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Special non uniform lattice (*snul*) orthogonal polynomials on discrete dense sets of points.

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"Il n'est pas nécessaire d'espérer pour entreprendre,
ni de réussir pour persévérer."
(Begin, even without hope; Proceed, even without success.)
William of Orange ("William the Silent"), murdered in Delft in 1584.

Abstract. Difference calculus compatible with polynomials (i.e., such that the divided difference operator of first order applied to any polynomial must yield a polynomial of lower degree) can only be made on special lattices well known in contemporary $q-$calculus. Orthogonal polynomials satisfying difference relations on such lattices are presented. In particular, lattices which are dense on intervals ($|q| = 1$) are considered.

1. Introduction.

Many works have been devoted to orthogonal polynomials satisfying remarkable differential or difference relations.

For instance, the classical orthogonal polynomials are characterized by the existence of a differential relation of the form

$$W(x)p_n'(x) = \omega_n(x)p_n(x) + \vartheta_n p_{n-1}(x), \tag{1}$$

where W is a fixed polynomial of degree ≤ 2, ω_n is a polynomial of degree ≤ 1,and where ϑ_n is constant (in x) (cf. [1], p.8).

This differential relation interacts most efficiently with the recurrence relation

$$a_{n+1}p_{n+1}(x) = (x - b_n)p_n(x) + a_n p_{n-1}(x) \tag{2}$$

in the production of various useful identities.

If we accept higher degrees polynomials in (1):

$$W(x)p_n'(x) = \Omega_n(x)p_n(x) - a_n\Theta_n(x)p_{n-1}(x), \tag{3}$$

with W, Ω_n and Θ_n polynomials of degrees $\leq s+2, s+1$ and $s \geq 0$, we get the *semi-classical* class, studied by Laguerre (the notations of (3) are [almost] Laguerre's ones [19]), Hendriksen and van Rossum [10], and Maroni [22] who coined this name. See also [4,5] for determination of the relevant measure.

If we try to extend (3) to a difference operator of first order, we expect to see the derivative $p_n'(x)$ replaced by some combination of $p_n(y(s))$ and $p_n(y(s+1))$, where $y(s)$ and $y(s+1)$ are two consecutive points on a lattice associated to the difference operator.

We will explore here the extension of the semi-classical property (3) to the remarkable difference operators and the corresponding nonuniform lattices studied by many people recently [2,6,12,16,17,21, 24,25,29,30]. In particular, those lattices which happen to fill densely an interval will be considered with special care.

2. The difference operator and the related lattices.

We consider here a first-order difference operator involving the values of a function at two points. For each x, let $\varphi_1(x)$ and $\varphi_2(x)$ be these still unknown points. The first-order *divided difference* operator at x is

$$(\mathcal{D}f)(x) = \frac{f(\varphi_2(x)) - f(\varphi_1(x))}{\varphi_2(x) - \varphi_1(x)}\,. \tag{4}$$

If we impose the condition that $\mathcal{D}f$ is a polynomial of degree $n-1$ if f has degree n, then $\varphi_1(x)$ and $\varphi_2(x)$ must be the two roots in y of a quadratic equation

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0. \tag{5}$$

(see [6,12, 21]).

Indeed, applying $\mathcal{D}$ to $f(x) = x^2$ and $f(x) = x^3$ readily yields that $\varphi_1 + \varphi_2$ and $\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2$ must be polynomials of degrees 1 and 2, which implies (5). Conversely, if (5) holds, any symmetric polynomial in φ_1 and φ_2 is a polynomial in x.

Let us figure the conic (5) and one of its parametric representations $\{x(s), y(s)\}$ such that $y(s)$ and $y(s+1)$ appear naturally as the two ordinates associated to the abscissa $x = x(s)$: one starts from some point $\{x_1 = x(s_1), y_1 = y(s_1)\}$ on the conic, and one looks for the points $\{x_k = x(s_1+k-1), y_k = y(s_1+k-1)\}$, $k = 1, 2, \ldots$

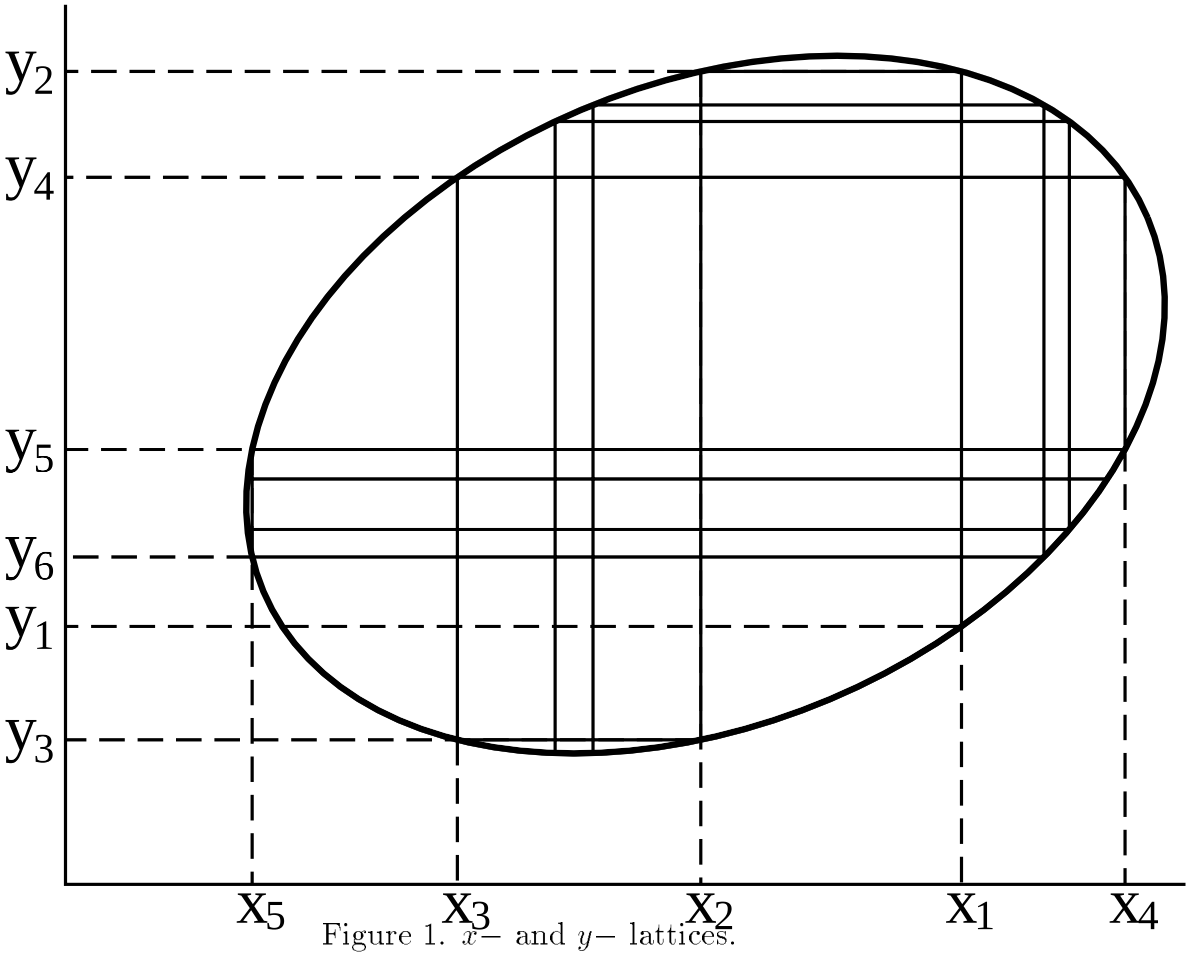

Figure 1. $x-$ and $y-$ lattices.

To achieve this, let us consider the familiar parametric representations in the two following cases:

1. The conic (5) has a center $B^2 - AC \neq 0$. With the center coordinates $x_c = (AE - BD)/(B^2 - AC)$ and $y_c = (CD - BE)/(B^2 - AC)$, one has $A(y-y_c)^2 + 2B(x-x_c)(y-y_c) + C(x-x_c)^2 + \widetilde{F} = 0$, with $\widetilde{F} = F - Ay_c^2 - 2Bx_cy_c - Cx_c^2 = F + Dy_c + Ex_c = F + (CD^2 - 2BDE + AE^2)/(B^2 - AC)$,

$$x = x(s) = x_c + \zeta\sqrt{A}(q^s + q^{-s}) \, , y = y(s) = y_c + \zeta\sqrt{C}(q^{s-1/2} + q^{-s+1/2}) \, , \qquad (6)$$

is a valid parametric representation of (5), (if $AC \neq 0$), where
$\zeta^2 = \widetilde{F}/(4(B^2 - AC))$ and

$$q^{1/2} + q^{-1/2} = -\frac{2B}{\sqrt{AC}} \text{ i.e., } q + q^{-1} = \frac{4B^2}{AC} - 2. \qquad (7)$$

Indeed, $x(s) = x(-s)$ is kept by the transformation $s \leftrightarrow -s$ in (6) but y becomes $y(s+1)$.
If $AC = 0$, the scheme of Fig. 1 and Fig. 2 does not work: horizontal and/or vertical lines do not meet the conic in two points any more.

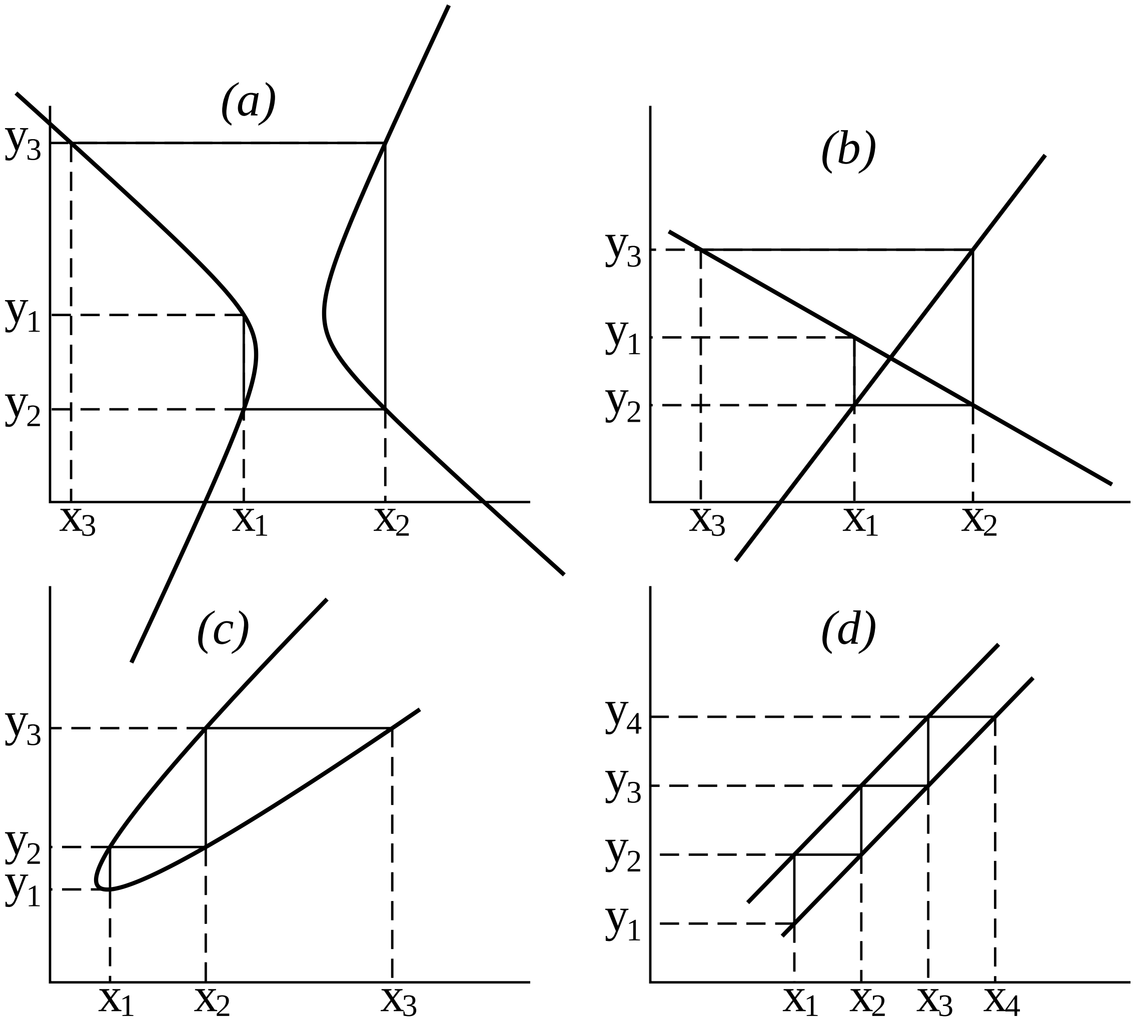

Figure 2. Other kinds of lattices.

The generic (also rightly called *hyperbolic*) case $|q| \neq 1$ gives a hyperbola (Fig. 2(a)). Remark that the asymptotes are given by (x and y large) $y \sim (C/A)^{1/2} q^{\pm 1/2} x$:

q is the ratio of the slopes of the asymptotes of the conic.

If $\widetilde{F} = 0$, one finds $x - x_c = X\sqrt{A}q^s$, $y - y_c = X\sqrt{C}q^{s \pm 1/2}$, the "old" $q-$lattice (Fig. 2(b)).

2. The conic (5) has no center, $B^2 - AC = 0$. Then,

$$x = x(s) = \sqrt{A}\left\{\frac{D^2 - AF}{2A(D\sqrt{C} + E\sqrt{A})} - 2\frac{D\sqrt{C} + E\sqrt{A}}{AC}s^2\right\} ,$$
$$y = y(s) = \sqrt{C}\left\{\frac{E^2 - CF}{2C(D\sqrt{C} + E\sqrt{A})} - 2\frac{D\sqrt{C} + E\sqrt{A}}{AC}(s - 1/2)^2\right\} ,$$

found directly to satisfy (5) in the parabolic case $B^2 = AC$, or by taking the limit of (6) when $q \to 1$: let $q = \exp(\varepsilon)$, then, $q^s = 1 + s\varepsilon + s^2\varepsilon^2/2 + \cdots$, $B = -\sqrt{AC}(q^{1/2} +$

$q^{-1/2})/2 = -\sqrt{AC}(1+\varepsilon^2/8+\cdots)$, etc. (Fig. 2(c)).
If $D\sqrt{C}+E\sqrt{A}=0$, one redefines s through the translation

$$s_{old} = \lim_{D\sqrt{C}+E\sqrt{A}\to 0} \pm\left\{ s_{new} + \frac{\sqrt{(D^2-AF)C}}{2(D\sqrt{C}+E\sqrt{A})}\right\} ,$$

to get

$$x = -2\sqrt{\frac{D^2-AF}{AC}}s \; , y = \frac{E}{\sqrt{AC}} - 2\frac{\sqrt{D^2-AF}}{A}(s\pm 1/2) \; ,$$

the simplest lattice (this one is uniform (Fig. 2(d))).

These points form one of the special non uniform lattices (*snul*) I-VI of [24,25]. If the quadratic equation (5) describes an ellipse, as in the figure, this lattice fills densely an interval, unless it is finite (periodic: $q^N=1$).

3. Semi-classical orthogonal polynomials on *snul*s.

Semi-classical *snul* orthogonal polynomials may be defined through a $\mathcal{D}-$difference equation of the form

$$W(x)(\mathcal{D}S)(x) = 2V(x)(\mathcal{M}S)(x) + U(x) \tag{8}$$

for the Stieltjes function

$$S(x) = \int_{\text{Supp. }\mu} (x-t)^{-1}d\mu(t) = \sum_0^\infty \mu_k/x^{k+1}$$

(Stieltjes transform of the orthogonality measure $d\mu$) where W, V and U are polynomials and $\mathcal{M}$ is the arithmetic mean operator:

$$(\mathcal{M}f)(x) = (f(\varphi_1(x)) + f(\varphi_2(x)))/2.$$

If μ has a jump (pole of S) at some y_n, it must have jumps at the other points $y_{n\pm1}$, $y_{n\pm2}, \ldots$ of the corresponding lattice.

It is possible to recover a difference relation extending (3) to the present difference operators, as attempted and *min of meer* (less or more) achieved in the paper [21], see here an attempt to achieve the converse:

Theorem. *Orthogonal polynomials are $\mathcal{D}-$semi-classical, i.e., their Stieltjes function satisfies an equation (8) with polynomials W, V, and U, if each p_n satisfies a linear first-order difference relation connecting p_n with p_{n-1}*

$$W_n(x)(\mathcal{D}p_n)(x) = \Omega_n(x)(\mathcal{M}p_n)(x) - a_n\Theta_n(x)(\mathcal{M}p_{n-1})(x), \tag{9}$$

where W_n, Ω_n, and Θ_n are polynomials of fixed (independent of n) degrees.

Indeed, let y_1 and y_2 be the ordinates corresponding to $x = x_1$, then (9) is a linear relation involving $p_n(y_1)$, $p_n(y_2)$, $p_{n-1}(y_1)$, and $p_{n-1}(y_2)$:

$$A_n p_n(y_1) + B_n p_n(y_2) + C_n p_{n-1}(y_1) + D_n p_{n-1}(y_2) = 0, \tag{10}$$

where A_n, B_n, C_n, and D_n are rational functions of fixed degrees of $x = x_1$ and y_1 (y_2 may be replaced by $-2(Bx + D)/A - y_1$ in $A_n = -W_n(x)/(y_2 - y_1) - \Omega_n(x)/2$ etc.). We come to $n+1$:

$$A_{n+1} p_{n+1}(y_1) + B_{n+1} p_{n+1}(y_2) + C_{n+1} p_n(y_1) + D_{n+1} p_n(y_2) = 0,$$

and use the recurrence relation (2):

$$\begin{aligned}[][(y_1 - b_n)A_{n+1} + a_{n+1}C_{n+1}]\, p_n(y_1) + [(y_2 - b_n)B_{n+1} + a_{n+1}D_{n+1}]\, p_n(y_2) - \\ a_n A_{n+1} p_{n-1}(y_1) - a_n B_{n+1} p_{n-1}(y_2) = 0,\end{aligned} \tag{11}$$

which is also a linear relation involving $p_n(y_1)$, $p_n(y_2)$, $p_{n-1}(y_1)$, and $p_{n-1}(y_2)$!

1. Either (10) and (11) are dependent for each n, then $A_n = K_n[(y_1 - b_n)A_{n+1} + a_{n+1}C_{n+1}]$ and $C_n = K_n a_n A_{n+1}$, i.e., the recurrence relation $A_n = K_n[(y_1 - b_n)A_{n+1} - a_{n+1}^2 K_{n+1} A_{n+2}]$ which looks somewhat like (2): actually, $K_1 \dots K_{n-1} a_1 \dots a_{n-1} A_n$ would be a solution of (2), and this is incompatible with the requirement that A_n keeps a finite degree. So,
2. or (10) and (11) are independent for some n. Then, it is possible to extract $p_n(y_1)$ and $p_n(y_2)$ in terms of $p_{n-1}(y_1)$ and $p_{n-1}(y_2)$:

   $$p_n(y_1) = X_n p_{n-1}(y_1) + Y_n p_{n-1}(y_2)\ , \quad p_n(y_2) = Z_n p_{n-1}(y_1) + U_n p_{n-1}(y_2),$$

   where X_n, Y_n, Z_n, and U_n are again rational functions of fixed degrees. We get the relation for $n+1$ using (2):

   $$\begin{aligned}\begin{bmatrix} p_{n+1}(y_1) \\ p_{n+1}(y_2) \end{bmatrix} &= \frac{1}{a_{n+1}} \begin{bmatrix} y_1 - b_n & 0 \\ 0 & y_2 - b_n \end{bmatrix} \begin{bmatrix} p_n(y_1) \\ p_n(y_2) \end{bmatrix} - \frac{a_n}{a_{n+1}} \begin{bmatrix} p_{n-1}(y_1) \\ p_{n-1}(y_2) \end{bmatrix} \\ &= \left\{ \frac{1}{a_{n+1}} \begin{bmatrix} y_1 - b_n & 0 \\ 0 & y_2 - b_n \end{bmatrix} - \frac{a_n}{a_{n+1}} \begin{bmatrix} X_n & Y_n \\ Z_n & U_n \end{bmatrix}^{-1} \right\} \begin{bmatrix} p_n(y_1) \\ p_n(y_2) \end{bmatrix}\ ,\end{aligned}$$

   showing that we can construct similar relations for $n+1$, $n+2$, etc. as far as x is not a zero of the determinants $\delta_n = X_n U_n - Z_n Y_n$, etc.:

   $$\begin{bmatrix} p_{n+1}(y_1) \\ p_{n+1}(y_2) \end{bmatrix} = \begin{bmatrix} X_{n+1} & Y_{n+1} \\ Z_{n+1} & U_{n+1} \end{bmatrix} \begin{bmatrix} p_n(y_1) \\ p_n(y_2) \end{bmatrix}\ ,$$

   whence the recurrence relations for the X's, Y's, Z's, and U's:

   $$\begin{aligned} a_{n+1} X_{n+1} &= y_1 - b_n - a_n U_n/\delta_n\ , \\ a_{n+1} U_{n+1} &= y_2 - b_n - a_n X_n/\delta_n\ , \\ a_{n+1} Y_{n+1} &= a_n Y_n/\delta_n\ , \\ a_{n+1} Z_{n+1} &= a_n Z_n/\delta_n\ , \\ a_{n+1}^2 \delta_{n+1} &= (y_1 - b_n)(y_2 - b_n) - a_n[(y_1 - b_n)X_n + (y_2 - b_n)U_n]/\delta_n + a_n^2/\delta_n\ . \end{aligned}$$

Let $\delta_n = \Theta_n/\Theta_{n-1}$ (yes, this is the Θ_n which will appear in (9)), then, with $\Theta_{n-1}X_n = \Upsilon_n$ and $\Theta_{n-1}U_n = \chi_n$,

$$\begin{aligned}
a_{n+1}\Upsilon_{n+1} &= (y_1 - b_n)\Theta_n - a_n\chi_n \ , \\
a_{n+1}\chi_{n+1} &= (y_2 - b_n)\Theta_n - a_n\Upsilon_n \ , \\
a_{n+1}^2\Theta_{n+1} &= (y_1 - b_n)(y_2 - b_n)\Theta_n - a_n[(y_1 - b_n)\Upsilon_n + (y_2 - b_n)\chi_n] + a_n^2\Theta_{n-1} \ .
\end{aligned}$$

Remark that $a_n\Theta_{n-1}Y_n$ and $a_n\Theta_{n-1}Z_n$ are independent of n.

These recurrence relations for Υ_n, χ_n and Θ_n are exactly the recurrence relations satisfied by *products* of solutions of (2) at y_1 and y_2! Indeed, if $a_{n+1}\xi_{n+1} = (y_1 - b_n)\xi_n - a_n\xi_{n-1}$, and $a_{n+1}\eta_{n+1} = (y_2 - b_n)\eta_n - a_n\eta_{n-1}$, one finds for $[\xi_n\eta_{n-1}, \xi_{n-1}\eta_n, \xi_n\eta_n]$ exactly the recurrence for $[\Upsilon_n, \chi_n, \Theta_n]$ (Actually, this is a recurrence of *fourth* order, one should work with vectors $[\xi_n\eta_{n-1}, \xi_{n-1}\eta_n, \xi_n\eta_n, \xi_{n-1}\eta_{n-1}]$ and $[\Upsilon_n, \chi_n, \Theta_n, \Theta_{n-1}]$).

Now, any solution of (2) is a combination of $p_n(x)$ and $q_n(x)$ defined by

$$q_n(x) = \int_{\text{Supp. }\mu} p_n(t)(x-t)^{-1}\,d\mu(t) = 1/(\gamma_n x^{n+1}) + \cdots \ ,$$

a useful well-known identity is

$$p_n q_{n-1} - p_{n-1} q_n = \gamma_n/\gamma_{n-1} = 1/a_n,$$

so, ξ_n is some combination of $p_n(y_1)$ and $q_n(y_1)$, η_n is some combination of $p_n(y_2)$ and $q_n(y_2)$, and

$$\begin{aligned}
\Upsilon_n &= \alpha p_n(y_1)p_{n-1}(y_2) + \beta p_n(y_1)q_{n-1}(y_2) + \gamma q_n(y_1)p_{n-1}(y_2) + \delta q_n(y_1)q_{n-1}(y_2) \ , \\
\chi_n &= \alpha p_{n-1}(y_1)p_n(y_2) + \beta p_{n-1}(y_1)q_n(y_2) + \gamma q_{n-1}(y_1)p_n(y_2) + \delta q_{n-1}(y_1)q_n(y_2) \ , \\
\Theta_n &= \alpha p_n(y_1)p_n(y_2) + \beta p_n(y_1)q_n(y_2) + \gamma q_n(y_1)p_n(y_2) + \delta q_n(y_1)q_n(y_2) \ .
\end{aligned}$$

As we want the degrees of the left sides to remain bounded when n increases, we must have $\alpha = 0$. Let us show that $\delta = 0$ as well: from $p_n(y_1) = X_n p_{n-1}(y_1) + Y_n p_{n-1}(y_2)$, $\Theta_{n-1}p_n(y_1) - \Theta_{n-1}X_n p_{n-1}(y_1) = \Theta_{n-1}p_n(y_1) - \Upsilon_n p_{n-1}(y_1) = [\beta p_{n-1}(y_1)q_{n-1}(y_2) + \gamma q_{n-1}(y_1)p_{n-1}(y_2) + \delta q_{n-1}(y_1)q_{n-1}(y_2)]p_n(y_1) - [\beta p_n(y_1)q_{n-1}(y_2) + \gamma q_n(y_1)p_{n-1}(y_2) + \delta q_n(y_1)q_{n-1}(y_2)]p_{n-1}(y_1) = \gamma p_{n-1}(y_2)[q_{n-1}(y_1)p_n(y_1) - q_n(y_1)p_{n-1}(y_1)] +$
$\delta q_{n-1}(y_2)[q_{n-1}(y_1)p_n(y_1) - q_n(y_1)p_{n-1}(y_1)] = [\gamma p_{n-1}(y_2) + \delta q_{n-1}(y_2)]/a_n = \Theta_{n-1}Y_n p_{n-1}(y_2)$,
or
$\gamma p_{n-1}(y_2) + \delta q_{n-1}(y_2) = a_n\Theta_{n-1}Y_n p_{n-1}(y_2) =$ constant $p_{n-1}(y_2)$, possible only if the constant (with respect to n) $a_n\Theta_{n-1}Y_n = \gamma$, and $\delta = 0$.

One finds similarly $a_n\Theta_{n-1}Z_n = \beta$.

Finally, at $n = 0$, $\Theta_0 = \beta p_0(y_1)q_0(y_2) + \gamma q_0(y_1)p_0(y_2) = [\beta S(y_2) + \gamma S(y_1)]/\mu_0$, and this relation between $S(y_1)$ and $S(y_2)$ is exactly (8)! One has $U = \Theta_0$, $W = (y_2 - y_1)(\beta - \gamma)/(2\mu_0)$, and $V = -(\beta + \gamma)/(2\mu_0)$.

□

4. Semi-classical measures on *snul*s.

Let us consider meromorphic Stieltjes functions S, corresponding therefore to discrete (atomic [8]) measures. From (8), or from the equivalent form $\beta S(y_2)+\gamma S(y_1)=\mu_0 U$, we have a recurrence

$$\beta(x_k,y_k)S(y_{k+1})+\gamma(x_k,y_k)S(y_k)=\mu_0 U(x_k),$$

showing that poles occur at some available lattice $\dots,y_k,y_{k+1},\dots$ with residues (masses of the measure) satisfying

$$\beta(x_k,y_k)\mu(y_{k+1})+\gamma(x_k,y_k)\mu(y_k)=0, \tag{12}$$

a Pearson-like equation like this is discussed in [29,30].

The masses will usually not make an infinite convergent sequence, so that they must remain in finite number if one wants a discrete measure. This is possible only if some value of $\beta(x,y)$ on a lattice, say $\beta(x_0,y_0)$ vanishes, so that we may start the nonzero masses at y_1 (and put $\mu(y_0)=0$), and also if some value of $\gamma(x,y)$ vanishes on the same lattice, say $\gamma(x_N,y_N)=0$, so that we stop the nonzero masses at y_N (and put $\mu(y_{N+1})=0$), see [29] (the "uninteresting case", p.655).

For instance, if $V=0$ in (8), we simply have equal masses: $\beta(x,y)=-\gamma(x,y)=W(x)$, and W must vanish at two points of the $x-$lattice at least. If this does not happen, the measure is approximated by a discrete measure with many small equal masses at more and more lattice points, and tends towards the limit distribution of these lattice points. Take for instance (6) with $x_c=y_c=0$, $\zeta\sqrt{A}=\zeta\sqrt{C}=1/2$, and $q=\exp(i\theta)$, then, $x_k=\cos k\theta$, $y_k=\cos(k-1/2)\theta$, the lattice VI of [25], distributed (if θ/π is not rational) like $d\mu(x)=(1-x^2)^{-1/2}dx$: we recover the Chebyshev polynomials! This is not surprising, as these polynomials have interesting properties with respect to the $\mathcal{D}-$operator [6, 12].

5. Dense discrete measures and combinations of Chebyshev polynomials.

We keep $x_k=\cos k\theta$, $y_k=\cos(k-1/2)\theta$, the lattice VI of [25] with θ/π not rational, and try to find a discrete measure with jumps at each of these y_k. "Strange" supports, er, *carriers*, have been well worked [11,20,23,31,32,33], there is a reference to Stieltjes himself in p.202 of [18].

As semi-classical orthogonal polynomials do not seem to be related to dense discrete measures, one tries combinations of the simplest such items, i.e., Chebyshev polynomials:

Theorem. *Let the measure μ be discrete with jumps $\mu(y_k)=1/(k-1/2)^2$ at $y_k=\cos((k-1/2)\theta)$, $k=\dots,-2,-1,0,1,2,\dots$, with θ/π irrational. Then, the orthonormal polynomials are*

$$p_0=\pi^{-1},$$

$$p_n=\frac{(\varepsilon_{2n-1}+\iota_{2n-1})T_n-(-1)^{\lfloor\xi_{2n-1}\theta/(2\pi)\rfloor}\iota_{2n-1}T_{|n-\xi_{2n-1}|}+(-1)^{\lfloor\eta_{2n-1}\theta/(2\pi)\rfloor}\varepsilon_{2n-1}T_{|n-\eta_{2n-1}|}}{\sqrt{2\pi^2\varepsilon_{2n}\iota_{2n}(\varepsilon_{2n-1}+\iota_{2n-1})}},$$

$$n\geq 1 \tag{13}$$

where ξ_j is the value of p which minimizes $p\theta/(2\pi) - \lfloor p\theta/(2\pi)\rfloor$ on $p = 1, 2, \ldots, j$, η_j is the value of p which minimizes $\lfloor p\theta/(2\pi)\rfloor + 1 - p\theta/(2\pi)$ on $p = 1, 2, \ldots, j$, (where $\lfloor x\rfloor$ is the largest integer smaller or equal than x), $\varepsilon_j = \xi_j\theta/(2\pi) - \lfloor \xi_j\theta/(2\pi)\rfloor$, $\iota_j = \lfloor \eta_j\theta/(2\pi)\rfloor + 1 - \eta_j\theta/(2\pi)$.

Remark that ε_j and ι_j are positive and decreasing with j. These ξ's and η's are known as denominators of remarkable rational approximants to the irrational number $\theta/(2\pi)$ ("Nebennäherungsbrüche" in [26] § 16) and are linked to the continued fraction expansion of $\theta/(2\pi)$. Each new ξ or η is the sum of the two last ones: if $\varepsilon_j > \iota_j$, $\ell = \xi_\ell = \xi_j + \eta_j$ and $\varepsilon_\ell = \varepsilon_j - \iota_j$, $\iota_\ell = \iota_j$; if $\varepsilon_j < \iota_j$, $\ell = \eta_\ell = \xi_j + \eta_j$ and $\iota_\ell = \iota_j - \varepsilon_j$, $\varepsilon_\ell = \varepsilon_j$. For instance, with the golden ratio $\theta/(2\pi) = (5^{1/2} + 1)/2 = 1.6180339887498948481\ldots$, one encounters the... Fibonacci numbers:

ξ or η	$\varepsilon = \xi\theta/(2\pi) - \lfloor\xi\theta/(2\pi)\rfloor$	$\iota = \lfloor\eta\theta/(2\pi)\rfloor + 1 - \eta\theta/(2\pi)$
1	0.6180339887498948481	0.3819660112501051518
2	0.2360679774997896963	
3		0.1458980337503154555
5	0.0901699437494742407	
8		0.0557280900008412147
13	0.0344418537486330259	
21		0.0212862362522081887
34	0.0131556174964248372	
55		0.0081306187557833515

With $\theta/(2\pi) = \sqrt{2}$, one finds new values of ξ at 1, 3, 5, 17, 29,..., new η values at 1, 2, 7, 12, 41,...

Remark also that the present discrete measure is definitely *not* semi-classical, as $\mu(y_{k+1})/\mu(y_k) = ((k - 1/2)/(k + 1/2))^2$ is not a rational function of $x_k = \cos k\theta$ and $y_k = \cos((k - 1/2)\theta)$, as it should have been according to (12).

Proof of the theorem. In order to show that a form like $p_n = A_nT_n + B_nT_{n-\xi_{2n-1}} + C_nT_{n-\eta_{2n-1}}$ is valid (using $T_{|p|} = T_p$), one must show that the scalar product

$$\begin{aligned}(p_n, T_m) &= \sum_{k=-\infty}^{\infty} \mu(y_k)p_n(y_k)T_m(y_k)\\ &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \mu(y_k)\left[A_n(T_{n+m}(y_k) + T_{n-m}(y_k)) + B_n(T_{n+m-\xi_{2n-1}}(y_k) + T_{n-m-\xi_{2n-1}}(y_k))\right.\\ &\qquad\left. + C_n(T_{n+m-\eta_{2n-1}}(y_k) + T_{n-m-\eta_{2n-1}}(y_k))\right]\end{aligned}$$

vanishes for $m = 0, 1, \ldots, 2n-1$, and has the value $2^{n-1}/\gamma_n = 1/A_n$ when $m = 2n$.

Let

$$\begin{aligned}
\tau_p &= \sum_{k=-\infty}^{\infty} \mu(y_k) T_p(y_k) \\
&= \sum_{k=-\infty}^{\infty} (k-1/2)^{-2} \cos(p(k-1/2)\theta) \\
&= 8[\cos(p\theta/2) + \cos(3p\theta/2)/9 + \cos(5p\theta/2)/25 + \cdots] \\
&= 2\pi^2 (-1)^{\lfloor p\theta/(2\pi)\rfloor} \left(\lfloor p\theta/(2\pi)\rfloor + 1/2 - p\theta/(2\pi)\right) \qquad (14') \\
&= -2\pi^2 (-1)^{\lceil p\theta/(2\pi)\rceil} \left(\lceil p\theta/(2\pi)\rceil - 1/2 - p\theta/(2\pi)\right) \ , \qquad (14'')
\end{aligned}$$

from elementary Fourier series, where $\lceil x \rceil$ is the smallest integer larger or equal than x. The form of $(14'')$ will sometimes be more convenient than $(14')$. The importance of using Chebyshev moments of measures with discrete masses has been shown by Prévost ([27,28]), remark in particular that τ_p does *not* tend to zero when $p \to \infty$, as it should with absolutely continuous measures. So, we have $(p_n, T_m) = (A_n\tau_{n+m} + B_n\tau_{n+m-\xi_{2n-1}} + C_n\tau_{n+m-\eta_{2n-1}} + A_n\tau_{n-m} + B_n\tau_{n-m-\xi_{2n-1}} + C_n\tau_{n-m-\eta_{2n-1}})/2$. Remark that $\tau_0 = \pi^2$, so that $p_0 = (\tau_0)^{-1/2} = \pi^{-1}$. For $n \geq 1$, let us show that it is possible to find A_n, B_n, and C_n such that $A_n\tau_N + B_n\tau_{N-\xi_{2n-1}} + C_n\tau_{N-\eta_{2n-1}} = 0$ for $N = 1, 2, \ldots, 2n-1$:
let $\rho = \theta/(2\pi)$. As $0 < \xi_{2n-1}\rho - \lfloor \xi_{2n-1}\rho \rfloor \leq N\rho - \lfloor N\rho \rfloor$, $(N - \xi_{2n-1})\rho - 1 < \lfloor N\rho \rfloor - \xi_{2n-1}\rho < \lfloor N\rho \rfloor - \lfloor \xi_{2n-1}\rho \rfloor \leq (N - \xi_{2n-1})\rho$, one has $\lfloor N\rho \rfloor - \lfloor \xi_{2n-1}\rho \rfloor = \lfloor (N - \xi_{2n-1})\rho \rfloor$. As $\lfloor \eta_{2n-1}\rho \rfloor + 1 - \eta_{2n-1}\rho \leq N\rho\rfloor + 1 - N\rho < 1$, $(N - \eta_{2n-1})\rho \leq \lfloor N\rho \rfloor - \lfloor \eta_{2n-1}\rho \rfloor < -\lfloor \eta_{2n-1}\rho \rfloor + N\rho < 1 + (N - \eta_{2n-1})\rho$, one has $\lfloor N\rho \rfloor - \lfloor \eta_{2n-1}\rho \rfloor = \lceil (N - \eta_{2n-1})\rho \rceil$.
In summary:

$$\lfloor N\rho \rfloor - \lfloor \xi_{2n-1}\rho \rfloor = \lfloor (N - \xi_{2n-1})\rho \rfloor \ , \ \lfloor N\rho \rfloor - \lfloor \eta_{2n-1}\rho \rfloor = \lceil (N - \eta_{2n-1})\rho \rceil, \qquad (15)$$

for $N = 1, 2, \ldots, 2n-1$.

So, $\tau_{N-\xi_{2n-1}} = 2\pi^2(-1)^{\lfloor N\rho \rfloor - \lfloor \xi_{2n-1}\rho \rfloor} \left(\lfloor N\rho \rfloor - \lfloor \xi_{2n-1}\rho \rfloor + 1/2 - N\rho \ + \xi_{2n-1}\rho\right) = (-1)^{\lfloor \xi_{2n-1}\rho \rfloor}\tau_N + 2\pi^2(-1)^{\lfloor N\rho \rfloor - \lfloor \xi_{2n-1}\rho \rfloor}\varepsilon_{2n-1}$,
$\tau_{N-\eta_{2n-1}} = -2\pi^2(-1)^{\lfloor N\rho \rfloor - \lfloor \eta_{2n-1}\rho \rfloor} \left(\lfloor N\rho \rfloor - \lfloor \eta_{2n-1}\rho \rfloor - 1/2 - N\rho \ + \eta_{2n-1}\rho\right) = -(-1)^{\lfloor \eta_{2n-1}\rho \rfloor}\tau_N + 2\pi^2(-1)^{\lfloor N\rho \rfloor - \lfloor \xi_{2n-1}\rho \rfloor}\iota_{2n-1}$,
$A_n\tau_N + B_n\tau_{N-\xi_{2n-1}} + C_n\tau_{N-\eta_{2n-1}} = [A_n + (-1)^{\lfloor \xi_{2n-1}\rho \rfloor}B_n - (-1)^{\lfloor \eta_{2n-1}\rho \rfloor}C_n]\tau_N + 2\pi^2(-1)^{\lfloor N\rho \rfloor}[(-1)^{\lfloor \xi_{2n-1}\rho \rfloor}\varepsilon_{2n-1}B_n + (-1)^{\lfloor \eta_{2n-1}\rho \rfloor}\iota_{2n-1}C_n]$, which vanishes indeed for $N = n \pm m = 1, 2, \ldots, 2n-1$ if $B_n = -K_n(-1)^{\lfloor \xi_{2n-1}\rho \rfloor}\iota_{2n-1}$, $C_n = K_n(-1)^{\lfloor \eta_{2n-1}\rho \rfloor}\varepsilon_{2n-1}$, and $A_n = K_n(\varepsilon_{2n-1} + \iota_{2n-1})$.

We now have to look at $(p_n, T_n) = (A_n\tau_{2n} + B_n\tau_{2n-\xi_{2n-1}} + C_n\tau_{2n-\eta_{2n-1}} + A_n\tau_0 + B_n\tau_{\xi_{2n-1}} + C_n\tau_{\eta_{2n-1}})/2 = 1/A_n$.

1. If $\xi_{2n} = \xi_{2n-1} < 2n$ and $\eta_{2n} = \eta_{2n-1} < 2n$, nothing changes in the evaluation of $A_n\tau_N + B_n\tau_{N-\xi_{2n-1}} + C_n\tau_{N-\eta_{2n-1}}$ when we replace N by $2n$, so $A_n\tau_{2n} + B_n\tau_{2n-\xi_{2n-1}} + C_n\tau_{2n-\eta_{2n-1}} = 0$, and we have only to look at $A_n\tau_0 + B_n\tau_{\xi_{2n-1}} + C_n\tau_{\eta_{2n-1}}$. One has $\tau_0 = \pi^2$, $\tau_{\xi_{2n-1}} = 2\pi^2(-1)^{\lfloor \xi_{2n-1}\rho \rfloor}(1/2 - \varepsilon_{2n-1})$, $\tau_{\eta_{2n-1}} = -2\pi^2(-1)^{\lfloor \eta_{2n-1}\rho \rfloor}(1/2 - \iota_{2n-1})$, yielding $A_n\tau_0 + B_n\tau_{\xi_{2n-1}} + C_n\tau_{\eta_{2n-1}} = 4K_n\pi^2\varepsilon_{2n-1}\iota_{2n-1}$, which must be equal

to $2/A_n = 2/(K_n(\varepsilon_{2n-1}+\iota_{2n-1}))$, whence $K_n = [(\varepsilon_{2n-1}+\iota_{2n-1})/(2\pi^2\varepsilon_{2n-1}\iota_{2n-1})]^{1/2}$, and this gives (13), as we still have $\varepsilon_{2n} = \varepsilon_{2n-1}$ and $\iota_{2n} = \iota_{2n-1}$.

2. $\xi_{2n} = 2n$ or $\eta_{2n} = 2n$, which happens only if $\xi_{2n-1} + \eta_{2n-1} = 2n$. Then, $A_n\tau_0 + B_n\tau_{\xi_{2n-1}} + C_n\tau_{\eta_{2n-1}} + A_n\tau_{2n} + B_n\tau_{2n-\xi_{2n-1}} + C_n\tau_{2n-\eta_{2n-1}} = A_n(\tau_0+\tau_{2n}) + (B_n + C_n)(\tau_{\xi_{2n-1}} + \tau_{\eta_{2n-1}})$.
   An interesting consequence of (15) is that $\lfloor\xi_{2n-1}\rho\rfloor$ and $\lfloor\eta_{2n-1}\rho\rfloor$ have now the same evenness: if we subtract the two equations of (15) with $N = n = (\xi_{2n-1}+\eta_{2n-1})/2$, one finds $-\lfloor\xi_{2n-1}\rho\rfloor + \lfloor\eta_{2n-1}\rho\rfloor = \lfloor(\eta_{2n-1}-\xi_{2n-1})\rho/2\rfloor - \lceil(\xi_{2n-1}-\eta_{2n-1})\rho/2\rceil$, which is an even integer, as $\lfloor x\rfloor = -\lceil -x\rceil$ ([15] § 1.2.4).
   So, let $\sigma = (-1)^{\lfloor\xi_{2n-1}\rho\rfloor} = (-1)^{\lfloor\eta_{2n-1}\rho\rfloor}$. One has $B_n + C_n = \sigma K_n(\varepsilon_{2n-1} - \iota_{2n-1})$, $\tau_{\xi_{2n-1}} + \tau_{\eta_{2n-1}} = 2\pi^2\sigma(1/2-\varepsilon_{2n-1}) - 2\pi^2\sigma(1/2-\iota_{2n-1}) = 2\pi^2\sigma(\iota_{2n-1}-\varepsilon_{2n-1})$.
   - 2a. If $\xi_{2n} = 2n$, $\varepsilon_{2n} = \varepsilon_{2n-1} - \iota_{2n-1}$, $2n\rho - \lfloor 2n\rho\rfloor = \xi_{2n-1}\rho - \lfloor\xi_{2n-1}\rho\rfloor - (\lfloor\eta_{2n-1}\rho\rfloor + 1 - \eta_{2n-1}\rho)$, so, $\lfloor 2n\rho\rfloor = \lfloor\xi_{2n-1}\rho\rfloor + \lfloor\eta_{2n-1}\rho\rfloor - 1$, $\tau_{2n} = 2\pi^2(-1)^{\lfloor 2n\rho\rfloor}(1/2-\varepsilon_{2n}) = -2\pi^2(1/2-\varepsilon_{2n})$, $A_n(\tau_0+\tau_{2n}) + (B_n+C_n)(\tau_{\xi_{2n-1}} + \tau_{\eta_{2n-1}}) = 4\pi^2 K_n\varepsilon_{2n}\iota_{2n-1}$, whence (13), as one still has $\iota_{2n} = \iota_{2n-1}$.
   - 2b. If $\eta_{2n} = 2n$, $\iota_{2n} = \iota_{2n-1} - \varepsilon_{2n-1}$, $\lfloor 2n\rho\rfloor + 1 - 2n\rho = \lfloor\eta_{2n-1}\rho\rfloor + 1 - \eta_{2n-1}\rho - (\xi_{2n-1}\rho - \lfloor\xi_{2n-1}\rho\rfloor)$, so $\lfloor 2n\rho\rfloor = \lfloor\eta_{2n-1}\rho\rfloor + \lfloor\xi_{2n-1}\rho\rfloor$, $\tau_{2n} = 2\pi^2(-1)^{\lfloor 2n\rho\rfloor}(-1/2 + \iota_{2n}) = -2\pi^2(1/2-\iota_{2n})$, $A_n(\tau_0+\tau_{2n}) + (B_n+C_n)(\tau_{\xi_{2n-1}} + \tau_{\eta_{2n-1}}) = 4\pi^2 K_n\iota_{2n}\varepsilon_{2n-1}$, whence (13), as one still has $\varepsilon_{2n} = \varepsilon_{2n-1}$. □

A numerical check has been performed with $\rho = \theta/(2\pi) = (5^{1/2}+1)/2 = 1.6180339887498948481\ldots$, the recurrence coefficients have been computed by Gautschi's `sti` (Stieltjes, of course!) subroutine [9].

For each n, one compares the computed $\pi\gamma_n 2^{-n} = 1/(2^n a_1 \ldots a_n)$ with the formula predicted from (13), i.e., $((\varepsilon_{2n-1}+\iota_{2n-1})/(8\varepsilon_{2n}\iota_{2n}))^{1/2}$. The agreement is satisfactory, taking into account that all the series involving $\mu(y_k) = 1/(k-1/2)^2$ have been truncated to 20000 terms, so that relative errors of about 10^{-3} may be expected.

We have longer and longer intervals where $a_n = 1/2$ and $b_n = 0$. On this example, $\liminf_{n\to\infty} a_n > 0$, although the vanishing of this lim inf could have been expected from pure discrete (atomic) measures [8], but other results have been published about singular measures [7,11,13,14,18,20,23,31,32,33].

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The saying of William of Orange, the most remarkable statesman of the 16$^{\text{th}}$ century [3] (could be compared to N. Mandela nowadays), applies quite well to scientific research (try to explain that to a contemporary state(?)sman).

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n	a_n	b_n	ε_{2n-1}	ι_{2n-1}	ε_{2n}	ι_{2n}	$\dfrac{1}{2^n a_1 \dots a_n}$	$\sqrt{\dfrac{\varepsilon_{2n-1}+\iota_{2n-1}}{8\varepsilon_{2n}\iota_{2n}}}$
0		0.2361						
1	0.4247	−0.5451	0.6180	0.3820	0.2361	0.3820	1.177	1.177
2	0.5	0.6180	0.2361	0.1459	0.2361	0.1459	1.177	1.177
3	0.3931	−0.3090	0.0902	0.1459	0.0902	0.1459	1.498	1.498
4	0.3090	0	0.0902	0.1459	0.0902	0.0557	2.423	2.423
5	0.6360	0	0.0902	0.0557	0.0902	0.0557	1.905	1.905
6	0.5	−0.3090	0.0902	0.0557	0.0902	0.0557	1.905	1.905
7	0.3931	0.3090	0.0344	0.0557	0.0344	0.0557	2.423	2.423
8	0.5	0	0.0344	0.0557	0.0344	0.0557	2.424	2.423
9	0.5	0	0.0344	0.0557	0.0344	0.0557	2.423	2.423
10	0.5	0.3090	0.0344	0.0557	0.0344	0.0557	2.423	2.423
11	0.3931	−0.3090	0.0344	0.0213	0.0344	0.0213	3.083	3.082
12	0.5	0	0.0344	0.0213	0.0344	0.0213	3.083	3.082
	...	...	...	...	...	...	...	...
17	0.3090	0	0.0344	0.0213	0.0132	0.0213	4.989	4.988
18	0.6360	0	0.0132	0.0213	0.0132	0.0213	3.922	3.921
19	0.5	0	0.0132	0.0213	0.0132	0.0213	3.922	3.921
	...	...	...	...	...	...	...	...
27	0.5	−0.3091	0.0132	0.0213	0.0132	0.0213	3.922	3.921
28	0.3930	0.3091	0.0132	0.0081	0.0132	0.0081	4.989	4.988
29	0.5	0	0.0132	0.0081	0.0132	0.0081	4.989	4.988
	...	...	...	...	...	...	...	...
44	0.5	0.3091	0.0132	0.0081	0.0132	0.0081	4.989	4.988
45	0.3930	−0.3091	0.0050	0.0081	0.0050	0.0081	6.348	6.344
46	0.5	0	0.0050	0.0081	0.0050	0.0081	6.348	6.344
	...	...	...	...	...	...	...	...
71	0.5	0	0.0050	0.0081	0.0050	0.0081	6.348	6.344
72	0.3089	0	0.0050	0.0081	0.0050	0.0031	10.277	10.265
73	0.6361	0	0.0050	0.0031	0.0050	0.0031	8.078	8.070
74	0.5	0	0.0050	0.0031	0.0050	0.0031	8.078	8.070

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