SELF-SIMILARITY IN SPECTRAL PROBLEMS AND q-SPECIAL FUNCTIONS [1]

Igor Loutsenko* and Vyacheslav Spiridonov+

* *Department of Physics, Joseph Henry Laboratories, Jadwin Hall, Princeton University, P.O. Box 708, Princeton, NJ 08544-0708, USA; e-mail: loutseni@feynman.princeton.edu*

+ *Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow region 141980, Russia; e-mail: svp@thsun1.jinr.ru*

Abstract

Similarity symmetries of the factorization chains for one-dimensional differential and finite-difference Schrödinger equations are discussed. Properties of the potentials defined by self-similar reductions of these chains are reviewed. In particular, their algebraic structure, relations to q-special functions, infinite soliton systems, supersymmetry, coherent states, orthogonal polynomials, one-dimensional Ising chains and random matrices are outlined.

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1. Introduction. Spectral problems of various types play an important role in classical and quantum mechanics and in the theory of integrable systems. Depending on the situation they may be based upon differential, finite-difference or integral equations, etc. Roughly speaking all such problems can be approximated by a search of eigenvalues of some matrices [1]. This leads to a number of common features in the methods of treatment of different spectral problems. We choose as a basic object of discussion the ordinary one-dimensional Schrödinger equation

$$L\psi(x) = -\psi_{xx}(x) + u(x)\psi(x) = \lambda\psi(x). \tag{1}$$

The eigenvalue problem for the operator L is completely defined by imposing some boundary conditions upon $\psi(x)$. However, it is convenient to consider (1) without such additional constraints and assume that $x, \lambda \in C$. When a non-formal meaning of the eigenvalues and eigenfunctions will be necessary, we assume that $\psi(x) \in L^2(R)$.

There are several connections of (1) with finite-difference equations and discrete systems — the main subject of the present workshop. First, the scheme of treatment of (1) to be described below is universal, e.g., it can be easily reformulated for the second order difference equation. Second, finite-difference equations appear already in the consideration of (1) for different potentials. In the space of all equations of this form finite-differences enter through the symmetry transformations mapping Schrödinger equations onto each other:

$$u(x) \to \tilde{u}(x), \qquad \psi(x) \to \tilde{\psi}(x), \qquad \lambda \to \tilde{\lambda}.$$

Third, in some cases the discrete spectrum eigenfunctions satisfy simple discrete equations. E.g., for $u(x) \propto x^2$ one gets from the condition of square integrability of eigenfunctions $\lambda_n \propto n$, $\psi_n(x) \propto H_n(x)e^{-x^2/2}$, where $H_n(x)$ are Hermite polynomials satisfying three-term recurrence relation. Fourth, there is a direct relation of (1) to some lattice models of statistical mechanics.

The paper is organized as follows. In the next several sections we review briefly some old results on self-similar potentials, their coherent states and the relation to q-special functions.

[1] Invited talk by the second author at the Third International Workshop on Symmetries and Integrability of Difference Equations (16-22 May, 1998, Sabaudia, Italy).

Then we outline briefly a generalization of the construction to orthogonal polynomials. In the last two sections we present our recent results on the appearance of one-dimensional Ising chains and random matrices in the context of the Schrödinger equation (1). The list of references given in the end does not pretend to be complete, it contains mostly the papers which influenced somehow the work of authors.

2. Self-similar spectra. Under the similarity transformations in d-dimensional Euclidean space one counts usually the rotations, dilations and shifts of the coordinates, $x_i \to q_{ij}x_j + a_i$, $i, j = 1, \ldots, d$, i.e. the affince transformations. In the $d = 1$ case this is reduced to $x \to qx + a$, which will be used below under the assumption that the parameters q and a are *fixed* (for $q \neq 1$ one may set $a = 0$ without loss of generality).

Let us consider separately the effects of translations and scalings upon the Schrödinger operator L with potential $u(x)$, eigenfunctions $\{\psi_j(x)\}$ and the spectrum $\{\lambda_j\}$ (the use of the subscript j for notation of the spectrum is symbolic). After application of some symmetry transformation we get the operator $\tilde{L} = -d^2/dx^2 + \tilde{u}(x)$ with the potential $\tilde{u}(x)$, unnormalized eigenfunctions $\{\tilde{\psi}_j(x)\}$ and the eigenvalues $\{\tilde{\lambda}_j\}$. So, the shift $x \to x + a$ brings in

$$\tilde{u}(x) = u(x + a), \qquad \{\tilde{\psi}_j(x)\} = \{\psi_j(x + a)\}, \qquad \{\tilde{\lambda}_j\} = \{\lambda_j\}.$$

The spectrum does not change under this transformation due to the absence of distinguished points in the boundary conditions. In the case of scaling transformations one replaces in (1) the variable x by qx, then multiplies the resulting equation by q^2 in order to bring it to the canonical form and gets the operator $\tilde{L}$ with

$$\tilde{u}(x) = q^2u(qx), \qquad \{\tilde{\psi}_j(x)\} = \{\psi(qx)\}, \qquad \{\tilde{\lambda}_j\} = \{q^2\lambda_j\}.$$

Now one gets a scaled form of the spectrum $\lambda \to q^2\lambda$. For $|q| > 1$ the spectral points are stretched with respect to the $\lambda = 0$ point, and, vice versa, for $|q| < 1$ we get a squeezed spectrum. The complete affine transformation of the spectral parameter requires joining of a shift $\lambda \to \lambda + h$ which is easily reached by the shift of the potential

$$\tilde{u}(x) = u(x) + h, \qquad \{\tilde{\psi}_j(x)\} = \{\psi_j(x)\}, \qquad \{\tilde{\lambda}_j\} = \{\lambda_j + h\}.$$

The combination of these three transformations gives

$$\tilde{u}(x) = q^2u(qx + a) + h, \qquad \{\tilde{\psi}_j(x)\} = \{\psi_j(qx + a)\}, \qquad \{\tilde{\lambda}_j\} = \{q^2\lambda_j + h\},$$

i.e. 'trivial' similarity transformations of the spectral data. The key constructive idea, bringing in a non-trivial content into the above considerations, consists in a sticking to the systems with *self-similar* spectra for which $\{\tilde{\lambda}_j\} = \{\lambda_j\}$. Note that such a constraint is applicable to any spectral problem, not just to the Schrödinger equation.

Analyze first the $q = 1$ case. The condition $\{\lambda_j + h\} = \{\lambda_j\}$ shows that the discrete spectrum should appear in the form of a number of independent bilateral arithmetic progressions with the increment equal to h. In the simplest case the number of such progression is finite ($= N$) and one can parametrize the eigenvalues as solutions of the equation $\lambda_j + h = \lambda_{j+N}$ or $\lambda_{pN+k} = hp + \lambda_k$, $k = 1, \ldots N$, $p \in Z$. The continuous spectrum may appear in the form of arithmetic progressions of permitted bands. For instance, if one requires that the addition of h to potential is equivalent to an isospectral transformation $x \to x + a$, i.e. $u(x) + h = u(x + a)$, then $u(x)$ is a sum of a periodic potential $u_p(x + a) = u_p(x)$ and the Airy part $\propto x$ known to lead to continuous spectrum filling the whole line, $u(x) = u_p(x) + hx/a$.

Consider now the $\{q^2\lambda_i\} = \{\lambda_i\}$ case. Formally, the discrete spectrum appears in the form of bilateral geometric progressions accumulating near the $\lambda = 0$ point from below or above. Enumeration of pure point spectra in case of the finite number of such progressions may be based upon the relation $q^2\lambda_j = \lambda_{j+N}$ giving $\lambda_{pN+k} = \lambda_k q^{2p}$, $k = 1, \ldots, N$, $p \in Z$. The continuous spectrum may arise in the form of infinite number of permitted bands concentrating near the zero.

For instance, the described situation may take place if $q^2 u(qx) = u(x)$. A solution of this equation is given by $u(x) = h(x)/x^2$, where $h(x)$ is a function periodic in the logarithmic scale $h(qx) = h(x)$. For $h = const.$ the wave functions $\psi(x)$ are given in terms of the Bessel functions. For $h < -1/4$ there is the "fall onto the center" phenomenon and under certain conditions discrete spectrum consists of one bilateral geometric progression of negative eigenvalues [2]. The bad point of this and of the previous example is that the potentials $u(x)$ are not bounded from below. Let us impose such a demand and assume that $u(x)$ is bounded from below and non-singular. Evidently this condition does not permit the infinite negative spectrum, i.e. it is necessary to delete the lower tails of our geometric (or arithmetic) progressions. However, the formula $q^2\lambda_j = \lambda_{j+N}$ still works and gives $\lambda_{pN+k} = \lambda_k q^{2p}, p = 0, 1, \ldots$. We have thus a partially self-similar spectrum:

$$\{q^2\lambda_j\} = \{\lambda_j\}/(\lambda_1, \ldots, \lambda_N),$$

i.e. the scaling $x \to qx$ leads to the deletion of N lowest eigenvalues from the spectrum. Below we outline a procedure for building a class of potentials with this property.

Note that for the $q^2 \neq 1$ case the geometric progressions of positive eigenvalues of the abstract operator L are bounded from below and no truncation is necessary. However, one has to go beyond the Schrödinger equation for building systems with such completely self-similar spectrum.

3. Infinite-soliton systems. The inverse scattering method and the general theory of solitons (see, e.g., [3, 4, 5]) provide us a guide for construction of potentials with prefixed spectral properties. The compatibility condition of two linear equations

$$L\psi(x,t) = \lambda\psi(x,t), \qquad \psi_t(x,t) = A\psi(x,t)$$

has the operator form $L_t = [A, L]$. When $L = -\partial_x^2 + u(x,t)$ and $A = -\partial_x^3 + 6u\partial_x + 3u_x$ one arrives at the KdV equation $u_t + u_{xxx} - 6uu_x = 0$.

If the potential $u(x) \to 0$ for $x \to \infty$ sufficiently fast then according to the inverse scattering method the potential $u(x)$ can be uniquely reconstructed from the reflection coefficient $R(k)$, discrete spectrum eigenvalues $\{\lambda_j\}$ and the normalization constants $\{\theta_j^{(0)}\}$ for corresponding eigenfunctions $\{\psi_j(x)\}$. For $R(k) = 0$ and finite number of eigenvalues one arrives at the N-soliton potentials. The corresponding solution of the KdV equation can be represented in the form

$$u(x,t) = -2\partial_x^2 \ln \tau_N(x,t), \tag{2}$$

where τ_N is the determinant of a $N \times N$ matrix C,

$$\tau_N = \det C, \qquad C_{ij} = \delta_{ij} + \frac{2\sqrt{k_i k_j}}{k_i + k_j} e^{(\theta_i + \theta_j)/2}, \tag{3}$$

$$\theta_i = k_i x - k_i^3 t + \theta_i^{(0)}, \qquad i, j = 1, 2, \ldots, N.$$

Parameters k_i describe amplitudes of solitons related to eigenvalues of L as $\lambda_i = -k_i^2/4$, $\theta_i^{(0)}/k_i$ are zero time phases of solitons and k_i^2 are their velocities.

Let us extend this solution to the pN-solitonic one by imposing the constraint $k_{j+N} = qk_j, q < 1$, and take the $p \to \infty$ limit. This leads to the self-similar spectrum $k_{pN+m} = q^p k_m, m = 1, \ldots, N$, i.e. to the infinte-soliton wavetrains whose amplitudes form N geometric progressions. The question of convergency of such a limit requires a special consideration.

Applying scaling transformations to derived potentials one has to scale the time t too, which gives $\tilde{u}(x,t) = q^2 u(qx, q^3 t)$. The phases of this potential are $\theta_j(qx, q^3 t) == k_{j+N}x - k_{j+N}^3 t + \theta_j^{(0)}$. There are two "drawbacks" of the resulting expression. First, there are infinitely many arbitrary parameters $\theta_j^{(0)}$, i.e. there is a functional freedom in the definition of $u(x)$. Second, there is a mixup between soliton amplitudes and phases — the solitons with the amplitudes k_{j+N} have now the zero time phases $\theta_j^{(0)}$ instead of $\theta_{j+N}^{(0)}$. Both these shortcomings are removed if we impose the constraint $\theta_{j+N}^{(0)} = \theta_j^{(0)}$, i.e. $\theta_j(qx, q^3 t) = \theta_{j+N}(x,t)$, which leads to truly self-similar infinite-soliton systems — the dilation $x \to qx, t \to q^3 t$ just deletes N solitons, corresponding to the lowest eigenvalues of L. Such potential is contains a finite number of parameters $k_1, \ldots, k_N, q, \theta_1^{(0)}, \ldots, \theta_N^{(0)}$ and should be characterized by some "finite" equations. There are two equivalent approaches to the construction of these equations. The algebraic one, based upon the factorization method, and the functional-analytic one, relying upon the Darboux transformations.

4. The factorization method. This method is well known in quantum mechanics [6], it was discussed already by Schrödinger. Within this approach one takes a chain of Schrödinger operators, $L_j = -d^2/dx^2 + u_j(x)$, and factorizes them as products of the first-order differential operators up to some constants λ_j:

$$L_j = A_j^+ A_j^- + \lambda_j, \qquad A_j^\pm = \mp\, d/dx + f_j(x). \tag{4}$$

Then the neighboring L_j are tied to each other through the abstract factorization chain

$$L_{j+1} = A_{j+1}^+ A_{j+1}^- + \lambda_{j+1} = A_j^- A_j^+ + \lambda_j,$$

i.e. one passes from L_j to L_{j+1} just by the permutation of operator factors. This gives the intertwining relations $A_j^- L_j = L_{j+1} A_j^-$, $L_j A_j^+ = A_j^+ L_{j+1}$. As a result, if one has $L_j \psi^{(j)} = \lambda \psi^{(j)}$, then $\psi^{(j+1)} \propto A_j^- \psi^{(j)}$. Indeed, $L_{j+1}(A_j^- \psi^{(j)}) = A_j^- L_j \psi^{(j)} = \lambda (A_j^- \psi^{(j)})$. Under particular conditions the action of the operator A_j^- leads to removing or addition of an eigenvalue in the spectrum of the operator L_{j+1} with respect to the spectrum of L_j. There may take place an isospectral situation as well. If zero modes of all A_j^- are normalizable, then $\lambda_j, \lambda_{j+1}, \ldots$ form the discrete spectrum of L_j.

Let us define the operators

$$M_j^- = A_{j+N-1}^- \ldots A_{j+1}^- A_j^-, \qquad M_j^+ = A_j^+ A_{j+1}^+ \ldots A_{j+N-1}^+.$$

They relate eigenfunctions of the operators L_j and L_{j+N} due to the intertwining relations

$$L_{j+N} M_j^- = M_j^- L_j, \qquad M_j^+ L_{j+N} = L_j M_j^+.$$

It is easy to see that the products of operators $M_j^\pm$ should commute either with L_j or with L_{j+N}. Indeed, one has the equalities

$$M_j^+ M_j^- = \prod_{k=0}^{N-1} (L_j - \lambda_{j+k}), \qquad M_j^- M_j^+ = \prod_{k=0}^{N-1} (L_{j+N} - \lambda_{j+k}), \tag{5}$$

which, together with the previous relations, look almost as an algebra of symmetries.

The main advantage of this method is that it is not tied to any particular type of spectral problem. One may realize the abstract factorization chain with the help of differential, difference, integral, etc operators.

5. Darboux transformations. One starts from the semi-disrecte LA-pair

$$L_j\psi^{(j)} = \lambda\psi^{(j)}, \qquad L = -d^2/dx^2 + u_j(x),$$
$$\psi^{(j+1)} = A_j^-\psi^{(j)}, \qquad A_j^- = d/dx + f_j(x).$$

The compatibility condition yields the intertwining relation $A_j^-L_j = L_{j+1}A_j^-$, the resolution of which gives $u_j = f_j^2 - f_{jx} + \lambda_j, u_{j+1} = u_j + 2f_{jx} = f_j^2 + f_{jx} + \lambda_j$. The substitution $f_j = -\phi_x^{(j)}/\phi^{(j)}$ converts the relation between u_j and f_j into the equation $-\phi_{xx}^{(j)} + u_j\phi^{(j)} = \lambda_j\phi^{(j)}$. As a result one gets the original Darboux transformation: $\psi^{(j+1)} = \psi_x^{(j)} - (\phi_x^{(j)}/\phi^{(j)})\psi^{(j)}$.

The compatibility condition (or the factorization chain) is equivalent to the following infinite chain of nonlinear differential-difference equations

$$(f_j(x) + f_{j+1}(x))_x + f_j^2(x) - f_{j+1}^2(x) = \mu_j \equiv \lambda_{j+1} - \lambda_j. \tag{6}$$

One may search solutions of this equation in the form of power series in j. The finite term expansion occurs only if $f_j(x) = a(x)j + b(x) + c(x)/j$, where a, b, c are some elementary functions of x [6]. The resulting Schrödinger equation is solved in terms of the ${}_2F_1$ hypergeometric function.

6. Polynomial supersymmetry. Upon the two neighboring operators L_j, L_{j+1} of the factorization chain a simple superalgebra describing the boson-fermion symmetry is realized. Let us introduce the 2×2 matrix Hamiltonian $H = -d^2/dx^2 + f_j^2(x) - f_{jx}(x)\sigma_3$ and define the supercharges $Q^\pm = A_j^\pm(\sigma_1 \pm i\sigma_2)/2$ (σ_k are the Pauli matrices). Then one has $\{Q^+, Q^-\} = H$, $[H, Q^\pm] = (Q^\pm)^2 = 0$. This construction was generalized in [7] to a symmetry between particles with parastatistics (para-supersymmetry). In the simplest case one takes the Hamiltoninan given by 3×3 diagonal matrix with the entries L_{j-1}, L_j, L_{j+1}. A symmetry between the boson and a parafermion of the second order is described now by some polynomial (cubic) relations between H and the correspodning generalized supercharges.

As a next step in the "nonlinearization" of supersymmetry the following polynomial supersymmetry algebra has been derived in [8]

$$\{Q^+, Q^-\} = P(H), \qquad [H, Q^\pm] = (Q^\pm)^2 = 0,$$

where $Q^\pm = M_1^\pm(\sigma_1 \pm i\sigma_2)/2$ and $P(H)$ is a polynomial of H:

$$P(H) = \prod_{k=1}^{N}(H - \lambda_k), \qquad H = \begin{pmatrix} L_1 & 0 \\ 0 & L_{N+1} \end{pmatrix}.$$

Since the supercharges have now more than one zero mode, the Witten index does not characterize the supersymmetry breaking. One may deform this algebra via the affine transformations and impose various natural constraints upon H. This gives an alternative way of derivation (which we are not describing here) of self-similar potentials considered in the next section.

7. Self-similar potentials. The infinite soliton potentials with self-similar spectra considered above appear as self-similar solutions of the chain (6). The simplest $N = 1$ case has been considered in [9, 10]. The general class of these potentials is defined by the following q-periodic reduction [11]:

$$f_{j+N}(x) = qf_j(qx), \qquad \mu_{j+N} = q^2\mu_j. \tag{7}$$

Note that the solitonic interpretation is valid only if $0 < q^2 < 1$ and there are no singularities on the whole line $-\infty < x < \infty$, which is not true for arbitrary $q, x, \lambda_j \in C$. What are the properties of these potentials ?

Algebraically, these are the systems whose symmetries are described by the quantum algebras. Indeed, the equations (5) form a closed algebra in the cases when the operators L_j and L_{j+N} are related to each other in some way, e.g., if they are similar to each other $L_j \sim L_{j+N}$. The q-periodic closure corresponds to the following constraint $L_{j+N} = q^2 T L_j T^{-1}$, where T is the scaling operator $T\psi(x) = \sqrt{|q|}\psi(qx)$. This gives $u_{j+N}(x) = q^2 u_j(qx)$.

For real $q \neq 0$, T is the unitary operator $T^\dagger = T^{-1}$. Substituting the operator constraint into (5) one arrives at the following polynomial algebra

$$LB^\pm = q^{\pm 2} B^\pm L, \qquad B^+ B^- = \prod_{k=1}^{N} (L - \lambda_k), \qquad B^- B^+ = \prod_{k=1}^{N} (q^2 L - \lambda_k), \tag{8}$$

where we have denoted $L \equiv L_1, B^- \equiv T^{-1} M_1, B^+ \equiv M_1^+ T$. For $N = 1$ this is a q-analog of the harmonic oscillator algebra [12], for $N = 2$ one gets the $su_q(1,1)$ algebra, etc.

Analytically, we deal with a class of functions appearing from solutions of the following system of nonlinear differential-q-difference equations:

$$(f_1(x) + f_2(x))_x + f_1^2(x) - f_2^2(x) = \mu_1, \quad \ldots\ldots$$

$$(f_N(x) + q f_1(qx))_x + f_N^2(x) - q^2 f_1^2(qx) = \mu_N.$$

A flavor of the structure of the general solution is obtained from consideration of various limiting cases, when $f_j(x)$ are expressed through known functions. The simplest situation is obtained when $f_j(x) = c_j = const.$ and arbitrary q. This leads to $L = -d^2/dx^2$, i.e. to the free nonrelativistic particle, which is interpreted in this way as a q-algebraic system. When $f_1(x)$ is not singular at $x = 0$, the crystal base limit $q \to 0$ results in the general N-soliton potential. One can take the limit $q \to 1$ in such a way that the geometric progressions in the spectrum are converted into the arithmetic ones. Then for $N = 1, 2$ one gets the potentials $u(x) \propto x^2, ax^2 + b/x^2$. For $N = 3, 4$ the functions $f_j(x)$ are expressed through the Painlevé IV and V transcendents [13, 14]. For $q = -1$ one gets a similar situation but now the functions are constrained to obey certain structure under the parity transformation. It is possible to consider the limit from Schrödinger equation to classical mechanics, in which case the self-similar potentials are determined by the much more simple functions [15, 16].

For q a primitive root of unity, $q^n = 1, q \neq \pm 1$, one gets the finite gap potentials with additional (quasi)crystallographic symmetries [17]. The appearance of Painlevé functions and of the finite-gap potentials in a similar setting was discussed also in [18, 19]. It is natural to refer to the whole class of functions emerging for general initial conditions and $0 < |q| < 1$ as the continuous q-Painlevé functions. Because of the emergence of quantum algebras, the derived class of self-similar functions represents a new set of q-special functions defined upon the differential equations. They differ in structure from the basic hypergeometric series [20], but there is a connection with them via the coherent states.

Let us remark that the Darboux transformations allow one to remove or add only a finite number of levels from/to the spectrum of a given Schrödinger operator. If one finds an intertwining operator deleting or adding permitted bands from/to the given spectrum, then the systems with a self-similar infinite gap spectrum can be constructed from the requirement that the action of this intertwining operator is equivalent to a simple dilation of the coordinate x.

8. Coherent states. There are many definitions of coherent states in physics. The purely group-theoretical approach uses the orbits of groups lying behind the symmetries of a taken physical system. For simple spectrum generating algebras, as in the harmonic oscillator or singular oscillator cases, these states may be defined as eigenfunctions of the lowering operator. In this approach coherent states play the role of generating functions of irreducible representations of the underlying algebra. Below we outline briefly the results of application of the latter definition to the nonlinear algebra of symmetries (8). A more detailed discussion of the structure of these coherent states can be found in [21].

The action of operators $B^{\pm}$ upon abstract eigenstates of the operator L, $L|\lambda\rangle = \lambda|\lambda\rangle$, has the form:

$$B^-|\lambda\rangle = \prod_{k=1}^{N} \sqrt{\lambda - \lambda_k}|\lambda q^{-2}\rangle, \qquad B^+|\lambda\rangle = \prod_{k=1}^{N} \sqrt{\lambda q^2 - \lambda_k}|\lambda q^2\rangle.$$

Let $0 < q < 1$ and $\lambda_k < 0$. Then the $\lambda < 0$ eigenstates of L are formed from up to N lowest weight unitary irreducible representations of the algebra (8). Indeed, B^- is the lowering operator for $\lambda < 0$ states and $\sqrt{\lambda - \lambda_k}$ becomes complex for $\lambda < \min\{\lambda_k\}$. Since $B^-|\lambda_k\rangle = 0$, this problem does not arise for a special choice of λ. Namely, if the point $\lambda < 0$ belongs to the spectrum of L, it must be of the form $\lambda_{pN+k} \equiv \lambda_k q^{2p}$ and the corresponding state is $|\lambda_{pN+k}\rangle \propto (B^+)^p |\lambda_k\rangle$ (we assume that $|\lambda_k\rangle$ are normalizable).

Coherent states for these series are defined as eigenstates of the lowering operator B^-:

$$B^-|\alpha\rangle_-^{(k)} = \alpha|\alpha\rangle_-^{(k)}, \quad k = 1, \ldots, N. \tag{9}$$

There are N such states since they are defined for each lowest weight series separately. Representing $|\alpha\rangle_-^{(k)}$ as a superposition of the states $|\lambda_{pN+k}\rangle$ one finds

$$|\alpha\rangle_-^{(k)} \propto \sum_{p=0}^{\infty} C_p^{(k)} \alpha^p |\lambda_{pN+k}\rangle \propto {}_N\varphi_{N-1}\left(\begin{matrix} 0, \ldots, 0 \\ b_1^k, \ldots, b_{N-1}^k \end{matrix}; q^2, z\right)|\lambda_k\rangle,$$

where ${}_N\varphi_{N-1}$ is a basic hypergeometric series with the operator argument $z = (-1)^N \alpha B^+/\lambda_1 \ldots \lambda_N$ and the parameters $\{q^2, b_j^k\} = \{q^2\lambda_k/\lambda_j\}$. The general definition of the series of this type is [20]

$${}_r\varphi_s\left(\begin{matrix} a_1,\ a_2,\ \ldots,\ a_r \\ b_1,\ b_2,\ \ldots,\ b_s \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n}[(-1)^n q^{n(n-1)/2}]^{1+s-r} z^n,$$

where r and s are arbitrary positive integers, and $a_1, \ldots, a_r, b_1, \ldots, b_s$ are free parameters. We use also the following compact notations $(a; q)_n = (1-a)\ldots(1-q^{n-1})$ and $(a_1, a_2, \ldots, a_n; q)_n = (a_1; q)_n (a_2; q)_n \ldots (a_n; q)_n$. These coherent states are normalizable if the complex variable α lies inside of the circle $|\alpha|^2 < |\lambda_1 \ldots \lambda_N|$. General ${}_r\varphi_s$ series are related to coherent states of the rational generalization of the algebra (8) [22].

The $\lambda = 0$ eigenstates of L are simultaneously eigenstates of the $B^{\pm}$ operators, i.e. they have a direct interpretation as coherent states. This is a special degenerate representation of the algebra (8).

Unusual coherent states appear from the non-highest weight representations of (8) corresponding to the $\lambda > 0$ eigenstates of L. Let $E_0 > 0$ be a discrete spectrum point. Then the action of $B^{\pm}$ generates discrete spectrum of L in the form of one bilateral geometric progression $E_0 q^{2n}, n \in Z$ (such situation cannot take place for the Schrödinger equation). In principle there may be an *arbitrary* number of such progressions because $\sqrt{\lambda - \lambda_k}$ is real for arbitrary $\lambda > 0$ and there are no truncation conditions as in the $\lambda < 0$ case.

Since for $\lambda > 0$ the lowering operator is B^+, the coherent states should be defined as eigenstates of this operator (instead of B^-):

$$B^+|\alpha\rangle_+ = \alpha|\alpha\rangle_+. \tag{10}$$

These coherent states can be expanded over the series of eigenstates of L with positive eigenvalues $|E_0 q^{2n}\rangle$:

$$|\alpha\rangle_+ \propto \sum_{n=-\infty}^{\infty} C_n \alpha^n |E_0 q^{2n}\rangle \propto {}_0\psi_N \left(\begin{matrix} 0, \dots, 0 \\ \lambda_1/E_0, \dots, \lambda_N/E_0 \end{matrix} ; q^2, z \right) |E_0\rangle,$$

where ${}_0\psi_N$ is a bilateral q-hypergeometric series with the operator argument $z = \alpha B^-/(-E_0)^N$. The general representative of these series is defined as follows [20]:

$${}_r\psi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, z \right) = \sum_{n=-\infty}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left((-1)^n q^{n(n-1)/2} \right)^{s-r} z^n.$$

Coherent states $|\alpha\rangle_+$ are normalizable if the parameter α lies outside of the previous indicated circle $|\alpha|^2 > |\lambda_1 \dots \lambda_N|$.

When the $\lambda > 0$ region is occupied by continuous spectrum, coherent states are defined as eigenfunctions of B^+ again, but in this case it is necessary to use integrals in the expansion of $|\alpha\rangle_+$ over the states $|\lambda\rangle$. Since the continuous spectrum may be considered as a continuous direct sum of irreducible representations with bilateral geometric progressions formed by spectral points, there are now infinitely many coherent states. Let $N = 1$ and the states $|\lambda\rangle$ are not degenerate. Then, normalizing $\lambda_1 = 1/(q^2 - 1)$, we have

$$|\alpha\rangle_+^{(s)} = C(\alpha) \int_0^{\infty} \frac{\lambda^{\gamma_s} |\lambda\rangle d\lambda}{\sqrt{(-\lambda q^2(1-q^2); q^2)_\infty}},$$

where

$$\gamma_s = \frac{2\pi i s - \ln(\alpha q^2 \sqrt{1-q^2})}{\ln q^2}, \qquad s \in \mathbf{Z}.$$

These states are normalizable for $|\alpha|^2 > 1/(1-q^2)$, and have the unit norm for

$$|C|^{-2} = \int_0^{\infty} \frac{\lambda^{-\tau} d\lambda}{(-\lambda q^2(1-q^2); q^2)_\infty} = \frac{\pi}{\sin \pi\tau} \frac{(q^{2\tau}; q^2)_\infty (q^2(1-q^2))^{\tau-1}}{(q^2; q^2)_\infty},$$

where $\tau = \ln|\alpha q \sqrt{1-q^2}|/\ln q$. The last integral is calculated exactly being a particular subcase of a Ramanujan q-beta integral [20].

Schrödinger operators with self-similar potentials have the continuous spectrum for $\lambda > 0$. It is doubly degenerate, which leads to duplication of the number of coherent states. An instructive example is provided by the free nonrelativistic particle model for which $L = -d^2/dx^2$. The corresponding q-harmonic oscillator algebra generators have the form

$$B^- = T^{-1}(d/dx + 1/\sqrt{1-q^2}), \qquad B^+ = (-d/dx + 1/\sqrt{1-q^2})T,$$

$$B^- B^+ - q^2 B^+ B^- = 1, \qquad L = B^+ B^- - 1/(1-q^2).$$

The eigenvalue problem $B^- \psi_\alpha^-(x) = \alpha \psi_\alpha^-(x)$ leads to the pantograph equation, which was analyzed in detail in [23]. The corresponding results show that in accordance with the purely

algebraic consideration there are no normalizable coherent states of this type. However, there are infinitely many normalizable eigenstates of the operator B^+ determined as appropriate solutions of the advanced pantograph equation [23]

$$\frac{d}{dx}\psi_\alpha^+(x) = -\alpha q^{-3/2}\psi_\alpha^+(q^{-1}x) + \frac{q^{-1}}{\sqrt{1-q^2}}\psi_\alpha^+(x).$$

An important property of these states is that they are determined by C^∞ but not analytical functions at the $x = 0$ point. Consideration of the $N > 1$ symmetry algebras in this realization leads to the generalized pantograph equations [24].

As a general conclusion to this section we would like to stress that all three types of objects associated with q-special functions in their standard meaning [20]: the ordinary and bilateral basic hypergeometric series, and Ramanujan type integrals, show up in the context of self-similar spectral problems.

9. Second order difference equation. Let us describe briefly a realization of the algebra of symmetries (8) upon the discrete Schrödinger equation or three term recurrence relation for orthogonal polynomials [25, 26, 27]. One considers an infinite chain of Jacobi matrices L_j and the corresponding eigenvalue problems:

$$L_j\psi_n^j \equiv \psi_{n+1}^j + u_n^j\psi_{n-1}^j + b_n^j\psi_n^j = \lambda\psi_n^j, \qquad n, j \in Z. \tag{11}$$

In the case of orthogonal polynomials, one considers (11) only for $n > 0$ and imposes the boundary conditions $\psi_0^j(\lambda) = 0$, $\psi_1^j(\lambda) = \lambda - b_1^j$. Simlar to the continuous case the forward discrete time step is defined by the Christoffel's transformation to kernel polynomials [28]:

$$\psi_n^{j+1} = \frac{\psi_{n+1}^j + C_n^{j+1}\psi_n^j}{\lambda - \lambda_{j+1}} \equiv \frac{S_{j+1}\psi_n^j}{\lambda - \lambda_{j+1}}. \tag{12}$$

The backward transformation was analyzed by Geronimus [29] and it has the form:

$$\psi_n^{j-1} = \psi_n^j + A_n^j\psi_{n-1}^j \equiv R_j\psi_n^j. \tag{13}$$

Here A_n^j and C_n^j are discrete analogs of the superpotentials $f_j(x)$ and S_j, R_j are the first order difference operators. The compatibility conditions of the $j \to j\pm1$ moves yield the factorizations $L_j = S_jR_j+\lambda_j$, $L_{j-1} = R_jS_j+\lambda_j$, or $u_n^j = A_n^jC_n^j$, $b_n^j = A_{n+1}^j+C_n^j+\lambda_j$. The abstract factorization chain in this case is equivavlent to the following set of nonlinear finite-difference equations:

$$A_n^jC_{n-1}^j = A_n^{j-1}C_n^{j-1}, \quad A_n^j + C_n^j + \lambda_j = A_{n+1}^{j-1} + C_n^{j-1} + \lambda_{j-1}, \tag{14}$$

known to define a discrete-time Toda lattice. Note that this is not an isospectral flow since the constants λ_j determine the character of the change of the spectrum of the taken Jacobi matrix after the Christoffel or Geronimus transformations.

Discrete Schrödinger equation analogs of the self-similar reductions (7) were described in [25]:

$$A_n^{j+N} = qA_{n+k}^j, \qquad C_n^{j+N} = qC_{n+k}^j, \qquad \lambda_{j+N} = q\lambda_j, \tag{15}$$

where k is an integer. When n is considered as continuous, k may be taken as a continuous variable as well. This closure is associated with classical, semi-classical and, so-called, Laguerre-Hahn class of orthogonal polynomials on linear and q-linear grids [30]. The corresponding recurrence coefficients are related to ordinary and q-analogs of some discrete Painlevé transcendents [25, 31]. The algebra of symmetries of these systems is derived along the same

lines as in the continuous case, however, in the present case the compact versions of algebras, like $su_q(2)$, are allowed as well. The coherent states are defined in the same way as in the continuous case. For some explicit examples see, e.g., [32].

The following discrete-time Volterra lattice has been derived in [26]:

$$D_n^j \left(D_{n-1}^j - \beta_j \right) = D_n^{j-1} \left(D_{n+1}^{j-1} - \beta_{j-1} \right). \tag{16}$$

It can be mapped upon the discrete-time Toda lattice (14) via the following quadratic relation:

$$A_n^j = D_{2n}^j D_{2n+1}^j, \qquad C_n^j = (D_{2n+1}^j - \beta_j)(D_{2n+2}^j - \beta_j), \tag{17}$$

$$\lambda_j = const. - \beta_j^2,$$

which generalizes well-known relation between the ordinary Toda and Volterra lattices. There is also the second similar mapping

$$A_n^j = D_{2n-1}^j D_{2n}^j, \qquad C_n^j = (D_{2n}^j - \beta_j)(D_{2n+1}^j - \beta_j), \tag{18}$$

with the same connection between λ_j and β_j as given before. As shown in [26], this discrete-time Volterra lattice is related to the g-algorithm proposed by Bauer in numerical analysis [33]. There exists an ansatz of semi-separation of discrete variables in (16) which leads [26] to recurrence coefficients of the Askey-Wilson polynomials — the most general set of classical orthogonal polynomials [34] and, simultaneously, of the polynomials considered by Askey and Ismail [35].

An interesting discrete symmetry for the chain (6) has been described in [14]. It is associated with a freedom in the intermediate steps of two-step discrete-time shifts $j \to j+2$ and related to the statement on permutability of a sequence of Bäcklund-Darboux transformations for fixed set of corresponding parameters. In [27] an analog of this symmetry for the discrete time Toda and Volterra lattices has been derived.

Let us describe briefly this refactorization symmetry for the equation (16). Let $\beta_j \neq 0$ for any j, then:

$$\tilde{D}_n^j = \frac{1}{\beta_{j-1}} \left(\beta_j D_n^j + \frac{(\beta_j^2 - \beta_{j-1}^2)(D_n^j D_{n+1}^j + D_n^{j-1} D_{n+1}^{j-1})}{\beta_j(\beta_j - D_{n-1}^j - D_{n+1}^j) + \beta_{j-1}(\beta_{j-1} - D_n^{j-1} - D_{n+2}^{j-1})} \right), \tag{19}$$

$$\tilde{D}_n^{j-1} = \frac{1}{\beta_j} \left(\beta_{j-1} D_n^{j-1} - \frac{(\beta_j^2 - \beta_{j-1}^2)(D_{n-1}^j D_n^j + D_{n-1}^{j-1} D_n^{j-1})}{\beta_j(\beta_j - D_{n-2}^j - D_n^j) + \beta_{j-1}(\beta_{j-1} - D_{n-1}^{j-1} - D_{n+1}^{j-1})} \right). \tag{20}$$

The change of spectral parameters β_j looks as follows:

$$\tilde{\beta}_j = \beta_{j-1}, \qquad \tilde{\beta}_{j-1} = \beta_j. \tag{21}$$

If one substitutes into (16) instead of D_n^j, D_n^{j-1} and β_j, β_{j-1} the tilded variables and keeps all other $D_n^k, \beta_k, k \neq j, j-1$, fixed, then the resulting equation will be satisfied automatically, i.e. we have a discrete symmetry. The transformation laws for superpotentials A_n^j, C_n^j, describing a similar symmetry for the discrete-time Toda lattice, follow from the maps (17), (18). If $\beta_j = 0$ or $\beta_{j-1} = 0$ (j is fixed), then there appears some additional freedom, we refer for details to [27].

We have sketched only the simplest types of discrete-time integrable systems. For an analysis of ordinary Toda lattice and its self-similar reductions see, e.g., [36, 37]. For more complicated examples and their applications see [38, 39, 40, 41]. Recently self-similar reductions of spectral transformation chains have been considered for biorthogonal rational functions in [42].

10. One-dimensional Ising chains. Tau-function of the N-soliton solution of the KdV equation (3) can be represented in the following Hirota form, which was widely discussed in the literature (see, e.g., [43, 5] and references therein):

$$\tau_N = \sum_{\mu_i=0,1} \exp\left(\sum_{1\le i<j\le N} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \theta_i\mu_i \right). \tag{22}$$

Here the phase shifts A_{ij} are expressed via the spectral variables k_i in a simple way

$$e^{A_{ij}} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}. \tag{23}$$

In [44] it was noticed that this expression for τ_N defines the grand partition function of the lattice gas model if $\theta_i = \theta^{(0)} = const$. A comparison with [45] shows that μ_i play the role of the filling factors of the lattice sites by molecules and $\theta^{(0)}$ is a chemical potential. The constants A_{ij} are proportional to the interaction energies of molecules.

Simultaneously, τ_N describes the partition function of a particular one-dimensional Ising chain:

$$Z_N = \sum_{\sigma_i=\pm1} e^{-\beta E}, \qquad \beta = \frac{1}{kT}, \tag{24}$$

$$E = \sum_{1\le i<j\le N} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} H_i\sigma_i,$$

where N is the number of spins $\sigma_i = \pm1$, J_{ij} are the exchange constants, H_i is an external magnetic field, T is the temperature and k is the Boltzmann constant. Indeed, after the substitution into (22) of the spin variables $\mu_i = (\sigma_i + 1)/2$, some simple calculations yield

$$\tau_N = e^{\Phi} Z_N, \qquad \Phi = \frac{1}{4}\sum_{i<j} A_{ij} + \frac{1}{2}\sum_{j=1}^{N} \theta_j, \tag{25}$$

with the following relations between soliton parameters and Ising chain characteristics

$$\beta J_{ij} = -\frac{1}{4}\, A_{ij}, \qquad \beta H_i = \frac{1}{2}\theta_i + \frac{1}{4}\sum_{j=1, i\ne j}^{N} A_{ij}. \tag{26}$$

A similar relation with Ising chains is valid for the whole KP hierarchy and some other differential and difference nonlinear integrable equations. The corresponding tau-functions have the form (22) for different choice of the phase shifts A_{ij} and the phases θ_i, a list of such equations can be found, e.g., in [43, 5].

A crucial observation of [44] is that the condition of self-similarity of the spectrum is related to translational invariance of the spin exchange constants J_{ij} of the infinite Ising chain induced by the KdV equation. In the simplest case one demands translational invariance of the system with respect to the shift by one site $j \to j+1$ which means that $J_{i+1,j+1} = J_{ij}$. As a result, the exchange J_{ij} or phase shifts A_{ij} depend only on the distance between the sites $|i-j|$. This natural physical requirement forces k_i to form one geometric progression

$$k_i = k_1 q^{i-1}, \qquad q = e^{-2\alpha}, \qquad A_{ij} = 2\ln|\tanh\alpha(i-j)|, \tag{27}$$

where k_1 and $q < 1$ are free parameters. One may demand also the invariance with respect to shifts by a multiple of the lattice site, when $J_{i+M,j+M} = J_{ij}$, and this leads to the general self-similar spectra $k_{j+M} = qk_j$.

In fact the translational invariance in not exact for finite chain $1 \le j \le N$. The limit $N \to \infty$ corresponds to the thermodynamic limit. It gives an infinite soliton potential with self-similar discrete spectra. The coordinate x and time t (and higher "times" of the corresponding hierarchies) are interpreted as parameters of the magnetic field H_i. The x, t dependent part of H_i decays exponentially fast for $i \to \infty$, since $q < 1$. Therefore in the thermodynamic limit only the values of constants $\theta_i^{(0)}$ are relevant for the partition function. The formalism allows us to treat the M-periodic magnetic fields, $H_{i+M} = H_i$, which for $M = 1$ is just a homogeinity condition.

For the KdV equation case one has $0 < |\tanh \alpha(i-j)| < 1$ and $J_{ij} = -A_{ij}/4\beta > 0$, i.e. an antiferromagnetic Ising chain. A similar situation takes place for general M-periodic case. Such interaction has a long distance character but the intensity falls off exponentially fast. The absence of phase transitions in such systems for nonzero temperatures is well known.

For $\alpha \to 0$ or $q \to 1$ the phse shifts $A_{ij} \propto J_{ij}/kT$ are diverging. If one renormalizes the exchange constants $J_{ij}^{ren} = J_{ij}(q^{-1} - q)$ and the temperature $kT_{ren} = kT(q^{-1} - q)$, then the interaction energy of a single spin with all other ones will be finite for $q \to 1$. As a result, in this limit one actually gets a nonlocal interaction model with a low effective temperature. Note that for imitation of the change of the temperature it is necessary to change simultaneously the magnetic field $H = H^{ren}/(q^{-1} - q)$.

The limit $q \to 0$ gives $J_{ij}^{ren} \propto \delta_{i+1,j}$, or the high temperature nearest neighbor interaction Ising chain (if H is renormalized). If H is kept finite then this limit corresponds to the non-interacting spins. So, the formalism provides only a partial description of the partition function in a two-dimensional subspace of (T, H, q). For fixed q the temperature T has a prescribed value and one may normalize the "KdV temperature" to $\beta = 1$.

Using the Wronskian formula for the representation of N-soliton potential [46], we were able to calculate the partition function for the translationally invariant Ising chains in a homogeneous magnetic field. Omitting technical details, which can be found in [44], for $M = 1$ we get $Z_N \to \exp(-N\beta f_I)$ for $N \to \infty$, where the free energy per site f_I has the form

$$-\beta f_I(q, H) = \ln \frac{2(q^4; q^4)_\infty \cosh \beta H}{(q^2; q^2)_\infty^{1/2}} + \frac{1}{4\pi} \int_0^{2\pi} d\nu \ln(|\rho(\nu)|^2 - q \tanh^2 \beta H), \tag{28}$$

$$|\rho(\nu)|^2 = \frac{(q^2 e^{i\nu}; q^4)_\infty^2 (q^2 e^{-i\nu}; q^4)_\infty^2}{(q^4 e^{i\nu}; q^4)_\infty^2 (q^4 e^{-i\nu}; q^4)_\infty^2} \frac{1}{4 \sin^2(\nu/2)} = q \frac{\theta_4^2(\nu/2, q^2)}{\theta_1^2(\nu/2, q^2)}.$$

Here $\theta_{1,4}(y, q)$ are the standard Jacobi theta-functions of the argument y and base q (our base is q^2). The original expression for the density function $\rho(\nu)$ is

$$\rho(\nu) = \frac{(q^2; q^4)_\infty^2}{(q^4; q^4)_\infty^2} \sum_{k=-\infty}^{\infty} \frac{e^{i\nu k - \epsilon k}}{1 - q^{4k+2}} = \frac{(q^2 e^{i\nu - \epsilon}; q^4)_\infty (q^2 e^{-i\nu + \epsilon}; q^4)_\infty}{(e^{i\nu - \epsilon}; q^4)_\infty (q^4 e^{-i\nu + \epsilon}; q^4)_\infty}, \tag{29}$$

where a small regularization parameter $\epsilon \to 0$ is introduced for absolute convergency of the infinite sum. The last equality in (29) was obtained with the help of the ${}_1\psi_1$ Ramanujan's bilateral basic hypergeometric series sum [20]:

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}.$$

This fact demonstrates again the general statement given earlier that the q-special functions are always behind the self-similar spectral problems.

The derivative of $f_I(H)$ with respect to H yields the total magnetization of the lattice:

$$m(H) = -\partial_H f_I = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\langle\sigma_i\rangle$$

$$= \tanh\beta H\left(1 - \frac{1}{\pi}\int_0^\pi \frac{\theta_1^2(\nu,q^2)d\nu}{\theta_4^2(\nu,q^2)\cosh^2\beta H - \theta_1^2(\nu,q^2)\sinh^2\beta H}\right). \tag{30}$$

The function $m(H)$ grows monotonically with H, its qualitative properties are predicted by the general theorems on the behavior of 1D systems with the fast decaying interactions.

The main drawback of the described scheme is that the exact calculation is valid only for a fixed temperature in the partition function (q is fixed). One may try to replace (27) by $A_{ij} = 2n\ln|(k_i - k_j)/(k_i + k_j)|$, where n is a sequence of integers, to change the temperature and look for integrable models with such phase shifts. This is not a simple task since there are many constraints involved into the resolution of this problem. We have found only one more example [44] for $n = 2$ corresponding to a subcase of the N-soliton solution of the KP-equation of B-type (BKP). The BKP equation has the form (see, e.g., [47, 48])

$$\frac{\partial}{\partial x_1}\left(9\frac{\partial u}{\partial x_5} - 5\frac{\partial^3 u}{\partial x_3\partial x_1^2} + \frac{\partial^5 u}{\partial x_1^5} - 30\frac{\partial u}{\partial x_3}\frac{\partial u}{\partial x_1} + 30\frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3} + 60\left(\frac{\partial u}{\partial x_1}\right)^3\right) - 5\frac{\partial^2 u}{\partial x_3^2} = 0. \tag{31}$$

τ-function of this integrable equation generates partition functions of quite general Ising chains. The exchange constants have now the form

$$\beta J_{ij} = -\frac{1}{4}A_{ij}, \quad e^{A_{ij}} = \frac{(a_i - a_j)(b_i - b_j)(a_i - b_j)(b_i - a_j)}{(a_i + a_j)(b_i + b_j)(a_i + b_j)(b_i + a_j)} \tag{32}$$

with the energy and partition function given by (24). For $a_i = b_i = k_i/2$ one gets the KdV-inspired model for $\beta = n = 2$.

Translational invariance of the spin lattice, $J_{ij} = J(i - j)$, results in the following spectral self-similarity

$$a_i = q^{i-1}, \quad b_i = bq^{i-1}, \quad q = e^{-2\alpha}, \tag{33}$$

where we set $a_1 = 1$ and assume that $\alpha > 0$. This gives the exchange

$$\beta J_{ij} = -\frac{1}{4}\ln\frac{\tanh^2\alpha(i-j) - (b-1)^2/(b+1)^2}{\coth^2\alpha(i-j) - (b-1)^2/(b+1)^2}.$$

Since this expression is invariant with respect to the inversion $b \to b^{-1}$, the parameter b is restricted to the unit disk $|b| \le 1$. It is not difficult to see that for $-1 < b < -q$ one has now the ferromagnet, i.e. $J_{ij} < 0$. For two other physical regions $q < b \le 1$ and $b = e^{i\phi} \ne -1$, one has the antiferromagnet, i.e. $J_{ij} > 0$.

In the thermodynamic limit $N \to \infty$ the partition function can be represented as the determinant of a Toeplitz matrix, which is diagonalized by the discrete Fourier transformation. Using again the ${}_1\psi_1$ sum, we have found the free energy per site for the homogeneous magnetic field $H_i = H$:

$$-\beta f_I(H) = \frac{1}{4}\ln\frac{(q,q,bq,q/b;q)_\infty}{(-q,-q,-bq,-q/b;q)_\infty} + \frac{1}{4\pi}\int_0^{2\pi} d\nu\ln|2\rho(\nu)|,$$

where

$$\rho(\nu) = \cosh 2\beta H + \frac{(-q;q)_\infty^2}{(-e^{i\nu}, -qe^{-i\nu}; q)_\infty} \left(\frac{(b^{-1}e^{i\nu}, qbe^{-i\nu}; q)_\infty}{(b^{-1}, qb; q)_\infty} + \frac{(be^{i\nu}, qb^{-1}e^{-i\nu}; q)_\infty}{(b, qb^{-1}; q)_\infty} \right). \quad (34)$$

The magnetization if obtained after taking the derivative with respect to H

$$m(H) = \tanh 2\beta H \ \left(1 - \frac{1}{\pi} \int_0^\pi \frac{d\nu}{1 + d(\nu) \cosh 2\beta H} \right), \quad (35)$$

where

$$d(\nu) = \frac{(qb, q/b; q)_\infty (b^{-1/2} - b^{1/2}) \theta_2(\nu, q^{1/2})}{(-q;q)_\infty^2 2\mathrm{Im}\ \theta_1(\nu - (i/2)\ln b, q^{1/2})}$$

for $q < b \leq 1$,

$$d(\nu) = \frac{(qb, q/b; q)_\infty (|b|^{-1/2} + |b|^{1/2}) \theta_2(\nu, q^{1/2})}{(-q;q)_\infty^2 2\mathrm{Re}\ \theta_2(\nu - (i/2)\ln |b|, q^{1/2})}$$

for $-1 < b < -q$, and

$$d(\nu) = \frac{(qe^{i\phi}, qe^{-i\phi}; q)_\infty 2\sin(\phi/2)\ \theta_2(\nu, q^{1/2})}{(-q;q)_\infty^2 [\theta_1(\nu + \phi/2, q^{1/2}) - \theta_1(\nu - \phi/2, q^{1/2})]}$$

for $b = e^{i\phi}$. Here $\theta_2(\nu, q^{1/2})$ is another Jacobi θ-function. Taking the limit $b \to 1$ one may find the magnetization for the second value of discrete temperature of the KdV equation inspired spin chain, $n = 2$. The details of calculations and graphical representation of $m(H)$ for some values of the parameters can be found in [44]. In a somewhat similar way one may consider the $M > 1$ periodic systems as well.

11. Random matrices. Random matrices were employed by Wigner and Dyson for studying spectra of complex systems with many degrees of freedom. When the density of levels is high enough, excitation energies can be described statistically. The detailed structure of hamiltonians is not known and statistical characteristics of the system are described by averaging over ensembles of random matrices. A fundamental constraint is that the probability distributions should be invariant under basic symmetries such as parity, rotation and time-reversal transformations [49].

Gaussian ensembles use a real symmetric, Hermitian or self-dual Hermitian random matrix H with the statistically independent elements H_{ik}. The probability density $P(H)$ for H_{ik} to lie in a unit volume is proportional to $\exp(-a\,\mathrm{tr}\,H^2 + b\,\mathrm{tr}\,H)$, with the measure invariant under orthogonal, unitary or symplectic transformations. Abandoning the condition of statistical independence one may pass to ensembles with more complicated $P(H)$.

Dyson has introduced circular ensembles of unitary random $n \times n$ matrices S with eigenvalues $\epsilon_j = e^{i\phi_j}$, $j = 1, \ldots, n$, such that the behavior of the phases ϕ_i is equivalent to the distribution of eigenvalues of a system. The condition of invariance of the measure under all unitary authomorphisms $S \to USW$, where U, W are arbitrary unitary matrices, determines the probability distribution. Any S can be represented in the form $S = U^{-1}EU$, E a diagonal matrix with the eigenvalues ϵ_j and U some unitary matrix. Then, using the invariance of probability measure, one can gauge away the U-dependent part of S and integrate out the "angular variables" in the probability distribution. As a result, the eigenvalue distribution becomes

$$P d\phi_1 \ldots d\phi_n \propto \prod_{i<j} |\epsilon_i - \epsilon_j|^2 d\phi_1 \ldots d\phi_n.$$

One can weaken the condition of invariance of the measure under arbitrary left or right translations and leave only pure unitary transformations. Then the invariant measure is not uniquely defined and the eigenvalue distribution may be written as

$$Pd\phi_1 \dots d\phi_n = f(\phi_1, \dots, \phi_n) \prod_{i<j} |\epsilon_i - \epsilon_j|^2 d\phi_1 \dots d\phi_n,$$

where the function f is symmetric under permutation of its arguments. One of such ensembles has been considered by Gaudin long time ago [50]. In this model one has the following probability law for eigenvalues

$$Pd\phi_1 \dots d\phi_n \propto \prod_{i<j} \left| \frac{\epsilon_i - \epsilon_j}{\epsilon_i - z\epsilon_j} \right|^2 d\phi_1 \dots d\phi_n, \tag{36}$$

interpolating between the Dyson unitary ensemble ($z = 0$) and the uniform distribution ($z = 1$).

The same model can be interpreted also as a Coulomb gas on a circle with the partition function

$$Z_n \propto \int_0^{2\pi} \dots \int_0^{2\pi} d\phi_1 \dots d\phi_n \exp\left(-\beta \sum_{i<j} V(\phi_i - \phi_j) \right),$$

where $\beta = 1/kT$ is fixed and the potential energy is

$$\beta V(\phi_i - \phi_j) = \ln\left(1 + \frac{\sinh^2 \gamma}{\sin^2((\phi_i - \phi_j)/2)} \right), \quad z = e^{-2\gamma}. \tag{37}$$

The grand partition function of this model corresponds to the τ-function of the KP hierarchy in the specific infinite soliton limit [44]. The finite soliton solutions provide thus a discretization of the system, namely, a lattice gas on the circle. For example, one may take unitary $n \times n$ matrices S with the eigenvalues equal to N-th roots of unity, i.e. $\phi_j = 2\pi m_j/N, m_j = 1, \dots, N$. The measure is taken to be continuous in the "angular" variables and discrete in eigenvalues, i.e. one takes the sums over ϕ_j instead of the integrals. The completely continuous model is recovered for $N \to \infty$, ϕ_j fixed:

$$\left(\frac{2\pi}{N}\right)^n \sum_{m_1=1}^{N} \dots \sum_{m_n=1}^{N} \underset{N\to\infty}{\to} \int_0^{2\pi} d\phi_1 \dots \int_0^{2\pi} d\phi_n. \tag{38}$$

For $n \times n$ matrices the lattice partition function is

$$Z_n(N, z) = \left(\frac{2\pi}{N}\right)^n \sum_{m_1=1}^{N} \dots \sum_{m_n=1}^{N} \prod_{1\le i<j\le n} \left| \frac{\epsilon_i - \epsilon_j}{\epsilon_i/\sqrt{z} - \sqrt{z}\epsilon_j} \right|^2,$$

$$\epsilon_j = \exp \frac{2i\pi m_j}{N}, \qquad z = e^{-2\gamma}.$$

The grand canonical ensemble partition function is

$$Z(z, \theta) = \sum_{n=0}^{N} \frac{Z_n(N, z) e^{\theta n}}{n!} = \sum_{\mu_m = 0,1} \exp\left(\sum_{1\le m<k\le N} A_{mk}\mu_m\mu_k + (\theta + \eta) \sum_{m=1}^{N} \mu_m \right), \tag{39}$$

where $\eta = \ln(2\pi/N)$ enters as an addition to the chemical potential θ and

$$A_{mk} = \ln \frac{\sin^2(\pi(m-k)/N)}{\sin^2(\pi(m-k)/N) + \sinh^2 \gamma} = \ln \frac{(a_m - a_k)(b_m - b_k)}{(a_m + b_k)(b_m + a_k)},$$

is the KP phase shift with the restricted choice of parameters

$$a_m = e^{2i\pi m/N}, \qquad b_m = -za_m, \quad m = 1, 2, \ldots, N. \tag{40}$$

So, the grand partition function of this discrete circular unitary matrix model coincides with the particular KP N-soliton τ-function at zero hierarchy "times". In the thermodynamical limit $N \to \infty$ the relation (38) takes place and we get the matrix model (36). In December 1998 we have known that this root-of-unity discretization was considered by Gaudin himself [51], where the connection with Ising chains was noticed as well but the relation with integrable equations was not established.

The BKP equation suggests a further generalization of this matrix model with the probability law

$$Pd\phi_1 \ldots d\phi_n \propto \prod_{i<j} \left|\frac{\epsilon_i - \epsilon_j}{\epsilon_i + \epsilon_j}\right|^2 \left|\frac{\epsilon_i + z\epsilon_j}{\epsilon_i - z\epsilon_j}\right|^2 d\phi_1 \ldots d\phi_n. \tag{41}$$

The grand partition function is also defined by (39) where A_{mk} is given by (32). In order to escape singularities appearing from zeros $\epsilon_i + \epsilon_j = 0$ for $\phi_j = \phi_i + \pi$, the parameters a_m, b_m have to be restricted. The choice (40) works for odd N only, which makes the $N \to \infty$ limit problematic. Another option is to replace m/N in (40) by $m/2N$, which corresponds to the half-circle system. Comparing the connection of BKP equation with Ising chains described in the previous section we see, that now we have spins on the circle, q is a root of unity and the exchange $J_{mk} = J(m-k)$ is

$$\beta J(m) = -\frac{1}{4} \ln \frac{1 + \sinh^2 \gamma / \cos^2(\pi m/N)}{1 + \sinh^2 \gamma / \sin^2(\pi m/N)}.$$

An important difference between the $|q| < 1$ and $|q| = 1$ Ising chains is that in the former case the thermodynamic limit $N \to \infty$ does not lead necessarily to the continuous space models as in the circular case. Also the relation of non-compact Ising chains to hermitian random matrices is not so straightforward. In particular, in order to get nontrivial weight functions one has to take inhomogeneous magnetic field corresponding to nonzero hierarchy "times". Calculation of the partition function is more involved in this situation.

12. Conclusions. Self-similarity in spectral problems serves as a key to special functions, in particular, to q-special functions. We hope that the brief review given in this paper gives a sufficient number of arguments for convincing in this vague statement. As to the notion of exact solvability of a given equation, it requires a more rigorous group-theoretical definition in terms of the operator (differential, difference, etc) Galois theory. In this approach one fixes in advance the basic fields of functions entering spectral problems and asks about the transcendence of solutions of the resulting equations over these fields. In the scheme we have presented the situation is slightly different — the basic fields of functions are defined as solutions of complicated nonlinear equations appearing after self-similar reductions of some infinite-dimensional systems of equations. Usually this gives simultaneously a part of solutions of the underlying linear spectral equations in quadratures over the basic field of functions, but determination of the trancendence of the general solution remains open. Therefore these two approaches should be used in conjunction. One of the problems appearing on this route is to give a complete characterization of the changes in the structure of corresponding (differential, ...) Galois groups resulting from the Darboux-type spectral transformations [21].

Self-similar spectra considered here are based upon linear map of the spectral parameter $\lambda \to q\lambda + h$. This leads naturally to spectra containing a finite number of geometric progressions

and "Painlevé-type" transcendental functions on the background. A natural generalization involves polynomial maps $\lambda \to P(\lambda)$ leading in general to chaotic structure of the spectrum. The situation now is much more complicated and only the simplest possibilities were considered in the literature. Note that, similar to the linear case, superpositions of several spectral sequences generated by the map $\lambda \to P(\lambda)$ are permitted too, but strong difficulties arise in the characterization of functions involed in these systems already in the simplest cases (see, e.g., [52, 53]). The scalings of discrete grids, or decimations entering at this stage hint upon the relevance of the self-similarity hidden in wavelets and quasicrystals. Perhaps all these subjects are naturally unified within an extended appoach to the concept of self-similarity.

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