Two-Stage Stochastic Linear Programs

1. **Short Reviews**
   - Probability Spaces and Random Variables
   - Convex Analysis

2. **Deterministic Linear Programs**

3. **Two-Stage Stochastic Linear Programs**
Outline

1 Short Reviews
   - Probability Spaces and Random Variables
   - Convex Analysis

2 Deterministic Linear Programs

3 Two-Stage Stochastic Linear Programs
A **probability space** is the triplet \((\Omega, \mathcal{A}, P)\), where

- **\(\Omega\)**: set of all random outcomes (example: six possible results of throwing die)
- **\(\mathcal{A}\)**: collection of random events, where events are subsets of \(\Omega\) (example: odd numbers, \(\leq 4\))
- Mapping \(P : A \rightarrow [0, 1]\) with \(P(\emptyset) = 0\), \(P(\Omega) = 1\) and \(P(A_1 \cup A_2) = P(A_1) + P(A_2)\) if \(A_1 \cap A_2 = \emptyset\)
Distributions

- **Cumulative distribution function** of a random variable $\xi$ is defined as $F_\xi(x) = P(\xi \leq x)$
- For discrete random variables, **probability mass distribution** $f$ is defined as $f(\xi^k) = P(\xi = \xi^k)$, $k \in K$ with $\sum_{k \in K} f(\xi^k) = 1$
- For continuous random variables, **density function** $f$ is defined by $P(a \leq \xi \leq b) = \int_a^b f(\xi) d\xi = \int_a^b dF(\xi)$ with $\int_{-\infty}^{\infty} dF(\xi) = 1$
The **expectation** of a random variable is \( \mu = \sum_{k \in K} \xi^k f(\xi^k) \) (discrete) or \( \int_{-\infty}^{\infty} \xi dF(\xi) \) (continuous)

The **variance** of a random variable is \( \mathbb{E}[(\xi - \mu)^2] \)

The **rth moment** of \( \xi \) is \( \bar{\xi}(r) = \mathbb{E}[\xi^r] \)

The **\( \alpha \)-quantile** of \( \xi \) is a point \( \eta \) such that for \( 0 < \alpha < 1 \), \( \eta = \min\{x | F(x) \geq \alpha\} \)
A **convex combination** of points $x_i, i = 1, \ldots, n$, is a point 
$\sum_{i=1}^{n} \lambda_i x_i$ where $\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, n$

$X$ is a **convex set** if it contains any convex combination of points $x_i \in X$

**Extreme point** of a convex set: a point which cannot be expressed as the convex combination of two distinct points in the set

**Convex hull** of a set of points: the set of all convex combinations of these points
Affine Sets and Hyperplanes

- \( h(x) = 0 \) is an **affine constraint** if it can be expressed as \( h(x) = Ax - b \)

- **Affine space**: the set \( h(x) = 0 \)

- Each set \( h_i(x) = A_i \cdot x - b = 0 \) is a **hyperplane** (where \( A_i \) is the \( i \)-th row of \( A \))

- \( Ax = 0 \) is the **parallel subspace**

- The **dimension** of an affine space is the dimension of its parallel subspace
Convex Functions

- $f$ is a **convex function** if for all $0 \leq \lambda \leq 1$ and any $x_1, x_2$ we have

  \[ f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2) \]

- The **domain** of $f$, $\text{dom} \ f$ is the set where $f$ is finite

- $f$ is **concave** if $-f$ is convex
The **epigraph** of $f$ is the set $\text{epi}(f) = \{(x, \beta) | \beta \geq f(x)\}$

- $f$ is convex if and only if its epigraph is convex
A continuous function $f$ is **piecewise linear** if it is linear over regions defined by linear inequalities.

A function is **separable** when it can be written as $f = \sum_{i=1}^{n} f_i(x)$.

Both of these properties can be exploited in computation.
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Deterministic Linear Programs

\[
\begin{align*}
\text{min } z &= c^T x \\
\text{s.t. } Ax &= b \\
x &\geq 0
\end{align*}
\]

- \( x \in \mathbb{R}^n, \ c \in \mathbb{R}^n, \ A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m \)
- A solution is a vector \( x \) such that \( Ax = b \)
- A feasible solution is a solution with \( x \geq 0 \)
- An optimal solution is a feasible solution \( x^* \) such that \( c^T x^* \leq c^T x \) for all feasible solutions \( x \)
A **basis** is a choice of $n$ linearly independent columns of $A$, with $A = [B, N]$ (after rearrangement)

Associated with a basis, we have a **basic solution** $x_B = B^{-1}b$, $x_N = 0$, $z = c_B^T B^{-1}b$

Basic feasible solutions correspond to extreme points of
\[
\{x | Ax = b, x \geq 0\}
\]

Condition for basis to be

- feasible: $B^{-1}b \geq 0$
- optimal: $c_N^T - c_B^T B^{-1}N \geq 0$ (can you prove this?)
Dual Problem

\[
\begin{align*}
\text{max } & \pi^T b \\
\text{s.t. } & \pi^T A \leq c^T
\end{align*}
\]

- Unbounded dual \(\Rightarrow\) infeasible primal
- Unbounded primal \(\Rightarrow\) infeasible dual
- Primal and dual can be infeasible
- If \(x\) is primal feasible and \(\pi\) is dual feasible, \(c^T x \leq \pi^T b\)
- There exists primal optimal solution \(x^* \Leftrightarrow\) there exists dual optimal solution \(\pi^*\)
- Strong duality: \(c^T x^* = (\pi^*)^T b\)
- Complementary slackness: \((\pi^*)^T A - c^T \perp x^*\)
### Linear Programming Duality Mnemonic Table

<table>
<thead>
<tr>
<th>Primal</th>
<th>Minimize</th>
<th>Maximize</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constraints</td>
<td>≥ $b_i$</td>
<td>≥ 0</td>
<td>Variables</td>
</tr>
<tr>
<td></td>
<td>≤ $b_i$</td>
<td>≤ 0</td>
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<tr>
<td></td>
<td>= $b_i$</td>
<td></td>
<td>Free</td>
</tr>
<tr>
<td>Variables</td>
<td>≥ 0</td>
<td>≤ $c_j$</td>
<td>Constraints</td>
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2-Stage Stochastic Linear Programs

- **Stochastic linear programs**: linear programs with uncertain data
- **Recourse programs**: some decisions (recourse actions) can be taken after uncertainty is disclosed
- **First-stage decisions**: decisions taken before uncertainty is revealed
- **Second-stage decisions**: decisions taken after uncertainty is revealed
- Sequence of events: $x \rightarrow \xi(\omega) \rightarrow y(\omega, x)$
- First and second stage may contain sequences of decisions
Sequence of Events

First stage

$\omega$

Uncertainty

$y(x, \omega)$

Second stage
min \ z = c^T x + \mathbb{E}_\xi [\min q(\omega)^T y(\omega)]

s.t. \ Ax = b
\ T(\omega)x + W y(\omega) = h(\omega)
\ x \geq 0, \ y(\omega) \geq 0

- First stage decisions \ x \in \mathbb{R}^{m_1}, \ c \in \mathbb{R}^{m_1}, \ b \in \mathbb{R}^{m_1}, \ A \in \mathbb{R}^{m_1 \times n_1}
- For a given realization \ \omega, \ second-stage \ data \ are \ q(\omega) \in \mathbb{R}^{m_2}, \ h(\omega) \in \mathbb{R}^{m_2}, \ T(\omega) \in \mathbb{R}^{m_2 \times n_1}
- All random variables of the problem are assembled in a single random vector \ \xi^T(\omega) = (q(\omega)^T, h(\omega)^T, T_1.(\omega), \ldots, T_{m_2}.(\omega))
Example: Capacity Expansion Planning

\[
\min_{x, y \geq 0} \sum_{i=1}^{n} (l_i \cdot x_i + \mathbb{E}_\xi \sum_{j=1}^{m} C_i \cdot T_j \cdot y_{ij}(\omega))
\]

s.t. \[\sum_{i=1}^{n} y_{ij}(\omega) = D_j(\omega), j = 1, \ldots, m\]

\[\sum_{j=1}^{m} y_{ij}(\omega) \leq x_i, i = 1, \ldots n\]

- \(l_i, C_i\): fixed/variable cost of technology \(i\)
- \(D_j(\omega), T_j\): height/width of load block \(j\)
- \(y_{ij}(\omega)\): capacity of \(i\) allocated to \(j\)
- \(x_i\): capacity of \(i\)

Why is \(D_j\) not dependent on \(\omega\)?
Example: Procrastination

\[
\max_{x, y \geq 0} c_1 x + \mathbb{E}_\xi c_2 y(\omega)
\]

s.t. \( x + y(\omega) \leq T \)

\( x + y(\omega) \geq T/2 \)

\( y(\omega) \leq N(\omega) \)

- \( x, y(\omega) \): amount of work before/after \( T \)
- \( c_1, c_2 \): quality increase per day for work prior to/after deadline
- \( T \): deadline
- \( N(\omega) \): deadline extension
Let $Q(x, \xi(\omega)) = \min_y \{ q(\omega)^T y \mid W y = h(\omega) - T(\omega)x, y \geq 0 \}$ be the second stage value function.

Let $V(x) = \mathbb{E}_\xi Q(x, \xi(\omega))$ be the expected second-stage value function.

The deterministic equivalent program is

$$\min z = c^T x + V(x)$$
$$\text{s. t. } A x = b$$
$$x \geq 0$$

If $V(x)$ is given, the problem is just a linear program (we will see why soon).
Convexity of $Q(x, \xi(\omega))$

$Q(x, \xi(\omega))$ is a convex function of $x$

Proof: If $y_1$ is optimal reaction to $x_1$ and $y_2$ is optimal reaction to $x_2$, then $\lambda y_1 + (1 - \lambda y_2)$ is feasible reaction to $\lambda x_1 + (1 - \lambda x_2)$

What can we say about $V(x)$?
Making Constraints Implicit

\[ Q(x, \xi) \] can be represented as
\[ Q(x, \xi) = \min_y \{ q(\omega)^T y + \delta(y | Y(x, \xi)) \}, \]
where
\[ Y(x, \xi) = \{ y \geq 0 : T(\omega)x + Wy = h(\omega) \} \]
and \( \delta(\cdot | \cdot) \) is an **indicator function**: 
\[
\delta(y | Y) = \begin{cases} 
0 & \text{if } y \in Y \\
+\infty & \text{otherwise}
\end{cases}
\]
Some Notations and Terminology

- \( K_1 = \{ x : Ax = b, x \geq 0 \} \)
- \( K_2(\xi) = \{ x : \exists y, T(\omega)x + Wy(\omega) = h(\omega), y(\omega) \geq 0 \} \)
- \( K_2 = \{ x : V(x) < \infty \} \)
- **Relatively complete recourse:** \( K_1 \subset K_2 \)
- **Complete recourse:** pos\( W = \mathbb{R}^{m_2} \), where pos\( W = \{ Wy : y \geq 0 \} \)
- **Fixed recourse:** \( W \) does not depend on \( \omega \)

Does example 1 have complete recourse? fixed recourse?

Does example 2 have complete recourse? fixed recourse?
§2.1 BL: probability spaces and random variables
§2.2 BL: deterministic linear programs
§2.3 BL: decisions and stages
§2.4 BL: 2-stage stochastic program with fixed recourse
§2.6 BL: implicit representation of the second stage
§2.11 BL: short reviews