Path-Based Conditions for Local Network Identifiability

Antoine Legat and Julien M. Hendrickx

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UCLouvain

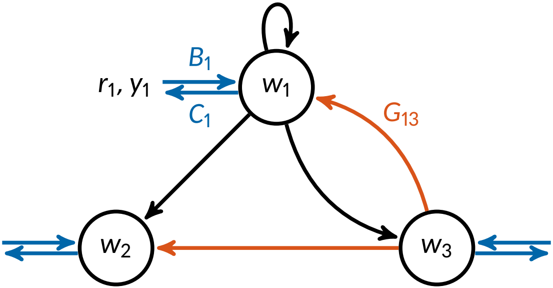
Model

state ⟵ $w = G\,w + B\,r$ ⟶ excitation

measure ⟵ $y = C\,w$

From given exc/meas, can we recover the unknown transfer fcts?

i.e. From r at B and y at C, can we recover the unknown G_{ij}?

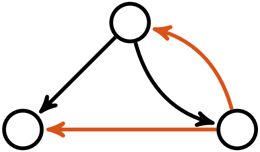


Assumptions: Network topology is known

Not all nodes are excited/measured

Global transfer matrix $C\,\underbrace{(I-G)^{-1}}_{T(G)}\,B$ is known

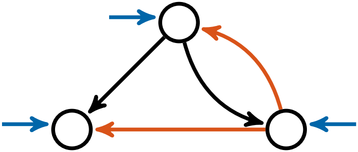
Identifiability



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B$ admit a unique solution G ?

Identifiability



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Does $\underbrace{C\,T\,\not{B}}_{\text{known}} = C\,(I - G)^{-1}\,\not{B}$ admit a unique solution G ?

All nodes excitated: i.e. $B = I$

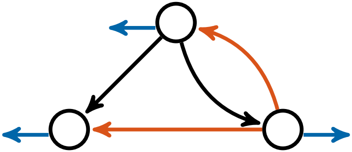
Necessary and sufficient path-based condition

[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

[Cheng, Shi, Van den Hof 2019]

Identifiability



Network *identifiable* if the unknown transfer fcts can be recovered.

> Does $\underbrace{\not{C}\, T\, B}_{\text{known}} = \not{C}\,(I - G)^{-1} B$ admit a unique solution G ?

All nodes excited: i.e. $B = I$

Necessary and sufficient path-based condition

[Hendrickx, Gevers, Bazanella 2017]

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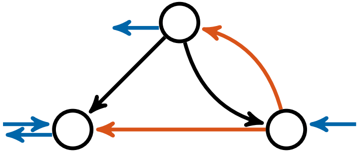
All nodes measured: i.e. $C = I$

Dual results

Identifiability



Network *identifiable* if the unknown transfer fcts can be recovered.

$$\text{Does } \underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B \text{ admit a unique solution } G\text{ ?}$$

All nodes excitated: i.e. $B = I$	Necessary and sufficient path-based condition [Hendrickx, Gevers, Bazanella 2017]
	Algo allocating measurements in the graph [Cheng, Shi, Van den Hof 2019]
All nodes measured: i.e. $C = I$	Dual results
General case:	Need to linearize

From the definition of identifiability...

Definition: Network identifiable at G if for all $\tilde{G}$:

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$

Network *generically* identifiable if it holds at *almost*[1] *all* G.

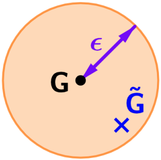
[1]except possibly on a lower-dimensional set

… we introduce *local* identifiability

Definition: Network *locally* identifiable at G if for all $\tilde{G}$ *on an ϵ—ball:*

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$

Network *generically* *locally* identifiable if it holds at *almost all* G.



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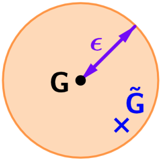
- Necessary for generic identifiability
- No counter-example to sufficiency known



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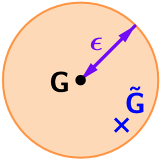
Network *generically locally* identifiable if it holds at *almost all* G.

- Necessary for generic identifiability
- No counter-example to sufficiency known



Theorem 1 [Legat, Hendrickx CDC2020]

G generically locally identif $\iff$ $\boxed{CT\Delta TB = 0 \Rightarrow \Delta = 0}$ $\forall\, \Delta$

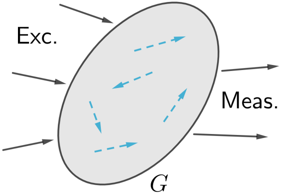
almost everywhere

Contribution

1. Definition of decoupled identifiability (necessary)
   Allows a novel approach based on a larger graph
2. Ingredients for our path-based condition
3. Path-based necessary condition and a sufficient one

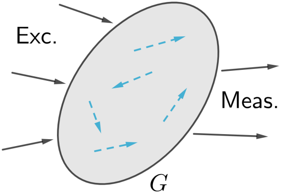
From Theorem 1 ...

Generic ...	Network G	
identifiable	$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$	
	$\Downarrow$	
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	

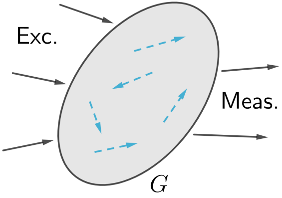
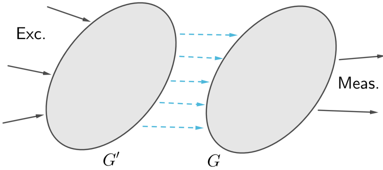


… we introduce decoupled identifiability

Generic …	Network G	
identifiable	$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$	
	$\Downarrow$	
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	
	$\Downarrow$	
decoupled-identif	$CT\Delta T'B = 0 \Rightarrow \Delta = 0$	

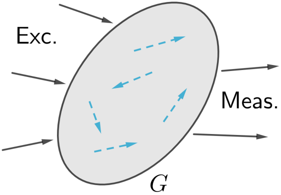


… and the decoupled network

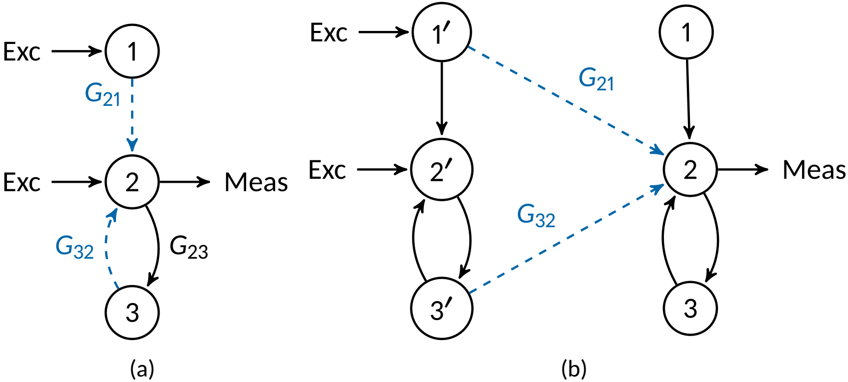
Generic …	Network G	Decoupled network
identifiable	$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$	$C T \Delta T' B = 0 \Rightarrow \Delta = 0$
	$\Downarrow$	
local identif	$C T \Delta T B = 0 \Rightarrow \Delta = 0$	
	$\Downarrow$	
decoupled-identif	$C T \Delta T' B = 0 \Rightarrow \Delta = 0$	
		

Necessary... and sufficient?

Generic ...	Network G	
identifiable	$CTB = C\tilde{T}B \Rightarrow G = \tilde{G}$	
	$\Downarrow \quad \Uparrow ?$	No counter-ex known
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	
	$\Downarrow \quad \Uparrow ?$	No counter-ex known
decoupled-identif	$CT\Delta T'B = 0 \Rightarrow \Delta = 0$	(10^6 systematic trials)



Basic example



(a): Unknowns in dashed blue

(b): Decoupled network: unknowns in the middle

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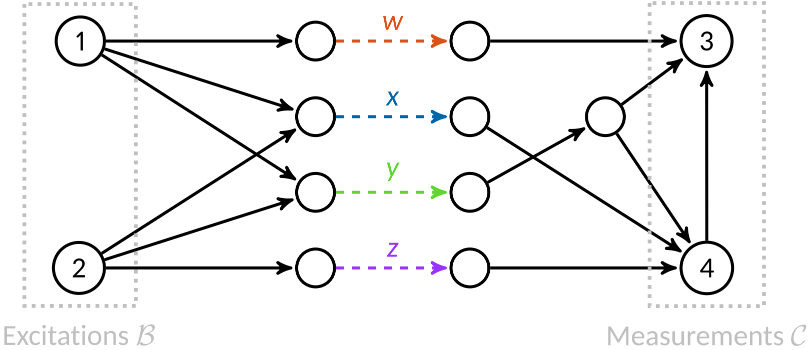
Ingredients for our path-based condition – 1

Theorem 1 can be formulated as a determinant, which can be expressed as the sum over all *assignations:*

$$\sigma : \text{unknown edges} \longrightarrow (\text{excitation, measurement})$$

Example:



Ingredients for our path-based condition – 1

Theorem 1 can be formulated as a determinant, which can be expressed as the sum over all *assignations:*

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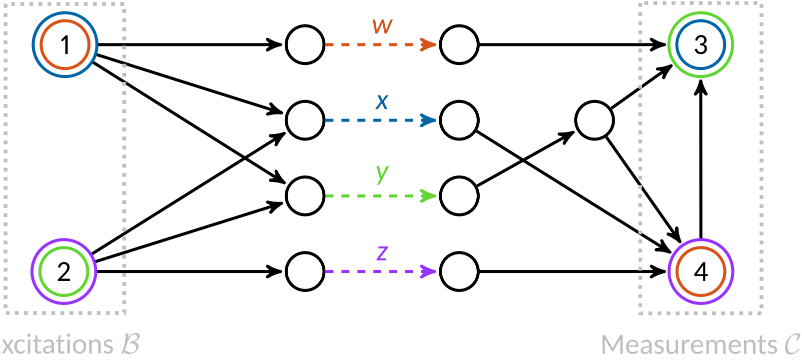
Example: $w \longrightarrow (1,4),\ x \longrightarrow (1,3),\ y \longrightarrow (2,3),\ z \longrightarrow (2,4)$

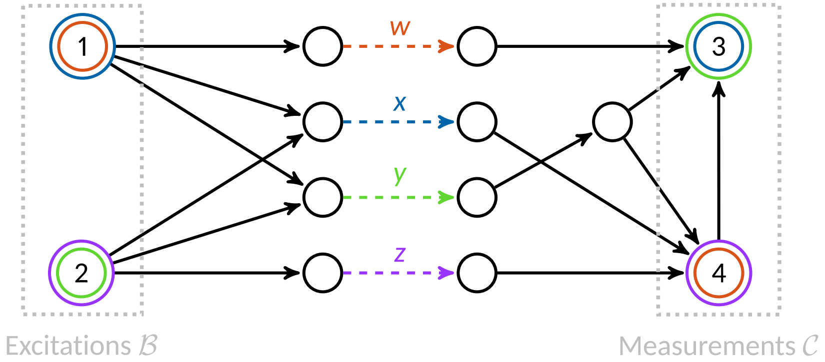


Ingredients for our path-based condition – 1

σ : unknown edges $\longrightarrow$ (excitation, measurement)

σ is *connected* if for each unknown edge, there is a path from its assigned exc. to its assigned meas., including the unknown edge.

Example: $w \to (1, 4),\ x \to (1, 3),\ y \to (2, 3),\ z \to (2, 4)$



Ingredients for our path-based condition – 1

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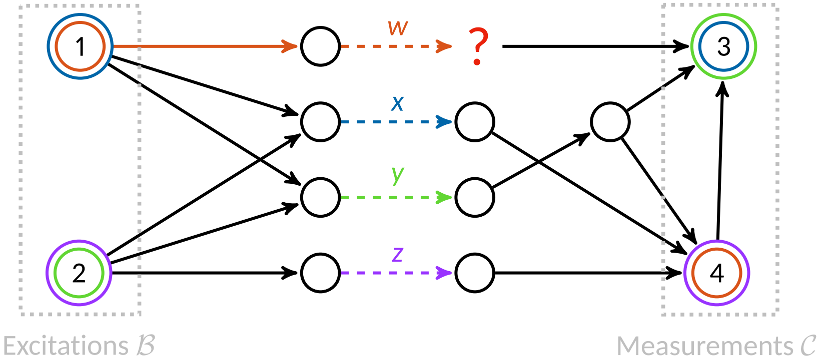
Example: ~~$w \to (1,4)$~~, $x \to (1,3)$, $y \to (2,3)$, $z \to (2,4)$ KO



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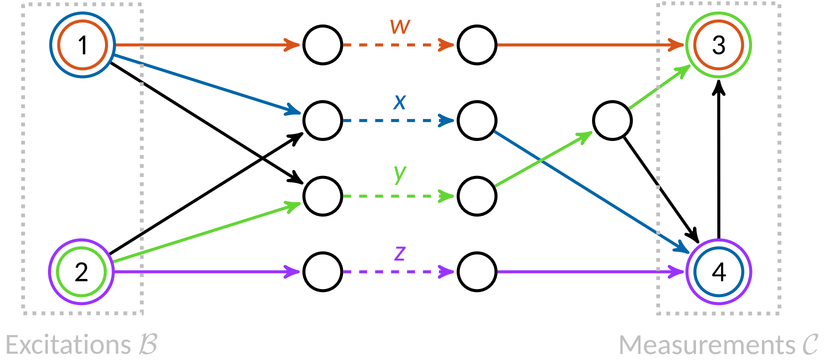
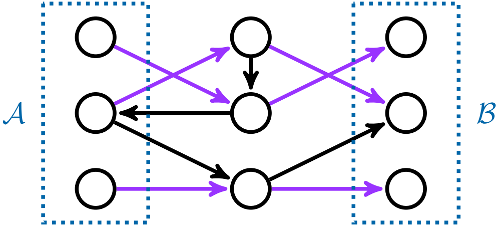
Example: $w \to (1, 3)$, $x \to (1, 4)$, $y \to (2, 3)$, $z \to (2, 4)$ OK



Ingredients for our path-based condition – 2

Theorem 1 can be formulated in terms of *generic rank* of T.

> The generic rank of a matrix T between two sets $\mathcal{A}$ and $\mathcal{B}$ equals the max number of vertex-disjoint paths from $\mathcal{A}$ to $\mathcal{B}$.



3 v-d paths

Contribution

1. Definition of decoupled identifiability (necessary)
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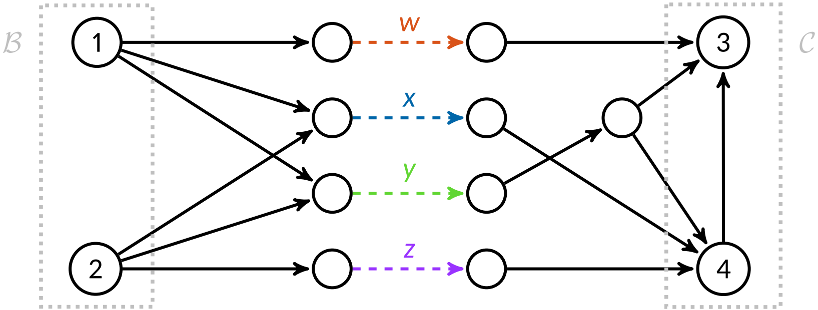
Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected assignation σ such that:

- For each excitation b,
  - (a) $|\mathcal{C}|$ unknown edges are assigned to b
  - (b) there are $|\mathcal{C}|$ vertex-disjoint paths between the edges assigned to b and the measures $\mathcal{C}$.
- For each measurement, dual of (a) and (b) with $\mathcal{B}$.



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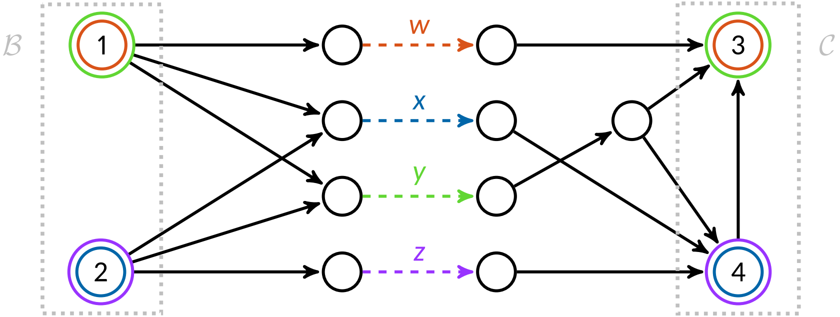

Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected ($\longrightarrow$ OK) assignation σ such that:

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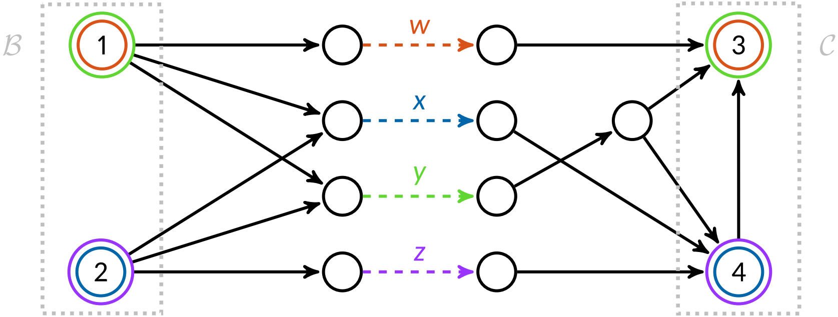

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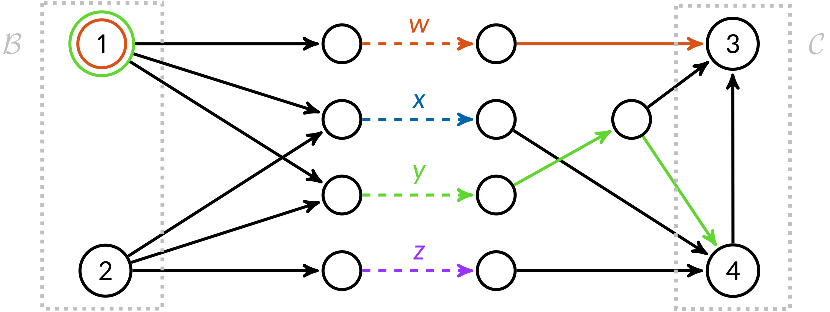

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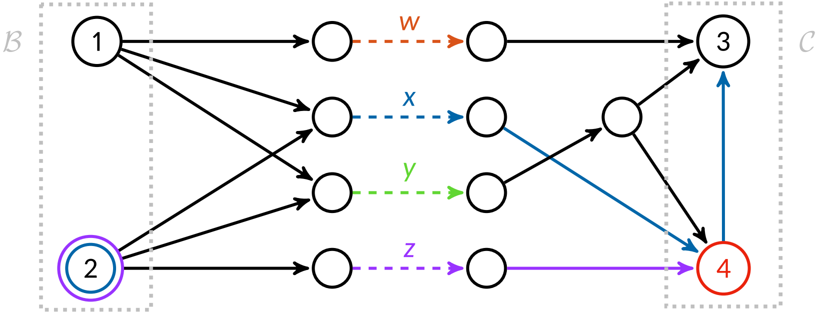

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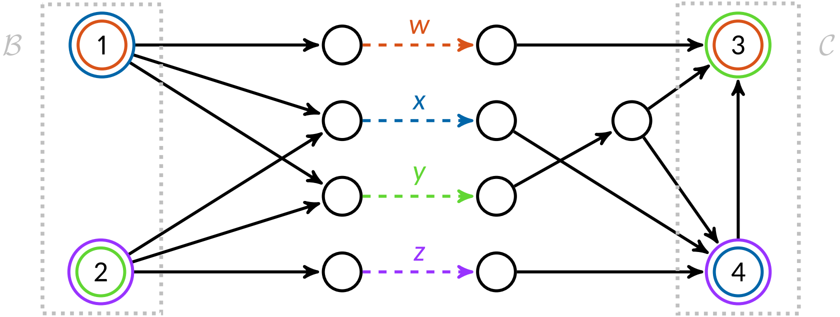

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- For each measurement, dual of (a) and (b) with $\mathcal{B}$

This condition is also *necessary* for generic *(local) identifiability* since:

Generic identif $\Rightarrow$ Generic local identif $\Rightarrow$ Generic decoupled-identif

A necessary condition and a sufficient one

Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected assignation σ such that:

- For each excitation b,
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- For each measurement, dual of (a) and (b) with $\mathcal{B}$

If there is only one such assignation, then this condition is also sufficient for generic decoupled identifiability.

Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected assignation σ such that:

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- For each measurement, dual of (a) and (b) with $\mathcal{B}$

If there is only one such assignation, then this condition is also sufficient for generic decoupled identifiability.

- σ is *not necessarily bijective:* two unknown edges can be assigned to the same (excitation, measure) pair
- The vertex-disjoint paths of condition (b) do not necessarily match the assigned measurements.

⟶ There *could be a stronger version* of Theorem 2.

Take-home message

- Introduced generic *decoupled-*identifiability,
  - Necessary for generic (local) identifiability
  - New: larger graph which decouples excitations and measures
- Derived a *path-based necessary condition* which also applies to generic (local) identifiability
- Whether the sufficient condition extends as well remains an open question
- There *could be a stronger version* of our theorem, extending previous results under full excitation/measurement
- **Further work:** when not all edges are identifiable, obtain a path-based condition for the recovery of some edges