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Problem

for Mathematicaists, Reducectionists, Macsymalists, and other Mapleators

Painlevé equations for semi-classical recurrence coefficients.

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A shorter version is in *Constructive Approximation*, **12** (1996) 303–306.

Background: orthonormal polynomials $p_n(x) = \gamma_n x^n + \cdots$:

$$a_{n+1}p_{n+1}(x) = (x - b_n)p_n(x) - a_n p_{n-1}(x) \tag{1}$$

related to a semi-classical weight w : $$\frac{w'(x)}{w(x)} = \frac{2V(x)}{W(x)} \tag{2}$$
where V and W are polynomials, satisfy differential relations

$$Wp'_n = (\Omega_n - V)p_n - a_n\Theta_n p_{n-1} \tag{3}$$

where Ω_n and Θ_n are polynomials of degrees $\le m-1$ and $m-2$ (if degrees V and $W \le m-1$ and m), with $\quad \Omega_{n+1}(x) = (x - b_n)\Theta_n(x) - \Omega_n(x), \quad$ (4)

$$(x - b_n)(\Omega_{n+1}(x) - \Omega_n(x)) = W(x) + a_{n+1}^2\Theta_{n+1}(x) - a_n^2\Theta_n(x)$$

representing nonlinear recurrence relations for the a_n's and b_n's [Laguerre, 19$^{\text{th}}$ century]. See for instance [2,7,12].

Statement: *if, in addition to (2), the weight w depends on a parameter t so that*

$$\frac{\partial w(x,t)/\partial t}{w(x,t)} = \frac{T(x,t)}{W(x,t)}, \tag{5}$$

where T and W are polynomials in x and t (the same W, which will now be written $W(x,t)$), each recurrence coefficient $a_n^2(t)$ and $b_n(t)$ satisfy a nonlinear differential equation having Painlevé property (movable singular points are only poles) . [many people, 20$^{\text{th}}$ century]. See, for instance, [1,3 (pp. 399–402), 4,5,6,8,11 (bottom of p. 363)].

I tried in [9] to show this statement for generalized Jacobi weights.

I have an (incomplete) elementary proof of this for **generalized Jacobi weights**

$$\left.\begin{aligned} w(x,t) &= 0, \qquad x < x_1(t) \text{ or } x > x_m(t) \\ &= C_j \prod_{k=1}^{m} |x - x_k(t)|^{\alpha_k}, \quad x_j(t) < x < x_{j+1}(t),\ \ j = 1,\dots,m-1. \end{aligned}\right\} \tag{6}$$

where therefore $W(x,t)=\prod_1^m(x-x_k(t))$ and (2) and (5) become

$$\frac{w'(x,t)}{w(x,t)}=\sum_{k=1}^m\frac{\alpha_k}{x-x_k(t)}\ \text{and}\ \frac{\partial w(x,t)/\partial t}{w(x,t)}=-\sum_{k=1}^m\frac{\alpha_k\dot{x}_k(t)}{x-x_k(t)} \tag{7}$$

(N.B. so, the C_j's and the α_j's do *not* depend on t; remark also that $\alpha_k=2V(x_k)/W'(x_k)$, $\Omega_n(x)=(n+(\sum_1^m\alpha_k)/2)x^{m-1}+\cdots$, $\Theta_n(x)=(2n+1+\sum_1^m\alpha_k)x^{m-2}+\cdots$). A differential system involving a_n and b_n is then

$$\frac{\dot{a}_n}{a_n}=\frac{1}{2}\sum_{k=1}^m\frac{(\Theta_n(x_k)-\Theta_{n-1}(x_k))\dot{x}_k}{W'(x_k)}, \tag{8}$$

$$\dot{b}_n=\sum_{k=1}^m\frac{(\Omega_{n+1}(x_k)-\Omega_n(x_k))\dot{x}_k}{W'(x_k)}=\sum_{k=1}^m\frac{((x_k-b_n)\Theta_n(x_k)-2\Omega_n(x_k))\dot{x}_k}{W'(x_k)}, \tag{9}$$

$$\frac{\dot{\gamma}_n}{\gamma_n}=-\frac{1}{2}\sum_{k=1}^m\dot{x}_k\Theta_n(x_k)/W'(x_k) \tag{10}$$

$$\frac{d}{dt}\frac{\Theta_n(x_j)}{W'(x_j)}=2\frac{\dot{\gamma}_n}{\gamma_n}\frac{\Theta_n(x_j)}{W'(x_j)}-2\sum_{k\neq j}\frac{\dot{x}_j-\dot{x}_k}{x_j-x_k}\frac{\Theta_n(x_k)\Omega_n(x_j)-\Theta_n(x_j)\Omega_n(x_k)}{W'(x_j)W'(x_k)}, \tag{11}$$

$$\frac{d}{dt}\frac{\Theta_{n-1}(x_j)}{W'(x_j)}=-2\frac{\dot{\gamma}_{n-1}}{\gamma_{n-1}}\frac{\Theta_{n-1}(x_j)}{W'(x_j)}+2\sum_{k\neq j}\frac{\dot{x}_j-\dot{x}_k}{x_j-x_k}\frac{\Theta_{n-1}(x_k)\Omega_n(x_j)-\Theta_{n-1}(x_j)\Omega_n(x_k)}{W'(x_j)W'(x_k)}, \tag{12}$$

$$\frac{d}{dt}\frac{\Omega_n(x_j)}{W'(x_j)}=a_n^2\sum_{k\neq j}\frac{\dot{x}_j-\dot{x}_k}{x_j-x_k}\frac{\Theta_n(x_j)\Theta_{n-1}(x_k)-\Theta_n(x_k)\Theta_{n-1}(x_j)}{W'(x_j)W'(x_k)}, \tag{13}$$

Problem: eliminate the Θ's and the Ω's in $(8-13)$ so to exhibit the scalar differential equations for $a_n(t)$ and $b_n(t)$, at least in the $m=3$ case (W of degree 3 in x and 1 in t), and in the even $m=5$ case ($w(x,t)=|x|^\alpha|x^2-x_1^2|^\beta|x^2-x_2^2|^\gamma, b_n=0$), see [8] for confluent cases and [9] for the even case.

Proof of $(8-14)$

The proof is valid only when $\alpha_k>0$, so that

$$q_n(x)=\int_{\mathbb{R}}\frac{w(u)p_n(u)}{x-u}du \tag{14}$$

is finite at the x_k's. Then, $\frac{d}{dt}\int_{\mathbb{R}}w(u,t)p_n(u,t)p_{n-i}(u,t)du=0,\quad i=0,1,\ldots$:

$$\sum_{k=1}^m\alpha_k\dot{x}_kq_n(x_k)p_{n-i}(x_k)+\int_{\mathbb{R}}w(u,t)\frac{\partial p_n(u,t)}{\partial t}p_{n-i}(u,t)du+\frac{\dot{\gamma}_n}{\gamma_n}\delta_{i,0}=0,$$

yielding the coefficients of the orthogonal expansion of

$$\frac{\partial p_n(x,t)}{\partial t} = -\frac{\dot\gamma_n}{\gamma_n} p_n(x,t) - \sum_{k=1}^{m} \alpha_k \dot x_k q_n(x_k,t) \sum_{i=0}^{n} p_{n-i}(x_k,t) p_{n-i}(x,t) \tag{15}$$

$$= -\frac{\dot\gamma_n}{\gamma_n} p_n(x,t) - \sum_{k=1}^{m} \alpha_k \dot x_k q_n(x_k,t) \left[a_n(t) \frac{p_n(x,t)p_{n-1}(x_k,t) - p_n(x_k,t)p_{n-1}(x,t)}{x - x_k} + p_n(x_k,t)p_n(x,t) \right] \tag{16}$$

(Christoffel-Darboux). From the coefficient of p_n:

$$\frac{\dot\gamma_n}{\gamma_n} = -\frac{1}{2} \sum_{k=1}^{m} \alpha_k \dot x_k q_n(x_k) p_n(x_k)$$

Now, from (14), (2), (3), (4), and (1)

$W q_n' = (\Omega_n + V) q_n - a_n \Theta_n q_{n-1}, W p_{n-1}' = a_n \Theta_{n-1} p_n - (\Omega_n + V) p_{n-1}, W q_{n-1}' = a_n \Theta_{n-1} q_n - (\Omega_n - V) q_{n-1},$

so that, using $p_n q_{n-1} - p_{n-1} q_n = 1/a_n$,

$$\Theta_n(x_k) = 2V(x_k) p_n(x_k) q_n(x_k) = \alpha_k W'(x_k) p_n(x_k) q_n(x_k), \Omega_n(x_k) - V(x_k) = a_n \alpha_k W'(x_k) q_n(x_k) p_{n-1}(x_k), \tag{17}$$

giving (10), (8) follows from $\gamma_{n-1} = a_n \gamma_n$: $\dot a_n / a_n = \dot\gamma_{n-1}/\gamma_{n-1} - \dot\gamma_n/\gamma_n$. Now, (16) becomes

$$\frac{\partial p_n(x,t)}{\partial t} = \frac{\dot\gamma_n}{\gamma_n} p_n(x,t) - \sum_{k=1}^{m} \dot x_k \, \frac{(\Omega_n(x_k) - V(x_k)) p_n(x) - a_n \Theta_n(x_k) p_{n-1}(x)}{W'(x_k)(x - x_k)}. \tag{18}$$

For (9), consider

$$\frac{\partial p_n(x,t)}{\partial t} = \frac{\partial[\gamma_n(t) x^n - \gamma_n(t)(b_0(t) + \cdots + b_{n-1}(t)) x^{n-1} + \cdots]}{\partial t} = \frac{\dot\gamma_n}{\gamma_n} p_n - \frac{\dot b_0 + \cdots + \dot b_{n-1}}{a_n} p_{n-1} + \cdots$$

and the coefficient of p_{n-1} in (15):

$$-\frac{\dot b_0 + \cdots + \dot b_{n-1}}{a_n} = -\sum_{1}^{m} \alpha_k \dot x_k q_n(x_k) p_{n-1}(x_k) = -\frac{1}{a_n} \sum_{1}^{m} \dot x_k \frac{\Omega_n(x_k) - V(x_k)}{W'(x_k)} .$$

For (11-13), we need the $t-$derivatives of $p_n(x_j(t),t)$, $q_n(x_j(t),t)$, etc.

$dp_n(x_j(t),t)/dt = (\partial p_n(x,t)/\partial t)(x_j) + \dot x_j p_n'(x_j)$

$$(3) \Rightarrow \; p_n'(x) = \frac{\Omega_n(x) - V(x)}{W(x)} p_n(x) - a_n \frac{\Theta_n(x)}{W(x)} p_{n-1}(x) = \sum_{k=1}^{m} \frac{(\Omega_n(x_k) - V(x_k)) p_n(x) - a_n \Theta_n(x_k) p_{n-1}(x)}{W'(x_k)(x - x_k)},$$

and use (18):

$$dp_n(x_j(t),t)/dt = \frac{\dot\gamma_n}{\gamma_n} p_n(x_j,t) + \sum_{\substack{k=1 \\ k\neq j}}^{m} (\dot x_j - \dot x_k) \, \frac{(\Omega_n(x_k) - V(x_k)) p_n(x_j) - a_n \Theta_n(x_k) p_{n-1}(x_j)}{W'(x_k)(x_j - x_k)}, \tag{19}$$

$$\frac{\partial q_n(x,t)}{\partial t} = \frac{\partial \int_{\mathbb{R}} \frac{w(u,t)p_n(u,t)}{x-u}du}{\partial t}$$

$$= -\sum_1^m \alpha_k \dot{x}_k \int_{\mathbb{R}} \frac{w(u)p_n(u)}{(u-x_k)(x-u)}du + \int_{\mathbb{R}} w(u)\frac{\dot{\gamma}_n p_n(u)}{\gamma_n(x-u)}du - \sum_1^m \dot{x}_k \int_{\mathbb{R}} w(u)\frac{[\Omega_n(x_k)-V(x_k)]p_n(u) - a_n\Theta_n(x_k)p_{n-1}(u)}{W'(x_k)(u-x_k)(x-u)}du$$

$$= \frac{\dot{\gamma}_n}{\gamma_n} q_n(x) - \sum_1^m \dot{x}_k \frac{[\Omega_n(x_k)+V(x_k)](q_n(x)-q_n(x_k)) - a_n\Theta_n(x_k)(q_{n-1}(x)-q_{n-1}(x_k))}{W'(x_k)(x-x_k)}$$

$$dq_n(x_j(t),t)/dt = \frac{\dot{\gamma}_n}{\gamma_n} q_n(x_j,t) + \sum_{\substack{k=1\\k\neq j}}^m (\dot{x}_j - \dot{x}_k)\ \frac{(\Omega_n(x_k)+V(x_k))q_n(x_j) - a_n\Theta_n(x_k)q_{n-1}(x_j)}{W'(x_k)(x_j-x_k)},$$

and we get (11) as $\alpha_j d(p_n(x_j)q_n(x_j))/dt$ (from (17)). For (12): use (11) with $n-1$ instead of n, use $\Omega_{n-1}(x) = (x - b_{n-1}\Theta_{n-1}(x) - \Omega_n(x)$ from (4), and (12) follows from (10) and also $\sum_1^m \Theta_{n-1}(x_k)/W'(x_k) =$ coefficient of x^{m-1} of $\Theta_{n-1} = 0$). Finally, we need $dp_{n-1}(x_j(t),t)/dt$ for (13): take (19) with $n-1$, replace $-a_{n-1}p_{n-2}(x_j)$ by $a_n p_n(x_j) - (x_j - b_{n-1})p_{n-1}(x_j)$:

$$dp_{n-1}(x_j(t),t)/dt = -\frac{\dot{\gamma}_{n-1}}{\gamma_{n-1}} p_{n-1}(x_j,t) - \sum_{\substack{k=1\\k\neq j}}^m (\dot{x}_j - \dot{x}_k)\ \frac{(\Omega_n(x_k)+V(x_k))p_{n-1}(x_j) - a_n\Theta_{n-1}(x_k)p_n(x_j)}{W'(x_k)(x_j-x_k)}.$$

Exercise: *show that the moment $\mu_n = \int_{\mathbb{R}} w(u)(u-x_1)^n du$ of (6) enters the solution of the following differential system as $\mu_n = \nu_{n,1}$,*
where $\nu_{n,j} = \int_{\mathbb{R}} w(u)\prod_1^m (u-x_k)^{\beta_k}(u-x_j)^{-1}du$:

$$\dot{\nu}_{n,j} = \sum_{\substack{k=1\\k\neq j}}^m (\dot{x}_j - \dot{x}_k)(\alpha_k+\beta_k)\frac{\nu_{n,j}-\nu_{n,k}}{x_j-x_k}, \qquad j=1,\dots,m$$

with $\beta_1 = n+1, \beta_2 = \ldots = \beta_m = 1$.

This shows that the moments of these generalized Jacobi weights are regular functions of t in any region where the $x_k(t)$'s are distinct. As $a_n^2(t)$ and $b_n(t)$ are ratios of determinants of moments, the movable singular points of their equations can only be poles (the zeros of the denominator determinants).

References.

[1] A.I. Aptekarev, A. Branquinho, F. Marcellán, Toda-type differential equations for the recurrence coefficients of orthogonal polynomials and Freud transformation, *Pré-Publicaçoes Univ. Coimbra, Dep. Matemática* 96–04 (Feb. 1996).

[2] S. BELMEHDI, A. RONVEAUX, About non linear systems satisfied by the recurrence coefficients of semiclassical orthogonal polynomials, *J. Approx. Th.* **76** (1994), 351-368.

[3] D.V. CHUDNOVSKY, Riemann monodromy problem, isomonodromy deformation eqations and completely integrable systems, pp.385-447 *in Bifurcation Phenomena in Mathematical Physics and Related Topics, Proceedings Cargèse, 1979* (C.BARDOS & D.BESSIS, editors), NATO ASI series C, vol. **54**, D.Reidel, Dordrecht, 1980.

[4] G.V. CHUDNOVSKY, Padé approximation and the Riemann monodromy problem, *ibidem*, pp.449-510.

[5] D.V. Chudnovsky, G.V. Chudnovsky, Explicit continued fractions and quantum gravity, *Acta Applicandae Math.* **36** (1994) 167–185.

[6] A.S. FOKAS, A.R. ITS, A.V. KITAEV, Discrete Painlevé equations and their appearance in quantum gravity, *Commun. Math. Phys.* **142** (1991) 313-344; The isomonodromy approach to matrix models in 2D quantum gravity, *ibidem* **147** (1992) 395-430.

[7] E. LAGUERRE, Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels, *J. Math. Pures Appl. (4)* **1** (1885), 135-165 = pp. 685-711 *in Oeuvres*, Vol.II, Chelsea, New-York 1972.

[8] A.P. MAGNUS, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, *J. Comp. Appl. Math.* **57** (1995) 215–237. Preliminary preprint: `ftp://unvie6.un.or.at/siam/opsf/pailevemagnus.tex`

[9] A.P. MAGNUS, Asymptotics for the simplest generalized Jacobi polynomials recurrence coefficients from Freud's equations: numerical explorations, *Annals of Numerical Mathematics* **2** (1995) 311–325. Preprint in directory `ftp://unvie6.un.or.at/siam/opsf/magnus/`

[10] A.P. MAGNUS, Problem: Painlevé equations for semi-classical recurrence coefficients, preprint `ftp://unvie6.un.or.at/siam/opsf/magnus_painleve_problem.tex`

[11] J.NUTTALL, Asymptotics of diagonal Hermite-Padé polynomials, *J.Approx. Th.* **42** (1984) 299-386.

[12] J.A. SHOHAT, A differential equation for orthogonal polynomials, *Duke Math. J.* **5** (1939),401-417.

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