An Asynchronous Distributed Algorithm for Solving Stochastic Unit Commitment

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Solving the sustainability puzzle

ENGIE Chair research problems and methodology

Integrated transmission/distribution operation

Renewable integration

Parallel computing

Demand response

Resource planning

Market design

Stochastic programming

ColorPower: residential demand response business models based on quality differentiated service

Rewarding flexible capacity in the Belgian electricity market
Outline

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2. Preliminaries
3. Asynchronous distributed block-coordinate subgradient method for dual minimization
4. Primal recovery
5. High performance computing implementation
6. Numerical results
   - Western Electricity Coordinating Council (WECC) system
   - Central Western European (CWE) system
7. Conclusions
Motivation
Renewables making headlines

Germany: Nuclear power plants to close by 2022

Denmark aims for 100 percent renewable energy in 2050

California to nearly double wind, solar energy output by 2020 -regulator
Sequential electricity markets

Day-ahead market model is a large-scale mixed-integer linear program, known as the **unit commitment problem**
Stochastic unit commitment models uncertainty endogenously in power system operations.
Motivation of the present work

- Stochastic unit commitment offers advantages over deterministic reserve policies for coping with uncertainty (Takriti and Birge, 1996), (Papavasiliou and Oren, 2013)

- Stochastic unit commitment has failed to become an industry standard:
  - Market design compatible with treatment of uncertainty
  - Difficulty and scale of stochastic unit commitment models

- Decomposition and parallelization have shown promise to solve stochastic unit commitment models (Cheung et al., 2015), (Kim and Zavala, 2015)
The two-stage stochastic unit commitment problem can be formulated as

$$\max_{u,v,w} \sum_{i=1}^{N} (c_i^T v_i + d_i^T w_i)$$

s.t. $v_i - u = 0, \quad i = 1, \ldots, N$

$$(v_i, w_i) \in D_i, \quad i = 1, \ldots, N$$

$u \in U$

- $u$ corresponds to non-anticipative commitment and production variables of generators
- $v_i$ are local copies of $u$ for each scenario $i$
- $w_i$ are commitment, production and transmission recourse variables

Scenario decomposition: relax non-anticipativity constraints in order to compute a solution iteratively
Parallelization of common decomposition methods leads to synchronous parallel algorithms, e.g. (Papavasiliou and Oren, 2013), (Cheung et al., 2015)

Execution of a synchronous parallel decomposition method.
Synchronous parallel decomposition method

Load profile of a synchronous parallel decomposition method. Red area indicates idle processors.

- Evaluation times of subproblems ($SP_i$) can vary significantly:
  - Up to 48x across subproblems within the same iteration
  - Up to 25x across iterations for the same subproblem
Preliminaries
Block-coordinate descent method

Each iteration performs a line search on a subset of variables (Tseng and Yun, 2009), (Fercoq and Richtárik, 2013), (Wright, 2015)

\[
\min_{x \in \mathbb{R}^N} f(x_1, x_2, \ldots, x_n)
\]

1. Let \( k := 0, \mathbf{x}^k := \bar{x}^0 \)
2. Select block-coordinate \( j(k) \) (cyclically, at random)
3. Compute \( \mathbf{t}^k := \arg \min_t f(x_1^k, \ldots, x_{j(k)-1}^k, t, x_{j(k)+1}^k, \ldots, x_n^k) \)
4. Let \( x_{j}^{k+1}(k) := t^k, \mathbf{x}_i^{k+1} := x_i^k \) \( \forall i \neq j(k) \)
5. Let \( k := k + 1 \) and return to 2.
Block-coordinate descent method

\[ f(x, y) = 5x^2 - 6xy + 5y^2 \]
The stochastic unit commitment problem

\[
\max_{u,v,w} \sum_{i=1}^{N} (c_i^T v_i + d_i^T w_i)
\]

s.t. \( v_i - u = 0, \quad i = 1, \ldots, N \) \((x_i)\)

\((v_i, w_i) \in D_i, \quad i = 1, \ldots, N\)

\( u \in U \)

- \(U\) is a bounded convex set, described by linear constraints
- \(D_i\) is a bounded non-convex set, described by mixed integer-linear constraints
- \(x_i\) are dual multipliers associated to non-anticipativity constraints
Scenario decomposition

- Lagrange relaxation of non-anticipativity constraints leads to the following separable dual problem

$$\min_{x \in \mathbb{R}^m} f_0(x) + \sum_{i=1}^{N} f_i(x_i)$$

where $f_0$ and $f_i$ are defined according to

$$f_0(x) := \max_{u \in \mathcal{U}} \left( - \sum_{i=1}^{N} x_i^T \right) u$$

$$f_i(x_i) := \max_{(v,w) \in \mathcal{D}_i} \left( (c_i^T + x_i^T)v + d_i^T w \right) \quad i = 1, \ldots, N$$

- $f_0$ and $f_i$ are non-differentiable convex functions

- Evaluation of $f_0$ requires to solve an LP, while evaluation of $f_i$ involves solving a large-scale MILP
Solution strategy for SUC

- Each evaluation of the dual function gives upper bounds to the SUC problem

- \( v_i \in \partial f_i(x_i), i = 1, \ldots, N \) are (typically) primal feasible first-stage solutions for which we can evaluate their second-stage cost

- Primal feasibility recovery gives us a lower bound to the SUC problem

- Lower and upper bounds are used for deciding on termination
Smooth approximation of $f_0$

- In order to obtain convergence guarantees, following (Nesterov, 2005), (Fercoq and Richtárik, 2013), we replace the non-decomposable part of the objective $f_0$ by the following smooth approximation

$$f_0^\mu(x) := \max_{u \in \mathcal{U}} \left( \left( - \sum_{i=1}^{N} x_i^T u \right) u - \frac{1}{2} \mu \|u - u_0\|_2^2 \right)$$

- $f_0^\mu$ is a differentiable function with Lipchitz gradient with constant $L_0^\mu$

- We focus then on minimizing the following approximation of the dual problem of stochastic unit commitment

$$\min_{x \in X} f(x) = f_0^\mu(x) + \sum_{i=1}^{N} f_i(x_i)$$
Contributions of the present work

- We use scenario decomposition and propose an asynchronous block-coordinate subgradient method for minimizing the Lagrangian dual of stochastic unit commitment (convex, non-differentiable).

  The proposed method does not perform a line search along coordinates on each iteration (Tseng and Yun, 2009), (Fercoq and Richtárik, 2013).

- We propose primal recovery heuristics

- We implement the proposed asynchronous algorithm on a high performance computing cluster. The algorithm is able to solve industry-scale instances within the same time frame of deterministic unit commitment.
Asynchronous distributed block-coordinate subgradient method
Serial block-coordinate subgradient method

\[
\min_{x \in X} f(x) = f^\mu_0(x) + \sum_{i=1}^{N} f_i(x_i)
\]

Consider the randomized coordinate descent method (Nesterov, 2012):

1. Let \( k := 0 \), \( x^k := \bar{x}^0 \)
2. Select component \( j(k) \) uniformly at random from \( \{1, \ldots, N\} \)
3. Compute \( \nabla f^\mu_0(x^k) \) and \( g(j(k), x^k_{j(k)}) \in \partial f_{j(k)}(x^k_{j(k)}) \)
4. Perform update according to

\[
x^{k+1} := P_X \left[ x^k - \lambda_k \cdot I^T_{j(k)} \left( I_{j(k)} \nabla f^\mu_0(x^k) + g(j(k), x^k_{j(k)}) \right) \right]
\]
5. Let \( k := k + 1 \) and return to 2.
Consider the expected update direction of the serial method,

\[
\mathbb{E}[I_J^T (I_J \nabla f_0^\mu (x^k) + g(J, x^k_j)) | x^k],
\]

where \( J \) is a discrete uniformly distributed variable on \( \{1, \ldots, N\} \).

The expected update direction coincides with the direction of a subgradient of \( f \) at \( x^k \):

- This property requires \( f_0^\mu \) to be smooth
- The serial method is a **stochastic subgradient method** (Ermoliev, 1983)

Provided we choose a diminishing, non-summable and square summable stepsize \( \lambda_k \), the algorithm will converge to an optimal solution with probability 1.
Asynchronous method: computation model

Conceptual computation model (Nedić et al., 2001).

- The **Subgradient computation system** evaluates $\nabla f_0^\mu$ and subgradients of component functions in parallel.

- The **Updating system** stores the last gradient evaluated $\nabla f_0^\mu(x^{k-l(k)})$ and the latest subgradients evaluated $g(i, x_i^{k-\ell(i,k)}) \ \forall i$.

- Updates are performed using the lastly available information after each evaluation of a subgradient.

- Delays $l(k)$ and $\ell(i, k)$ appear because evaluation of gradients and subgradients is not instantaneous.
The Updating system performs the following operations:

1. Wait for next subgradient evaluation. Store new information when received.

2. Select component $j(k)$ uniformly at random from $\{1, \ldots, N\}$

3. Perform update according to

$$x^{k+1} := P_X \left[ x^k - \lambda_k \cdot I_{j(k)}^T \left( I_{j(k)} \nabla f_0 \left( x^{k-l(k)} \right) + g(j(k), x^{k-l(j(k),k)}) \right) \right]$$

4. Let $k := k + 1$ and return to 1.
Built-in load balancing for subgradient computation

\[ f_1(x_1^0) \quad f_2(x_2^0) \quad f_3(x_3^0) \quad f_4(x_4^0) \quad f_5(x_5^0) \]

\[ k \quad j(k) \]
Start: $f_1(x_1^0)$, $f_2(x_2^0)$, $f_3(x_3^0)$, $f_4(x_4^0)$, $f_5(x_5^0)$

\[
\begin{array}{c}
k \\
1 \\
4
\end{array}
\]

\[
\begin{array}{c}
\text{j(k)}
\end{array}
\]
Start

\[ f_1(x_1^0) \]
\[ f_2(x_2^0) \]
\[ f_3(x_3^0) \]
\[ f_4(x_4^0) \]
\[ f_5(x_5^0) \]

\[ f_4(x_4^1) \]
\[ f_3(x_3^2) \]

\[ \begin{array}{c|c}
  k & j(k) \\
  \hline
  1 & 4 \\
  2 & 3 \\
\end{array} \]
Start

\[
\begin{align*}
    f_1(x_1^0) & \quad f_2(x_2^0) & \quad f_3(x_3^0) & \quad f_4(x_4^0) & \quad f_5(x_5^0) \\
    f_4(x_4^1) & \quad f_3(x_3^2) & \quad f_2(x_2^3) \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>$k$</th>
<th>$j(k)$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
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<tr>
<td>2</td>
<td>3</td>
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<tr>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>
Built-in load balancing for subgradient computation

\[
\begin{align*}
&f_1(x_1^0) \\
&f_2(x_2^0) \\
&f_4(x_4^1) \\
&f_3(x_3^2) \\
&f_3(x_3^4) \\
&f_3(x_3^2) \\
&f_5(x_5^0) \\
&f_4(x_4^0) \\
&f_2(x_2^3)
\end{align*}
\]

\[
\begin{array}{c|c}
 k & j(k) \\
\hline
 1 & 4 \\
 2 & 3 \\
 3 & 2 \\
 4 & 3 \\
\end{array}
\]
Start

\[ f_1(x_1^0) \quad f_2(x_2^0) \quad f_3(x_3^0) \quad f_4(x_4^0) \quad f_5(x_5^0) \]

\[ f_4(x_4^1) \quad f_3(x_3^2) \quad f_1(x_1^5) \quad f_3(x_3^4) \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( j(k) )</th>
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<tbody>
<tr>
<td>1</td>
<td>4</td>
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<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>
Built-in load balancing for subgradient computation

\begin{align*}
\text{Start} & \quad f_1(x^0_1) \quad f_2(x^0_2) \quad f_3(x^0_3) \quad f_4(x^0_4) \quad f_5(x^0_5) \\
& \quad f_4(x^1_4) \quad f_3(x^2_3) \quad f_3(x^4_3) \quad f_3(x^6_3) \\
& \quad f_1(x^5_1) \quad f_3(x^3_3) \\
\end{align*}

<table>
<thead>
<tr>
<th>(k)</th>
<th>(j(k))</th>
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<tbody>
<tr>
<td>1</td>
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<td>4</td>
<td>3</td>
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<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>
Built-in load balancing for subgradient computation

\[
\begin{align*}
\text{Start} & \quad f_1(x_1^0) \quad f_2(x_2^0) \quad f_3(x_3^0) \quad f_4(x_4^0) \quad f_5(x_5^0) \\
& \quad f_4(x_4^1) \quad f_3(x_3^2) \quad f_3(x_3^4) \quad f_3(x_3^6) \\
& \quad f_1(x_1^5) \quad f_3(x_3^1) \quad f_3(x_3^3) \\
\end{align*}
\]

\[
\begin{array}{c|c}
\text{k} & \text{j(k)} \\
1 & 4 \\
2 & 3 \\
3 & 2 \\
4 & 3 \\
5 & 1 \\
6 & 3 \\
7 & 2 \\
\end{array}
\]
Built-in load balancing for subgradient computation

\[
\begin{align*}
&f_1(x_1^0) \quad f_2(x_2^0) \quad f_3(x_3^0) \quad f_4(x_4^0) \quad f_5(x_5^0) \\
f_4(x_4^1) \quad f_3(x_3^2) \quad f_3(x_3^4) \quad f_3(x_3^6) \quad f_2(x_2^3) \\
f_1(x_1^5) \quad f_3(x_3^5) \quad f_3(x_3^8) \quad f_2(x_2^7) \\
f_4(x_4^8) \\
\end{align*}
\]

\[
\begin{array}{c|c}
  k & j(k) \\
  \hline
  1 & 4 \\
  2 & 3 \\
  3 & 2 \\
  4 & 3 \\
  5 & 1 \\
  6 & 3 \\
  7 & 2 \\
  8 & 4 \\
\end{array}
\]
Built-in load balancing for subgradient computation

Start

\[
\begin{align*}
&f_1(x_1^0) \\
&f_4(x_4^1) \\
&f_1(x_1^5) \\
&f_1(x_1^9) \\
&f_2(x_2^0) \\
&f_3(x_3^0) \\
&f_3(x_3^2) \\
&f_3(x_3^4) \\
&f_3(x_3^6) \\
&f_4(x_4^0) \\
&f_4(x_4^8) \\
&f_5(x_5^0) \\
&f_2(x_2^3) \\
&f_2(x_2^7) \\
\end{align*}
\]

\[
\begin{array}{c|c}
  k & j(k) \\
  \hline
  1 & 4 \\
  2 & 3 \\
  3 & 2 \\
  4 & 3 \\
  5 & 1 \\
  6 & 3 \\
  7 & 2 \\
  8 & 4 \\
  9 & 1 \\
\end{array}
\]
Built-in load balancing for subgradient computation

\[
\begin{array}{c}
\text{Start} \\
\begin{array}{c}
f_1(x_1^0) \\
f_4(x_4^1) \\
f_1(x_1^5) \\
f_1(x_1^9) \\
f_5(x_5^0) \\
f_3(x_3^0) \\
f_3(x_3^2) \\
f_3(x_3^4) \\
f_3(x_3^6) \\
f_3(x_3^{10}) \\
f_2(x_2^3) \\
f_2(x_2^7) \\
f_2(x_2^9) \\
\end{array}
\end{array}
\]

\[
\begin{array}{cc}
k & j(k) \\
1 & 4 \\
2 & 3 \\
3 & 2 \\
4 & 3 \\
5 & 1 \\
6 & 3 \\
7 & 2 \\
8 & 4 \\
9 & 1 \\
10 & 3 \\
\end{array}
\]
Assumptions

1. **Subgradient boundedness**

   There exist $C$ and $D$ such that
   \[
   \sup_{j \in \{1, \cdots, N\}, \ x, y \in X} \| I_j \nabla f^\mu_0(x) + g(j, y) \|_2 \leq C, \quad \sup_{j \in \{1, \cdots, N\}, \ x \in X} \| g(j, x) \|_2 \leq D. \]

2. **Delay boundedness**

   There exists $L$ such that $l(k) \leq L$ and $\ell(i, k) \leq L, \ \forall i, k$

3. **Diminishing-bounded stepsize**

   $\lambda_k$ is independent from $j(k)$ and there exist positive constants $\hat{G}$ and $\hat{G}$, such that
   \[
   \hat{G} \gamma_k \leq \lambda_k \leq \hat{G} \gamma_k, \quad \gamma_k = \frac{1}{(1 + r_k)^q} \quad \forall k, \quad \sum_{k=0}^{\infty} \gamma_k = \infty, \quad \sum_{k=0}^{\infty} \gamma_k^2 < \infty
   \]
Convergence of the asynchronous method

- Under assumptions 1 and 2, we can show that the expected update direction of the asynchronous method is an approximate subgradient of $f$ at $x^k$.

- In particular, the expected update direction
  $\mathbb{E}[I_J^T (I_J \nabla f_0^\mu (x^{k-l(k)}) + g(J, x^{k-\ell(j,k)})) | F_k]$, where $J$ is a discrete uniform random variable on $\{1, \ldots, N\}$ and $F_k = \{x^k, x^{k-1}, \ldots, x^0\}$, complies with

$$N \cdot (x^k - y)^T \mathbb{E} \left[ I_J^T \left( I_J \nabla f_0^\mu (x^{k-l(k)}) + g(J, x^{k-\ell(j,k)}) \right) \mid F_k \right] \geq$$

$$f(x^k) - f(y) - \left( C^2 L_0^\mu \sum_{m=k-L}^{k-1} \lambda_m^2 + 2CDN \sum_{m=k-L}^{k-1} \lambda_m \right)$$

- The asynchronous method can be shown to converge to an optimal solution of the dual problem with probability 1, see (Ermoliev, 1983) and (Nedić, 2002).
Unlike most asynchronous algorithm applications, we require computing bounds on the dual objective $\rightarrow$ \textbf{duality gap}.

Recall we are trying to solve the following problem,

$$ \min_{x \in \mathbb{R}^m} f_0(x) + \sum_{i=1}^{N} f_i(x_i) $$

We can compute an upper bound, after each subgradient evaluation, at the cost of evaluating $f_0$ for a composite of the evaluated $x_i$'s as follows

$$ U^k := f_0 \left( \left[ (x_1^{k-\ell(1,k)})^T \ldots (x_N^{k-\ell(N,k)})^T \right]^T \right) + \sum_{j=1}^{N} f_j(x_j^{k-\ell(j,k)}) $$
Primal recovery
We consider 4 methods for generating primal candidates:

1. **First-in-first-out (FIFO)**
   Evaluating solutions to scenario subproblems as they arrive

2. **Random (RND)**
   Selecting solutions to scenario subproblems at random

3. **Last-in-first-out (LIFO)**
   Evaluating solutions to scenario subproblems in reverse order

4. **Importance sampling recombination heuristic (IS)**
   Recall that $f_i(x_i) := \sup_{(v,w) \in D_i} \left( (c_i^T + x_i^T)v + d_i^T w \right)$ and let $\bar{v}_i^*$ be the last solution to scenario subproblem $i$, $\forall i$
   
   (a) Pick a sample from $\{1, \ldots, N\}$ based on the **estimated importance** of each scenario.

   (b) Average the $\bar{v}_i^*$'s of the sample and project the result onto the feasible set of $\nu$. 

---

Primal recovery

□ We consider 4 methods for generating primal candidates:

1. **First-in-first-out (FIFO)**
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Aravena, Papavasiliou
High performance computing implementation
HPC implementation diagram

\[\sum_{i=1}^{N} \mathbf{x}^k_i, \bar{\mathbf{u}}_{\text{best}}, \sum_{i=1}^{N} \mathbf{x}^\ell_i, \nabla f_0^\mu(\mathbf{x}^k), f(\mathbf{x}^\ell)\]

\[\sum_{i=1}^{N} f_i(\mathbf{x}^\ell_i), \sum_{i=1}^{N} \mathbf{x}^\ell_i, \sum_{i=1}^{N} f_i(\mathbf{x}^\ell_i)\]

\[\mathbf{x}^k_j(k), g(j(k), \mathbf{x}^k_j(k)), f_j(k)(\mathbf{x}^k_j(k))\]

\[\bar{\mathbf{v}}, \bar{\mathbf{u}}_{\bar{\mathbf{u}}}, \bar{\mathbf{u}}, \bar{\mathbf{u}}_{\bar{\mathbf{u}}}\]

\[\mathbf{v} \leftarrow \bar{\mathbf{v}} \leftarrow \bar{\mathbf{u}}_{\bar{\mathbf{u}}}, \bar{\mathbf{u}} \leftarrow \bar{\mathbf{u}}_{\bar{\mathbf{u}}}, \bar{\mathbf{u}}_{\bar{\mathbf{u}}} \leftarrow \bar{\mathbf{u}}_{\bar{\mathbf{u}}}\]

\[\mathbf{h}_i(\bar{\mathbf{u}}_{\bar{\mathbf{u}}})\]

\[\mathbf{f}_j(k)(\mathbf{x}^k_j(k))\]

\[\mathbf{x}^k_j(k)\]

\[\mathbf{f}_j(k)(\mathbf{x}^k_j(k))\]

\[\sum_{i=1}^{N} \mathbf{x}^k_i, \bar{\mathbf{u}}_{\text{best}}\]

\[\mathbf{x}^\ell_i\]

\[\mathbf{f}_j(k)(\mathbf{x}^k_j(k))\]

\[\mathbf{x}^k_j(k)\]

\[\mathbf{f}_j(k)(\mathbf{x}^k_j(k))\]

\[\sum_{i=1}^{N} \mathbf{x}^k_i, \bar{\mathbf{u}}_{\text{best}}\]

\[\sum_{i=1}^{N} \mathbf{x}^\ell_i\]

\[\mathbf{f}_j(k)(\mathbf{x}^k_j(k))\]

\[\sum_{i=1}^{N} \mathbf{x}^\ell_i\]

\[\mathbf{f}_j(k)(\mathbf{x}^k_j(k))\]

\[\sum_{i=1}^{N} \mathbf{x}^k_i, \bar{\mathbf{u}}_{\text{best}}\]

\[\sum_{i=1}^{N} \mathbf{x}^\ell_i\]

\[\mathbf{f}_j(k)(\mathbf{x}^k_j(k))\]
Slave operations

1. Wait for next task from Master.
2. Execute task.
3. Send result to Master.
4. Return to 1.
Primal projection

Slave $s_1$
Evaluating $f_0^\mu, f_0$ ...

Slave $s_2$
Evaluating $f_{j(k)}$ ...

Slave $s_3$
Projecting onto $\mathcal{V}$ ...

Slave $s_4$
Evaluating 2nd stage ...

\[ \sum_{i=1}^{N} x_i^k, \tilde{u}^\text{best}, \sum_{i=1}^{N} \sum_{i=1}^{N} x_i^\ell \]

\[ \nabla f_0^\mu(x^k), f(x^\ell) \]

\[ x_j^k(k), g(j(k), x_j^k(k)), f_j(k)(x_j^k(k)) \]

\[ \tilde{v}, \tilde{u}, \vec{u}, i \]

\[ h_i(\tilde{u}^i) \]
Second stage subproblem

Slave $s_1$
Evaluating
$f_0^\mu, f_0$ ...

\[ \sum_{i=1}^{N} x_i^k, \tilde{u}_{\text{best}} \]
\[ \sum_{i=1}^{N} f_i(x_i^e), \sum_{i=1}^{N} x_i^e \]
\[ \nabla f_0^\mu (x_k), f(x^e) \]
\[ x_j^k(k) \]
\[ g(j(k), x_j^k(k)), f_j(k)(x_j^k(k)) \]

Master

Slave $s_2$
Evaluating
$f_{j(k)}$ ...

Slave $s_3$
Projecting
onto $\mathcal{V}$ ...

Slave $s_4$
Evaluating
2nd stage ...

\[ \tilde{v} \]
\[ \tilde{u} \]
\[ \tilde{u}, i \]
\[ h_i(\tilde{u}) \]
Dual scenario subproblem

Slave \( s_1 \) Evaluating \( f_0^\mu, f_0 \) ...

\[ \sum_{i=1}^{N} x_i^k, \tilde{u}^{\text{best}}, \sum_{i=1}^{N} f_i(x_i^\ell), \nabla f_0^\mu(x_k^k), f(x^\ell) \]

Slave \( s_2 \) Evaluating \( f_{j(k)} \) ...

\[ g(j(k), x^k_{j(k)}), f_{j(k)}(x^k_{j(k)}) \]

Slave \( s_3 \) Projecting onto \( \mathcal{V} \) ...

\[ h_i(\tilde{u}^t), \tilde{v} \]

Slave \( s_4 \) Evaluating 2nd stage ...

\[ \sum_{i=1}^{N} x_i^k, \tilde{u}^{\text{best}}, \sum_{i=1}^{N} x_i^\ell, \nabla f_0^\mu(x_k^k), f(x^\ell) \]

Equations:

- \[ \sum_{i=1}^{N} x_i^k, \tilde{u}^{\text{best}} \]
- \[ \sum_{i=1}^{N} f_i(x_i^\ell) \]
- \[ \nabla f_0^\mu(x_k^k), f(x^\ell) \]
- \[ g(j(k), x^k_{j(k)}), f_{j(k)}(x^k_{j(k)}) \]
- \[ h_i(\tilde{u}^t) \]

- \[ \sum_{i=1}^{N} x_i^k, \tilde{u}^{\text{best}}, \sum_{i=1}^{N} x_i^\ell, \nabla f_0^\mu(x_k^k), f(x^\ell) \]
Non-scenario dual subproblems

Slave $s_1$
Evaluating $f^\mu_0, f_0$ ...

Slave $s_2$
Evaluating $f_{j(k)}$ ...

Master

Slave $s_3$
Projecting onto $\mathcal{V}$ ...

Slave $s_4$
Evaluating 2nd stage ...

\[ \sum_{i=1}^{N} x^k_i, \tilde{u}^\text{best}, \sum_{i=1}^{N} f_i(x^\ell_i), \sum_{i=1}^{N} x^\ell_i, \nabla f^\mu_0(x^k), f(x^\ell) \]

\[ \sum_{i=1}^{N} x^k_i, \tilde{u} \]

\[ \tilde{v} \]

\[ \tilde{u}, i \]

\[ h_i(\tilde{u}^t) \]
Intensively parallel tasks

Slave $s_1$
Evaluating $f_0^\mu, f_0$ ...

Slave $s_2$
Evaluating $f_{j(k)}$ ...

Master

Slave $s_3$
Projecting onto $\mathcal{V}$ ...

Slave $s_4$
Evaluating 2nd stage ...

\[
\sum_{i=1}^{N} x_i^k, \bar{u}^{\text{best}}, \sum_{i=1}^{N} x_i^\ell, \nabla f_0^\mu(x^k), f(x^\ell)
\]

\[
\sum_{i=1}^{N} f_i(x_i^\ell), \sum_{i=1}^{N} x_i^\ell,
\]

\[
\nabla f_{j(k)}(x_{j(k)}^k), f_{j(k)}(x_{j(k)}^k)
\]

\[
\tilde{v}, \bar{u}^t, i, h_i(\bar{u}^t), g(j(k), x_{j(k)}^k), f_{j(k)}(x_{j(k)}^k)
\]
Master operations

1. Wait for next result from any Slave
2. Receive result from Slave $s$ and post-process it
3. Check bounds: if $UB - LB < \epsilon$ terminate
4. Decide next task
5. If task is dual scenario:
   (a) Select $j$ at random from $\{1, \ldots, N\}$ and update $x_j$
   (b) Send new $x_j$ to Slave $s$
Else, pre-process task accordingly and send it to Slave $s$
6. Return to 1.
Numerical results and conclusions
Numerical results

- The asynchronous algorithm is implemented in C using the Xpress C API and using MPI to handle communications between subprocesses.

- Numerical experiments ran on the Cab cluster, hosted at the Lawrence Livermore National Laboratory, and at the Lemaitre3 cluster, hosted at UC Louvain.

- We ran experiments using:
  - Stochastic unit commitment instances of the WECC system (Papavasiliou et al., 2015) and CWE system (Aravena and Papavasiliou, 2017)
  - Stochastic mixed-integer programming instances with 50+ scenarios from SIPLIB (Ahmed et al., 2015)
### Problem sizes

Sizes per scenario of stochastic unit commitment instances in the literature.

<table>
<thead>
<tr>
<th>Instance</th>
<th>Rows</th>
<th>Columns</th>
<th>Non-zeros</th>
<th>Integers</th>
<th>Subproblem solution time [s], avg. (max.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WECC(^1)</td>
<td>69 447</td>
<td>28 943</td>
<td>240 724</td>
<td>4 080</td>
<td>9.4 (25.7)</td>
</tr>
<tr>
<td>WECC(^2)</td>
<td>34 441</td>
<td>23 090</td>
<td>139 394</td>
<td>3 074</td>
<td>8.3 (67.9)</td>
</tr>
<tr>
<td>EDF(^3)</td>
<td>812 906</td>
<td>73 562</td>
<td>–</td>
<td>26 122</td>
<td>–</td>
</tr>
<tr>
<td>CWE(^4)</td>
<td>609 589</td>
<td>390 075</td>
<td>1 941 270</td>
<td>9 753</td>
<td>3 383.2 (7 851.8)</td>
</tr>
</tbody>
</table>

Largest instances solved in the present study.

<table>
<thead>
<tr>
<th>Instance</th>
<th>Scenarios</th>
<th>Rows</th>
<th>Columns</th>
<th>Integers</th>
</tr>
</thead>
<tbody>
<tr>
<td>WECC(^2)</td>
<td>1000</td>
<td>36 267 000</td>
<td>23 091 826</td>
<td>3 074 000</td>
</tr>
<tr>
<td>CWE(^4)</td>
<td>120</td>
<td>74 755 320</td>
<td>46 822 372</td>
<td>1 170 360</td>
</tr>
<tr>
<td>SSLP10.50.2000(^5)</td>
<td>2000</td>
<td>120 001</td>
<td>1 020 010</td>
<td>1 000 010</td>
</tr>
</tbody>
</table>

\(^1\)(Cheung et al., 2015), \(^2\)(Papavasiliou et al., 2015), \(^3\)(van Ackooij and Malick, 2016), \(^4\)(Aravena and Papavasiliou, 2017), \(^5\)(Ahmed et al., 2015)
130 thermal generators, 182 nodes, 319 lines, hourly resolution, 24 hour horizon, generation and transmission contingencies, multi-area renewable production
## WECC: Solution times

<table>
<thead>
<tr>
<th>$N$</th>
<th>Step size</th>
<th>Recovery Type</th>
<th>Cores</th>
<th>Share</th>
<th>Solution time [s], avg. (max.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2% optimality</td>
</tr>
<tr>
<td>10</td>
<td>Dim. 1/k</td>
<td>FIFO</td>
<td>16</td>
<td>0.5</td>
<td>228.1 (792.9)</td>
</tr>
<tr>
<td></td>
<td>Dim. 1/k</td>
<td>RND</td>
<td>16</td>
<td>0.5</td>
<td>229.4 (856.1)</td>
</tr>
<tr>
<td></td>
<td>Dim. 1/k</td>
<td>LIFO</td>
<td>16</td>
<td>0.5</td>
<td>200.6 (739.7)</td>
</tr>
<tr>
<td></td>
<td>Dim. 1/k</td>
<td>IS</td>
<td>16</td>
<td>0.5</td>
<td>178.0 (638.0)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>FIFO</td>
<td>16</td>
<td>0.5</td>
<td>148.2 (469.3)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>RND</td>
<td>16</td>
<td>0.5</td>
<td>117.8 (392.6)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>LIFO</td>
<td>16</td>
<td>0.5</td>
<td>131.2 (446.2)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>IS</td>
<td>16</td>
<td>0.5</td>
<td>118.4 (441.4)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>FIFO</td>
<td>160</td>
<td>0.5</td>
<td>267.6 (325.1)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>RND</td>
<td>160</td>
<td>0.5</td>
<td>113.2 (345.6)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>LIFO</td>
<td>160</td>
<td>0.5</td>
<td>99.4 (268.3)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>IS</td>
<td>160</td>
<td>0.5</td>
<td>95.5 (289.8)</td>
</tr>
<tr>
<td>100</td>
<td>Polyak</td>
<td>IS</td>
<td>160</td>
<td>0.5</td>
<td>517.9 (1126.1)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>FIFO</td>
<td>256</td>
<td>0.75</td>
<td>723.9 (2155.2)</td>
</tr>
<tr>
<td></td>
<td>Polyak</td>
<td>LIFO</td>
<td>256</td>
<td>0.75</td>
<td>411.7 (1354.5)</td>
</tr>
<tr>
<td>1000</td>
<td>Polyak</td>
<td>IS</td>
<td>256</td>
<td>0.75</td>
<td>2535.0 (6427.0)</td>
</tr>
</tbody>
</table>

Experiments ran at Cab cluster, hosted at Lawrence Livermore National Laboratory.
Central Western European system

656 thermal generators, 679 nodes, 1073 lines, quarterly resolution, 24 hour horizon, multi-area renewable production
CWE: Solution times

Solution time statistics for WECC instances, over 8 representative day types. All instances use the Polyak stepsize to perform dual updates, the IS primal recovery heuristic and a *DualShare* of 0.75.

<table>
<thead>
<tr>
<th>$N$</th>
<th># Cores</th>
<th>Solution time [s], avg. (max.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>2% optimality</td>
</tr>
<tr>
<td>30</td>
<td>96</td>
<td>2 580.3 (5 908.2)</td>
</tr>
<tr>
<td>60</td>
<td>192</td>
<td>2 563.7 (5 593.3)</td>
</tr>
<tr>
<td>120</td>
<td>384</td>
<td>2 696.5 (5 973.0)</td>
</tr>
</tbody>
</table>

Experiments ran at Cab cluster, hosted at Lawrence Livermore National Laboratory.
### SIPLIB: Solution times and gaps

<table>
<thead>
<tr>
<th>Instance</th>
<th>$N$</th>
<th>Dual Share</th>
<th>LB</th>
<th>UB</th>
<th>Solution time [s]</th>
<th>Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>dcap243_200</td>
<td>200</td>
<td>0.5</td>
<td>2311.2</td>
<td>2334.5</td>
<td>9.20</td>
<td>1.00%</td>
</tr>
<tr>
<td>dcap243_300</td>
<td>300</td>
<td>0.5</td>
<td>2551.4</td>
<td>2574.8</td>
<td>15.79</td>
<td>0.91%</td>
</tr>
<tr>
<td><strong>dcap243_500</strong></td>
<td><strong>500</strong></td>
<td><strong>0.5</strong></td>
<td><strong>2161.6</strong></td>
<td><strong>2180.3</strong></td>
<td><strong>43.76</strong></td>
<td><strong>0.86%</strong></td>
</tr>
<tr>
<td>dcap342_200</td>
<td>200</td>
<td>0.5</td>
<td>1605.9</td>
<td>1621.9</td>
<td>8.76</td>
<td>0.99%</td>
</tr>
<tr>
<td>dcap342_300</td>
<td>300</td>
<td>0.5</td>
<td>2063.3</td>
<td>2084.2</td>
<td>29.33</td>
<td>1.00%</td>
</tr>
<tr>
<td><strong>dcap342_500</strong></td>
<td><strong>500</strong></td>
<td><strong>0.5</strong></td>
<td><strong>1899.5</strong></td>
<td><strong>1916.3</strong></td>
<td><strong>51.88</strong></td>
<td><strong>0.88%</strong></td>
</tr>
<tr>
<td>sslp.5.25.50</td>
<td>50</td>
<td>0.5</td>
<td>-122.8</td>
<td>-121.6</td>
<td>4.22</td>
<td>1.00%</td>
</tr>
<tr>
<td>sslp.5.25.100</td>
<td>100</td>
<td>0.5</td>
<td>-128.7</td>
<td>-127.4</td>
<td>6.01</td>
<td>1.00%</td>
</tr>
<tr>
<td>sslp.10.50.50</td>
<td>50</td>
<td>0.5</td>
<td>-368.3</td>
<td>-364.6</td>
<td>11.50</td>
<td>0.99%</td>
</tr>
<tr>
<td>sslp.10.50.100</td>
<td>100</td>
<td>0.5</td>
<td>-357.8</td>
<td>-354.2</td>
<td>28.66</td>
<td>1.00%</td>
</tr>
<tr>
<td>sslp.10.50.500</td>
<td>500</td>
<td>0.9</td>
<td>-352.6</td>
<td>-349.1</td>
<td>81.80</td>
<td>1.00%</td>
</tr>
<tr>
<td>sslp.10.50.1000</td>
<td>1000</td>
<td>0.9</td>
<td>-355.3</td>
<td>-351.7</td>
<td>129.09</td>
<td>1.00%</td>
</tr>
<tr>
<td><strong>sslp.10.50.2000</strong></td>
<td><strong>2000</strong></td>
<td><strong>0.9</strong></td>
<td><strong>-350.8</strong></td>
<td><strong>-347.3</strong></td>
<td><strong>226.15</strong></td>
<td><strong>1.00%</strong></td>
</tr>
</tbody>
</table>

All instances use Polyak stepsize, IS primal recovery and **96 cores**. Experiments ran at Lemaitre3 cluster, hosted at Université catholique de Louvain.
Parallel efficiency of the asynchronous algorithm

Plot drawn using WECC, spring weekdays, **100-scenario stochastic unit commitment instance**, solved using \textit{Dual Share} 0.75, Polyak stepsize and IS primal recovery.
Estimated idle time of processors when solving SUC using a parallel synchronous algorithm. Average and maximum over 8 day types.

<table>
<thead>
<tr>
<th>System</th>
<th>$N$</th>
<th># Cores</th>
<th>Dual Share</th>
<th>Synchronous idle time [%], avg. (max.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Same conditions</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td># cores</td>
</tr>
<tr>
<td>WECC</td>
<td>10</td>
<td>16</td>
<td>0.5</td>
<td>48.0 (53.2)</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>160</td>
<td>0.5</td>
<td>70.6 (80.4)</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>256</td>
<td>0.75</td>
<td>37.6 (57.0)</td>
</tr>
<tr>
<td>CWE</td>
<td>30</td>
<td>96</td>
<td>0.75</td>
<td>36.4 (47.0)</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>192</td>
<td>0.75</td>
<td>43.6 (62.2)</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>384</td>
<td>0.75</td>
<td>46.9 (61.2)</td>
</tr>
</tbody>
</table>
Conclusions

- Randomized block-coordinate descent converges without the need for line search
- Synchronous algorithms can lead to idle times of up to 80.4% of the total wall time
- The asynchronous algorithm allows us to solve industrial-scale instances within operationally acceptable time frames
- The asynchronous algorithm effectively solves generic stochastic mixed-integer programming instances
Thank you

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http://perso.uclouvain.be/anthony.papavasiliou/

Ignacio Aravena, aravenasolis1@llnl.gov
http://sites.google.com/site/iaravenasolis/
Appendix
WECC: Dual algorithm progress

Convergence of dual function, winter weekday

\[
1 - \frac{|UB|}{LB^*}
\]

Solution time $\times \left( \frac{\#\text{dual processors}}{\#\text{scenarios}} \right)$ [secs]

- 10 scenarios
- 100 scenarios
- 1000 scenarios
Variation of solution time with the *Dual Share*. Results in the table correspond to WECC, spring weekdays, 100 scenario instance, solved using 8 nodes (96 cores) and a Polyak stepsize.

<table>
<thead>
<tr>
<th>Primal Recovery</th>
<th>Dual Share</th>
<th>Solution time [s], avg. (max.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k = 0$</td>
<td>$k = 200N$</td>
</tr>
<tr>
<td>IS</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>LIFO</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>IS</td>
<td>0.75</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Incremental method

Each iteration uses gradient information of a part of the objective function (Nedić et al., 2001), (Gürbüzbalaban et al., 2015)

\[
\min_{x \in \mathbb{R}^N} \sum_i f_i(x)
\]

1. Let \( k := 0, x^k := \bar{x}^0 \)
2. Select component \( j(k) \) (cyclically, at random)
3. Compute \( g^k \in \partial f_{j(k)}(x^k) \)
4. Let \( x^{k+1} := x^k - \lambda_k g^k \)
5. Let \( k := k + 1 \) and return to 2.
Incremental method

\[ f(x, y) = 3x^2 - 2xy + 2y^2 + 2x^2 - 4xy + 3y^2 \]
Incremental method

\[ f(x, y) = 3x^2 - 2xy + 2y^2 + 2x^2 - 4xy + 3y^2 \]
Block-coordinate descent method

\[ f(x, y) = 5x^2 - 6xy + 5y^2 \]
Asynchronous BCD vs asynchronous incremental method

- The asynchronous incremental method (Nedić, 2001) could be readily applied to minimize $f$. We lean towards coordinate descent because
  - As $x$ approaches an optimal value, the gradient of $f_0^\mu$ will tend to point in the opposite direction to the subgradient of $f_i$. This makes the incremental method susceptible to oscillations.
  - Every time the asynchronous incremental method updates $x$ using the gradient of $f_0^\mu$, it will introduce an additional delay on all subgradients currently being computed. Each block-coordinate update causes an additional delay only on block $j(k)$, if $g(j(k), x_{j(k)}^{k-l'(j(k),k)})$ is being computed.
The Master decides which task to execute next, based on the following directives:

1. Use at most $N$ concurrent processes for dual scenario tasks.

2. Maintain the proportion of Slaves engaged in dual tasks (dual scenario or dual $f_0$) as close as possible to the value of the configuration parameter DualShare.

3. A dual $f_0$ task must be executed every certain number of dual scenario tasks.

4. (Opt.) If using the IS primal recovery heuristic, a new candidate must be generated and a primal projection task must be executed whenever the number of (candidate, scenario) pairs pending for second stage evaluation is deemed small.
Initialization stage

- No (good) subgradient estimates available for free at $x^0$
- _dual scenario_ tasks can take up to 12 hours to be completed for our realistic test case
  - Primal recovery can only start after the first _dual scenario_ task has concluded
  - The first full upper bound can be computed only after concluding one _dual scenario_ for every scenario
- Initialization: solve a _period-relaxation_ of the scenario subproblem for each scenario with $x^0$
- The initialization provides initial subgradient estimates, an upper bound and $N$ primal candidates
Execution profile

- 10 scenarios, 11 processors: 1 master + 10 slaves
- 65% DualShare, dual $f_0$ executed every 2 updates, LIFO primal recovery

Start

---------------------
Execution profile

- 10 scenarios, 11 processors: 1 master + 10 slaves
- 65% *DualShare*, *dual* $f_0$ executed every 2 updates, LIFO primal recovery

Start

$$\bar{f}_1(x_1^0), \bar{f}_2(x_2^0), \bar{f}_3(x_3^0), \bar{f}_4(x_4^0), \bar{f}_5(x_5^0), \bar{f}_6(x_6^0), \bar{f}_7(x_7^0), \bar{f}_8(x_8^0), \bar{f}_9(x_9^0), \bar{f}_{10}(x_{10}^0)$$
Execution profile

- 10 scenarios, 11 processors: 1 master + 10 slaves
- 65% DualShare, dual $f_0$ executed every 2 updates, LIFO primal recovery
Execution profile

- 10 scenarios, 11 processors: 1 master + 10 slaves
- 65% *DualShare*, *dual* $f_0$ executed every 2 updates, LIFO primal recovery

\[
\begin{align*}
\bar{f}_1(x_1^0) & \quad \bar{f}_2(x_2^0) & \quad \bar{f}_3(x_3^0) & \quad \bar{f}_4(x_4^0) & \quad \bar{f}_5(x_5^0) & \quad \bar{f}_6(x_6^0) & \quad \bar{f}_7(x_7^0) & \quad \bar{f}_8(x_8^0) & \quad \bar{f}_9(x_9^0) & \quad \bar{f}_{10}(x_{10}^0) \\
\bar{f}_6(x_6^1) & \quad \bar{f}_3(x_3^2) & \quad \bar{f}_7(x_7^3) & \quad \bar{f}_8(x_8^4) & \quad \bar{f}_4(x_4^5) & \quad \bar{f}_1(x_1^6) & \quad \bar{f}_9(x_9^7) & \quad \bar{f}_0(x_0^8) & \quad h_1(u^1) & \quad h_2(u^1) & \quad h_3(u^4) & \quad h_4(u^4)
\end{align*}
\]
Execution profile

- 10 scenarios, 11 processors: 1 master + 10 slaves
- 65% *DualShare*, dual $f_0$ executed every 2 updates, LIFO primal recovery
Execution profile

- 10 scenarios, 11 processors: 1 master + 10 slaves
- 65% *DualShare*, *dual* $f_0$ executed every 2 updates, LIFO primal recovery

```
Start

\[
\begin{align*}
\bar{f}_1(x_1^0) & \quad \bar{f}_2(x_2^0) & \quad \bar{f}_3(x_3^0) & \quad \bar{f}_4(x_4^0) & \quad \bar{f}_5(x_5^0) & \quad \bar{f}_6(x_6^0) & \quad \bar{f}_7(x_7^0) & \quad \bar{f}_8(x_8^0) & \quad \bar{f}_9(x_9^0) & \quad \bar{f}_{10}(x_{10}^0) \\
\hline
\hline
f_1(x_1^1) & \quad f_2(x_2^1) & \quad f_3(x_3^1) & \quad f_4(x_4^1) & \quad f_5(x_5^1) & \quad f_6(x_6^1) & \quad f_7(x_7^1) & \quad f_8(x_8^1) & \quad f_9(x_9^1) & \quad f_{10}(x_{10}^1) \\
\hline
h_1(u^1) & \quad h_2(u^1) & \quad h_3(u^1) & \quad h_4(u^1) & \quad h_5(u^1) & \quad h_6(u^1) & \quad h_7(u^1) & \quad h_8(u^1) & \quad h_9(u^1) & \quad h_{10}(u_{10}^1)
\end{align*}
\]
```
10 scenarios, 11 processors: 1 master + 10 slaves

65% *DualShare*, *dual* $f_0$ executed every 2 updates, LIFO primal recovery

![Diagram of execution profile]
- 10 scenarios, 11 processors: 1 master + 10 slaves

- 65% *DualShare*, *dual* $f_0$ executed every 2 updates, LIFO primal recovery
Execution profile

- 10 scenarios, 11 processors: 1 master + 10 slaves
- 65% DualShare, dual $f_0$ executed every 2 updates, LIFO primal recovery
10 scenarios, 11 processors: 1 master + 10 slaves

65% DualShare, dual $f_0$ executed every 2 updates, LIFO primal recovery
10 scenarios, 11 processors: 1 master + 10 slaves

65% DualShare, dual $f_0$ executed every 2 updates, LIFO primal recovery
 Execution profile

- 10 scenarios, 11 processors: 1 master + 10 slaves
- 65% DualShare, dual $f_0$ executed every 2 updates, LIFO primal recovery

```
Start

f_1(x_1^0) f_2(x_2^0) f_3(x_3^0) f_4(x_4^0) f_5(x_5^0) f_6(x_6^0) f_7(x_7^0) f_8(x_8^0) f_9(x_9^0) f_10(x_10^0)

f_6(x_6^1) f_3(x_3^2) f_7(x_7^3) f_8(x_8^4) f_4(x_4^5) f_5(x_5^6) f_1(x_1^7) f_9(x_9^8) h_1(u^1) h_2(u^1) h_3(u^1)

h_6(u^1) h_7(u^1) f_6^\mu(x^7) f_9(x_9^8) f_3(x_3^9) h_5(u^1) f_5^\mu(x^8)
```

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