t+\vartheta_n}\right)(\zeta_n - \vartheta_n(\vartheta_n+1)/2) + \frac{\vartheta_n(\nu_n+\vartheta_n)(\nu_n t+\vartheta_n)}{4t(t-1)}\right\}$$

whence, at last, with $\zeta_n - \vartheta_n(\vartheta_n+1)/2 = t(t-1)\dot\vartheta_n/2$:

$$\ddot\vartheta_n = \frac{1}{t(t-1)}(-2t\dot\vartheta_n - 2\vartheta_n\dot\vartheta_n + 2\dot\zeta_n) = \\ = \frac{1}{2}\left(\frac{1}{\nu_n+\vartheta_n} + \frac{1}{\vartheta_n} + \frac{1}{\nu_n t+\vartheta_n}\right)\dot\vartheta_n^2 - \left(\frac{1}{t} + \frac{1}{t-1} - \frac{\nu_n}{\nu_n t+\vartheta_n}\right)\dot\vartheta_n + \\ + \frac{\alpha^2\vartheta_n(\nu_n t+\vartheta_n)}{2t^2(t-1)(\nu_n+\vartheta_n)} - \frac{\beta^2(\nu_n+\vartheta_n)(\nu_n t+\vartheta_n)}{2t(t-1)^2\vartheta_n} + \frac{(1-\gamma^2)\vartheta_n(\nu_n+\vartheta_n)}{2t(t-1)(\nu_n t+\vartheta_n)} + \\ + \frac{\vartheta_n(\nu_n+\vartheta_n)(\nu_n t+\vartheta_n)}{2t^2(t-1)^2}.$$

which is a Painlevé equation of the sixth kind ([In] § 14.4) in $-\vartheta_n/\nu_n$ (the zero of Θ_n) ([Chua] p.399-402, explaining works of R.Fuchs on equations of form (21)).

We can return to a_n and b_n as functions of ϑ_n and ζ_n by using again $a_n^2\nu_{n-1}$ as a function of ϑ_n, κ_n and ω_n (and $\nu_{n-1} = 2n+\alpha+\beta+\gamma-1$ is known) and taking b_n from $2\kappa_n - \vartheta_n = -(2\nu_n-1)(1+t)-(\nu_n+1)b_n$. Inverting the connection should give a (probably algebraic) differential system involving only a_n and b_n (will somebody do that?)

6. Example 2. $\exp(x^3/3+tx)$ on $\{x : x^3 < 0\}$.

Much simpler identities occur when the weight w is the exponential of a polynomial, so that w'/w is a polynomial itself. Recall (end of Section 2) that $W(x)w(x) \to 0$ when x tends to the endpoints (if any) of the support S. We want the simplest case ($W(x) = 1$), so that the support cannot have finite endpoints, but must end on directions where $w(x) \to 0$, with at least one complex direction (or else all the moments vanish). So, we can take the set $\{x : x^3 < 0\}$, or for instance only $\{x : \arg x = \pm 2\pi/3\}$, or also some equivalent contour, as $\{x : \mathrm{Re}\, x = \text{a positive constant}\}$ leading to Airy functions and integrals ([Chu2] [Mar1]). The weight can be considered as a confluent generalized Jacobi weight with singular points at ∞: $w(z) = \lim_{N\to\infty}[1 + (z^3/3 + tz)/N]^N$, with an exponent N independent of the parameter t. As (25) and (26) hold for distinct finite singular points, we return to (24) assumed to be still valid: here, $W(z) = 1$, $2V(z) = w'(z)/w(z) = z^2 + t$. Working (13) and (15) about ∞, we have

$$\Theta_n(z) = z + b_n, \qquad \Omega_n(z) = (z^2+t)/2 + a_n^2.$$

Pushing (13) and (15) up to the z^{-1} term, one finds the corresponding Laguerre-Freud equations, i.e., the identities

$$a_n^2 + a_{n+1}^2 + b_n^2 + t = 0, \qquad n + a_n^2(b_n + b_{n-1}) = 0. \tag{30}$$

We compute H in (23) up to the $O(1)$ term, as H is now expected to be a polynomial (see [Fed] § 2), taking care of $\dot{w}/w = z$:

$$A = \begin{bmatrix} a_n^2 & -a_n(z+b_n) \\ a_n(z+b_{n-1}) & -a_n^2 - z^2 - t \end{bmatrix}, \quad H = \begin{bmatrix} \dot\gamma_n/\gamma_n & -a_n \\ a_n & -\dot\gamma_{n-1}/\gamma_{n-1} - z \end{bmatrix},$$

The diagonal elements of (24) yield $2\dot{a}_n = a_n(b_n - b_{n-1})$, and the off-diagonal elements: $2\dot\gamma_n/\gamma_n + b_n = 0$, $\dot{a}_n b_n + a_n\dot{b}_n = a_n(\dot\gamma_n/\gamma_n + \dot\gamma_{n-1}/\gamma_{n-1})b_n - 2a_n^3 - a_n t$ and $\dot{a}_n b_{n-1} + a_n\dot{b}_{n-1} = -a_n(\dot\gamma_n/\gamma_n + \dot\gamma_{n-1}/\gamma_{n-1})b_{n-1} + 2a_n^3 + a_n t$. Using (30), all these equations are compatible with the differential system

$$\begin{cases} \dfrac{\dot{a}_n}{a_n} &= \quad b_n + \dfrac{n}{2a_n^2}, \\ \dot{b}_n &= \quad -b_n^2 - 2a_n^2 - t, \end{cases}$$

which is the differential system equivalent to the second Painlevé equation for $(-b_n, 4a_n^2)$ ([Chu2], [Ge] p.339). The connection with Painlevé transcendents can lead to advances in the solution of the problem posed by Maroni in [Mar1]: when do we have $a_1, a_2, \ldots \neq 0$ in (30)? The problem is now to localize the zeros of solutions of special Painlevé equations.

7. Example 3. $\exp(-x^4/4 - tx^2)$ on $\mathbb{R}$.

This is the simplest nontrivial Freud's weight, and the corresponding orthogonal polynomials have been much worked ([BoN] [Fr2] [LeQ] [Lub] [Mag2] [Mag3] [NeV] [GFOPCF] [NeV2] [Sho]). As for example 2, we expand (13) and (15) with $W(z) = 1$ and $2V(z) = -z^3 - 2tz$:

$$\Theta_n(z) = -z^2 - 2t - a_n^2 - a_{n+1}^2, \qquad \Omega_n(z) = -z^3/2 - (a_n^2 + t)z.$$

A relation between the a_n's is found by expanding (20), equating the z^2 terms gives

$$a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2) + 2ta_n^2 = n, \; n = 1, 2, \ldots \; (a_0 = 0) \tag{31}$$

a relation which seems to have been found by Shohat ([Sho]), rediscovered by Freud [Fr2] and Bessis [Bes]. Remark that we have a degree of freedom on a_1: this is because the weight can have the real axis *and* the pure imaginary axis in its support, with $w(x) = \lambda \exp(-x^4/4 - tx^2)$ on the pure imaginary axis, and the preceding results hold for any λ, so a_1 is some function (which can be computed from first moments) of λ. However, if it is requested that all the a_n's are positive, the solution is unique and can be computed efficiently ([LeQ], see also [Nev2] p.470). Now, (17) and (23) are computed:

$$A = \begin{bmatrix} -a_n^2 z & a_n(z^2 + 2t + a_n^2 + a_{n+1}^2) \\ -a_n(z^2 + 2t + a_{n-1}^2 + a_n^2) & z^3 + (a_n^2 + 2t)z \end{bmatrix},$$

$$H = \begin{bmatrix} \dot\gamma_n/\gamma_n - a_n^2 & a_n z \\ -a_n z & -\dot\gamma_{n-1}/\gamma_{n-1} + z^2 + a_n^2 \end{bmatrix},$$

The equations from (24) amount to be equivalent to

$$\frac{\dot\gamma_n}{\gamma_n} = \frac{a_n^2 + a_{n+1}^2}{2}, \tag{32}$$

which, with $a_n\gamma_n = \gamma_{n-1}$, gives

$$\frac{\dot a_n}{a_n} = \frac{a_{n-1}^2 - a_{n+1}^2}{2}. \tag{33}$$

Actually, (32) *(and* (33)*) can be recovered by quite elementary means:* let $\{p_n(x;t)\}$ be the polynomials orthonormal with respect to an even measure of the form $d\sigma(x;t) = \exp(-tx^2)d\sigma(x;0)$ on some support S, we have then for the monic orthogonal polynomials p_n/γ_n:

$$\frac{\partial}{\partial t}\frac{1}{\gamma_n^2} = \frac{\partial}{\partial t}\int_S \left(\frac{p_n(x;t)}{\gamma_n}\right)^2 \exp(-tx^2)\, d\sigma(x;0) =$$

$$= -\int_S x^2 \left(\frac{p_n(x;t)}{\gamma_n}\right)^2 \exp(-tx^2)\, d\sigma(x;0) = -\frac{a_n^2 + a_{n+1}^2}{\gamma_n^2},$$

using $x^2 p_n = a_n a_{n-1} p_{n-2} + (a_n^2 + a_{n+1}^2) p_n + a_{n+1} a_{n+2} p_{n+2}$ from (2) when $b_n = 0$, and that the derivative in t of a monic polynomial must be of degree $< n$. *Conversely*, it has been shown that (33) implies that the a_n's are the coefficients of the recurrence of orthogonal polynomials with respect to a measure of the form $\exp(-tx^2) d\sigma(x;0)$ where $d\sigma(x;0)$ does not depend on t [KvM] [Mo] (see also [Fra1], [Fra2]), [Y], [93, pp. 45–46].

It is even probably possible to recover the information given by (24) for all the semi-classical orthogonal polynomials by more elementary means, but the connection with monodromy theory, interesting on its own right, has more advantages: for instance, it is known that the differential equations produced by (24) have the Painlevé property (foreword of [Pain], see [Mal] for a modern proof), i.e., movable singular points can only be poles (see [Cha], [In] chap. 14). No wonder that the classical Painlevé transcendants appear in these examples.

Now, we get an equation for the single a_n using (31): let $u_n = a_n^2$, from $\dot{u}_n = u_n(u_{n-1} - u_{n+1})$,

$$\begin{aligned}
\ddot{u}_n &= \dot{u}_n(u_{n-1} - u_{n+1}) + u_n(\dot{u}_{n-1} - \dot{u}_{n+1}) \\
&= u_n(u_{n-1} - u_{n+1})^2 + u_n[u_{n-1}(u_{n-2} - u_n) - u_{n+1}(u_n - u_{n+2})] \\
&= u_n(u_{n-1} - u_{n+1})^2 + u_n[n - 1 - 2tu_{n-1} - u_{n-1}(u_{n-1} + 2u_n) + \\
&+ n + 1 - 2tu_{n+1} - u_{n+1}(u_{n+1} + 2u_n)] \\
&= u_n[2n - 2(u_n + t)(u_{n-1} + u_{n+1}) - 2u_{n-1}u_{n+1}] \\
&= u_n[2n - 2(u_n + t)(u_{n-1} + u_{n+1}) - (u_{n-1} + u_{n+1})^2/2 + (u_{n-1} - u_{n+1})^2/2] \\
&= u_n[2n + 2(u_n + t)^2 - (u_{n-1} + 2u_n + u_{n+1} + 2t)^2/2] + (\dot{u}_n)^2/(2u_n) \\
&= u_n[2n + 2(u_n + t)^2 - (n/u_n + u_n)^2/2] + (\dot{u}_n)^2/(2u_n) \\
&= \frac{u_n}{2}\left[4(u_n + t)^2 - \left(\frac{n}{u_n} - u_n\right)^2\right] + \frac{(\dot{u}_n)^2}{2u_n} \\
&= \frac{(\dot{u}_n)^2}{2u_n} + \frac{1}{2u_n}\left(3u_n^2 + 2tu_n - n\right)\left(u_n^2 + 2tu_n + n\right),
\end{aligned}$$

which is a special case of the 4$^{\text{th}}$ *Painlevé equation*

$$\ddot{y} = \frac{\dot{y}^2}{2y} + \frac{3y^3}{2} + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \tag{34}$$

with $\alpha = -n/2$ and $\beta = -n^2/2$ [Bu] [Fok1] [Fok2] [Ge] [Ok]. The connection was first discovered by Kitaev [Fok1, 91,92, 93 p. 35].

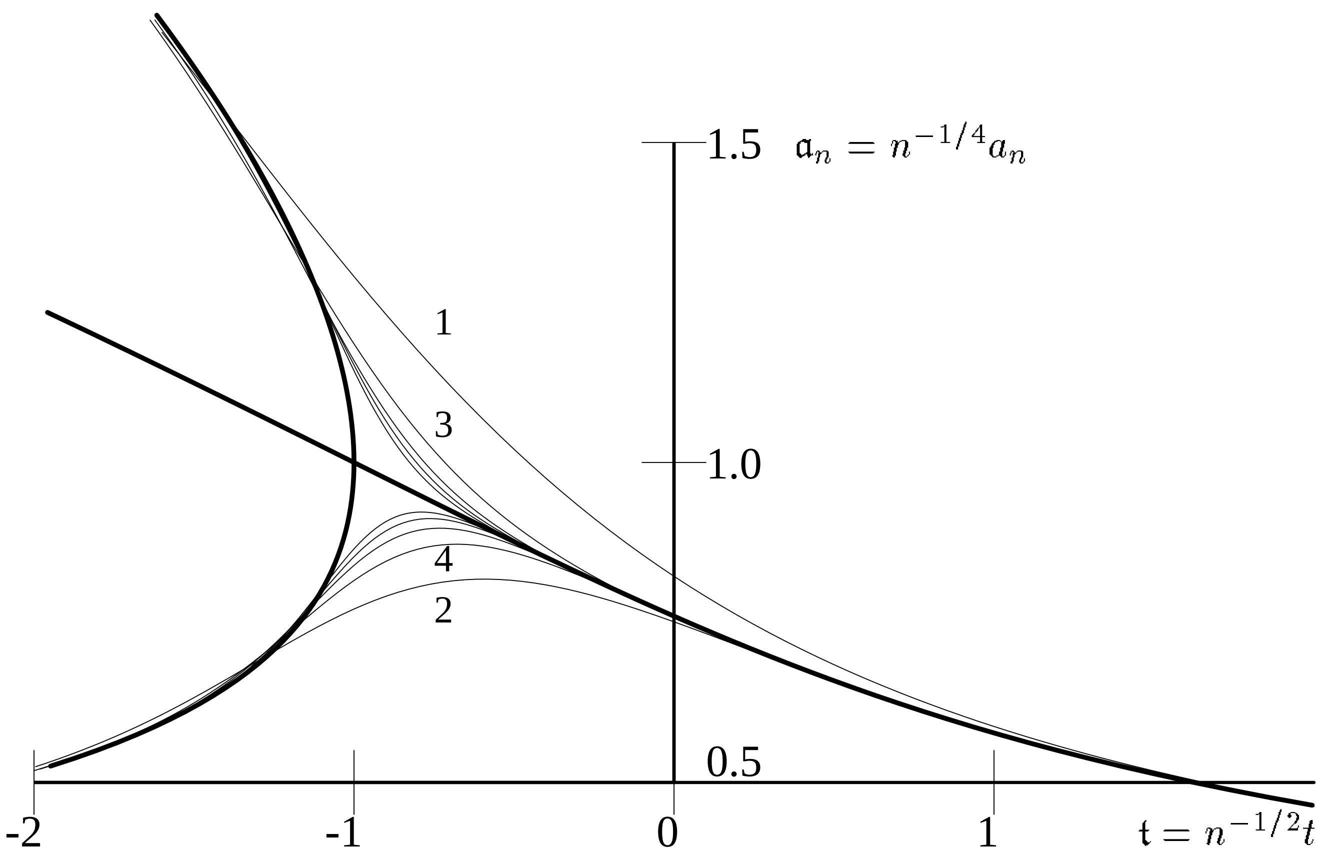
For $a_n = \sqrt{u_n}$, one has a form without first derivative:

$$4a_n^3 \ddot{a}_n = (3a_n^4 + 2ta_n^2 - n)(a_n^4 + 2ta_n^2 + n). \tag{35}$$

Let $\mathfrak{a}_n = n^{-1/4} a_n$ and $\mathfrak{t} = n^{-1/2} t$, then we have another form

$$4\mathfrak{a}_n^3 \frac{d^2}{d\mathfrak{t}^2}\mathfrak{a}_n = n^2(3\mathfrak{a}_n^4 + 2\mathfrak{t}\mathfrak{a}_n^2 - 1)(\mathfrak{a}_n^4 + 2\mathfrak{t}\mathfrak{a}_n^2 + 1). \tag{36}$$

What can be the use of these equations? To explore these things, let us first look at the graph of some $\mathfrak{a}_n$'s computed with the Lew & Quarles method [LeQ]:



$\mathfrak{a}_1, \ldots, \mathfrak{a}_{10}$ (only $\mathfrak{a}_1, \ldots, \mathfrak{a}_4$ are marked) tend to be close to the zeros of the right-hand side of (36) (thick line). In particular, $\mathfrak{a}_n(t) \sim 1/\sqrt{2\mathfrak{t}}$ when $\mathfrak{t} \to +\infty$: $a_1^2 = \mu_2/\mu_0 = \int_{-\infty}^{\infty} x^2 w(x)dx / \int_{-\infty}^{\infty} w(x)dx$, where $w(x) = \exp(-x^4/4 - tx^2)$. From [Erd] p.119, $\mu_0 = \sqrt{\pi\sqrt{2}}\exp(t^2/2)D_{-1/2}(t\sqrt{2})$ (parabolic cylinder function). When $t \to +\infty$, $\mu_0 \sim \sqrt{\pi/t}$ ([Erd] p.122), so $a_1^2 = \mu_2/\mu_0 = -\dot{\mu}_0/\mu_0 \sim 1/(2t)$. From (31), if $a_1, a_2, \ldots, a_{n-1}$

are $O(t^{-1/2})$, $a_n \sim \sqrt{n/(2t)}$ when $t \to \infty$. When $t \to -\infty$, $\mu_0 \sim$ constant $t^{-1/2}\exp(t^2)$ ([Erd] p.123), so $a_1 \sim \sqrt{-2t}$. The figure suggests that $\mathfrak{a}_n(\mathfrak{t}) \sim \sqrt{-2\mathfrak{t}}$ when $t \to -\infty$ and n is odd, while $\mathfrak{a}_n(\mathfrak{t}) \sim \sqrt{-1/(2\mathfrak{t})}$ when n is even.

One of the most interesting uses of expressions of recurrence coefficients where n is not bound to be an integer is to define **general associated orthogonal polynomials**, i.e., polynomials defined by $a_{n+\nu+1}p_{n+1}^{(\nu)}(z) = (z - b_{n+\nu})p_n^{(\nu)}(z) - a_{n+\nu}p_{n-1}^{(\nu)}(z)$, and degree $p_n^{(\nu)} = n$ (as in [AW], [ILVW]). So, let us define a_ν as some solution of (35) with n replaced by ν:

$$4a_\nu^3\ddot{a}_\nu = (3a_\nu^4 + 2ta_\nu^2 - \nu)(a_\nu^4 + 2ta_\nu^2 + \nu), \tag{37}$$

where ν is a given complex number. Then, $y = \left[\dfrac{\nu}{2a_\nu^2} - \dfrac{a_\nu^2}{2} - t \mp \dfrac{\dot{a}_\mu}{a_\mu}\right]^{1/2}$ satisfies the same equation (37), but with ν replaced by $\nu \pm 1$ (Schlesinger transformation, [Fok1] §3.3). Indeed, derivating $y^2 + t + (a_\nu^2 - \nu/a_\nu^2)/2 = \mp\dot{a}_\nu/a_\nu$ yields $2y\dot{y} = \mp(\nu \pm 1 - 2a_\nu^2y^2 - y^4 - 2ty^2)$ and a new derivation establishes the property. So, the definition makes sense and (31) still holds with ν. There are still two degrees of freedom in (37), but they are removed when suitable boundary conditions are fixed ([DeC1] [DeC2]). Here, we just have to impose $a_\nu = O(t^{-1/2})$ when $t \to +\infty$ ([Yos], quoting Malmquist; the point being that $a_\nu(t)$ must have an asymptotic series when $t \to +\infty$ for fixed ν, the relation with the dual situation, i.e., t fixed and $\nu \to +\infty$ is striking, see Section 4 of [Wi]). In summary:

For any real or complex ν, the associated Freud orthogonal polynomials p_n^ν (which are related to the weight $\exp(-x^4/4 - tx^2$ on $\mathbb{R}$ when $\nu = 0$) have recurrence coefficient $a_{\nu+1}(t), a_{\nu+2}(t), \ldots$, where $a_\mu(t)$ is completely defined as the solution of

$$4a_\mu^3\ddot{a}_\mu = (3a_\mu^4 + 2ta_\mu^2 - \mu)(a_\mu^4 + 2ta_\mu^2 + \mu),$$

which remains $O(t^{-1/2})$ when $t \to +\infty$.

For the associated polynomials themselves, we can now construct $\Theta_{\nu+n}$ and $\Omega_{\nu+n}$, therefore a differential equation (18) with index $\nu + n$. Let $\varphi_{\nu+n}$ and $\psi_{\nu+n}$ be two independent solutions of this differential equation (in z). Following Hahn ([Ha1] eq. (17)), $p_n^{(\nu)} = (\varphi_{\nu+n}\psi_{\nu-1} - \psi_{\nu+n}\varphi_{\nu-1})/(\varphi_\nu\psi_{\nu-1} - \psi_\nu\varphi_{\nu-1})$. It can then be shown that $f_\nu = \lim_{n\to\infty} p_{n-1}^{(\nu+1)}/p_n^{(\nu)}$ satisfies a *Riccati equation* (Laguerre-Hahn class [Mag1]).

8. Example 4. $(x - t)^\rho \exp(-x^2)$ on $[t, \infty)$.

The corresponding orthogonal polynomials are called (when $t = \rho = 0$) the Maxwell polynomials in [BeR], where other references can be found ($\rho = 1$: speed polynomials in [ClS]). This case is closely related to the preceding one: put $x = t + u^2/2$ in $\int_t^\infty p_n(x)p_m(x)(x - t)^\rho \exp(-x^2)\,dx = \delta_{m,n}$ to find that $\tilde{p}_{2n}(u) =$

$2^{-(\rho+1)/2}\exp(-t^2/2)p_n(t+u^2/2)$ is the orthonormal polynomial of degree $2n$ with respect to the weight $\tilde{w}(u)=|u|^{2\rho+1}\exp(-u^4/4-tu^2)$ on $\mathbb{R}$. So, we have $a_n=\tilde{a}_{2n}\tilde{a}_{2n-1}/2$ and $b_n=t+(\tilde{a}_{2n}^2+\tilde{a}_{2n+1}^2)/2$ ([Chi] , etc.).

For the $\tilde{a}_n$'s, we still have $\dot{\tilde{a}}_n=\tilde{a}_n(\tilde{a}_{n-1}^2-\tilde{a}_{n+1}^2)/2$ as before, but a slightly different recurrence relation $\tilde{a}_n^2(\tilde{a}_{n-1}^2+\tilde{a}_n^2+\tilde{a}_{n+1}^2+2t)=n+(2\rho+1)\,\mathrm{odd}(n)$, where $\mathrm{odd}(n)=(1-(-1)^n)/2$ [Fr2] [Mag2]. Working this yields now ($u_n=\tilde{a}_n^2$).

$$\ddot{u}_n=\frac{\dot{u}_n^2}{2u_n}+\frac{3u_n^3}{2}+4tu_n^2+2\left(t^2+\frac{n}{2}+(2\rho+1)\frac{1+3(-1)^n}{4}\right)u_n-\frac{(n+(2\rho+1)\mathrm{odd}(n))^2}{2u_n},$$

i.e., the Painlevé 4^{th} equation (34) with $\alpha=-n/2-(2\rho+1)(1+3(-1)^n)/4$ and $\beta=-(n+(2\rho+1)\mathrm{odd}(n))^2/2$.

Many almost-classical orthogonal polynomials (see [Chin], [ClS] and references in [BeR] and [Gau]) could still be worked, and the simplest of them will likely be related to other Painlevé transcendents (perhaps not the *first* one... although [Fok2] finds first Painlevé transcendents as solutions of a limit case of (31)) At least a new case is briefly presented now:

9. Example 5. Beyond Painlevé: $\exp(-x^6-tx^2)$ on $\mathbb{R}$.

With $u_n=a_n^2$, $\dot{u}_n=u_n(u_{n-1}-u_{n+1})$ still holds, but the recurrence relation is somewhat more complicated than before [Fr2] [Mag2] [Mag3]: $u_n(u_{n-2}u_{n-1}+u_{n-1}^2+2u_{n-1}u_n+u_n^2+2u_nu_{n+1}+u_{n-1}u_{n+1}+u_{n+1}^2+u_{n+1}u_{n+2}+2t)=n$, for $n=1,2,\ldots$ As a first step, one has a differential system for $u_{n-1},\ldots,u_{n+2}$ by eliminating u_{n-2} and u_{n+3} from the recurrence relation:

$$\begin{cases}\dot{u}_{n-1}=u_{n-1}u_{n-2}-u_{n-1}u_n=\\ \quad=\dfrac{n}{u_n}-2t-u_{n-1}^2-3u_{n-1}u_n-u_n^2-2u_nu_{n+1}-u_{n-1}u_{n+1}-u_{n+1}^2-u_{n+1}u_{n+2},\\ \dot{u}_n=u_n(u_{n-1}-u_{n+1}),\\ \dot{u}_{n+1}=u_{n+1}(u_n-u_{n+2}),\\ \dot{u}_{n+2}=u_{n+1}u_{n+2}-u_{n+2}u_{n+3}=\\ \quad=-\dfrac{n+1}{u_{n+1}}+2t+u_{n+2}^2+u_nu_{n+2}+3u_{n+1}u_{n+2}+u_{n+1}^2+2u_nu_{n+1}+u_n^2+u_{n-1}u_n,\end{cases}$$

which can still be transformed... This case in considered in [Fok2].

Acknowledgements.

Many thanks to R.Askey, R.Caboz, D.& G.Chudnovsky, C. De Coster, L.Haine, J.Meinguet, A.Ronveaux, P. van Moerbeke, M.Willem, F. Marcellán, A. Its, the referees and editors.

References.

[Ak] N.I. AKHIEZER, *Elements of the Theory of Elliptic Functions*, translated from the 2^{nd} Russian edition (Nauka, Moscow, 1970), *Transl. Math. Monographs* **79**, A.M.S., Providence, 1990.

[Al] F. MARCELLÁN, I. ALVAREZ ROCHA, On semiclassical linear functionals: integral representations. This volume pp. 239–249.

[Apt] A.I. APTEKAREV, Asymptotic properties of polynomials orthogonal on a system of contours and periodic motions of Toda lattices, *Mat. Sb.* **125** (1984) 231-258, = *Math. USSR Sbornik* **53** (1986) 233-260.

[AW] R.ASKEY, J.WIMP, Associated Laguerre and Hermite polynomials, *Proc. Royal Soc. Edinburgh* **96A** (1984) 15-37.

[AtE] F.V.ATKINSON, W.N.EVERITT, Orthogonal polynomials which satisfy second order differential equations, pp. 173-181 *in E.B.Christoffel* (P.L.BUTZER and F.FEHÉR, editors), Birkhäuser, Basel, 1981.

[Bel] S. BELMEHDI, On semi-classical linear functionals of class $s = 1$. Classification and integral representations. *Indag. Mathem.* N.S. **3** (3) (1992), 253-275.

[BeR] S. BELMEHDI, A. RONVEAUX, Laguerre-Freud's equations for the recurrence coefficients of semi-classical orthogonal polynomials, to appear in *J. Approx. Theory*.

[BeR2] S. BELMEHDI, A. RONVEAUX, On the coefficients of the three-term recurrence relation satisfied by some orthogonal polynomials. *Innovative Methods in Numerical Analysis*, Bressanone, Sept. 7-11^{th}, 1992.

[Bes] D. BESSIS, A new method in the combinatorics of the topological expansion, *Comm. Math. Phys.* **69** (1979), 147-163.

[BIZ] D. BESSIS, C. ITZYKSON, J.B. ZUBER, Quantum field theory techniques in graphical enumeration, *Adv. in Appl. Math.* **1** (1980), 109-157.

[BoN] S. BONAN, P.NEVAI, Orthogonal polynomials and their derivatives,I, *J. Approx. Theory* **40** (1984), 134-147.

[BLN] S.S. BONAN, D.S. LUBINSKY, P.NEVAI, Orthogonal polynomials and their derivatives,II, *SIAM J. Math. An.* **18** (1987), 1163-1176.

[Bre] C. BREZINSKI, *Padé-type Approximation and General Orthogonal Polynomials ISNM* **50**, Birkhäuser-Verlag, Basel, 1980.

[Brez] C. BREZINSKI, *History of Continued Fractions and Padé Approximants*, Springer-Verlag, Berlin, 1991.

[dBvR] M.G. de BRUIN, H. van ROSSUM, Formal Padé approximation, *Nieuw Arch. Wisk.* (3) **23** (1975), 115-130.

[Bu] F.J. BUREAU, Les équations différentielles du second ordre à points critiques fixes, II. Les intégrales de l'équation A4 de Painlevé, *Bull. Cl. Sci. Acad. Roy. Belg.* **69** (1983) 397-433.

[Cha] R. CHALKLEY, New contributions to the related works of Paul Appell, Lazarus Fuchs, Georg Hamel, and Paul Painlevé on nonlinear differential equations whose solutions are free of movable branch points. *J. Diff. Eq.* **68** (1987) 72-117.

[Chi] T.S. CHIHARA, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.

[Chin] R.C.Y. CHIN, A domain decomposition method for generating orthogonal polynomials for a Gaussian weight on a finite interval, *J. Comp. Phys.* **99** (1992) 321-336.

[Chua] D.V. CHUDNOVSKY, Riemann monodromy problem, isomonodromy deformation eqations and completely integrable systems, pp.385-447 *in Bifurcation Phenomena in Mathematical Physics and Related Topics, Proceedings Cargèse, 1979* (C.BARDOS & D.BESSIS, editors), NATO ASI series C, vol. **54**, D.Reidel, Dordrecht, 1980.

[Chub] G.V. CHUDNOVSKY, Padé approximation and the Riemann monodromy problem, pp.449-510 *in Bifurcation Phenomena in Mathematical Physics and Related Topics, Proceedings Cargèse, 1979* (C.BARDOS & D.BESSIS, editors), NATO ASI series C, vol. **54**, D.Reidel, Dordrecht, 1980.

[Chu0] D.V.CHUDNOVSKY, G.V.CHUDNOVSKY, Introduction to *The Riemann Problem, Complete Integrability and Arithmetic Applications* (D. Chudnovsky and G. Chudnovski, Eds.), pp.1-11, Springer-Verlag (Lecture Notes Math. **925**), Berlin, 1982.

[Chu1] D.V.CHUDNOVSKY, G.V.CHUDNOVSKY, Laws of composition of Bäcklund transformations and the universal form of completely integrable systems in dimensions two and three, *Proc. Nat. Acad. Sci. USA* **80** (1983) 1774-1777.

[Chu2] D.V.CHUDNOVSKY, G.V.CHUDNOVSKY, High precision computation of special function in different domains, talk given at Symbolic Mathematical Computation conference, Oberlech, July 1991.

[ClS] A.S. CLARKE, B. SHIZGAL, On the generation of orthogonal polynomials using asymptotic methods for recurrence coefficients. *J. Comp. Phys.* **104** (1993) 140-149.

[DeC1] C.DE COSTER, M.WILLEM, private communication, 18 September 1992.

[DeC2] C.DE COSTER, M.WILLEM, Density, spectral theory and homoclinics for singular Sturm-Liouville systems, to appear in *J. Comp. Appl. Math.*

[Dra] A. DRAUX, *Polynômes orthogonaux formels – Applications*, Lect. Notes Math. **974**, Springer-Verlag, Berlin, 1983.

[Erd] A. ERDÉLYI *et al.*, editors, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.

[Fed] M.V. FEDORYUK, Isomonodromy deformations of equations with irregular singularities, *Mat. Sb.* **181** (1990) = *Math. USSR Sb.* **71** (1992) 463-479.

[Fok1] A.S. FOKAS, U. MUGAN, M.J. ABLOWITZ, A method of linearization for Painlevé equations. Painlevé IV,V. *Physica D* **30** (1988) 247-283.

[Fok2] A.S. FOKAS, A.R. ITS, A.V. KITAEV, Discrete Painlevé equations and their appearance in quantum gravity, *Commun. Math. Phys.* **142** (1991) 313-344.

[Fra1] J.P. FRANCOISE, Symplectic geometry and integrable m-body problems on the line, *J. Math. Phys.***29** (1988) 1150-1153.

[Fra2] J.P. FRANCOISE, Systèmes intégrables à m corps sur la droite, *in Analyse Globale et Physique Mathématique*, Dec. 1989, preprint.

[Fr1] G. FREUD, *Orthogonal Polynomials*, Akadémiai Kiadó/Pergamon Press, Budapest/Oxford, 1971.

[Fr2] G.FREUD, On the coefficients in the recursion formulæ of orthogonal polynomials, *Proc. Royal Irish Acad. Sect. A* **76** (1976), 1-6.

[RFu] R. FUCHS, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singuläre Stellen, *Math. Ann.* **63** (1907) 301-321.

[GaN] J.L. GAMMEL, J. NUTTALL, Note on generalized Jacobi polynomials, *in* "The Riemann Problem, Complete Integrability and Arithmetic Applications" (D. Chudnovsky and G. Chudnovski, Eds.), pp.258-270, Springer-Verlag (Lecture Notes Math. **925**), Berlin, 1982.

[GaCL] J.P. GASPARD, F. CYROT-LACKMANN, Density of states from moments. Application to the impurity band, *J. Phys. C: Solid State Phys.* **6** (1973) 3077-3096.

[Gau] W.GAUTSCHI, Computational aspects of orthogonal polynomials, pp.181-216 *in Orthogonal Polynomials: Theory and Practice* (P.NEVAI, editor) *NATO ASI Series C* **294**, Kluwer, Dordrecht, 1990.

[Ge] R.GERARD, La géométrie des transcendantes de P.Painlevé, pp.323-352 *in Mathématique et Physique, Séminaire de l'Ecole Normale Supérieure 1979-1982* (L. BOUTET de MONVEL, A.DOUADY & J.L.VERDIER, editors), *Progress in Mathematics* **37**, Birkhäuser, Boston, 1983.

[GV1] J.S.GERONIMO, W.VAN ASSCHE, Orthogonal polynomials with asymptotically periodic recurrence coefficients, *J. Approx. Th.* **46** (1986) 251-283.

[GV2] J.S.GERONIMO, W.VAN ASSCHE, Approximating the weight function for orthogonal polynomials on several intervals, *J. Approx. Th.* **65** (1991), 341-371.

[Gr] C.C. GROSJEAN, The measure induced by orthogonal polynomials satisfying a recursion formula with either constant or periodic coefficients. Part I: Constant coefficients, *Acad. Analecta, Kon. Acad. Wet. Lett. Sch. Kunsten Belg.* **48**, Nr. 3, (1986), 39-60. Part II: Pure or mixed periodic coefficients (general theory), *ibid.* **48** Nr. 5 (1986) 55-94.

[GrM1] D.J. GROSS, A.A. MIGDAL, A nonperturbative treatment of two-dimensional quantum gravity. Princeton preprint PUPT 1159 (1989).

[GrM2] D.J. GROSS, A.A. MIGDAL, Nonperturbative two-dimensional quantum gravity. *Phys. Rev. Letters* **64** (1990) 127-130.

[Ha1] W.HAHN, On differential equations for orthogonal polynomials, *Funk. Ekvacioj*, **21** (1978) 1-9.

[Ha2] W.HAHN, Über Orthogonalpolynome, die linearen funktionalgleichungen genügen, pp. 16-35 *in Polynômes Orthogonaux et Applications, Proceedings, Bar-le-Duc 1984*, (C.BREZINSKI & al., editors), *Lecture Notes Math.* **1171**, Springer, Berlin 1985.

[HH] L. HAINE, E. HOROZOV, Toda orbits of Laguerre polynomials and representations of the Virasoro algebra. Preprint Institut Mathématique Université Catholique de Louvain 217 (1992).

[Hay] R. HAYDOCK, The recursive solution of the Schrödinger equation, pp.215-294 *in* H. EHRENREICH *et al.*, editors: *Solid State Physics* **35** , Ac. Press, N.Y. , 1980.

[HayN] R. HAYDOCK, C.M.M. NEX, A general terminator for the recursion method, *J. Phys. C: Solid State Phys.* **18** (1985) 2235-2248.

[HvR1] E. HENDRIKSEN, H. van ROSSUM, A Padé-type approach to non-classical orthogonal polynomials, *J. Math. An. Appl.* **106** (1985) 237-248.

[HvR2] E. HENDRIKSEN, H. van ROSSUM, Semi-classical orthogonal polynomials, pp. 354-361 *in Polynômes Orthogonaux et Applications, Proceedings, Bar-le-Duc 1984*, (C.BREZINSKI & al., editors), *Lecture Notes Math.* **1171**, Springer, Berlin 1985.

[In] E.L. INCE, *Ordinary Differential Equations*, Longmans Green 1928 = Dover 1956.

[I] M.ISMAIL, On sieved orthogonal polynomials III: orthogonality on several intervals, *Trans. Amer. Math. Soc.* **294** (1986), 89-111.

[ILVW] M.ISMAIL, J.LETESSIER, G.VALENT, J.WIMP, Some results on associated Wilson polynomials, pp.293-298 *in Orthogonal Polynomials and their Applica-*

tions (C.BREZINSKI *et al.*, editors), *IMACS Annals on Computing and Applied Mathematics* **9** (1991), Baltzer AG, Basel.

[KvM] M.KAC, P. van MOERBEKE, On an explicitly soluble system on nonlinear differential equations related to certain Toda lattices, *Adv. Math.* **16** (1975) 160-169.

[Lag] E. LAGUERRE, Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels, *J. Math. Pures Appl. (4)* **1** (1885), 135-165 = pp. 685-711 *in Oeuvres*, Vol.II, Chelsea, New-York 1972.

[LaG] Ph. LAMBIN, J.P. GASPARD, Continued-fraction technique for tight-binding systems. A generalized-moments approach, *Phys. Rev. B/* **26** (1982) 4356-4368.

[LD] J.A.LAPPO-DANILEVSKY, *Mémoires sur la théorie des systèmes des équations différentielles linéaires*, vol. I,II,III bound as one volume, Chelsea Pub. Co. ,1953.

[LW] D. LEVI, P. WINTERNITZ, editors: *Painlevé Transcendents. Their Asymptotics and Physical Applications. NATO ASI Series: Series B: Physics* **278**, Plenum Press, N.Y., 1992.

[LeQ] J.S.LEW, D.A.QUARLES, Nonnegative solutions of a nonlinear recurrence, *J. Approx. Th.* **38** (1983), 357-379.

[LiMu] J.-M. LIU, G. MÜLLER, Infinite-temperature dynamics of the equivalent-neighbor XYZ model, *Phys. Rev. A* **42** (1990) 5854-5864.

[Lub] D.S. LUBINSKY, A survey of general orthogonal polynomials for weights on finite and infinite intervals, *Acta ApplicandæMathematicæ* **10** (1987) 237-296.

[Mag1] A.P. MAGNUS, Riccati acceleration of Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials, pp. 213-230 *in Padé Approximation and its Applications, Proceedings, Bad Honnef 1983*, Lecture Notes Math. **1071** (H.WERNER & H.T.BÜNGER, editors), Springer-Verlag, Berlin, 1984.

[Mag2] A.P. MAGNUS, A proof of Freud's conjecture about orthogonal polynomials related to $|x|^\rho \exp(-x^{2m})$ for integer m, pp. 362-372 *in Polynômes Orthogonaux et Applications, Proceedings Bar-le-Duc 1984.* (C.BREZINSKI *et al.*, editors), *Lecture Notes Math.* **1171**, Springer-Verlag, Berlin 1985.

[Mag3] A.P. MAGNUS, On Freud's equations for exponential weights, *J. Approx. Th.* **46** (1986) 65-99.

[Mag4] A.P. MAGNUS, Associated Askey-Wilson polynomials as Laguerre-Hahn orthogonal polynomials, pp. 261-278 *in Orthogonal Polynomials and their Applications, Proceedings Segovia 1986.* (M.ALFARO *et al.*, editors), *Lecture Notes Math.* **1329**, Springer-Verlag, Berlin 1988.

[Mal] B.MALGRANGE, Sur les déformations isomonodromiques. I. Sigularités régulières, pp.401-426 *in Mathématique et Physique, Séminaire de l'Ecole Normale Supérieure 1979-1982* (L. BOUTET de MONVEL, A.DOUADY & J.L.VERDIER, editors), *Progress in Mathematics* **37**, Birkhäuser, Boston, 1983. II. Singularités irrégulières, *ibid.*, 427-438.

[Mar] P. MARONI, Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques, pp. 279-290 *in Orthogonal Polynomials and their Applications. Proceedings, Segovia 1986* (M. ALFARO *et al.*, editors), *Lecture Notes Math.* **1329**, Springer, Berlin 1988.

[Mar1] P. MARONI, Un exemple d'une suite orthogonale semi-classique de classe un. *Polinomios ortogonales y applicaciones; Actas VI Simposium, Gijon* (1989), 234-241.

[Mar2] P. MARONI, Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques. pp. 95-130 *in Orthogonal Polynomials and their Applications* (C.BREZINSKI *et al.*, editors), *IMACS Annals on Computing and Applied Mathematics* **9** (1991), Baltzer AG, Basel.

[Mo] J.MOSER, Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. Math.* **16** (1975) 197-220.

[Nev] P.NEVAI, Two of my favorite ways of obtaining asymptotics for orthogonal polynomials, pp.417-436 *in Anniversary Volume on Approximation Theory and Functional Analysis*, (P.L.BUTZER, R.L.STENS and B.Sz.-NAGY, editors), **ISNM 65**, Birkhäuser Verlag, Basel, 1984.

[GFOPCF] P. NEVAI, Géza Freud, orthogonal polynomials and Christoffel functions. A case study, *J. Approx. Theory* **48** (1986), 3-167.

[Nev2] P.NEVAI, Research problems in orthogonal polynomials, pp. 449-489 *in Approximation Theory VI*, vol. **2** (C.K.CHUI, L.L.SCHUMAKER & J.D.WARD, editors), Academic Press, 1989.

[N] J.NUTTALL, Asymptotics of diagonal Hermite-Padé polynomials, *J.Approx. Th.* **42** (1984) 299-386.

[Ok] K. OKAMOTO, Studies on the Painlevé equations III. Second and fourth Painlevé equations, P_{II} and P_{IV}, *Math. Ann.* **275** (1986) 221-255.

[OW] E.P. O'REILLY, D. WEAIRE, On the asymptotic form of the recursion basis vectors for periodic Hamiltonians, *J. Phys. A: Math. Gen.* **17** (1984) 2389-2397.

[Pain] P. PAINLEVÉ, *Œuvres de Paul Painlevé*, vol. **3**, C.N.R.S. , Paris, 1975.

[Peh1] F.PEHERSTORFER, On Bernstein-Szegő orthogonal polynomials on several intervals. II. Orthogonal polynomials with periodic recurrence coefficients, *J. Approx. Th.* **64** (1991) 123-161.

[Peh] F.PEHERSTORFER, On orthogonal polynomials on several intervals, *VII Simposium sobre polinomios ortogonales y applicaciones*, Granada, España, 23-27 Sept. 1991.

[Per] O.PERRON, *Die Lehre von den Kettenbrüchen*, 2[nd] edition, Teubner, Leipzig, 1929 = Chelsea,

[Sho] J.A. SHOHAT, A differential equation for orthogonal polynomials, *Duke Math. J.* **5** (1939),401-417.

[StT] H.STAHL, V.TOTIK, *General Orthogonal Polynomials*, (*Encyc. Math. Appl.* **43**), Cambridge U.P., Cambridge, 1992.

[VA] W. VAN ASSCHE, *Asymptotics for Orthogonal Polynomials. Springer Lecture Notes Math.* **1265**, Springer-Verlag, Berlin 1987.

[Wi] J.WIMP, Current trends in asymptotics: some problems and some solutions, *J. Comp. Appl. Math.* **35** (1991) 53-79.

[Y] S.YAMAZAKI, The semi-infinite system of nonlinear differential equations $\dot{A}_k = 2A_k(A_{k+1} - A_{k-1})$; methods of integration and asymptotic time behaviours, *Nonlinearity* **3** (1990) 653-676.

[Yos] S.YOSHIDA, 2-parameter family of solutions for Painlevé (I)$\sim$(V) at an irregular singular point, *Funk. Ekvacioj*, **28** (1985) 233-248.

[Zu] J.B. ZUBER, L'invariance conforme et la physique à deux dimensions. *La Recherche* **24** (1993) 142-151.

[91] A.S. Fokas, A.R. Its and A.V. Kitaev, Isomonodromic approach in the theory of two-dimensional quantum gravity, *Uspekhi Mat. Nauk* **45** (6) (276) (1990) 135–136 (in Russian).

[92] A.S. Fokas, A.R. Its and A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Commun. Math. Phys.* **147** (1992) 395–430 (in Russian).

[93] A.S. Fokas, A.R. Its and Xin Zhou, Continuous and discrete Painlevé equations, pp. 33-47 *in* [LW]